Ballooning mode spectrum in general toroidal systems

R. L. Dewar and A. H. Glasser

Citation: Physics of Fluids (1958-1988) **26**, 3038 (1983); doi: 10.1063/1.864028
View online: http://dx.doi.org/10.1063/1.864028
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof1/26/10?ver=pdfcov
Published by the AIP Publishing

Ballooning mode spectrum in general toroidal systems

R. L. Dewar[a)] and A. H. Glasser[b)]
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08544

(Received 24 March 1982; accepted 20 June 1983)

A WKB formalism for constructing normal modes of short-wavelength ideal hydromagnetic, pressure-driven instabilities (ballooning modes) in general toroidal magnetic containment devices with sheared magnetic fields is developed. No incompressibility approximation is made. A dispersion relation is obtained from the eigenvalues of a fourth-order system of ordinary differential equations to be solved by integrating along a line of force. Higher-order calculations are performed to find the amplitude equation and the phase change at a caustic. These conform to typical WKB results. In axisymmetric systems, the ray equations are integrable, and semiclassical quantization leads to a growth rate spectrum consisting of an infinity of discrete eigenvalues, bounded above by an accumulation point. However, each eigenvalue is infinitely degenerate. In the nonaxisymmetric case, the rays are unbounded in a four-dimensional phase space, and semiclassical quantization breaks down, leading to broadening of the discrete eigenvalues and the accumulation point of the axisymmetric unstable spectrum into continuum bands. Analysis of a model problem indicates that the broadening of the discrete eigenvalues is numerically very small, the dominant effect being broadening of the accumulation point.

I. INTRODUCTION

Ideal hydromagnetic stability theory currently plays an important role in design studies for high-β (β = plasma pressure/magnetic pressure) toroidal magnetic confinement devices.[1]

Although the validity of the ideal model is limited to modes with wavelengths much greater than particle gyroradii, and with growth rates faster than the diamagnetic drift frequency, resistive rates, etc., its simplicity makes accurate inclusion of geometric effects possible. Thus, a practical way to proceed is, first, to determine the ideal hydromagnetic instability spectrum; then, to examine the eigenmodes for consistency with the use of the ideal model; and, finally, to use a more physically correct model where necessary.

Assuming that the plasma is not unstable to a gross free boundary mode (kink), one finds the rather surprising result that the most unstable ideal modes (interchanges and ballooning modes) always lie outside the domain of validity of the ideal model since they are localized to an indefinitely small region. Even if such localized modes survive (as drift waves) in a finite-Larmor-radius treatment, they would not be expected to be catastrophic for confinement.

Interest then shifts to more extended modes with wavelengths short compared with the machine size but long compared with a gyroradius. We shall call this the intermediate regime. In axisymmetric geometry (tokamaks), such modes do indeed exist,[2] and can be treated by a modified version of WKB theory.[3] The limitation on the validity of ideal theory comes from violation of the criterion that the growth rate be larger than the diamagnetic drift frequency.[4] Even with this limitation, there is still an intermediate regime where finite-Larmor-radius (FLR) effects are negligible.

One might assume that the intermediate regime exists quite generally in machines which are sufficiently close to axisymmetry. It is the main result of this paper that this is not strictly the case, despite the fact that locally the ballooning formalism generalizes naturally.[5] Due to a previously overlooked degeneracy in the axisymmetric spectrum, the breaking of axisymmetry has a profound effect, coupling energy to arbitrarily short wavelengths. The relevance of ideal ballooning theory to nonaxisymmetric systems is thus called into question. A similar conclusion has been reached in mirror geometry.[6] Numerical study of a model for a tokamak with strong toroidal ripple, however, shows that the coupling is very weak for low-order modes.

In Sec. II we set up the coordinates and function space in which solutions are to be found. The periodicity problem[7–11] is treated by using the concept of a covering space,[12] rather than through the use of the "ballooning transformation" (a critique of which is given in Appendix A).

