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BALANCED SUBGROUPS OF ABELIAN GROUPS

BY

ROGER H. HUNTER([1])

ABSTRACT. The balanced subgroups of Fuchs are generalised to arbitrary abelian groups. Projectives and injectives with respect to general balanced exact sequences are classified; a new class of groups is introduced in order to classify these projectives.

1. **Introduction.** L. Fuchs [2] defined the notion of balanced subgroups of abelian p-groups and showed the significant role they play in the study of totally projective abelian p-groups. This paper extends the definition of balanced subgroups to general abelian groups and contains an investigation of their properties, generalising the work of Fuchs [2]. In §4, injectives with respect to balanced exact sequences of general abelian groups are characterised, enabling us to answer a question of C. L. Walker [8] regarding the injectives for regular exact sequences. Cotorsion groups are characterised as being injective with respect to balanced exact sequences of torsion free abelian groups. Dually, projectives with respect to general balanced exact sequences are studied in §6 and are characterised by means of a class $\mathcal{A}$ whose members are extensions of cyclic groups by totally projective torsion groups. Precisely, an abelian group A is balanced projective if and only if A is a direct summand of a direct sum of members of the class $\mathcal{A}$. It is shown that there are enough balanced projectives, and that the torsion part of a balanced projective is a summand of an S-group in the sense of R. B. Warfield [10]. The properties of members of the class $\mathcal{A}$ are investigated in some detail and groups in $\mathcal{A}$ are classified up to isomorphism using pairs $(\mathbf{M}, H^*)$, where $\mathbf{M}$ is a height matrix and H^* is totally projective. The balanced projectives include the totally projective torsion groups, completely decomposable torsion free groups, and certain mixed groups with totally projective torsion part. Thus our work sheds some light on problems 81 and 83 of L. Fuchs [2].

2. **Preliminaries.** All groups under discussion will be abelian, so the word 'group' will always designate an abelian group. For all unexplained terminology and notation see Fuchs [1] and [2]. Throughout, the symbol A will be reserved for a group and P for the set of all primes. For a given prime p, *the reduced*

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p-length of A, denoted $l_p(A)$, is the least ordinal σ for which $p^{\sigma+1}A = p^\sigma A$. We follow C. L. Walker [8] in defining the *height* of an element a in A to be $H(a) = \langle\langle \beta_2, \beta_3, \ldots, \beta_p, \ldots\rangle\rangle$ where $p \in P$ and β_p is the p-height of a in A. The height matrix $\mathrm{H}_A(a)$ of an element a in A is the $\omega \times \omega$ matrix $[\sigma_{pk}], p \in P, k = 0, 1, \ldots$, where $\sigma_{pk} = h_p^A(p^k a)$, the p-height of $p^k a$ in A. The p-indicator $U_p^A(a) = (\sigma_0, \sigma_1, \ldots)$ of a in A is just the p-row of $\mathrm{H}_A(a)$. For a discussion of the properties of the height matrix, see Fuchs [2]. We will often write just $H(a)$, $\mathrm{H}(a)$ and $U_p(a)$ when there is no danger of confusion.

A *height* $K = \langle\langle \beta_2, \beta_3, \ldots, \beta_p, \ldots\rangle\rangle$, $p \in P$, is a sequence of ordinals and symbols ∞; if $L = \langle\langle \gamma_2, \gamma_3, \ldots, \gamma_p, \ldots\rangle\rangle$ is another height then $K \geqslant L$ if $\beta_p \geqslant \gamma_p$ for all p in P. For each height K we define $A(K) = \{a \in A\colon H(a) \geqslant K\}$.

A *height matrix* $\mathrm{M} = [\sigma_{pk}]$, $p \in P$, $k = 0, 1, \ldots$, is an $\omega \times \omega$ matrix whose entries σ_{pk} are ordinals and symbols ∞ satisfying $\sigma_{p,k+1} \geqslant \sigma_{pk} + 1$ for all p and k. We observe the convention $\infty + 1 = \infty = \infty - 1$ and say that $\infty - 1$ exists. Given a height matrix $\mathrm{M} = [\sigma_{pk}]$ and prime p define $p\mathrm{M}$ to be the matrix with p-row $(\sigma_{p1}, \sigma_{p2}, \ldots)$ and all other rows the same as M. For arbitrary positive integers n and k we define $(nk)\mathrm{M} = n(k\mathrm{M})$; this, together with the above definition of $p\mathrm{M}$ for p in P, yields a 'multiplication' of height matrices by arbitrary positive integers. Two height matrices M and N are said to be *equivalent* (we write $\mathrm{M} \sim \mathrm{N}$) if there are positive integers m and n such that $m\mathrm{M} = n\mathrm{N}$.

A *p-indicator* is a sequence $\mathbf{u}_p = (\sigma_0, \sigma_1, \ldots)$ of ordinals and symbols ∞ such that $\sigma_i + 1 \leqslant \sigma_{i+1}$ for $i = 0, 1, \ldots$. Multiplication of p-indicators by powers of a prime p and equivalence of p-indicators are defined by modifying the previous definitions for height matrices in the obvious way. When dealing with p-indicators for some fixed prime p, the reference to p may be dropped. An indicator $\mathbf{u} = (\sigma_0, \sigma_1, \ldots)$ has a *gap* if, for some $k \geqslant 0$, we have $\sigma_k + 1 < \sigma_{k+1}$; in this case the gap is said to *follow* σ_k and *precede* σ_{k+1}. For notational convenience we say that a gap precedes σ_0–however, we do not consider this gap to be in $\mathbf{u}$. A height matrix has a gap if one of its rows has a gap. From this point on, all unexplained notation or terminology applied to indicators will be taken as obvious from the appropriate definitions for height matrices.

Let $\mathrm{M} = [\sigma_{pk}]$ and $\mathrm{N} = [\rho_{pk}]$ be height matrices. Then we write $\mathrm{M} \geqslant \mathrm{N}$ to mean $\sigma_{pk} \geqslant \rho_{pk}$ for all p in P and $k = 0, 1, \ldots$. We denote the p-row of a height matrix M by M_p (thus M_p is a p-indicator). For each group A and height matrix M, define

$$A(\mathrm{M}) = \{a \in A\colon \mathrm{H}(a) \geqslant \mathrm{M}\} = \bigcap_{p \in P} A(\mathrm{M}_p).$$

Let B be a subgroup of A. An element a in $A \backslash B$ is *p-proper* (resp. *H-proper*, H*-proper*) with respect to B if $h_p^A(a) = h_p^{A/B}(a + B)$ (resp. $H_A(a) = H_{A/B}(a + B)$,

$\mathrm{H}_A(a) = \mathrm{H}_{A/B}(a + B)$), and B is *p-nice* (resp. *H-nice*, H-*nice*) in A if every coset $a + B$ contains an element p-proper (resp. H-proper, H-proper) with respect to B.

For each ordinal σ and prime p, denote the Z/pZ dimension of $p^\sigma A[p]/p^{\sigma+1}A[p]$ by $f_\sigma^p(A)$ and the Z/pZ dimension of

$$p^\sigma A[p]/((p^{\sigma+1}A + B) \cap p^\sigma A[p])$$

by $f_\sigma^p(A, B)$. We also write $f_\infty^p(A)$ for the Z/pZ dimension of $p^\infty A[p]$ and $f_\infty^p(A, B)$ for the Z/pZ dimension of $p^\infty A[p]/(B \cap p^\infty A[p])$. The cardinals $f_\sigma^p(A)$ are known as the *Ulm invariants* of A, and the $f_\sigma^p(A, B)$ are called the *relative Ulm invariants* of A with respect to B.

