PPPL--2719

DE91 002511

Ballooning Instabilities in Tokamaks with Sheared Toroidal Flows

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Abstract

The stability of ballooning modes in the presence of sheared toroidal flows is investigated. The eigenmodes are shown to be related by a Fourier transformation to the non-exponentially growing Floquet solutions found by Cooper [Plasma Phys. Controlled Fusion **30**, 1805 (1988)]. It is further shown that the problem cannot be reduced further than to a two dimensional partial differential equation. Next, the generalized ballooning equation is solved analytically for a circular tokamak equilibrium with sonic flows, but with a small rotation shear compared to the sound speed. With this ordering, the centrifugal forces are comparable to the pressure gradient forces driving the instability, but coupling of the mode with the sound wave is avoided. A new stability criterion is derived which explicitly demonstrates that flow shear is stabilizing at constant centrifugal force gradient.

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I. INTRODUCTION

The ability to attain high values of beta, where beta is the ratio of plasma to magnetic pressure, is limited by the class of instabilities known as ballooning modes.[1] These modes are now well understood for static equilibria.[2–6] However, new difficulties arise in the analysis when differential rotation is present in the equilibrium state.[7–9] This problem is of considerable practical importance, since strong toroidal flows are known to result from unbalanced neutral beam injection.[10,11]

The structure of ballooning modes is determined by the conjunction of dynamical and geometrical constraints.[12] In static equilibria, dynamical considerations lead to the requirement of large parallel wavelength. In the presence of flow, an additional dynamical constraint must be satisfied: in order to minimize the kinetic energy, the phase velocity of the perturbation must match the rotation speed of the plasma. In more physical terms, perturbations which cause the plasma to flow along a corrugated flux surface require a large amount of energy and are therefore proscribed.

This additional constraint can be readily accommodated within the framework of the WKB formalism by incorporating in the eikonal the Doppler shift associated with the flow.[13–15] In equilibria with differential rotation, however, the Doppler-shifted frequency varies from one flux surface to another. Thus, the eikonal solutions do not have pure exponential time dependence and are not eigenmodes of the system. This is a source of difficulties when one seeks to determine the radial structure of these solutions. In particular, the eikonal solutions are found to develop large radial gradients which eventually violate the WKB ordering.

In the present paper, we show that the eikonal solutions consist of a superposition of a large, quasi-periodic array of radially shifted eigenmodes. The eigenmodes are well-behaved and satisfy the large-wavelength ordering uniformly. They are related to the eikonal solutions by a transformation formula which we derive.

We then present an asymptotic solution of the generalized ballooning equation for large aspect ratio, axisymmetric tokamaks with circular cross-section. We assume that the flows are purely toroidal and sonic but that the flow shear is relatively small, $d\Omega/dq \sim \epsilon^{1/2}\omega_s$, where Ω is the rotation frequency, q is the safety factor, $\omega_s = c_s/qR$ is the sound frequency, and ϵ is the inverse aspect ratio. The purpose of this last assumption is to avoid coupling to the sound wave. We emphasize that no assumption is made as to the relative size of the growth rate compared to the shear of the rotation frequency. A new dispersion formula is derived, in which the stabilizing effect of the radial variation of the Doppler shift is displayed in a transparent fashion. The connection between this analysis and the conventional ballooning mode analysis is made in Appendix A.

The body of the paper is divided into two essentially independent sections. In Sec. II, we describe the two basic representations for perturbations of an equilibrium with sheared flow and derive the transformation formula relating them. In Sec. III, we present the solution of the generalized ballooning equation, and the results are discussed in Sec. IV.

II. REPRESENTATION OF THE PERTURBATION

A. Introduction

The essential features of large wavenumber perturbations in plasmas with flow are expressed by the heuristic dispersion relation

$$(\omega - \mathbf{k}\cdot\mathbf{v})^2 = v_A^2\left(k_{\|}^2 - \frac{\hat{\beta}}{L_p L_c}\right), \tag{1}$$

where ω and $\mathbf{k}$ are the mode frequency and wavevector, $\mathbf{v}$ is the equilibrium flow velocity and v_A is the Alfvén speed. The left hand side of this equation corresponds to the kinetic energy. On the right hand side, the first term represents the line bending energy and the last term models the destabilizing forces. Here $\hat{\beta}$ is a modified beta representing the combined effect of pressure and centrifugal forces, while L_p and L_c are the pressure gradient and curvature scale lengths, respectively.

For large toroidal wavenumber n, the dispersion relation is dominated by the stabilizing line bending and kinetic energy terms. In seeking the most unstable modes, one is led to require that the magnetic and flow resonance conditions, $k_{\|} = 0$ and $\omega = \mathbf{k}\cdot\mathbf{v}$, be satisfied simultaneously. In axisymmetric equilibria with toroidal flows, the flow resonance condition is simply $\omega = n\Omega$, and the magnetic resonance condition for a poloidal Fourier harmonic m is $m = nq$.

It is well known that coupling between poloidal Fourier harmonics prevents the magnetic shear from localizing ballooning modes. However, there is no coupling between different frequency components or eigenmodes. Flow shear, therefore, will localize the unstable eigenmodes around their flow-resonant surface. The localization width can be estimated from the dispersion relation to be $w \sim \omega_A (n\, d\Omega/dr)^{-1}$, where the poloidal Alfvén frequency ω_A is defined as $\omega_A = v_A/qR$. It is important to compare this localization width to the distance between magnetic resonant surfaces, given by $\delta = (n\, dq/dr)^{-1}$. For sonic flows, $\delta/w \sim \beta^{1/2} \ll 1$, so that the mode will extend over many magnetic resonant surfaces and have rich poloidal harmonic content.

Flow shear also has important consequences for the geometrical properties of the problem. In static equilibria, there is an approximate lattice symmetry between poloidal harmonics centered on nearby magnetic resonant surfaces.[16,17] This symmetry can be described more precisely as an invariance of the mode equation under a radial shift accompanied by a twist such that the field lines in each magnetic resonant surface are mapped onto the field lines of the next resonant surface.[16] The symmetry manifests itself in the dispersion relation as an invariance under the substitution $q \rightarrow q + 1/n$ and $m \rightarrow m + 1$.

In the presence of flow shear, by contrast, the poloidal harmonics on nearby resonant surfaces will experience different flow velocities. As a result, the purely spatial, "twisting slice" symmetry described above will be replaced by a dynamical symmetry between poloidal harmonics of *different* eigenmodes. This dynamical symmetry is reflected in the dispersion relation as an invariance under $\omega \rightarrow \omega + \dot{\Omega}$, $q \rightarrow q + 1/n$ and $m \rightarrow m + 1$, where $\dot{\Omega} = d\Omega/dq$.

It is clear that reduction of the stability problem to its simplest form depends critically on the effective use of the geometrical or symmetry properties of the equilibrium. In fact, we will see that the symmetries provide a simple classification of the three relevant representations for ballooning modes: namely, the classical ballooning representation, and the eikonal and eigenmode representations. We will begin by describing the latter two representations and deriving the transformation formulas relating one to the other. We will then conclude the section with a discussion of the role of symmetry and with a comparison of the classical ballooning representation with the representations for perturbations in equilibria with sheared flows.