In Secs. III and IV we review the linearized equation of motion, its relation to the energy principle, and its expansion under the WKB and ballooning (modified-WKB) orderings. In Sec. V we give an exact formal elimination of the longitudinal component of the plasma displacement, and we show that the structure of the equation is such that the ballooning expansion can (in principle) be carried through to all orders in the expansion parameter without encountering a violation of the ordering. This remains true in the boundary layer ordering needed in the vicinity of a bounding caustic, which is treated in Sec. X, after the dispersion relation, ray, and amplitude equations are discussed in Secs. VI–IX. A numerical model for a tokamak with ripple is presented in Sec. VIII. The periodicity problem is discussed in Secs. VII and XI, where the quantization condition for the axisymmetric case is derived, and a qualitative argument is given for the breakdown of the Kolmogorov–Arnol'd–Moser (KAM)

[a)] Present address: Department of Theoretical Physics, Research School of Physical Sciences, the Australian National University, Canberra A. C. T. 2601, Australia.

[b)] Permanent address: Physics Department, Auburn University, Auburn, Alabama 36849.

theorem[13] in the nonaxisymmetric case, based on the theory of one-dimensional mappings. This argument shows that ballooning modes are generically singular, and that there is an unstable continuous spectrum in ideal hydromagnetics in apparent contradiction with accepted belief.[14,15] The origin of the discrepancy is that the type of singularity normally considered in axisymmetric geometry is localized to a single magnetic surface, whereas the ballooning singularities span a range of surfaces. We show in Appendix B that the result for the surface singularities does generalize to the nonaxisymmetric case. The spectrum is computed for the model problem as a function of the ripple parameter in Sec. XI.

II. COORDINATES AND FUNCTION SPACES

There is considerable analytical and computational advantage to be gained from using coordinates (ψ,θ,ζ) in which the field lines are "straight,"[16] that is, in which the equilibrium magnetic field $\mathbf{B}$ can be represented in the form

$$\mathbf{B} = \nabla\zeta \times \nabla\psi + q(\psi)\nabla\psi\times\nabla\theta, \tag{1}$$

where $2\pi\psi$ is the poloidal magnetic flux between the magnetic axis and a magnetic surface, $\psi = \text{const}$ (assuming nested toroidal surfaces), and $\theta(\mathbf{r})$ and $\zeta(\mathbf{r})$ are, respectively, generalized poloidal and toroidal angles (Fig. 1). The function $q(\psi)$ is the safety factor$\equiv 2\pi$/rotational transform.

The existence of nested toroidal surfaces is not guaranteed in nonaxisymmetric geometry, but it is reasonable to assume that any containment device will be designed so that Eq. (1) can represent $\mathbf{B}$ to a good approximation, at least at low β. The study of high-β stability can be viewed as a first step towards ascertaining the existence of surfaces under fusion conditions.

The domain of the position vector $\mathbf{r}(\theta,\zeta)$ on a given magnetic surface is normally taken to be the square unit cell

$$D_1:\quad 0\leqslant\theta\leqslant 2\pi,\quad 0\leqslant\zeta<2\pi,$$

with the edges $\theta = 0, 2\pi, \zeta = 0, 2\pi$, topologically identified. However, $\mathbf{r}$ can be analytically continued to a function periodic in θ and ζ defined on the infinite domain

$$D_\infty:\quad -\infty<\theta<\infty,\quad -\infty<\zeta<\infty,$$

with the topology of the plane (i.e., $D_\infty \equiv \mathbb{R}^2$). The domain D_∞, together with the function $\mathbf{r}$, forms a *covering space*[12] for the torus.

The length element in both cases is $|d\mathbf{r}|$, so what we have done is to invent a fictitious plasma with the metric of a torus, but with the topology of a slab. The advantage of this trick is that a broader class of solutions of the equations of motion is admissible on the covering space than on the unit cell, thus making the $\mathbf{k}\cdot\mathbf{B} = 0$ constraint (Sec. IV) easy to satisfy. The physical solutions, those 2π periodic in θ and ζ, must lie within the completion of the solution space, so we have not lost anything by going over to the covering space. However, to recover the physical solutions we must first construct the *general solution* on the covering space. This approach is contrasted with the ballooning transformation approach in Appendix A.

Although we do not pretend to great rigor, we seek to construct the theory within the framework provided by standard functional analysis.[17] In particular, we suppose the space of functions defined on the covering space to be a Hilbert space; a space of normalizable functions with the norm being provided by an inner product. The inner product which arises naturally is that which makes the force operator Hermitian.