For a given ordinal σ, a subgroup B of a p-group A is said to be *σ-dense* in A if $B + p^\rho A = A$ whenever $\rho < \sigma$.

If X is a subset of A then $\langle X\rangle$ denotes the subgroup of A generated by X; if A is torsion free then $\langle X\rangle_*$ denotes the unique minimal pure subgroup containing X. A class of groups will always be understood to contain, with each member A, all groups isomorphic to A. If $\mathcal{B}$ is a class of groups, denote the class of all direct sums of groups in $\mathcal{B}$ by $\mathcal{B}^\Sigma$, and the class of all direct summands of groups in $\mathcal{B}^\Sigma$ by $\overline{\mathcal{B}}$.

Nunke [7] defined the class of totally projective groups and showed that a totally projective group is the direct sum of a free group and a totally projective p-group. Since the free groups introduce only a trivial perturbation, we ignore them so that by a *totally projective group* we will always mean a torsion group whose p-components are totally projective p-groups. Note that we are including the divisible torsion groups in the class of totally projective groups.

We conclude this section with a definition and a list of well-known results which will be required in the sequel.

2.1. Definition. Let B be a subgroup of A and p a prime. Then we denote by $A(p, B)$ the subgroup of A defined by

$$A(p, B) = \{a \in A : p^k a \in B \text{ for some integer } k \geqslant 0\}.$$

The following lemma is essentially due to Rotman.

2.2. Lemma. *Let A have torsion free rank* 1 *and suppose the element a in A has infinite order. Then $\langle a\rangle$ is p-nice in $A(p, \langle a\rangle)$ for all p in P.*

2.3. Lemma (Wallace [9]). *If A has torsion free rank* 1 *and A_p is totally projective then $(A/\langle a\rangle)_p$ is totally projective for every a in A.*

2.4. Theorem (Hill, E. A. Walker). *Let A and C be groups and let ϕ be an isomorphism between a p-nice subgroup G of A and a subgroup H of C which does not decrease heights. Suppose that*

(i) *A/G is a totally projective p-group, and*

(ii) *if σ is an ordinal or ∞ then $f^p_\sigma(A, G) \leqslant f^p_\sigma(C, H)$.*

Then ϕ extends to a monomorphism ϕ^* of A into C. If equality holds in (ii) *for every σ then ϕ^* can be chosen as an isomorphism of A onto C.*

In particular, two totally projective groups are isomorphic if and only if they have the same Ulm invariants.

The proof of Corollary 81.4 in [2] applies directly to give:

2.5. COROLLARY. *Let A and C be groups and η a homomorphism of a p-nice subgroup G of A into C which does not decrease heights. If A/G is a totally projective p-group then η can be extended to a homomorphism $\eta^*: A \longrightarrow C$.*

2.6. LEMMA (MEGIBBEN). *If B is a subgroup of A such that A/B has no p-torsion, then B is p-isotype in A.*

The following lemma will be useful on several occasions. We omit the proof which is straightforward.

2.7. LEMMA. *Let A and B be groups and let H and G_i, $i \in I$, be subgroups of A such that $A = \Sigma_{i\in I} G_i$ and $G_i \cap \Sigma_{j\in J;\, j\neq i} G_j = H$ for all i in I. If $\phi_i: G_i \longrightarrow B$ are homomorphisms satisfying $\phi_i|_H = \phi_j|_H$ for all i, j in I then there exists a homomorphism $\phi^*: A \longrightarrow B$ such that $\phi^*|_{G_i} = \phi_i$ for all i in I.*

3. **Balanced subgroups.** The balanced subgroups defined by Fuchs [2] in the context of p-groups play an important role in the theory of totally projective p-groups. In this section we define balanced subgroups of arbitrary abelian groups, together with a weaker concept which we call H-balanced.

3.1. DEFINITION. An exact sequence

$$(1) \qquad 0 \longrightarrow B \longrightarrow A \xrightarrow{\alpha} C \longrightarrow 0$$

is said to be *balanced* if the induced sequence $0 \longrightarrow B(\mathrm{M}) \longrightarrow A(\mathrm{M}) \longrightarrow C(\mathrm{M}) \longrightarrow 0$ is exact for every height matrix M. When convenient, we instead apply the adjective 'balanced' to the subgroup B, the homomorphism α or the element of $\mathrm{Ext}(C, A)$ corresponding to (1). A group G is *balanced projective* if the induced map $\mathrm{Hom}(G, A) \longrightarrow \mathrm{Hom}(G, C)$ is surjective for every balanced exact sequence (1), and *balanced injective* if $\mathrm{Hom}(A, G) \longrightarrow \mathrm{Hom}(B, G)$ is surjective.

3.2. DEFINITION. An exact sequence $0 \longrightarrow B \longrightarrow A \longrightarrow C \longrightarrow 0$ is called *H-balanced* if $0 \longrightarrow B(K) \longrightarrow A(K) \longrightarrow C(K) \longrightarrow 0$ is exact for every height K.

The H-balanced projectives and H-balanced injectives are defined in the obvious manner. We remark that Exercise 6 on p. 93 of [2] effectively shows that our definition of balanced specialises to that of Fuchs in the context of p-groups. The same specialisation is obvious for H-balanced.

We begin our discussion of balanced subgroups with a result giving several alternate characterisations.

3.3. LEMMA. *The following are equivalent for a subgroup B of A (here C denotes the quotient A/B and α the natural epimorphism $A \longrightarrow C$):*

(a) *B is balanced in A;*

(b) *B is both isotype and* H-*nice in A;*

(c) *$T(B)$ is balanced in $T(A)$, and B is* H-*nice in A;*

(d) *to each c in C there is an element a in A such that $\alpha a = c$, $\mathrm{H}(a) = \mathrm{H}(c)$ and $o(a) = o(c)$; and*

(e) *the sequence $0 \longrightarrow B/B(\mathrm{M}) \longrightarrow A/A(\mathrm{M}) \longrightarrow C/C(\mathrm{M}) \longrightarrow 0$ is exact for all height matrices* M.

PROOF. (a) $\Leftrightarrow$ (b). Immediate from the definitions.

(b) $\Leftrightarrow$ (c). Proposition 80.2 of [2] shows that $T(B)$ is balanced in $T(A)$ if and only if $\alpha(p^\sigma T(A)[p]) = p^\sigma T(C)[p]$ for all p in P and all ordinals σ. Let p be fixed. Given c in $p^\sigma T(C)[p]$, choose a in $p^\sigma A$ such that $\alpha a = c$. Then $pa \in p^{\sigma+1}A \cap B = p^{\sigma+1}B$ so $pa = pb$ with b in $p^\sigma B$. Clearly $a - b \in p^\sigma T(A)[p]$ and $\alpha(a-b) = c$.

(c) $\Rightarrow$ (d). When $o(a) = \infty$ there is nothing to prove, and when $o(a)$ is finite, we simply modify Exercise 6, p. 93 of [2] to ensure that orders are preserved.

(d) $\Rightarrow$ (b). Since (d) implies that $\alpha(p^\sigma A[p^k]) = p^\sigma C[p^k]$ for all primes p and integers $k \geqslant 0$, Lemma 2.8 of [7] shows that B is isotype in A.