B. Eikonal Representation

The primary motivation for the eikonal representation is to formulate the problem so as to automatically satisfy the dynamical constraints. Assuming the canonical form for the displacement,

$$\boldsymbol{\xi}(\mathbf{r},t) = \hat{\boldsymbol{\xi}}(\mathbf{r},t)\exp(iS(\mathbf{r},t)), \tag{2}$$

the magnetic and flow resonance conditions can be expressed as $\mathbf{B}\cdot\boldsymbol{\nabla}S = 0$ and $dS/dt = 0$, where $d/dt = \partial/\partial t + \mathbf{v}\cdot\boldsymbol{\nabla}$ is the convective time derivative along the unperturbed flow. The solution is[13]

$$S = n\left(\alpha - \Omega t + \int_{q_0}^{q}\theta_0(\hat{q})d\hat{q}\right), \tag{3}$$

where $\alpha = \zeta - q\theta$ is the usual field-line label and q_0 is a reference flux-surface. The wavevector associated with this eikonal is

$$\mathbf{k}_\perp = n\left[\boldsymbol{\nabla}\zeta - q\boldsymbol{\nabla}\theta - (\theta + \dot{\Omega}t - \theta_0)\boldsymbol{\nabla}q\right]. \tag{4}$$

The most important new feature of this wavevector is that it is nonstationary in time. As a result, the time translation invariance of the original mode equation is lost in the eikonal formalism. The eikonal or WKB approach thus leads to a partial differential equation in two variables: time, and the coordinate along the field line.[13–15] The explicit form of this equation will be given in Sec. III. Its most significant property is that it depends on time only through the wavevector $\mathbf{k}_\perp$. Note that, $\mathbf{k}_\perp$ also introduces the familiar secular dependence on θ.

The final, periodic solution is constructed as for static equilibria:

$$\boldsymbol{\xi}(\mathbf{r},t) = \sum_{j=-\infty}^{+\infty} \hat{\boldsymbol{\xi}}_n(\theta + 2\pi j, t \mid q) \exp[inS(q, \theta + 2\pi j, \zeta, t)]. \tag{5}$$

Equation (5) is Cooper's eikonal representation for perturbations in the presence of shear flow.[13] This representation yields solutions which are clearly not eigenmodes of the system, that is, these solutions are not invariant under time translations. They are, however, invariant under the dynamical lattice symmetry, as emphasized by Pegoraro.[14]

It is instructive to write the ballooning equation in a coordinate system drifting along the field lines at a speed such that the wavevector remains independent of time.[15] In this coordinate system the equilibrium coefficients become periodic functions of $\eta - \tau$, where $\eta = \theta + \dot{\Omega}t - \theta_0$ is the new field line coordinate and $\tau \neq \dot{\Omega}t - \theta_0$ is the new time variable. The time periodicity of the equation in this frame of reference implies that solutions can be found which behave like Floquet or Bloch functions, that is, these solutions are the product of a periodic and an exponential function of time:[15]

$$\hat{\boldsymbol{\xi}}(\eta, \tau) = \boldsymbol{\Phi}(\eta, \tau) \exp(-i\omega t), \tag{6}$$

where $\boldsymbol{\Phi}$ is a periodic function of its second argument.

We now turn to the eigenmode formulation of the problem, with which we will derive the transformation formula relating the Floquet solutions to the eigenmodes.

C. Eigenmode Representation

The eigenmode equation can be written schematically as

$$L(\nabla_{\|}, \boldsymbol{\nabla}_{\perp}, \partial_t - \mathbf{v} \cdot \boldsymbol{\nabla} \mid r, \theta)\boldsymbol{\xi}(\mathbf{r}, t) = 0. \tag{7}$$

The explicit form of this equation for magnetohydrodynamic instabilities is given in Sec. III. In its general form, however, Eq. (7) is equally applicable to other types of instabilities such as drift modes.

The eigenmode equation can be reduced to two dimensions by expressing the plasma displacement in terms of solutions which are invariant under the azimuthal and time translation symmetry,

$$\boldsymbol{\xi}_{n,k}(\mathbf{r}, t) = \exp(in\zeta - ik\theta - i\omega_{n,k}t)\hat{\boldsymbol{\xi}}_{n,k}(r, \theta), \tag{8}$$

where k is the principal poloidal mode-number. The eigenfrequency must satisfy $\omega_{n,k} = n\Omega_0 + O(1)$ where $\Omega_0 = \Omega(q_0)$ and $q_0 = k/n$ labels the flow-resonant surface. Note that k can also be interpreted as a radial mode-number for a given n.

We now derive the lowest order eigenmode equation. Recall that flow shear limits the radial extent of the mode to a width $w \sim \omega_A/n\Omega'$. We assume that the flow shear is finite, $\Omega' \sim 1$, so that $w \sim 1/n$. In this narrow interval, the radial variation of the equilibrium parameters can be neglected, except for the rotation frequency which must be expanded to first order around the flow-resonant surface q_0. The perpendicular gradient can be seen to be dominated by the gradient of the exponential factor and by the radial gradient of $\hat{\boldsymbol{\xi}}$. Introducing the microscopic radial variable $x = nq - k$, the lowest order mode equation is

$$L_0[-x + i\partial_\theta\,,\ i\hat{\mathbf{k}}_\perp + \hat{\mathbf{r}}\,nq'\partial_x\,,\ \tilde{\omega}_{n,k} - \dot{\Omega}x \mid \theta\,]\hat{\boldsymbol{\xi}}_{n,k}(x,\theta) = 0, \qquad (9)$$

where $\hat{\mathbf{k}}_\perp = \nabla(n\zeta - k\theta)$ and $\hat{\mathbf{r}} = \nabla r$. The lowest order rotation frequency has been incorporated into the Doppler shifted frequency, $\tilde{\omega} = \omega - n\Omega_0$. Note that this equation depends on the radial variable x only through the parallel gradient and inertial terms.

Equation (9) is the reduced equation for the perturbation in the "eigenmode representation." It should be emphasized that this equation, a two-dimensional partial differential equation, is formally of the same degree of complexity as the generalized ballooning equation which results from the WKB approach. Because of the similarity between the classical and the generalized ballooning equations, however, the latter is usually easier to solve than the eigenmode Eq. (9).

D. Transformation Formula

The distinguishing property of the eikonal solutions is that they are invariant, to lowest order, under the dynamical lattice symmetry. The eigenmodes evidently do not have this property since they are radially localized to a region of width $w \sim 1/n$. To construct an eikonal perturbation, one must then superpose a large number of eigenmodes centered on successive magnetic resonant surfaces. The amplitude of these modes must be bounded by an envelope which is slowly varying on the scale length of the resonant surface spacing but nonetheless narrow with respect to the equilibrium scale length. This latter condition is only necessary to justify the neglect of the variation of the equilibrium parameters. Therefore, we will henceforth ignore the slow variation of the envelope and extend the sum over an infinite array of eigenmodes centered around a reference surface q_0. We will also choose the initial conditions such that for $t = 0$, the Bloch shift $\theta_0 = 0$. The lattice-symmetric perturbation is then

$$\boldsymbol{\xi}(\mathbf{r}, t) = e^{in\zeta} \sum_{k=-\infty}^{\infty} e^{-i\omega_k t - ik\theta} \hat{\boldsymbol{\xi}}_k(q, \theta), \tag{10}$$

where $\omega_k = \omega_{k_0} + (k - k_0)\dot{\Omega}$ and the central mode number k_0 is such that $k_0 = nq_0$. We have dropped the subscript n to simplify the notation. The lattice symmetry implies that the different eigenfunctions are related to each other by

$$\hat{\boldsymbol{\xi}}_k(q, \theta) = \boldsymbol{\psi}(nq - k, \theta) + O(1/n). \tag{11}$$

In order to carry out the summation in Eq. (10), we replace $\boldsymbol{\psi}$ by its inverse Fourier transform $\boldsymbol{\Psi}$,

$$\boldsymbol{\psi}(x, \theta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boldsymbol{\Psi}(\eta, \theta) e^{-ix\eta} d\eta, \tag{12}$$

and make use of the identity

$$\sum_{k=-\infty}^{+\infty} e^{iky} = 2\pi \sum_{l=-\infty}^{+\infty} \delta(y - 2\pi l), \tag{13}$$

whence

$$\boldsymbol{\xi}(\mathbf{r}, t) = e^{-i\tilde{\omega}_{k_0} t} \sum_{l=-\infty}^{+\infty} e^{in(\zeta - q\theta - \Omega t - 2\pi l q)} \boldsymbol{\Psi}(\theta + \dot{\Omega} t + 2\pi l, \theta). \tag{14}$$