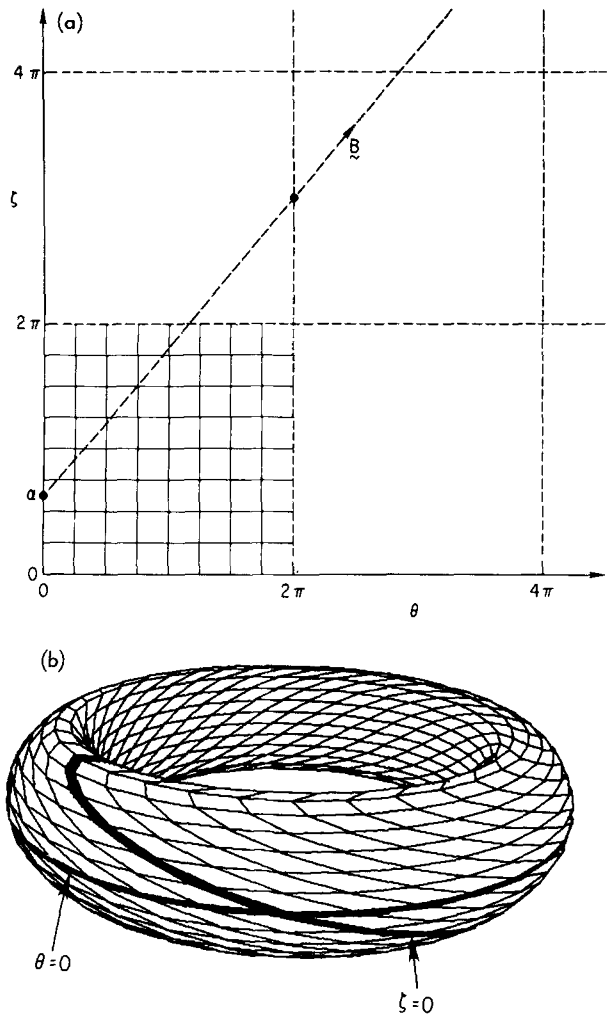

FIG. 1. A square grid on a unit cell D_1 of the covering space D_∞ (a) is mapped onto a grid on a toroidal magnetic surface by the position function $\mathbf{r}(\psi,\theta,\zeta)$, with $\psi = \text{const}$. The position function is periodic on the square lattice shown in (a). The oblique dotted line in (a) is a typical magnetic field line: $\zeta = \alpha + q(\psi)\theta$, with $\alpha = \text{const}$. (b) shows an axisymmetric case with $q = 1.185$.

Because $\mathbf{B}$ must satisfy the equilibrium condition

$$\nabla p = \mathbf{j}\times\mathbf{B}, \tag{2}$$

where $p(\psi)$ is the equilibrium pressure and $\mathbf{j} = \nabla\times\mathbf{B}$ the equilibrium current density, the angles ζ and θ are not entirely arbitrary. From Eq. (2) we see that $\mathbf{j}\cdot\nabla\psi = 0$, whence we have the condition

$$\nabla\cdot(|\nabla\psi|^2\nabla_s\zeta) = q\nabla\cdot(|\nabla\psi|^2\nabla_s\theta), \tag{3}$$

where

$$\nabla_s \equiv (\mathsf{I} - \nabla\psi\nabla\psi/|\nabla\psi|^2)\cdot\nabla, \tag{4}$$

I being the unit dyadic. Equation (3) shows that one of either θ or ζ may be arbitrarily specified, but that the angle not specified must be found by solving a two-dimensional Poisson's equation. Alternatively, we can specify the Jacobian

$$J = (\nabla\psi\cdot\nabla\theta\times\nabla\zeta)^{-1}, \tag{5}$$

and determine both ζ and θ. Although there is some analytical simplification to be had by requiring J to be a function of ψ alone, in the manner of Hamada,[16] computational experience[2,18] with high-β equilibria has shown that it is preferable to retain control over θ by allowing J to vary within a surface.

In (ψ,θ,ζ) coordinates, the operator $\mathbf{B}\cdot\nabla$ is represented as

$$\mathbf{B}\cdot\nabla = J^{-1}(\partial_\theta + q\,\partial_\zeta), \tag{6}$$

where ∂_θ denotes $(\partial/\partial\theta)_{\zeta,\psi}$.