(a) $\Leftrightarrow$ (e). The commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B(\mathrm{M}) \arrow[r] \arrow[d] & A(\mathrm{M}) \arrow[r] \arrow[d] & C(\mathrm{M}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B/B(\mathrm{M}) \arrow[r] \arrow[d] & A/A(\mathrm{M}) \arrow[r] \arrow[d] & C/C(\mathrm{M}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact columns and exact middle row shows that the first row is exact if and only if the last one is: that is, (a) and (e) are equivalent.

REMARKS. 1. An identical result is obtained by replacing balanced with H-balanced throughout.

2. Let $0 \longrightarrow B \longrightarrow A \longrightarrow C \longrightarrow 0$ be balanced (or H-balanced); then the sequence $0 \longrightarrow D_B \longrightarrow D_A \longrightarrow D_C \longrightarrow 0$ of maximal divisible subgroups is exact and therefore splits off.

Observing that the order of each element in a reduced group is determined by its height matrix, we have the following, perhaps interesting, consequence of (d) in 3.3.

3.4. LEMMA. *Let B be a subgroup of a reduced group A. Then B is balanced if (and only if) B is* H*-nice in A.*

The next lemma collects some properties of H-nice subgroups. The proof is routine and hence is omitted.

3.5. LEMMA. *Let B and C be subgroups of A such that $C \leqslant B$. Then:*

(i) *if B is* H*-nice in A then B/C is* H*-nice in A/C; and*

(ii) *if C is* H*-nice in A and B/C is* H*-nice in A/C then B is* H*-nice in A.*

Using 3.3 and 3.5 it is easy to prove that: *both the balanced and the H-balanced exact sequences form a proper class in the sense of Mac Lane* [5].

We list some properties of balanced subgroups that are readily proved.

(A) If B is balanced in A then $f^p_\sigma(A) = f^p_\sigma(A/B) + f^p_\sigma(B)$ for all primes p and ordinals σ.

(B) Let $\{A_i: i \in I\}$ be a set of groups and let B_i be a subgroup of A_i for each i in I. Then $\bigoplus_{i\in I}B_i$ is balanced in $\bigoplus_{i\in I}A_i$ if and only if B_i is balanced in A_i for every i in I.

(C) Every subgroup of A is balanced if and only if A is elementary torsion.

(D) If B is a balanced subgroup of A then $\mathrm{Tor}(B, X)$ is a balanced subgroup of $\mathrm{Tor}(A, X)$ for every X.

(E) If $0 \longrightarrow B \longrightarrow A \longrightarrow C \longrightarrow 0$ is balanced, then so are $0 \longrightarrow T(B) \longrightarrow T(A) \longrightarrow T(C) \longrightarrow 0$ and $0 \longrightarrow B/T(B) \longrightarrow A/T(A) \longrightarrow C/T(C) \longrightarrow 0$.

4. **Balanced injectives.** In this section we show that the balanced injectives are just the pure injectives (otherwise known as the algebraically compact groups).

4.1. LEMMA. *Let W be torsion free and homogeneous of type* $(0, \ldots, 0, \ldots)$. *Then every short exact sequence* $0 \longrightarrow U \longrightarrow V \xrightarrow{\eta} W \longrightarrow 0$ *is balanced.*

PROOF. We need only show U is H-nice in V as 2.6 takes care is isotypeness. Suppose $w \in W$. Then there is an element w' in W and a positive integer n such that $nw' = w$ and $H(w') = \langle\langle 0, \ldots, 0, \ldots\rangle\rangle$. Now any element v' in V for which $\eta v' = w'$ also satisfies $\eta n v' = w$ and $H(nv') = H(w)$.

4.2. THEOREM. *A group A is injective with respect to balanced exact sequences of torsion free groups if and only if A is cotorsion.*

PROOF. Let A be injective with respect to balanced exact sequences of torsion free groups. We use a counting argument to show A cotorsion. Choose a cardinal $\mathfrak{n}$ such that $|A| \leqslant \mathfrak{n}$ and such that $\mathfrak{n}^{\aleph_0} > \mathfrak{n}$; for instance $\mathfrak{n} = |A| + 2^{|A|} + 2^{2^{|A|}} + \cdots$ is such a cardinal (see [3] for details of proof). Note that since $2^{\mathfrak{m}} = \mathfrak{m}^{\mathfrak{m}}$ for every infinite cardinal $\mathfrak{m}$, we have $\mathfrak{m} \leqslant \mathfrak{m}^{\aleph_0} \leqslant 2^{\mathfrak{m}}$. Let $X = \Pi_{\mathfrak{n}} Z$, $Y = \bigoplus_{\mathfrak{n}} Z$ and let W be the full inverse image of the divisible part of X/Y under the natural epimorphism $X \longrightarrow X/Y$; then $|W| = \mathfrak{n}^{\aleph_0}$. Now let

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0 \tag{2}$$

be a free resolution of W. Applying $\mathrm{Hom}(-A)$ to (2) gives

$$\ldots \mathrm{Hom}(V, A) \xrightarrow{\phi} \mathrm{Hom}(U, A) \longrightarrow \mathrm{Ext}(W, A) \longrightarrow \mathrm{Ext}(V, A) = 0.$$

By 4.1, (2) is balanced so our assumption on A implies ϕ is epic and $\mathrm{Ext}(W, A) = 0$. Suppose A is not cotorsion, that is, $\mathrm{Ext}(Q, A) \neq 0$. The sequence $0 \longrightarrow Y \longrightarrow W \longrightarrow \bigoplus_{\mathfrak{n}^{\aleph_0}} Q \longrightarrow 0$ induces

$$\ldots \mathrm{Hom}(Y, A) \xrightarrow{\nu} \mathrm{Ext}\left(\bigoplus_{\mathfrak{n}^{\aleph_0}} Q, A\right) \longrightarrow \mathrm{Ext}(W, A) = 0. \tag{3}$$

Here $|\mathrm{Hom}(Y, A)| \leqslant \Pi_{\mathfrak{n}} |A| \leqslant \mathfrak{n}^{\mathfrak{n}} = 2^{\mathfrak{n}}$ while

$$\left|\mathrm{Ext}\left(\bigoplus_{\mathfrak{n}^{\aleph_0}} Q, A\right)\right| = \prod_{\mathfrak{n}^{\aleph_0}} |\mathrm{Ext}(Q, A)| \geqslant 2^{\mathfrak{n}^{\aleph_0}}$$

which is contrary to the epimorphism ν in (3).

Conversely, if A is cotorsion then $\mathrm{Ext}(W, C) = 0$ for every torsion free W and so A is injective with respect to balanced exact sequences of torsion free groups.

4.3. THEOREM. *A group A is balanced injective if and only if it is pure injective.*

PROOF. Let A be balanced injective. By 4.2, A is cotorsion. Clearly the torsion part $T(A)$ is injective with respect to balanced exact sequences of torsion groups. Griffith [3] has shown that $T(A)$ must be torsion complete, from which it follows that A is pure injective. Conversely, every pure injective group is balanced injective, since balanced exact sequences are pure.

Let B be a subgroup of A. C. L. Walker [8] defines B to be *regular* in A if for every x in A/B and every rank 1 subgroup C/B of A/B containing x there is a c in the coset x such that $o(x) = o(c)$ and $H_C(c) = H_{C/B}(x)$, and then asks for a description of injectives with respect to regular exact sequences. Since regularity lies between purity and balanced, 4.3 yields an immediate answer to this question.