The lattice-symmetric perturbation given in Eq. (14) is seen to be identical to the eikonal solution in Eqs. (5) and (6) after the simple change of variables

$$\boldsymbol{\Psi}(\eta, \theta) = \boldsymbol{\Phi}(\eta, \eta - \theta). \tag{15}$$

Conversely, the eigendisplacement can be expressed in terms of the Floquet solution of the eikonal problem by combining Eqs. (8), (11)-(12), and (15). One finds

$$\xi_k(q,\theta) = \frac{\dot{\Omega}}{2\pi}\int_{-\infty}^{+\infty} \Phi(\theta + \dot{\Omega}\nu, \dot{\Omega}\nu)\exp\left[in\left(\alpha - (\Omega(q) - \Omega_0)\nu\right) - i\omega_k t\right] d\nu. \tag{16}$$

The transformation formula, Eq. (16), is the main result of this section. It allows the eigenmodes to be recovered from the solution of the generalized ballooning equations. We emphasize that the consistency of this analysis depends on the finiteness of the flow shear. In the limit $\dot{\Omega} \to 0$, the lowest order solutions of the eigenmode equation are no longer localized and the sum in Eq. (10) will diverge.

The limit of vanishing flow shear is thus seen to be singular. The physical interpretation of this singularity is that even an arbitrarily small amount of flow shear will qualitatively modify the long-time response of the plasma. The singularity can be resolved by carrying out a classical ballooning analysis with the ordering $n\dot{\Omega} \sim 1$. This analysis is sketched in Appendix A, where it is shown that the large flow-shear limit of the classical ballooning formalism converges properly to the small flow-shear limit of the general theory. The $\dot{\Omega} \to 0$ limit will be discussed further in Sec. III.

E. Summary

The role of symmetry can now be summarized as follows: The eikonal representation results when the general perturbation is expanded on a "basis" of solutions which are invariant under the dynamic lattice symmetry; the eigenmode representation results from expanding the perturbation in terms of time-translation invariant solutions.

The classical ballooning representation, by contrast, expresses the perturbation in terms of functions which are invariant under *both* the time translation and lattice symmetries. The simultaneous application of the time translation and lattice symmetries allows the mode equation to be reduced to an ordinary differential equation.

It is easy to see, however, that in the presence of flow shear the time translation and lattice symmetry operations do not commute. As a result, it is not possible to find solutions which are invariant under both of these operations. A corollary of this statement, and the principal conclusion to be drawn from the foregoing discussion, is that the problem cannot be reduced further than to a two dimensional partial differential equation.

We conclude this section by some comments on the problem of higher order corrections in the expansion in powers of $1/n$. For ballooning modes in rigid equilibria, the $1/n$ corrections problem is of fundamental interest since the lowest order equation yields radially unbounded solutions which are clearly not appropriate eigenmodes.[6,18] This is not the case in sheared equilibria, since the global radial structure of the eigenmode is completely determined by the lowest order equations. There is thus no motivation for carrying the expansion to a higher order. In this respect, ballooning modes in sheared equilibria are somewhat analogous to resistive interchange modes.[19]

A related question concerns the comparison between our transformation equations and the inverse ballooning transformation derived by Hazeltine et al. and Newcomb.[20–22] The inverse ballooning transformation involves a filtering operation which does not appear in our formulation. The filtering serves to uniquely extract higher order information concerning the slow variation of the amplitude of the poloidal harmonics. The equivalent information for sheared-flow systems would consist of the variation in the amplitude of the eigenmodes which are superposed to construct the lattice-symmetric perturbation. These amplitudes have no physical significance and are, therefore, irrelevant.

In the next section, we will adopt the eikonal representation and solve the generalized ballooning equation.

III. STABILITY ANALYSIS

A. Equilibrium

Neoclassical transport theory for axisymmetric equilibria predicts that poloidal flows are strongly damped in axisymmetric tokamaks.[23,24] We therefore restrict consideration to purely toroidal flows,

$$\mathrm{v} = R\Omega(\psi)\hat{\boldsymbol{\zeta}},$$

where ψ is the poloidal magnetic flux.

The magnetohydrodynamic (MHD) force balance equation is

$$\nabla p + \rho(\mathrm{v}\cdot\nabla)\mathrm{v} = \mathbf{J}\times\mathbf{B}. \tag{17}$$

In addition to force balance, it is also necessary to satisfy the equation of state. We will not use the adiabatic equation of state, however, but will replace it by the temperature convection equation,

$$\frac{dT}{dt} = 0. \tag{18}$$

This choice of equation of state is motivated by the observation that strong electron parallel thermal conductivity will enforce isothermal flux surfaces at equilibrium. Note that while both the adiabatic and the temperature convection equations suffer from degeneracy for axisymmetric geometries with purely toroidal flows, so that equilibria with isothermal flux surfaces are in fact allowed by the adiabatic model,[25] such configurations are pathological for an adiabatic fluid. In particular, they will not have nearby bifurcated equilibria at marginal stability. This is an undesirable circumstance for the stability analysis; it is avoided here by the use of the temperature convection equation.

For isothermal flux surfaces, the parallel component of the equilibrium equation requires that[25,26]

$$\rho(\psi, R) = \rho_0(\psi) \exp\left(\frac{\Omega^2 R^2}{2T}\right), \tag{19}$$

while the component of the equilibrium equation perpendicular to the flux surfaces yields the Grad-Shafranov equation,

$$R^2 \nabla \left(R^{-2} \nabla \psi\right) = -I \frac{dI}{d\psi} - R^2 \left(\frac{\partial p}{\partial \psi}\right)_R, \tag{20}$$

where I is the poloidal current and the partial derivative of the pressure is to be taken at constant R. Analytical solutions of the Grad-Shafranov equation with flow have been given by Maschke and Perrin.[26]

B. Linearized Equations

The linearized MHD equations of motion are[27]

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} + 2\rho \mathbf{v} \cdot \nabla \frac{\partial \boldsymbol{\xi}}{\partial t} = \mathbf{F}(\boldsymbol{\xi}), \tag{21}$$

where the force operator $\mathbf{F}$ is the sum of the static force operator $\mathbf{F}_s$ and of terms related to the flow,

$$\mathbf{F}(\boldsymbol{\xi}) = \mathbf{F}_s(\boldsymbol{\xi}) + \nabla \cdot (\rho \boldsymbol{\xi} \mathbf{v} \cdot \nabla \mathbf{v} - \rho \mathbf{v}\mathbf{v} \cdot \nabla \boldsymbol{\xi}). \tag{22}$$

The static force operator is

$$\mathbf{F}_s(\boldsymbol{\xi}) = (\nabla \times \mathbf{Q}) \times \mathbf{B} + \mathbf{J} \times \mathbf{Q} + \nabla(p \nabla \cdot \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \nabla p),$$

where $\mathbf{Q}$ is the perturbed magnetic field,

$$\mathbf{Q} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}).$$

The equilibrium flow can be seen to affect the stability properties in two distinct ways. First, it produces centrifugal forces which affect stability both directly and through modifications of the equilibrium. Second, it modifies the inertial response of the plasma. This second effect is dominant for large wavenumbers and is the most interesting effect from the theoretical point of view due to its profound consequences for the structure of ballooning modes, as discussed in Sec. II.

It is important to note that these two effects can be modified separately.[13] That is, sequences of equilibria with identical centrifugal force and pressure profiles but varying flow shear can be constructed. One can also construct families of equilibria with the same flow frequency profiles but with varying centrifugal forces.