Equilibrium quantities are invariant under the discrete symmetry operations T (toroidal rotation through 2π radians), and P (poloidal rotation through 2π radians):

$$\begin{aligned} &T\colon\quad (\psi,\theta,\zeta)\mapsto(\psi,\theta,\zeta+2\pi),\\ &P\colon\quad (\psi,\theta,\zeta)\mapsto(\psi,\theta+2\pi,\zeta). \end{aligned} \tag{7}$$

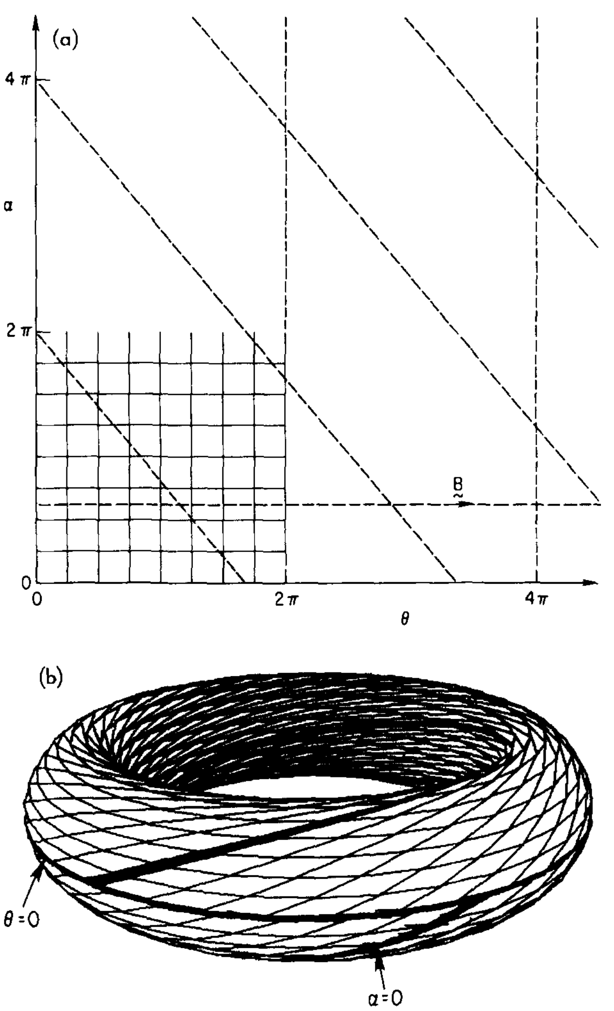

FIG. 2. A square grid on a portion of the covering space (a) is mapped onto a grid on the magnetic surface (b) by the position function $\mathbf{r}(\alpha,\psi,\theta)$. Since $\mathbf{r}$ is periodic on the skewed lattice shown in (a), the image mesh is broken along the image of $\theta = 2\pi$ [field lines do not close on themselves if q is irrational, and horizontal lines in (a) are field lines].

In the following work, it will be found convenient to introduce the Clebsch representation for the full magnetic field

$$\mathbf{B} = \nabla\alpha\times\nabla\psi, \tag{8}$$

where

$$\alpha = \zeta - q\theta, \tag{9}$$

and to use the set (α,ψ,θ) as our coordinate system (Fig. 2).

In (α,ψ,θ) coordinates, the $\mathbf{B}\cdot\nabla$ operator becomes

$$\mathbf{B}\cdot\nabla = J^{-1}\,\partial_\theta, \tag{10}$$

where now $\partial_\theta \equiv (\partial/\partial\theta)_{\alpha,\psi}$ and $J = (\nabla\alpha\cdot\nabla\psi\times\nabla\theta)^{-1} = (\nabla\psi\cdot\nabla\theta\times\nabla\zeta)^{-1}$, as in Eq. (5).