4.4. COROLLARY. *A group A is injective with respect to regular exact sequences if and only if A is pure injective.*

A natural question that arises is: can one imbed every group as a balanced subgroup of a balanced injective? The answer is no. This is clear from the following

4.5. PROPOSITION. *Every balanced subgroup of a balanced injective is again balanced injective.*

PROOF. Let B be a balanced subgroup of a pure injective group A. We may assume A is reduced. Now $\bigcap_{n<\omega} n!(A/B) = 0$ and Corollary 39.2 of [1] shows that B is also pure injective. As B is pure in A, we also have B a summand of A.

REMARKS. 1. 4.6 should be contrasted with 6.3 which asserts that every group is a balanced homomorphic image of a balanced projective.

2. The results of 4.1–4.5 hold for H-balanced exact sequences. Thus A is balanced injective if and only if A is H-balanced injective.

5. Two classes of groups. In this section we study two classes of groups (denoted by $\mathbb{A}$ and $\mathbb{C}$) whose members have torsion free rank at most 1. Groups in $\mathbb{A}$ are the 'building blocks' for our study of the balanced projectives, and groups in $\mathbb{C}$ are similarly fundamental to our work with H-projectives. We first examine arbitrary groups with torsion free rank 1, considering them as extensions: every group A with torsion free rank 1 containing an element a of infinite order is an extension

$$0 \longrightarrow \langle a \rangle \longrightarrow A \xrightarrow{\eta} T^* \longrightarrow 0 \tag{4}$$

of the free group $\langle a \rangle$ by the torsion group T^*. With each extension (4) we associate the pair (M, T^*) where $\mathrm{M} = \mathrm{H}(a)$. The asterisk is used throughout to indicate that a group is being considered, in some sense, as the last term of a sequence like (4).

5.1. DEFINITION. Let M be a height matrix, $\mathbf{u}$ a p-indicator and T^* a torsion group. The pair (M, T^*) (the pair $(\mathbf{u}, T^*)$) is said to be *admissible* if there is a group A of torsion free rank 1 containing an element a of infinite order such that $\mathrm{H}(a) = \mathrm{M}$ ($U_p(a) = \mathbf{u}$) and $A/\langle a \rangle \cong T^*$.

The problem of finding admissible pairs immediately localises to a single prime.

5.2. PROPOSITION. *Let* M *be a height matrix,* T^* *a torsion group. Then* (M, T^*) *is admissible if and only if the pairs* (M_p, T^*_p) *are admissible for every prime* p.

PROOF. Suppose a is an element of infinite order in a group A of torsion free rank 1 such that $\mathrm{H}(a) = \mathrm{M}$ and $A/\langle a \rangle \cong T^*$. Let $A_{(p)}$ be the complete inverse image of T^*_p in A; then $A/A_{(p)}$ is torsion with no p-component so by 2.6, $A_{(p)}$ is p-isotype in A and $U_p^{A_{(p)}}(a) = U_p^A(a) = \mathrm{M}_p$.

Conversely, suppose we are given, for each prime p, a group $A_{(p)}$ containing an element a_p of infinite order such that $U_p(a_p) = \mathrm{M}_p$ and $A_{(p)}/\langle a_p \rangle \cong T^*_p$. Since $A_{(p)}/\langle a_p \rangle$ has no q-torsion it is evident from 2.6 that $U_q(a_p) = (0, 1, 2, \ldots)$ for every $q \neq p$. Write $B = \bigoplus_p A_{(p)}$ and let C be the subgroup of B generated by $\{a_p - a_q : p, q \in P\}$. Setting $A = B/C$, the element $a = a_p + C$ ($= a_q + C$ for every other prime q) is such that $A/\langle a \rangle \cong T^*$ and $\mathrm{H}(a) = \mathrm{M}$.

Fundamental to our treatment of a group A of torsion free rank 1 as an extension (4) is the observation that η imbeds $T(A)$ as a subgroup of T^*. It is readily seen that $T^*/\eta T(A)$ is locally cyclic, so that $T(A)$ and T^* are closely related. In fact, the Ulm invariants of $T(A)$ (which are the same as those of A) and T^* can be related via H(a); we do this in the next three lemmas. The restriction of $T^* = A/\langle a\rangle$ to a p-group in the proofs of these lemmas affords no loss of generality—if $A_{(p)}$ is the group derived from A in 5.2 then $U_p^{A(p)}(a) = U_p^A(a)$ and $f_\sigma^p(A_{(p)}) = f_\sigma^p(A)$ for σ an ordinal or ∞. Thus we can use $A_{(p)}$ and M_p for each prime p separately to find the desired relationships. The proof of the first lemma is trivial.

5.3. LEMMA. *Let A have torsion free rank* 1 *and suppose $T(A)$ is reduced. Then the divisible part of $(A/\langle a\rangle)_p$ is $Z(p^\infty)$ if $U_p(a)$ contains ∞, and is* 0 *otherwise.*

5.4. LEMMA. *Let A have torsion free rank* 1 *and suppose a is an element of A having infinite order and p-indicator $(\sigma_0, \sigma_1, \ldots)$. Then for every ordinal σ we have*

$$f_\sigma^p(A/\langle a\rangle) = \begin{cases} f_\sigma^p(A, \langle a\rangle) + 1 & \textit{if } \sigma + 1 = \sigma_i \textit{ and a gap precedes } \sigma_i\textit{; and} \\ f_\sigma^p(A, \langle a\rangle) & \textit{otherwise.}\end{cases}$$

PROOF. Assume $A/\langle a\rangle$ is a p-group. Let $\overline{B}$ $(\overline{b})$ denote the image of a subset B (element b) under the natural map $A \longrightarrow A/\langle a\rangle$. Observe that $f_\sigma^p(A, \langle a\rangle) \leqslant f_\sigma^p(A/\langle a\rangle) \leqslant f_\sigma^p(A, \langle a\rangle) + 1$ for every ordinal σ, since A has torsion free rank 1 and $\langle a\rangle$ is p-nice in A (see 2.2). A little computation (using the p-niceness of $\langle a\rangle$ in A) shows that $f_\sigma^p(A, \langle a\rangle) < f_\sigma^p(A/\langle a\rangle)$ if and only if $\overline{p^\sigma A[p]} + p^{\sigma+1}\overline{A} < p^\sigma\overline{A}[p] + p^{\sigma+1}\overline{A}$.

Suppose σ is indeed such that $f_\sigma^p(A, \langle a\rangle) < f_\sigma^p(A/\langle a\rangle)$. Then p-niceness of $\langle a\rangle$ in A ensures the existence of a b in A satisfying $h_p(b) = \sigma$ and

$$b \in p^\sigma\overline{A}[p]\backslash(p^{\sigma+1}\overline{A} + \overline{p^\sigma A[p]}).$$

We have $pb = ra$ for some integer r. If $\sigma + 1 = h_p(b) + 1 < h_p(ra)$ then there exists c in $p^{\sigma+1}A$ with $pc = ra = pb$, so that $b - c \in p^\sigma A[p]$, whence $\overline{b} \in p^{\sigma+1}\overline{A} + \overline{p^\sigma A[p]}$, a contradiction. Thus $\sigma + 1 = h_p(ra) = \sigma_i$ for some i. If $\sigma = \sigma_{i-1}$ then $h_p(b) = \sigma = \sigma_{i-1} = h_p(r'a)$, where $pr' = r$ and $pb = p(r'a)$, so that $b = r'a + t$ such that $t \in p^\sigma A[p]$. Then $\overline{b} = \overline{t} \in \overline{p^\sigma A[p]}$, a contradiction. Thus $\sigma \neq \sigma_{i-1}$ and since $\sigma + 1 = \sigma_i$ it follows that a gap precedes σ_i.