C. Generalized Ballooning Equations

The generalized ballooning equations have been derived previously by other authors.[7,9] An alternative derivation based on the eigenmode formulation of Sec. II is given in Appendix B. After elimination of the longitudinal component of the displacement, one is left with two coupled equations for the parallel and perpendicular components of the transverse displacement:

$$\boldsymbol{\xi}_t = X\mathbf{N} + Z\mathbf{B}/\rho,$$

where $\mathbf{N} = (\mathbf{B}\times\mathbf{k})/B^2 k_\alpha$. In terms of these variables, the generalized ballooning equations take the form

$$\rho\frac{\partial}{\partial t}\left(\mathbf{N}^2\frac{\partial X}{\partial t}\right) - \mathbf{C}\cdot\mathbf{N}\frac{\partial Z}{\partial t} = \frac{1}{\mathcal{J}}\frac{\partial}{\partial\theta}\left(\frac{\mathbf{N}^2}{\mathcal{J}}\frac{\partial X}{\partial\theta}\right) - VX + AS^2\left(\frac{1}{\mathcal{J}}\frac{\partial Z}{\partial\theta} - AX\right), \tag{23}$$

$$\rho^{-1}B^2\frac{\partial^2 Z}{\partial t^2} + \mathbf{C}\cdot\frac{\partial}{\partial t}(\mathbf{N}X) = \frac{1}{\mathcal{J}}\frac{\partial}{\partial\theta}\left[S^2\left(\frac{1}{\mathcal{J}}\frac{\partial Z}{\partial\theta} - AX\right)\right], \tag{24}$$

where $\mathbf{C} = 2\Omega(\mathbf{B}\times\hat{\boldsymbol{\zeta}})$ is a Coriolis-like coefficient and $S^2 = pB^2/\rho^2(p+B^2)$ is proportional to the sound frequency. The "potential" term V driving the instability is given by $V = \boldsymbol{\nu}\cdot\mathbf{N}$, where

$$\boldsymbol{\nu} = -2\left.\frac{\partial p}{\partial\psi}\right|_R \boldsymbol{\kappa} - \left.\frac{\partial(\rho\Omega^2R^2)}{\partial\psi}\right|_R \boldsymbol{\kappa}_T + 2\rho\Omega\Omega'\frac{\mathbf{B}\cdot\hat{\mathbf{z}}}{B^2}\nabla\psi. \tag{25}$$

Here $\boldsymbol{\kappa}_T$ is the toroidal curvature, $\boldsymbol{\kappa}_T = -\hat{\mathbf{R}}/R$, and $\Omega' = d\Omega/d\psi$. The coupling term A is similarly given by $A = \boldsymbol{\alpha}\cdot\mathbf{N}$ with

$$\boldsymbol{\alpha} = \left(2p\boldsymbol{\kappa} + \rho\Omega^2R^2\boldsymbol{\kappa}_T\right)/T. \tag{26}$$

The potential V contains two new terms in addition to the pressure-gradient driving term. The first of these represents the effect of the centrifugal forces, while the second is directly related to flow shear. This latter term is small for sonic flows and will not be considered further. The centrifugal force term, however, is of the same magnitude as the pressure gradient driving term and plays an important role in determining the stability properties. It is destabilizing for centrally peaked centrifugal force profiles.[7,9,15]

When investigating the asymptotic properties of the ballooning equations, it is useful to separate the parts of V and A which are secular in $\eta = \theta + \dot{\Omega} t$. Thus, we write $V = -q'\eta V_g + V_n$, where

$$V_g = \boldsymbol{\nu} \cdot \frac{\mathbf{B} \times \nabla\psi}{B^2},$$

$$V_n = \boldsymbol{\nu} \cdot \frac{\mathbf{B} \times (\nabla\zeta - q\nabla\theta)}{B^2}.$$

The notation used here has its origin in the fact that, for static equilibria, V_g and V_n are proportional respectively to the geodesic and normal components of the field-line curvature κ. In the static case, one finds that the flux surface average of the geodesic curvature vanishes. The appropriate generalization of this fundamental property to rotating equilibria is that

$$V_g = -\mathbf{B} \cdot \nabla\sigma, \tag{27}$$

so that the flux surface average of the 'geodesic' component of the potential vanishes: $\langle V_g \rangle = 0$.

The same property holds for A. Namely, if $A = -q'\eta A_g + A_n$, then

$$A_g = -\mathbf{B} \cdot \nabla a, \tag{28}$$

where $a = \rho I/B^2$, and $\langle A_g \rangle = 0$.

We thus see that the generalized ballooning equations are almost identical in structure to the compressible ballooning equations for static equilibria.[5] Only the two Coriolis-like terms have no static equivalent. The most important difference, of course, is that N now depends on time through the wavevector k.

D. Asymptotic Solution

The ballooning equations can be solved analytically for marginally unstable modes in a large aspect ratio tokamak.[28,29] We adopt the "high beta" ordering,

$$\beta = \frac{p}{2B^2} \sim \epsilon,$$

where $\epsilon \ll 1$ is the inverse aspect-ratio, and assume that the Mach number $\mathcal{M}$ is of order unity:

$$\mathcal{M}^2 = \frac{\Omega^2 R^2}{2T} \sim 1.$$

The growth rate and flow shear, on the other hand, are assumed to be comparable to each other but somewhat smaller than the rotation frequency. Specifically,

$$\gamma \sim \dot{\Omega} \sim \epsilon^{1/2}\Omega \sim \epsilon^{1/2}\omega_s \sim \epsilon\omega_A.$$

The ballooning equations (23)-(24) are most easily solved by transforming to the coordinates (η, τ), where $\eta = \theta + \dot{\Omega}t$, and $\tau = \dot{\Omega}t$. Inspection of the ballooning equations then reveals two different asymptotic regimes for finite and large η. In the inner region, determined by $\eta \sim 1$, the inertial effects are negligible and the parallel displacement Z can be eliminated. Equation (23) is then reduced to the marginal stability equation

$$\frac{1}{\mathcal{J}}\frac{\partial}{\partial\eta}\left(\frac{N^2}{\mathcal{J}}\frac{\partial X}{\partial\eta}\right) - V(\eta,\eta-\tau)X = 0. \tag{29}$$

For large values of η, the solutions of Eq. (29) have the asymptotic behavior

$$X \sim A_\pm(\tau)\left(|\eta|^{-1/2-\nu} + \Delta_\pm(\tau)|\eta|^{-1/2+\nu}\right), \tag{30}$$

where

$$\nu = \left(\frac{1}{4} - D_I\right)^{1/2}.$$

Note that for each τ there is a unique $\Delta(\tau)$ and a unique solution of Eq.(29) such that $\Delta_- = \Delta_+ = \Delta$. We will see that this is the solution which is required by the asymptotic matching procedure. The matching parameter $\Delta(\tau)$ is the principal result of the solution of the inner problem for the purpose of the asymptotic analysis. For small values of Δ, one can show that $\Delta = 2\delta W(\xi_0,\xi_0)/(A_+^2+A_-^2)$, where δW is the usual energy functional and ξ_0 is the solution of Eq. (29) for a potential V_0 such that $\Delta = 0$.

The coefficient D_I is the analog of the coefficient which appears in the Mercier criterion, $D_I < 1/4$. Its general form is given in Appendix C. Near the magnetic axis of tokamaks with circular flux surfaces it reduces to

$$D_I = \frac{2rq^2}{q'^2 B_0^2}\left\{(q^2-1)p_0' - q^2\left[(1+\mathcal{M}^2)\mathcal{M}^2 p_0\right]'\right\}. \tag{31}$$

As a result of the small flow-shear assumption, our expression for D_I does not contain the resonances at $\dot{\Omega} = \omega_s$ and $\dot{\Omega} = \omega_A$ found by Bondeson et al. in their derivation of the Suydam criterion for cylindrical plasmas with flows.[30] The resonance at $\dot{\Omega} = \omega_s$, however, has been shown to be an artifact of the

MHD model by Bondeson and Iacono.[31] This resonance is removed by a kinetic treatment of the parallel motion.