In these coordinates the symmetry operations are represented as follows:

$$\begin{aligned} &T\colon\quad (\alpha,\psi,\theta)\mapsto(\alpha+2\pi,\psi,\theta),\\ &P\colon\quad (\alpha,\psi,\theta)\mapsto(\alpha-2\pi q,\psi,\theta+2\pi). \end{aligned} \tag{11}$$

III. LINEARIZED EQUATION OF MOTION

Following Bernstein *et al.*[19] we denote by $\boldsymbol{\xi}$ the infinitesimal displacement of a fluid element from its equilibrium position, and assume a normal mode to be excited, so that the time dependence of $\boldsymbol{\xi}$ is as $\exp(-i\omega t)$. Then the equation of motion is the eigenvalue equation

$$(\omega^2\rho\mathsf{I} + \mathsf{F})\cdot\boldsymbol{\xi} = 0, \tag{12}$$

where F is the force density operator,[19] and ρ is the mass density. To exclude kink modes from consideration, we assume there to be a conducting wall at the edge of the plasma, so that the boundary condition there is $\boldsymbol{\xi}\cdot\nabla\psi = 0$.

We use a representation of F corresponding to a form of δW introduced by Furth *et al.*[20,21]

$$\begin{aligned} \mathsf{F} = {} & \mathsf{P}_\perp\cdot(\nabla - 2\boldsymbol{\kappa})B^2(\nabla + 2\boldsymbol{\kappa})\cdot\mathsf{P}_\perp + \nabla\gamma p\nabla\\ & - [(\nabla\psi)\mathbf{B}\cdot\nabla(\nabla\alpha/B) - (\nabla\alpha)\mathbf{B}\cdot\nabla(\nabla\psi/B)]\\ & \cdot[(\nabla\psi/B)\mathbf{B}\cdot\nabla(\nabla\alpha) - (\nabla\alpha/B)\mathbf{B}\cdot\nabla(\nabla\psi)]\\ & + \sigma[(\nabla\psi)\mathbf{B}\cdot\nabla(\nabla\alpha) - (\nabla\alpha)\mathbf{B}\cdot\nabla(\nabla\psi)] + 2\boldsymbol{\kappa}\nabla p, \end{aligned} \tag{13}$$

where $\mathsf{P}_\perp$ projects onto the plane perpendicular to $\mathbf{e}_B \equiv \mathbf{B}/B$,

$$\mathsf{P}_\perp \equiv \mathsf{I} - \mathbf{e}_B\mathbf{e}_B, \tag{14}$$

$\boldsymbol{\kappa}$ is the equilibrium field line curvature vector

$$\boldsymbol{\kappa} \equiv \mathbf{e}_B\cdot\nabla\mathbf{e}_B = B^{-2}\mathsf{P}_\perp\cdot\nabla(p + B^2/2), \tag{15}$$

and $\sigma \equiv \mathbf{j}\cdot\mathbf{B}/B^2$ measures the parallel equilibrium current density. In Eq. (13), the ∇ operator acts on everything to its right (except in $\nabla\alpha$, $\nabla\psi$, and ∇p). Hermiticity of F under the inner product for functions defined on D_n ($= D_1$ or D_∞),

$$(\mathbf{f},\mathbf{g})_n = \int_0^{\psi_a} d\psi \int\!\!\int_{D_n} d\alpha\, d\theta\, J\,\mathbf{f}^*\cdot\mathbf{g}, \tag{16}$$

follows from the identity

$$\mathbf{B}\cdot\nabla\sigma = 2\mathbf{B}\times\nabla p\cdot\boldsymbol{\kappa}/B^2. \tag{17}$$

Since[19]

$$\delta W = -\frac{1}{2}\int d^3r\,\boldsymbol{\xi}^*\cdot\mathsf{F}\cdot\boldsymbol{\xi} = -\frac{1}{2}(\boldsymbol{\xi},\mathsf{F}\cdot\boldsymbol{\xi})_1, \tag{18}$$

there is a one-to-one correspondence between the terms of Eq. (13) and those of δW.[20,21] The first term of Eq. (13) corresponds to the stabilizing field line compression term $|\mathbf{Q}\cdot\mathbf{B} - \boldsymbol{\xi}\cdot\nabla p|^2/B^2$ [where $\mathbf{Q}\equiv\nabla\times(\boldsymbol{\xi}\times\mathbf{B})$], the second term corresponds to the stabilizing fluid compression term $\gamma p|\nabla\cdot\boldsymbol{\xi}|^2$, and the third to the stabilizing field line tension term $|\mathbf{Q}\times\mathbf{B}|^2/B^2$. The fourth term corresponds to $-\sigma\boldsymbol{\xi}^*\times\mathbf{B}\cdot\mathbf{Q}$, which represents the potentially destabilizing influence of the parallel current, giving rise to the kink instability. The fifth term corresponds to $-2\boldsymbol{\xi}\cdot\nabla p\boldsymbol{\xi}^*\cdot\boldsymbol{\kappa}$, which represents the potentially destabilizing interaction of plasma pressure gradient and field line curvature. It is this term which gives rise to ballooning instability.