Conversely, suppose $\sigma + 1 = \sigma_i$ for some i and a gap precedes σ_i. Then $h_p(p^ia) = \sigma + 1$ and $p^ia = pb$ with $h_p(b) = \sigma$. For this b it follows that $\overline{b} \notin p^{\sigma+1}\overline{A} + \overline{p^\sigma A[p]}$; if not, the p-niceness of a in A yields $b = c + x + sa$ such that s is an integer, $x \in p^\sigma A[p]$ and $c \in p^{\sigma+1}A$. This implies $h_p(sa) \geqslant \sigma$ and $h_p(psa) = \sigma + 1$ so in fact $h_p(sa) = \sigma$, contradicting $\sigma \neq \sigma_j$ for $j < i$.

5.5. LEMMA. *Let A have torsion free rank* 1 *and suppose a is an element of A having infinite order and p-indicator* $(\sigma_0, \sigma_1, \ldots)$. *Then for σ an ordinal or ∞ we have*

$$f_\sigma^p(A) = \begin{cases} f_\sigma^p(A/\langle a\rangle) + 1 & \text{if } \sigma = \sigma_n \text{ and a gap follows } \sigma_n; \\ f_\sigma^p(A/\langle a\rangle) - 1 & \text{if there is an } n \text{ such that } \sigma + 1 = \sigma_n \\ & \qquad \text{and a gap precedes } \sigma_n; \text{ and} \\ f_\sigma^p(A/\langle a\rangle) & \text{otherwise.} \end{cases}$$

PROOF. Assume $A/\langle a\rangle$ is a p-group. Wallace [9] has shown that when σ is an ordinal

$$f_\sigma^p(A, \langle a\rangle) = \begin{cases} f_\sigma^p(A) - 1 & \text{if } \sigma = \sigma_n \text{ and a gap follows } \sigma_n; \text{ and} \\ f_\sigma^p(A) & \text{otherwise.} \end{cases}$$

The case for σ an ordinal now follows from 5.4 while the case when $\sigma = \infty$ is immediate from 5.3.

For a group A with torsion free rank 1 containing an element a of infinite order, we have shown in 5.5 that the Ulm invariants of A (and therefore of $T(A)$) are uniquely determined by those of the quotient $A/\langle a\rangle$ in the presence of $\mathrm{H}(a)$ and vice versa. With this in mind, we review what is known about the connection between $\mathrm{H}(a)$ and the Ulm invariants of $T(A)$. Write $\mathrm{H}(a) = [\sigma_{pk}]$ and $T = T(A)$. Then the following two conditions must be satisfied by $\mathrm{H}(a)$ and T (see Fuchs [2, p. 200]):

(i) *if there is a gap following σ_{pk} then $f^p_{\sigma_{pk}}(T) \neq 0$; and*

(ii) *if $\sigma_{pk} \neq \infty$ for all k then $\sigma_{pk} < l_p(T) + \omega$, while if $\sigma_{pr} = \infty$ for some r then $\sigma_{pk} < l_p(T)$ whenever $\sigma_{pk} \neq \infty$.*

We seek corresponding conditions on a pair (M, T^*). First, it follows from 5.5 that we must have:

(a) *if $\sigma_{pk} - 1$ exists and a gap precedes σ_{pk} then $f^p_{\sigma_{pk}-1}(T^*) \neq 0$.*

Recall that our convention $\infty - 1 = \infty$ means that $\infty - 1$ exists. In order to translate condition (ii) into our setting, the following lemma is needed.

5.6. LEMMA. *Let A have torsion free rank* 1 *and contain an element a of infinite order. Write $A/\langle a\rangle = T^*$ and $T = T(A)$. Then $l_p(T^*) + \omega = l_p(T) + \omega$.*

PROOF. We may assume T is reduced. Suppose T^* is reduced. Then $T \cong \eta T \leqslant T^*$ implies $l_p(T) \leqslant l_p(T^*)$, while 5.3 shows $h_p(p^k a) = \sigma_k < \infty$ for $k = 0, 1, \ldots$. We consider the sequence $\sigma_0, \sigma_1, \ldots$, and distinguish two cases.

1. The sequence $(\sigma_0, \sigma_1, \ldots)$ has only a finite number of gaps. If σ_n is the largest term with a gap preceding it (recall that a gap always precedes σ_0) then let $\sigma_n = \gamma + r$ with γ zero or a limit ordinal and r finite. Condition (ii) implies that $l_p(T) \geqslant \gamma$ and 5.5 shows $f_\rho^p(T) = f_\rho^p(T^*)$ whenever $\rho \geqslant \sigma_n$. Therefore $l_p(T) + r \geqslant l_p(T^*) \geqslant l_p(T)$.

2. The sequence $(\sigma_0, \sigma_1, \ldots)$ has infinitely many gaps. If $\gamma = \sup_i \sigma_i$ then condition (i) implies $l_p(T) \geqslant \gamma$. Since $f_\rho^p(T) = f_\rho^p(T^*)$ when $\rho \geqslant \gamma$, it follows that $l_p(T) = l_p(T^*)$.

Suppose T^* is not reduced; then T^* has a (unique) summand $Z(p^\infty)$. Let $\eta\colon A \longrightarrow T^*$ be the natural homomorphism. Now

$$T^*/\eta T \cong Z(p^\infty) \cong (Z(p^\infty) + \eta T)/\eta T$$

implies $Z(p^\infty) + \eta T = T^*$. Writing $T^* = Z(p^\infty) \oplus B$ we have

$$\begin{aligned} B &\cong T^*/Z(p^\infty) = (\eta T + Z(p^\infty))/Z(p^\infty) \\ &\cong \eta T/(\eta T \cap Z(p^\infty)) = \eta T/C \end{aligned}$$

where C is cyclic because ηT is reduced. Let $\gamma = l_p(B) = l_p(T^*)$. As C is finite and therefore p-nice in ηT it follows that $l_p(\eta T) \geqslant \gamma$ and $p^\gamma \eta T \leqslant C$. Thus $l_p(T^*) \leqslant l_p(\eta T) \leqslant l_p(T^*) + k$ for some integer $k \geqslant 0$.

Condition (ii) and 5.6 together imply

(b) *if* $\sigma_{pk} \neq \infty$ *then* $\sigma_{pk} < l_p(T^*) + \omega$.

5.7. LEMMA. *Let A have torsion free rank* 1 *and contain an element a of infinite order such that* $\mathrm{H}(A) = \mathrm{M}$ *and* $A/\langle a\rangle \cong T^*$. *Write* $T = T(A)$ *and* $\mathrm{M} = [\sigma_{pk}]$. *Then conditions* (i) *and* (ii) *are satisfied exactly when* (a) *and* (b) *are satisfied.*

PROOF. We see from 5.5 that (i) and (a) are both satisfied automatically, and it remains only to show that (ii) is satisfied if and only if (b) is. In view of the fact that $l_p(T) + \omega = l_p(T^*) + \omega$, we have (ii) implies (b) and (b) implies the first part of (ii). In the remaining case when (b) is assumed and $\sigma_{pk} = \infty$ for some but not all k, one considers the greatest integer l for which $\sigma_{pl} \neq 0$ and recalls that 5.5 implies $f_{\sigma_{pl}}^p(T) \neq 0$ so $l_p(T) > \sigma_{pl}$.

In the light of 5.7, we make the following definition.

5.8. DEFINITION. Let $\mathrm{M} = [\sigma_{pk}]$ be a height matrix, T^* a torsion group. We say that M and T^* are *compatible* if they satisfy conditions (a) and (b) of 5.7.