In the outer region, determined by $\eta \gtrsim \epsilon^{-1}$, the ballooning equations are characterized by two scale lengths: namely, the connection length $\delta\eta \sim 2\pi$ which characterizes the variation of the equilibrium parameters and the inertial length $\delta\eta \sim \epsilon^{-1}$ over which the inertial terms vary appreciably. A two-scale analysis may thus be applied to reduce the system of equations to a single differential equation for the averaged function $\overline{X}$,

$$\overline{X}(\eta,\tau) = \frac{1}{2\pi}\int_{\eta-\pi}^{\eta+\pi} X(\hat{\eta},\tau)\, d\hat{\eta}. \tag{32}$$

The two-scale analysis is detailed in Appendix C. One finds

$$\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial\overline{X}}{\partial\eta}\right) - D_I\overline{X} = M\eta^2\left(\frac{\dot{\Omega}}{\omega_A}\right)^2\frac{\partial^2\overline{X}}{\partial\tau^2}, \tag{33}$$

where M is a constant related to the geometry. Near the magnetic axis of a tokamak with circular flux surfaces, it is given by

$$M = 1 + 2q^2(1 + 4\mathcal{M}^2 + \mathcal{M}^4). \tag{34}$$

Note that Eq. (33) is separable, but is subject to a time-dependent matching condition. It can be solved by Fourier transformation in the τ variable,[28,29]

$$\bar{x}(\eta,\omega) = \int_{-\infty}^{+\infty}\overline{X}(\eta,\tau)e^{i\omega\tau}\, d\tau.$$

The solution is

$$\bar{x}(\eta,\omega) = a_{\pm}(\omega)2^{1-\nu}(-i\lambda\omega)^{\nu}|\eta|^{-1/2}K_{\nu}(-i\lambda\omega|\eta|)/\Gamma(\nu), \tag{35}$$

where

$$\lambda = \left|\frac{\dot{\Omega}}{\omega_A}\right| M^{1/2}.$$

The small-η asymptotic form of $\bar{x}$ is

$$\bar{x}(\eta,\omega) \sim a_{\pm}(\omega)\left[|\eta|^{-1/2-\nu} - 2^{-2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}(-i\omega\lambda)^{2\nu}|\eta|^{-1/2+\nu}\right]. \qquad (36)$$

The inverse Fourier transform of $\bar{x}$, $\overline{X}$, is now matched to the large-η asymptotic form of the inner solution given in Eq. (30). It is acceptable to match $\overline{X}$ to X since the large-η limit of the inner solution does not have rapid variation in η. The result is an integral equation for the amplitudes $a_{\pm}(\omega)$:

$$2^{-2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\lambda^{2\nu}\int_{-\infty}^{\infty}(-i\omega)^{2\nu}a_{\pm}(\omega)e^{-i\omega\tau}\,d\omega = -\Delta(\tau)\int_{-\infty}^{\infty}a_{\pm}(\omega)e^{-i\omega\tau}\,d\omega. \qquad (37)$$

This equation can be solved for the special case $D_I = 0$, such as for the model circular equilibrium of Ref. 6. One finds

$$\frac{dA_{\pm}}{d\tau} = -\frac{\Delta(\tau)}{\lambda}A_{\pm}(\tau). \qquad (38)$$

Equation (38) can be integrated to find

$$A_{\pm}(\tau) = A_{0\pm}\exp\left(-\frac{1}{\lambda}\int_0^{\tau}\Delta(s)\,ds\right). \qquad (39)$$

The integral of the zeroth harmonic of Δ yields a secular term proportional to the time t. The proportionality constant is the growth rate of the eigenmode:

$$\gamma = -\frac{\omega_A}{2\pi M^{1/2}}\oint\Delta(s)\,ds. \qquad (40)$$

Equation (40) is the principal result of this section. In order to interpret this result and assess its significance it is necessary to compare the growth rate in

Eq. (40) to that of the equivalent rigid rotator. By equivalent rigid rotator, it is meant here the rigidly rotating equilibrium which has locally the same pressure and centrifugal force gradients as the sheared equilibrium under investigation.

In the case of a rigid rotator, the conventional ballooning formulation can be applied.[9] The resulting equations are similar to Eqs. (23) and (24), but the parameter τ now reduces to the Bloch phase shift θ_0. The growth rate for the rigid rotator is thus given by $\gamma = -\omega_A \Delta(\theta_{0M})/M^{1/2}$, where the Bloch phase shift θ_{0M} is determined by the condition that the growth rate be maximal.

Going back to the case with flow shear, one concludes that the growth rate is always smaller than that of the equivalent rigid rotator, as the perturbation is unable to maintain the most unstable Bloch phase-shift. More significantly, equilibria which are unstable for $\dot{\Omega} = 0$ may be completely stabilized by flow shear when the average of Δ is positive. This result has been conjectured by previous authors.[8,15] Eq. (40), however, provides an explicit criterion for the occurrence of flow-shear stabilization.

The difference between the growth rate for small flow shear calculated above and the growth rate for the equivalent rigid rotator reflects the fact that even a small amount of flow shear will destroy the phase coherence of the perturbation after a time $t \sim 1/\dot{\Omega}$. The coherence is regained when all the eigenmodes have drifted an entire number of times around the torus. The eikonal solutions thus display periodic bursts of ballooning activity with frequency $\omega_T = \dot{\Omega}$ and overall growth rate γ.[13,15]

If the growth rate $\gamma_0(\theta_{0M})$ is much larger than the burst frequency, however, a coherent perturbation will grow to large amplitude before the phase mixing effects become significant. In this case, the rigid rotator growth rate may be more physically relevant than the long-time growth rate in Eq. (40).

IV. DISCUSSION

The eikonal method for the stability analysis of equilibria with sheared flows has been placed on a firm physical basis. We have shown that the eikonal solutions can be interpreted as superpositions of eigenmodes centered on the lattice of mode-rational surfaces. This construction is analogous to the quasi-mode construction of Roberts and Taylor,[16] although here it results in solutions that are not approximate eigenmodes. The true eigenmodes are localized to a narrow radial layer, but for sonic flows they will still extend over many mode rational surfaces and will contain a large number of strongly coupled poloidal harmonics. Therefore, a purely local stability analysis such as that leading to the Mercier criterion is not sufficient.

We have presented a solution of the generalized Balloning equations for a model equilibrium with circular flux surfaces such as that used in Ref. 6. The result of this analysis can be summarized as follows.

Plasma flows affect stability in two ways: first, flow acts on perturbations directly through centrifugal forces; second, flow shear modifies the plasma response by introducing a mismatch between the mode frequency and the plasma rotation frequency.

The centrifugal forces act in essentially the same way in sheared and rigidly rotating equilibria. They are found to be destabilizing when their gradient is directed towards the magnetic axis, as has been reported previously by other authors.[9,15]

Flow shear, by contrast, is found to be always stabilizing for equivalent centrifugal force profiles. The stabilizing effect of flow shear can be understood as resulting from the precession of the Bloch shift around the flux surfaces. This precession causes the perturbation to experience alternatively stabilizing and destabilizing effects.[32,33] The resultant growth rate corresponds to the average over the Bloch phase-shift of the energy available to drive the instability and is therefore reduced from the achievable peak value at $\theta_0 = \theta_{0M}$.

The stability results derived here correspond to the zero flow-shear limit of the general, finite flow-shear theory developed in Sec. II. One expects that these results can be derived in an equivalent way from the classical ballooning formalism in the limit $n\dot{\Omega} \rightarrow \infty$. This is indeed the case, as shown in Appendix A. The time-dependent ballooning formulation, however, has the advantage of being more general. For example, the shear of the rotation frequency is comparable to the characteristic frequency for drift instabilities, so that the conventional ballooning formulation will not be applicable to those instabilities.