IV. BALLOONING ORDERING

In the standard WKB theory for hydromagnetic waves,[22] one orders both the wave vector $\mathbf{k}$ and the frequency ω to be large, thus finding the Alfvén wave

$$\omega_A^2\,\rho = (\mathbf{k}\cdot\mathbf{B})^2, \quad \boldsymbol{\xi}\sim\mathbf{k}\times\mathbf{B}, \tag{19}$$

and the slow and fast magnetosonic waves, whose respective dispersion relations and polarizations, in the limit $|\mathbf{k}\cdot\mathbf{B}/k| \ll 1$, are

$$\omega_S^2\,\rho \simeq \frac{\gamma p(\mathbf{k}\cdot\mathbf{B})^2}{B^2+\gamma p}, \quad \boldsymbol{\xi}\sim(B^2+\gamma p)k^2\mathbf{B} - \gamma p\mathbf{k}\cdot\mathbf{B}\mathbf{k}_\perp, \tag{20}$$

$$\omega_F^2\,\rho \simeq (B^2+\gamma p)k^2, \quad \boldsymbol{\xi}\sim\gamma p\mathbf{k}\cdot\mathbf{B}\mathbf{B} + (B^2+\gamma p)B^2\mathbf{k}_\perp, \tag{21}$$

where $\mathbf{k}_\perp$ is the projection of $\mathbf{k}$ perpendicular to $\mathbf{B}$. The field line and fluid compression terms in Eq. (13) enter symmetrically into both the slow and fast magnetosonic dispersion relations.

Because F is Hermitian, ω^2 must be real: Instability occurs if $\omega^2 < 0$. Thus, higher-order corrections can never destabilize the fast magnetosonic wave, but they can destabilize both the Alfvén and slow magnetosonic modes when $\omega^2 = 0$ to lowest order. That is, when $\mathbf{k}\cdot\mathbf{B} = 0$; in which case we also have $\mathbf{k}\cdot\boldsymbol{\xi} = 0$. Thus we conclude that short-wavelength instabilities are locally flute-like, approximately incompressible, and are a combination of Alfvén and slow magnetosonic waves.

The ballooning ordering, then, is one where $\mathbf{k}$ is large, but $\mathbf{k}\cdot\mathbf{B}$ and ω are finite. Equivalently, we can introduce an expansion parameter ϵ, write

$$\boldsymbol{\xi} = \hat{\boldsymbol{\xi}}(\mathbf{r},\epsilon)\exp[i\epsilon^{-1}S(\mathbf{r}) - i\omega t], \tag{22}$$

and define $\mathbf{k}\equiv\nabla S$ to be $O(\epsilon^0)$. Because the amplitude $\hat{\boldsymbol{\xi}}$ can describe slow deviations from flute-like behavior, we are free to constrain $\mathbf{k}\cdot\mathbf{B}$ to vanish to all orders. That is, we require

$$\mathbf{B}\cdot\nabla S = 0. \tag{23}$$

From Eq. (10) we see that the solution is $S = S(\alpha,\psi)$, so that $\exp(i\epsilon^{-1}S)$ is not a single-valued function of $\mathbf{r}$. Thus we are forced from the outset to consider solutions defined on the covering space.

In systems with magnetic shear [$q'(\psi)\neq 0$], it is convenient to use the representation

$$\mathbf{k} = k_\alpha\nabla\alpha + k_q\nabla q, \tag{24}$$

where $k_\alpha(\alpha,\psi)\equiv\partial_\alpha S$, and $k_q\equiv(\partial_\psi S)/q'$. Unlike the standard WKB case, $\mathbf{k}$ has only two degrees of freedom. Fortunately, this constraint does not lead to problems at any order in the WKB expansion which we are now about to develop.