We now restrict attention to pairs (M, H^*) where H^* is totally projective. One reason for this is that it enables us to use Ulm's theorem for totally projective groups in our calculations.

5.9. Definition. Let $\mathbb{A}$ be the class of groups A such that A is an extension of a cyclic (finite or infinite) group by a totally projective group. An admissible pair $(\mathbf{M}, H^*)$, where H^* is totally projective, is said to be *$\mathbb{A}$-admissible.* If $\mathbf{M}$ is a height matrix and H^* is a torsion group then we say that the pair $(\mathbf{M}, H^*)$ is *$\mathbb{A}$-compatible* if H^* is totally projective and $\mathbf{M}$ and H^* are compatible.

Observe that $\mathbb{A}$ includes all torsion free groups of rank 1, all totally projective groups (in fact, a torsion group is a member of $\mathbb{A}$ exactly when it is totally projective), and by 2.3, groups of torsion free rank 1 having totally projective torsion part (the latter have been classified by Wallace [9]). If $A \in \mathbb{A}$ and B is a finitely generated subgroup of A then $A/B \in \mathbb{A}$. In particular if g and h are elements of a group G such that $o(g) = o(h) = \infty$ and $G/\langle g\rangle$ is totally projective, then $G/\langle h\rangle$ is also totally projective.

The possibility of nonisomorphic groups being associated with a given $\mathbb{A}$-admissible pair is excluded by the following theorem.

5.10. Theorem. *If A and A' are two groups in $\mathbb{A}$ containing, respectively, elements a and a' of infinite order such that* $\mathrm{H}(a) = \mathrm{H}(a')$ *and* $A/\langle a\rangle \cong A'/\langle a'\rangle$ *then* $A \cong A'$.

Proof. We may assume that A (and therefore A') is reduced. It is clear from 5.5 that $f^p_\sigma(A, \langle a\rangle) = f^p_\sigma(A', \langle a'\rangle)$ for all primes p and ordinals σ, while $f^p_\infty(A, \langle a\rangle) = 0 = f^p_\infty(A', \langle a'\rangle)$ follows because A and A' are reduced. Now $f^p_\sigma(A(p, \langle a\rangle), \langle a\rangle) = f^p_\sigma(A, \langle a\rangle)$ for all p and σ, so 2.4 yields, for each prime p, an isomorphism $A(p, \langle a\rangle) \longrightarrow A'(p, \langle a'\rangle)$. The homomorphism $A \longrightarrow A'$ given by 2.7 is clearly an isomorphism.

5.11. Definition. Let $(\mathbf{M}, H^*)$ be an $\mathbb{A}$-admissible pair and A the unique (up to isomorphism) group in $\mathbb{A}$ containing an element a of infinite order such that $\mathrm{H}(a) = \mathbf{M}$ and $A/\langle a\rangle \cong H^*$. Then we say that $(\mathbf{M}, H^*)$ *represents* A.

In general there are many different $\mathbb{A}$-admissible pairs representing the same group in $\mathbb{A}$, and some way of equating such pairs would be useful. We begin by defining invariants for an arbitrary pair $(\mathbf{M}, T^*)$ in a manner suggested by 5.5.

5.12. Definition. Let $\mathbf{M} = [\sigma_{pk}]$ be a height matrix, T^* a torsion group. For each prime p, and for σ an ordinal or ∞ we define:

$$f^p_\sigma(\mathbf{M}, T^*) = \begin{cases} f^p_\sigma(T^*) + 1 & \text{if } \sigma = \sigma_{pk} \text{ and a gap follows } \sigma_{pk}; \\ f^p_\sigma(T^*) - 1 & \text{if there is an } n \text{ such that } \sigma + 1 = \sigma_{pn} \\ & \qquad\qquad \text{and a gap precedes } \sigma_{pn}; \text{ and} \\ f^p_\sigma(T^*) & \text{otherwise.} \end{cases}$$

5.13. Definition. Let $(\mathbf{M}, H^*)$ and $(\mathbf{N}, G^*)$ be two $\mathbb{A}$-compatible pairs. We say that $(\mathbf{M}, H^*)$ and $(\mathbf{N}, G^*)$ are *equivalent* (we write $(\mathbf{M}, H^*) \sim (\mathbf{N}, G^*)$) if

$\mathrm{M} \sim \mathrm{N}$ and $f_\sigma^p(\mathrm{M}, H^*) = f_\sigma^p(\mathrm{N}, G^*)$ for all σ and primes p.

It is clear that $\sim$ defined in 5.13 is an equivalence relation—our next result shows that the obvious correspondence is a bijection between equivalence classes of $\mathbb{A}$-admissible pairs and isomorphism classes in $\mathbb{A}$.

5.14. THEOREM. *Let A, A' be two groups in $\mathbb{A}$ containing, respectively, elements a, a' of infinite order such that* $\mathrm{H}(a) = \mathrm{M}$ *and* $\mathrm{H}(a') = \mathrm{N}$. *Then $A \cong A'$ if and only if* $(\mathrm{M}, A/\langle a\rangle) \sim (\mathrm{N}, A'/\langle a'\rangle)$.

PROOF. Suppose $(\mathrm{M}, A/\langle a\rangle) \sim (\mathrm{N}, A'/\langle a'\rangle)$. It is clear from 5.5 that the Ulm invariants of A and A' are the same. Replacing a and a', if necessary, by suitable multiples of themselves we may assume $\mathrm{H}(a) = \mathrm{H}(a')$. Then 5.5 and 2.4 yield $A/\langle a\rangle \cong A'/\langle a'\rangle$. By 5.10, $A \cong A'$. The converse is obvious.

We now show that $\mathbb{A}$-compatible pairs are $\mathbb{A}$-admissible in two stages—first for a restricted class of $\mathbb{A}$-compatible pairs and using this, for an arbitrary $\mathbb{A}$-compatible pair.

5.15. THEOREM. *Let* M *be a height matrix with only a finite number of gaps. Then a pair* (M, H^*) *is $\mathbb{A}$-admissible if and only if it is $\mathbb{A}$-compatible.*

PROOF. From the discussion leading up to 5.8 it is immediate that all $\mathbb{A}$-admissible pairs are $\mathbb{A}$-compatible. For the converse part, suppose (M, H^*) is $\mathbb{A}$-compatible. There is an integer $n \geqslant 1$ such that $n\mathrm{M}$ has no gaps, and a totally projective group G^* such that (M, H^*) and $(n\mathrm{M}, G^*)$ are equivalent. Now (M, H^*) is admissible if and only if $(n\mathrm{M}, G^*)$ is. In view of this we assume M contains no gaps and by 5.2 we replace M by $\mathrm{M}_p = \mathbf{u} = (\sigma, \sigma + 1, \sigma + 2, \dots)$ and H^* by H_p^*. Next we show that $l_p(H^*) = \sigma$ can also be assumed. Now $\mathbb{A}$-compatibility of $(\mathbf{u}, H^*)$ ensures that $f_{\sigma-1}^p(H^*) \neq 0$ when σ is not a limit ordinal. This fact, together with standard results on totally projective groups (see, for example, Fuchs [2]) now allows us to write $H^* = H_1^* \oplus H_2^*$ with $l_p(H_1^*) = \sigma$. Now $(\mathbf{u}, H_1^*)$ is $\mathbb{A}$-compatible and if $(\mathbf{u}, H_1^*)$ represents A' then $(\mathbf{u}, H^*)$ represents $A = A' \oplus H_2^*$. As the result is trivial when $\sigma = \infty$ we also assume that σ is an ordinal.