The results of our analytic solution of the ballooning equation are in qualitative agreement with the numerical results of Cooper.[13] Detailed comparison is not possible, however, as the calculations of Ref. 13 are based on an incompressible model. In static equilibria, marginally unstable displacements are divergence-free, so that such a model will correctly predict the marginal stability curves. In equilibria with flow, however, this is no longer the case: marginal eigenmodes have nonzero divergence and compressibility must be allowed to obtain quantitatively correct stability results.

Comparison of the predictions of the theory presented in this paper with experimental results is also difficult. First, it must be noted that the beta limits in current experiments are believed to be set primarily by the external kink modes. This difficulty can be avoided by assuming that the kink-mode and ballooning mode stability limits are similar, as they are for static equilibria. One may then regard the ballooning mode results as indicative of general stability trends. A more serious difficulty is that balanced beam injection, which produces nearly static equilibria, results in confinement degradation. Therefore, balanced beam plasmas are generally not capable of reaching the beta limit.

It has recently been proposed that sheared toroidal rotation might be used as a way of gaining access to the region of second stability.[8] Further analytic and numerical work is needed before this important issue can be resolved. The analysis presented here, however, leads to a somewhat pessimistic assessment of this possibility. This assessment is based on the observation that for extended ballooning modes in low shear equilibria (i.e., the "weak ballooning" limit) the stability criterion is independent of the Bloch shift θ_0. Thus, the stability will only be affected by the centrifugal force gradients, which are generally destabilizing.

It has also been suggested that stabilization might be achieved by inversion of the centrifugal force profile.[9,15] However, while this may be possible in the core region, the density and rotation speed must return to zero at the edge, so that stabilization of the core can only be achieved at the expense of the stability of the exterior region.

ACKNOWLEDGMENTS

The authors would like to thank T. M. Antonsen Jr., M. S. Chance, S. Cowley, S. M. Kaye, and E. Synakowski for useful discussions. This work was supported in part by U.S. Department of Energy contract DE-AC02-76--CH03073, and by an appointment to the postdoctoral program administered by Oak Ridge Universities.

Appendix A: Classical Ballooning Analysis

When the flow shear is small compared to the characteristic frequency of the instability under consideration, the classical ballooning formalism may be applied. In order to apply this formalism, the flow shear frequency is formally assumed to be of the order of $1/n$, and the subsidiary ordering $n\dot{\Omega} \sim \lambda^{-1}\omega_A$ is adopted, where $\lambda = (n\epsilon)^{-1} \ll 1$. A calculation similar to that in Sec. III leads to the dispersion relation

$$\omega - n\Omega(q) = \Lambda(\theta_0, q),$$

where $\Lambda = -i\omega_A \Delta(\theta_0, q)/M^{1/2}$.

In static equilibria, the most unstable modes are localized between two turning points in the vicinity of the maximum growth rate allowed by the dispersion relation. The maximum growth rate is usually, but not always, reached at a point where $\theta_0 = 0$.

In the presence of flow shear, by contrast, the dependence of the frequency on the radial coordinate q is dominated by the strong variation in the doppler shift, and no extremum can be found. Therefore, the only stationary modes present are the so-called "passing" modes localized between turning points at $\theta_0 = 0$ and $\theta_0 = \pm\pi$.[34] Note that these turning points are located in the complex q plane. The amplitude of the mode envelope is determined to lowest order by the equation

$$\left(\gamma - n\dot{\Omega}\frac{\partial}{\partial\theta_0}\right) a(\theta_0) = -\frac{\omega_A}{M^{1/2}}\Delta(\theta_0, q_0)a(\theta_0).$$

The quantization condition is then obtained by requiring that $a(\theta_0)$ be periodic. This condition yields the growth rate found previously and given in Eq. (40).

Note that the radial envelope of the mode is determined from $a(\theta_0)$ by a filtered Fourier transform, as described in Ref. 21.

Appendix B: Generalized Ballooning Equations

The equations of motion are most easily manipulated in their variational form. The action is given by

$$\mathcal{L} = \mathcal{T} - \mathcal{V},$$

where $\mathcal{T}$ is the kinetic energy,

$$\mathcal{T} = \int dt \int d^3r\, \rho \left| \frac{d\boldsymbol{\xi}}{dt} \right|^2,$$

and $\mathcal{V}$ is the potential energy,

$$\mathcal{V} = -\int dt \int d^3r\, \boldsymbol{\xi}^* \cdot [\mathbf{F}_s(\boldsymbol{\xi}) + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{\xi} (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v})] \, .$$

The action can be rearranged in order to "diagonalize" the potential in terms of the three basic modes, namely the shear Alfvén, and the slow and fast compressional Alfvén modes. The procedure is similar to that used for static equilibria. The parallel component of the perturbed magnetic field is eliminated in favor of the perpendicular divergence through

$$\frac{\mathbf{B} \cdot \mathbf{Q}}{B^2} = \boldsymbol{\nabla}_\perp \cdot \boldsymbol{\xi} - 2\, \boldsymbol{\xi} \cdot \boldsymbol{\kappa} + \frac{\boldsymbol{\xi} \cdot (\mathbf{J} \times \mathbf{B})}{B^2},$$

where $\boldsymbol{\kappa}$ is the field-line curvature,

$$\boldsymbol{\kappa} = \frac{\mathbf{J} \times \mathbf{B}}{B^2} + \frac{\boldsymbol{\nabla}_\perp B}{B}$$

and σ is proportional to the parallel component of the current

$$\sigma = \mathbf{J} \cdot \mathbf{B}/B^2.$$

After collecting the self-adjoint pairs of terms, integrating by parts and completing the squares, the potential becomes

$$\mathcal{V} = \int dt \int d^3r \Big[|\mathbf{Q}_\perp|^2 + p|\nabla \cdot \boldsymbol{\xi} - \frac{1}{T}\boldsymbol{\xi} \cdot (\mathbf{v} \cdot \nabla)\mathbf{v}|^2 + B^2|\nabla_\perp \cdot \boldsymbol{\xi} + 2\,\boldsymbol{\xi} \cdot \boldsymbol{\kappa}|^2$$

$$+\sigma(\mathbf{B} \times \boldsymbol{\xi}^*) \cdot \mathbf{Q} - 2(\boldsymbol{\xi}^* \cdot \boldsymbol{\kappa})\,(\boldsymbol{\xi} \cdot (\mathbf{J} \times \mathbf{B})) - \frac{\rho}{T}|\boldsymbol{\xi} \cdot (\mathbf{v} \cdot \nabla \mathbf{v})|^2$$

$$-\boldsymbol{\xi}^* \cdot (\boldsymbol{\xi} \cdot \nabla)\,(\rho \mathbf{v} \cdot \nabla \mathbf{v})) + \frac{\boldsymbol{\xi}^* \cdot \mathbf{B}}{B^2}(\boldsymbol{\xi} \times \mathbf{B}) \cdot \nabla \times (\rho \mathbf{v} \cdot \nabla \mathbf{v}) \Big],$$

where the terms have the following interpretation: the first term is the line-bending energy associated with the shear-Alfvén wave; the second term is responsible for the slow wave; the third for the fast wave; the fourth term is the kink term, which does not contribute to ballooning instabilities; the fifth term represents the destabilizing effect of unfavorable curvature, and the remaining terms are related to the centrifugal forces.

The variational form can now be simplified, for large wavenumber perturbations, by observing that the action is dominated by the perpendicular divergence terms unless the perturbation is chosen to be incompressible to the lowest order in $1/n$. That is, one must take

$$\boldsymbol{\xi} = \frac{\mathbf{B} \times \nabla \hat{X}}{nB^2} + \frac{\hat{Z}\mathbf{B}}{\rho} + O(1/n).$$

Note that $\nabla_\perp \cdot \boldsymbol{\xi}$ can be varied independently at constant $\hat{X}$ and $\hat{Z}$. The potential may, therefore, be minimized directly with respect to $\nabla_\perp \cdot \boldsymbol{\xi}$. The minimum is

$$\mathcal{V} = \int dt \int d^3r \left[\frac{\beta B^2}{\rho^2} \left| (\mathbf{B} \cdot \nabla)\hat{Z} - (2\rho\boldsymbol{\kappa} + (\rho/T)\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \boldsymbol{\xi}_\perp \right|^2 \right.$$

$$+|\mathbf{Q}_\perp|^2 - 2(\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa})(\boldsymbol{\xi}_\perp \cdot (\mathbf{J} \times \mathbf{B}))$$

$$-\frac{\hat{Z}^2}{\rho}\mathbf{B} \cdot (\mathbf{B} \cdot \nabla)(\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{\rho}{T}|\boldsymbol{\xi}_\perp \cdot (\mathbf{v} \cdot \nabla \mathbf{v})|^2$$

$$-\boldsymbol{\xi}_\perp^* \cdot (\boldsymbol{\xi}_\perp \cdot \nabla)(\rho \mathbf{v} \cdot \nabla \mathbf{v})) - 2\hat{Z}\boldsymbol{\xi}_\perp \cdot (\mathbf{B} \cdot \nabla)(\mathbf{v} \cdot \nabla \mathbf{v})\Big].$$

An explicit expression for $\mathbf{Q}_\perp$ in terms of $\hat{X}$ can be derived from

$$\mathbf{Q}_\perp = -B^{-2}\mathbf{B} \times [\nabla\psi(\mathbf{B} \cdot \nabla)\nabla\alpha - \nabla\alpha(\mathbf{B} \cdot \nabla)\nabla\psi] \cdot \boldsymbol{\xi}_\perp.$$

We find

$$\mathbf{Q}_\perp = \mathcal{J}^{-1}B^{-2}\mathbf{B} \times (\nabla\zeta - q\nabla\theta - i\nabla q\partial_x)(ix + \partial_\theta)\hat{X}.$$

It is clear from this result that the eigenmode representation will lead to a rather formidable partial differential equation containing fourth order derivatives. Completing the derivation of this equation presents no fundamental difficulty. For the sake of conciseness, however, we will instead transform to the eikonal representation, which yields much more compact equations. We, therefore, replace $\hat{X}$ by its inverse Fourier transform

$$\hat{X}(x,\theta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\eta,\theta)\, e^{-i\eta x}\, d\eta,$$

and similarly for $\hat{Z}$. We then change variables to (η, τ), where $\tau = \eta - \theta$. This results in $i\partial_x \to \eta$ and $ix + \partial_\theta \to \partial_\eta$, so that in the eikonal representation one has

$$\mathbf{Q}_\perp = \mathbf{N}\,\mathbf{B} \cdot \nabla X$$

and $\boldsymbol{\xi}_\perp = \mathbf{N}X$, where $\mathbf{N} = \mathbf{B} \times \mathbf{k}/(nB^2)$.

We next examine the kinetic energy $\mathcal{T}$. In the eikonal representation, the convective time derivative is

$$\frac{d}{dt} = \left[-i\tilde{\omega} - \dot{\Omega} \left(\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \tau} \right) - \Omega \hat{\mathbf{z}} \times \right],$$

where the two partial derivatives can be abbreviated as a constant-θ time derivative,

$$\left. \frac{\partial}{\partial t} \right|_{\theta} = \dot{\Omega} \left(\left. \frac{\partial}{\partial \eta} \right|_{\tau} + \left. \frac{\partial}{\partial \tau} \right|_{\eta} \right).$$

Note that the time derivative does not commute with the wavevector k.

The ballooning equations of motion Eqs. (23)-(24), are now easily obtained from the action $\mathcal{L}$.

Appendix C: Asymptotic Ballooning Equations

The derivation of the exterior ballooning equation presented in this appendix follows a similar derivation by Kotschenreuther.[29] The asymptotic equations for large η are derived by separating the displacement into the sum of its slowly and rapidly varying parts. The slowly varying part of the perpendicular displacement, $\overline{X}$, is given by Eq. (32); its fluctuating part is $\widetilde{X} = X - \overline{X}$. The parallel displacement Z is separated likewise into $\overline{Z}$ and $\tilde{Z}$.

The ballooning equations can themselves be separated in the same fashion. The aim of the derivation presented here is then to reduce the resulting system of equations to a single equation for $\overline{X}$.

We adopt the ordering described in Sec. III, and further assume that $\eta \sim \epsilon^{-1}$ and that, by convention, $\overline{X} \sim 1$. The relative order of the remaining unknowns is then determined by balancing the dominant terms in the equations. By inspection of the fluctuating part of the perpendicular equation one deduces that $\widetilde{X} \sim \epsilon$ and $\tilde{Z} \sim \epsilon^{-1}$. Note that in the averaged perpendicular equation the leading order contribution of $\tilde{Z}$ is compensated by $A\overline{X}$, and the higher order corrections to these terms are needed. It is convenient to introduce the auxilliary dependent variable F defined by

$$F = S^2(\mathbf{B} \cdot \boldsymbol{\nabla} Z - AX).$$

From the averaged part of the parallel equation, one finds that $\overline{F} \sim \epsilon^{5/2}$. Thus, $\overline{Z} \sim \epsilon^{-1/2}$ and neither $\overline{Z}$ nor $\overline{F}$ will contribute to the other equations.

The fluctuating part of the parallel equation determines $\tilde{F}$:

$$\frac{\partial \tilde{F}}{\partial \eta} = \Omega\Omega'\eta \, \frac{\partial (R^2)}{\partial \eta} \frac{\partial \overline{X}}{\partial \tau} + \frac{\mathcal{J} B^2}{\rho} \frac{\partial^2 \tilde{Z}}{\partial t^2} + O(\epsilon^{7/2}), \tag{C1}$$

where primes denote derivatives with respect to the flux ψ. The first term on the right hand side is of order $\epsilon^{3/2}$ and the second term is of order ϵ^2. We integrate this equation, keeping only the lowest order term:

$$\frac{\partial \tilde{Z}}{\partial \theta} - \eta \frac{\partial a}{\partial \eta} \overline{X} = \frac{\mathcal{J}\Omega\Omega'\eta}{S^2} \left(R^2 - \frac{\langle R^2/S^2 \rangle}{\langle 1/S^2 \rangle} \right) \frac{\partial \overline{X}}{\partial \tau} + O(1). \tag{C2}$$

The integration constant in Eq. (C2) has been determined so as to satisfy the integrability condition for $\tilde{Z}$. The brackets denote the flux surface average:

$$\langle g \rangle = \frac{1}{2\pi \overline{\mathcal{J}}} \oint \mathcal{J} g \, d\eta.$$

$\tilde{Z}$ is now determined, to lowest order by integrating Eq. (C2):

$$\tilde{Z} = \eta a \overline{X} + O(\epsilon^{-1/2}). \tag{C3}$$

This result can be used to eliminate $\tilde{Z}$ from Eq. (C1).