To complete the proof, we use a construction of Hill and Megibben [4]. The existence of a subgroup H of H^* such that H is isotype and σ-dense in H^* and $0 \neq H^*/H = Z(p^\alpha)$ where $\alpha \in \{0, 1, \dots\} \cup \{\infty\}$ is well known. Note that $\alpha = \infty$ whenever $\sigma \geqslant \omega$. Let R be a torsion free group of rank 1 containing an element a such that $h_p^R(a) = \alpha$ and such that $H^*/H \cong R/\langle a\rangle$. Define A to be the subdirect sum of H^* and R having kernels H and $\langle a\rangle$ respectively. Identifying H^* and R as subgroups of $H^* \oplus R$ in the natural way, we have $A + H^* = A + R = H^* \oplus R$ and $A \cap H^* = H$ and $A \cap R = \langle a\rangle$. Arguing as in Proposition 1.7 of Hill and Megibben [4] we have $p^\sigma A = A \cap p^\sigma(H^* \oplus R)$. Now $p^\sigma(H^* \oplus R) = P^\sigma R$ and since $p^\sigma R = \langle a\rangle$ when $\sigma < \omega$ and $p^\sigma R = R$ when $\sigma \geqslant \omega$, it follows that $p^\sigma A =$

$A \cap p^\sigma R = \langle a \rangle$. Therefore $h_p^A(a) = \sigma$ and

$$A/\langle a \rangle = A/A \cap R = (A + R)/R = (H^* \oplus R)/R \cong H^*$$

shows that A is represented by $(\mathbf{u}, H^*)$.

It will be convenient to have a notation for the groups described in 5.15.

5.16. Definition. Let $\mathcal{C}$ be the class of groups C in $\mathcal{A}$ such that the height matrix of every element c in C contains only finitely many gaps.

Before proceeding further we point out a modification of Theorem 103.3 [2]. Let A have torsion free rank 1 and totally projective torsion part T. Let λ_p be the p-length of T_p and $\mathrm{H}(a) = [\sigma_{pk}]$, where a is an element of A having infinite order. Then conditions (i) and (ii) of 5.7 together with

(iii) *if ρ is a limit ordinal not cofinal with ω and $\rho \leqslant \lambda_p < \rho + \omega$ then $\sigma_{pk} \neq \infty$ implies $\sigma_{pk} < \max(\lambda_p, \omega)$*

must be satisfied by $\mathrm{H}(a)$ and T. Condition (iii) is in fact equivalent to condition (iv), p. 201 of [2] (this equivalence is evident from Proposition 4 of [6]). Thus Theorem 103.3 of [2] becomes:

5.17. Theorem. *Let T be a reduced totally projective group,* M *a height matrix. There exists a mixed group A of torsion free rank* 1 *with $T(A) = T$ and which contains an element a of infinite order with* $\mathrm{H}(a) = \mathrm{M}$ *if and only if* M *satisfies* (i)–(iii).

Now the main theorem of this section.

5.18. Theorem. *Let* M *be a height matrix, H^* a torsion group. Then* (M, H^*) *is $\mathcal{A}$-admissible if and only if it is $\mathcal{A}$-compatible.*

Proof. Only half the statement requires verification and again we can replace M by $\mathrm{M}_p = \mathbf{u} = (\sigma_0, \sigma_1, \ldots)$ and H^* by H_p^*. Assume that $(\mathbf{u}, H^*)$ is $\mathcal{A}$-compatible. In view of 5.15 we need only consider the case when $\mathbf{u}$ has infinitely many gaps. There is a totally projective p-group H with Ulm invariants given by $f_\sigma^p(H) = f_\sigma^p(\mathrm{M}, H^*)$. Clearly $l_p(H) > \sigma_n$ for $n = 0, 1, \ldots$ so that $\mathbf{u}$ and H satisfy (iii) with $\mathbf{u}$ in place of M and H in place of T. Let A be a group of torsion free rank 1, torsion part H, and element a of infinite order such that $U_p(a) = \mathbf{u}$ (the existence of such an A is guaranteed by (5.17)). Since $A/\langle a \rangle$ is a totally projective p-group and has the same Ulm invariants as H^*, 2.4 yields $A/\langle a \rangle \cong H^*$.

Having solved the problem of which pairs (M, H^*) are admissible when H^* is totally projective, it will be useful to know that every height matrix occurs in some $\mathcal{A}$-admissible pair. We first prove a corresponding result for heights.

5.19. Proposition. *To each height $K = \langle\langle \beta_2, \beta_3, \ldots, \beta_p, \ldots \rangle\rangle$ there is a* *group C in* $\mathcal{C}$ *containing an element c of infinite order such that* $H(c) = K$. *Further, this c can be chosen so that* $\mathrm{H}(c)$ *contains no gaps.*

PROOF. Let M be the height matrix with no gaps and first column $\langle\langle\beta_2, \ldots, \beta_p, \ldots\rangle\rangle$ and let $H^* = \bigoplus_{p\in P} H^p_{\beta_p}$, where $H^p_{\beta_p}$ is the generalised Prüfer p-group of length β_p (if $\beta_p = \infty$, we set $H^p_{\beta_p} \cong Z(p^\infty)$). It is easy to check that (M, H^*) is $\mathcal{A}$-compatible.

5.20. PROPOSITION. *To each height matrix* M *there is a group A in* $\mathcal{A}$ *represented by* (M, H^*) *for some totally projective group* H^*.

PROOF. Let $\mathrm{M} = [\sigma_{pk}]$ and set $H^* = \bigoplus_{p\in P} \bigoplus_{k<\omega} H^p_{\sigma_{pk}}$; now argue as for 5.19.

Let λ be a limit ordinal not cofinal with ω. Warfield [10] defines a p-group G to be a λ-*elementary S-group* if G is a λ-dense isotype subgroup of corank 1 in some totally projective p-group. A p-group G is an S-group if G is the direct sum of a totally projective group and λ-elementary S-groups for various limit ordinals λ not cofinal with ω. By an *S-group* we mean a torsion group whose p-components are S-groups.

5.21. THEOREM. *If* $A \in \mathcal{A}$ *then* $T(A)$ *is an S-group.*

PROOF. We need only consider a reduced group A in $\mathcal{A}$ with torsion free rank 1 represented by (M, H^*) where H^* is a p-group. Put $\mathbf{u} = \mathrm{M}_p = (\sigma_0, \sigma_1, \ldots)$, let $\eta\colon A \twoheadrightarrow H^*$ be the natural homomorphism and T the torsion part of A. Recall that $\eta T \cong T$. We consider three cases.

1. $\mathbf{u}$ has infinitely many gaps. As we saw in the proof of 5.18 there is a group A' with torsion free rank 1 and represented by (M, H^*) such that $T(A')$ is totally projective. Now $A \cong A'$ implies $T(A')$ is totally projective.