We next turn to the perpendicular ballooning equation. Integration of the fluctuating part of this equation yields

$$\frac{\partial\widetilde{X}}{\partial\eta} = -\left(1 - \frac{\mathcal{J}B^2/|\nabla\psi|^2}{\langle B^2/|\nabla\psi|^2\rangle}\right)\frac{\partial\overline{X}}{\partial\eta} - \frac{\mathcal{J}B^2}{|\nabla\psi|^2}\left(\sigma - \frac{\langle\sigma B^2/|\nabla\psi|^2\rangle}{\langle B^2/|\nabla\psi|^2\rangle}\right)\frac{\overline{X}}{\eta q'}, \quad \text{(C4)}$$

while the slowly varying part is

$$q'^2\frac{\partial}{\partial\eta}\left(\eta^2\left\langle\frac{|\nabla\psi|^2}{\mathcal{J}^2B^2}\right\rangle\frac{\partial\overline{X}}{\partial\eta} + \eta^2\left\langle\frac{|\nabla\psi|^2}{\mathcal{J}^2B^2}\frac{\partial\widetilde{X}}{\partial\eta}\right\rangle\right) - \langle V\widetilde{X}\rangle - \langle V\rangle\overline{X} + \langle A\widetilde{F}\rangle$$

$$= -\Omega q'\eta\left\langle\frac{\partial}{\partial\eta}(R^2)\frac{\partial\widetilde{Z}}{\partial t}\right\rangle + \left\langle\rho\mathcal{J}\frac{\partial}{\partial t}\left(\frac{|\nabla\psi|^2}{B^2}\eta^2\frac{\partial\overline{X}}{\partial t}\right)\right\rangle. \quad \text{(C5)}$$

Equations (C1)-(C4) can now be used to eliminate $\widetilde{Z}$, $\widetilde{F}$ and $\widetilde{X}$ from Eq. (C5) in favor of the slowly varying part of the perpendicular component of the displacement, $\overline{X}$. The result is Eq. (33), with

$$D_I = E + F + H, \quad \text{(C6)}$$

$$E = \frac{1}{q'^2}\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle\left[q'\left(I' + \frac{\langle\sigma B^2\rangle}{\langle B^2\rangle}\right) + \left\langle R\frac{\partial R}{\partial\psi}\left(\frac{\partial(\Omega^2\rho)}{\partial\psi}\right)_R\right\rangle\right]$$

$$F = \frac{1}{q'^2}\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle\left[\left\langle\frac{\sigma^2B^2}{|\nabla\psi|^2}\right\rangle - \left\langle\frac{\sigma B^2}{|\nabla\psi|^2}\right\rangle^2\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle^{-1}\right] - \left\langle\frac{1}{B^2}\left(\frac{\partial p}{\partial\psi}\right)_R\right\rangle$$

$$H = \frac{1}{q'^2}\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle\left[\left\langle\frac{\sigma B^2}{|\nabla\psi|^2}\right\rangle\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle^{-1} - \frac{\langle\sigma B^2\rangle}{\langle B^2\rangle}\right]$$

$$M = \left\langle\frac{\rho|\nabla\psi|^2}{B^2}\right\rangle^{-1}\left[\left\langle\frac{\rho|\nabla\psi|^2}{B^2}\right\rangle + \left\langle\frac{a^2B^2}{\rho}\right\rangle - \langle a\rangle\left\langle\frac{aB^2}{\rho}\right\rangle\right.$$

$$\left. + \left\langle\frac{\Omega^2R^4}{S^2}\right\rangle - \left\langle\frac{\Omega R^2}{S^2}\right\rangle^2\left\langle\frac{1}{S^2}\right\rangle^{-1}\right].$$

References

[1]D. Dobrott, D. B. Nelson, J. M. Greene, A. H. Glasser, M. S. Chance, and E. A. Frieman, Phys. Rev. Lett. **39**, 943 (1977)

[2]J. W. Connor, R. J. Hastie, and J. B. Taylor, Phys. Rev. Lett. **40**, 396 (1978).

[3]Y. C. Lee and J. W. VanDam, in *Proceedings of the Finite Beta Theory Workshop*, edited by B. Coppi and W. Sadowski (U.S. Department of Energy, Washington, DC, 1979), p. 93.

[4]A. H. Glasser, in Ref. 3, p.55.

[5]R. L. Dewar and A. H. Glasser, Phys. Fluids **26**, 3038 (1983).

[6]J. W. Connor, R. J. Hastie, and J. B. Taylor, Proc. R. Soc. Lond. A. **365**, 1 (1979).

[7]E. Hameiri and P. Lawrence, J. Math. Phys. **25**, 396 (1984).

[8]A. Bhattacharjee, in *Theory of Fusion Plasmas*, edited by A. Bondeson, E. Sindoni, and F. Troyon (Editrice Compositori, Bologna, Italy, 1988), pp. 47-68.

[9]A. Bhattacharjee, R. Iacono, J. L. Milovich, and C. Paranicas, Phys. Fluids **B1**, 2207 (1989).

[10]S. D. Scott, P. H. Diamond, R. J. Fonck, R. J. Goldston, R. B. Howell, K. P. Jaehnig, G. Schilling, E. J. Synakowski, M. C. Zarnstorff, C. E. Bush, E. Fredrickson, K. W. Hill A. C. Janos, D. K. Mansfield, D. K. Owens, H.

Park, G. Pautasso, A. T. Ramsey, J. Schivell, G. D. Tait, W. M. Tang, and G. Taylor, Phys. Rev. Lett. **64**, 531 (1990)

[11]B. Grek, H. Park, R. J. Goldston, D. W. Johnson, D. K. Mansfield, and J. Schivell, in *Proceedings of the 14th European Conference on Controlled Fusion and Plasma Physics, Madrid*, edited by F. Englemann and J. L. Alvarez Rivaz (European Physical Society, Petit Lancy, Switzerland, 1987), Vol. I, pp. 132-135.

[12]R. D. Hazeltine and J. D. Meiss, Phys. Rep. **121**, 1 (1985)

[13]W. A. Cooper, Plasma Phys. Controlled Fusion, **30**, 1805 (1988).

[14]F. Pegoraro, Phys. Lett. A **142**, 384 (1989).

[15]E. Hameiri and S. T. Chun, Phys. Rev. A **41**, 1186 (1990).

[16]K. V. Roberts and J. B. Taylor, Phys. Fluids **8**, 315 (1965).

[17]F. Pegoraro and T. J. Schep, Phys. Fluids **24**, 478 (1981).

[18]R. L. Dewar, J. Manickam, R. C. Grimm, and M. S. Chance, Nucl. Fusion **21**, 493 (1981)

[19]J. W. Connor and J. B. Taylor, Phys. Fluids **30**, 3180 (1987).

[20]R. D. Hazeltine, D. A. Hitchcock, and S. M. Mahajan, Phys. Fluids **24**,180 (1981).

[21]W. A. Newcomb, Phys. Fluids B **2**, 86 (1990).

[22]R. D. Hazeltine and W. A. Newcomb, Phys. Fluids B **2**, 7 (1990).

[23]A. Hassam and R. M. Kulsrud, Phys. Fluids **21**, 2271 (1978).

[24]F. L. Hinton and S. K. Wong, Phys. Fluids **28**, 3082 (1985).

[25]E. Hameiri, Phys. Fluids **26**, 230 (1983).

[26]E. K. Maschke and H. Perrin, Plasma Phys. **22**, 679 (1980)

[27]E. Frieman and M. Rotenberg, Rev. Mod. Phys. **32**, 898 (1960).

[28]L. Chen, R. B. White, C. Z. Cheng, F. Romanelli, J. Weiland, R. Hay J. W. Van Dam, M. N. Rosenbluth, S. T. Tsai, and D. C. Barnes, in *Plasma Phys. Controlled Nuclear Fusion Research, 1984* (IAEA, Vienna, 1985), Vol. 2, pp. 59-66.

[29]M. Kotschenreuther, Phys. Fluids **29**,2898 (1986).

[30]A. Bondeson, R. Iacono, and A. Bhattacharjee, Phys.Fluids **30**, 2167 (1987).

[31]A. Bondeson and R. Iacono, Phys. Fluids B **1**, 1431 (1989).

[32]J. M. Greene and M. S. Chance, Nucl. Fusion **21**, 453 (1981).

[33]M. S. Chance, in *Theory of Fusion Plasmas*, edited by A. Bondeson, E. Sindoni and F. Troyon (Editrice Compositori, Bologna, Italy, 1987), pp. 87-105.

[34]R. L. Dewar, M. S. Chance, A. H. Glasser, J. M. Greene, and E. A. Frieman, private communication.

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