2. $\mathbf{u}$ has only finitely many gaps and $\sup_i \sigma_i = \rho < \infty$. Let λ be the limit ordinal such that $\lambda + \omega = \rho$. When λ is cofinal with ω, an argument similar to that used in 1 shows that T is totally projective. Suppose λ is not cofinal with ω. By changing to another representation if necessary, we can assume that $\sigma_0 \geqslant \lambda$. It is easily shown that ηT is λ-dense in H^* and that

$$p^\sigma \eta T = p^\sigma H^* \cap \eta T \quad \text{for all } \sigma \leqslant \lambda.$$

Then $\eta T/p^\lambda \eta T \cong (\eta T + p^\lambda H^*)/p^\lambda H^*$. Set $G = (\eta T + p^\lambda H^*)/p^\lambda H^*$. Then G is λ-dense and isotype of corank 1 in the totally projective p-group $H^*/p^\lambda H^*$. Thus $\eta T/p^\lambda \eta T$ is an S-group. Observe that $p^\rho \eta T = p^\rho H^*$. As $\lambda + \omega = \rho$, we see that $p^\lambda \eta T/p^\rho \eta T$ is a direct sum of cyclic p-groups and thence an S-group. Put $G' = \eta T/p^\rho \eta T$; then we have shown that $p^\lambda G'$ and $G'/p^\lambda G'$ are both S-groups, so by Warfield [10], G' is an S-group. However, $p^\rho \eta T = p^\rho H^*$ is also an S-group so that ηT is itself an S-group.

3. u contains ∞. Assume $\sigma_0 = \infty$; then H^* has a unique summand $Z(p^\infty)$ such that $\eta T \cap Z(p^\infty) = 0$. Since $\eta T + Z(p^\infty) = H^*$, we have $H^* = \eta T \oplus Z(p^\infty)$ and ηT is totally projective.

6. Balanced projectives. In this section we completely characterise the balanced projectives in the category of all abelian groups: they are just the direct summands of direct sums of members of the class $\mathfrak{A}$. We also explore the properties of balanced projectives. Every group is the balanced image of a balanced projective. If A is balanced projective, then $A/T(A)$ is completely decomposable and $T(A)$ is a summand of an S-group in the sense of Warfield. A torsion (torsion free) group is balanced projective in the category of all abelian groups if and only if it is totally projective (completely decomposable).

We begin with a generalisation of Lemma 80.3 of [2] to mixed groups. The proof requires little change and is therefore omitted.

6.1. LEMMA. *Given a commutative diagram*

$$\begin{array}{ccccccccc} & & 0 & \to & N & \to & A & & \\ & & & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \\ 0 & \to & U & \to & V & \xrightarrow{\alpha} & W & \to & 0 \end{array}$$

where U is H-balanced in V and the two rows are exact, suppose that ψ does not decrease heights in A. If the element a is p-proper with respect to N and $pa \in N$ then ψ can be extended to a map ψ^*: $\langle N, a\rangle \to V$ such that $\alpha\psi^* a = \phi a$ and ψ^* does not decrease heights.

6.2. THEOREM. *If $A \in \overline{\mathfrak{A}}$ then A is balanced projective, and if $C \in \overline{\mathfrak{C}}$ then C is H-projective.*

PROOF. We need only consider groups A in $\mathfrak{A}$ and groups C in $\mathfrak{C}$.

Suppose $A \in \mathfrak{A}$, let ϕ: $A \to W$ be a homomorphism and let $0 \to U \to V \xrightarrow{\alpha} W \to 0$ be a balanced exact sequence: we show that ϕ lifts to a homomorphism ψ: $A \to V$ such that $\alpha\psi = \phi$. When A is torsion, A is totally projective. Since $0 \to T(U) \to T(V) \to T(W) \to 0$ is balanced, ϕ lifts to ψ: $A \to T(V)$ such that $\alpha\psi = \phi$ and we are done. If A has torsion free rank 1, let a be an element of infinite order in A and choose v in V such that $\alpha v = \phi a$ and $\mathrm{H}(\phi a) = \mathrm{H}(v)$. The correspondence $a \mapsto v$ gives rise to a homomorphism ψ': $\langle a\rangle \to V$ which does not decrease heights in A. If ϕ_p is the restriction of ϕ to $A(p, \langle a\rangle)$ for each prime p, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \langle a\rangle & \to & A(p, \langle a\rangle) & \to & (A/\langle a\rangle)_p & \to & 0 \\ & & \downarrow{\scriptstyle\psi'} & & \downarrow{\scriptstyle\phi_p} & & & & \\ 0 \to U & \to & V & \longrightarrow & W & \longrightarrow & & \longrightarrow & 0 \end{array}$$

where both rows are exact and the bottom row is balanced. Now 2.2 implies $\langle a\rangle$ is p-nice in $A(p, \langle a\rangle)$; using 6.1 and a nice composition series for $(A/\langle a\rangle)_p$ (see [2]) we extend ψ' to $\psi_p\colon A(p, \langle a\rangle) \longrightarrow V$ in such a way that $\alpha\psi_p = \phi_p$ (this extension is done transfinitely, taking unions at limit ordinals and using 6.1 at nonlimit ordinals). An application of 2.7 yields a $\psi\colon A \longrightarrow V$ such that $\alpha\psi = \phi$.

The proof that groups C in C, and hence groups in C, are H-projective is similar: the case for torsion C is the same as above, while for C having torsion free rank 1 we must choose a c in C with infinite order such that $\mathrm{H}(c)$ has no gaps to ensure that the map $\psi'\colon \langle c\rangle \longrightarrow V$ does not decrease heights in C.

6.3. THEOREM. *There are enough balanced* (*H-balanced*) *projectives. In particular, every group G can be imbedded in a balanced* (*H-balanced*) *exact sequence* $0 \longrightarrow B \longrightarrow A \longrightarrow G \longrightarrow 0$ *where* $A \in \mathsf{A}^{\Sigma}$ $(A \in \mathsf{C}^{\Sigma})$, *and every balanced* (*H-balanced*) *projective is in* $\overline{\mathsf{A}}$ $(\overline{\mathsf{C}})$.

PROOF. Corresponding to each element g in G we choose a group A_g in A containing an element a_g such that $o(a_g) = o(g)$ and $\mathrm{H}(a_g) = \mathrm{H}(g)$; when $o(g) = \infty$ we refer to 5.20 and for the case when $o(g)$ is finite one readily constructs a direct sum of generalised Prüfer groups containing the required element. By 2.6, the height preserving map $\phi\colon \langle a_g\rangle \longrightarrow \langle g\rangle$ sending $a_g \longmapsto g$ extends to a map $A_g(p, \langle a_g\rangle) \longrightarrow G$ and 2.7 provides a homomorphism $\phi_g\colon A_g \longrightarrow G$ whose restriction to $\langle a_g\rangle$ is ϕ. The epimorphism $\bigoplus_{g\in G}\phi_g\colon \bigoplus_{g\in G}A_g \longrightarrow G$ satisfies condition (d) of 3.3 and is therefore balanced. The final statement of the theorem is now trivial. An identical argument to the above can be used to show the existence of enough H-projectives.

A summand of a balanced projective is again balanced projective. If A is balanced projective then $A/T(A)$ is completely decomposable; this follows from the fact that A is a direct summand of a direct sum of groups having torsion free rank 1. We see from 6.3 and 5.21 that $T(A)$ is a summand of an S-group. These results are summarized in the following:

6.4. PROPOSITION. (i) *The torsion part of every balanced projective* (*H-projective*) *is a direct summand of an S-group.*

(ii) *A torsion* (*torsion free*) *group is balanced projective if and only if it is totally projective* (*completely decomposable*); *the same holds for H-balanced.*

(iii) *If A is balanced projective* (*H-projective*) *then $A/T(A)$ is completely decomposable.*

(iv) *A torsion summand of a balanced* (*H-balanced*) *projective is totally projective.*

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF SYDNEY, SYDNEY, N.S.W. 2006, AUSTRALIA

Current address: Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003