The aim is to expand $\hat{\boldsymbol{\xi}}$ and Eq. (12) in a power series in ϵ. We can formally require Eq. (12) to be satisfied locally to all orders in ϵ, but this asymptotic equality is weaker than exact equality since an exponentially small error term can remain. In fact, we shall find a multitude of seemingly independent local solutions. By using asymptotic matching where the ϵ expansion breaks down, and requiring toroidal and poloidal periodicity, we can find the correct linear superposition of the locally independent asymptotic solutions to form a physically acceptable global normal mode.

V. ELIMINATION OF THE LONGITUDINAL COMPONENT

Substituting Eq. (22) in Eq. (12) and commuting the exponential factor with F (so that $\nabla\to i\epsilon^{-1}\mathbf{k}+\nabla$), we find

$$(\epsilon^{-2}\hat{\mathsf{F}}^{(-2)} + \epsilon^{-1}\hat{\mathsf{F}}^{(-1)} + \rho\omega^2\mathsf{I} + \hat{\mathsf{F}}^{(0)})\cdot\hat{\boldsymbol{\xi}} = 0, \tag{25}$$

where

$$\hat{\mathsf{F}}^{(-2)} = -(B^2+\gamma p)\mathbf{k}\mathbf{k}, \tag{26}$$

$$\hat{\mathsf{F}}^{(-1)} \equiv \hat{\mathsf{F}}_L + \hat{\mathsf{F}}_R, \tag{27}$$

$$\hat{\mathsf{F}}^{(0)} \equiv \mathsf{F}. \tag{28}$$

The two terms in $\hat{\mathsf{F}}^{(-1)}$ are defined by

$$\hat{\mathsf{F}}_L \equiv i[\mathsf{P}_\perp\cdot(\nabla - 2\boldsymbol{\kappa})B^2 + \nabla\gamma p]\mathbf{k}, \tag{29}$$

$$\hat{\mathsf{F}}_R \equiv i\mathbf{k}[B^2(\nabla + 2\boldsymbol{\kappa})\cdot\mathsf{P}_\perp + \gamma p\nabla], \tag{30}$$

where $\mathsf{P}_\perp$ is defined in Eq. (14).

It is also convenient to define a projector onto the plane transverse to $\mathbf{e}_k\equiv\mathbf{k}/k$ [where $k\equiv(\mathbf{k}\cdot\mathbf{k})^{1/2}$]

$$\mathsf{P}_t \equiv \mathsf{I} - \mathbf{e}_k\mathbf{e}_k. \tag{31}$$

This projector acts as an annihilator for the leading terms of $\hat{\mathsf{F}}$,

$$\mathsf{P}_t\cdot\hat{\mathsf{F}}^{(-2)} = \hat{\mathsf{F}}^{(-2)}\cdot\mathsf{P}_t = \mathsf{P}_t\cdot\hat{\mathsf{F}}_R = \hat{\mathsf{F}}_L\cdot\mathsf{P}_t = 0. \tag{32}$$

We decompose $\hat{\boldsymbol{\xi}}$ into its longitudinal and transverse parts

$$\hat{\boldsymbol{\xi}} = \hat{\xi}_l\mathbf{e}_k + \hat{\boldsymbol{\xi}}_t, \tag{33}$$

where $\hat{\boldsymbol{\xi}}_t \equiv \mathsf{P}_t\cdot\hat{\boldsymbol{\xi}}$.

Using Eqs. (32) we can now effect an exact formal elimination of the longitudinal component,

$$\hat{\xi}_l = \epsilon G\mathbf{e}_k\cdot[\hat{\mathsf{F}}_R + \epsilon(\rho\omega^2\mathsf{I} + \hat{\mathsf{F}}^{(0)})]\cdot\hat{\boldsymbol{\xi}}_t, \tag{34}$$

with Green's function G defined by inverting

$$G^{-1} \equiv G_0^{-1} - \epsilon\mathbf{e}_k\cdot\hat{\mathsf{F}}^{(-1)}\cdot\mathbf{e}_k - \epsilon^2\mathbf{e}_k\cdot(\rho\omega^2\mathsf{I} + \hat{\mathsf{F}}^{(0)})\cdot\mathbf{e}_k, \tag{35}$$

where

$$G_0^{-1} \equiv -\mathbf{e}_k\cdot\hat{\mathsf{F}}^{(-2)}\cdot\mathbf{e}_k = k^2(B^2+\gamma p). \tag{36}$$

Substituting back into Eq. (25) we obtain a formally exact equation for the transverse component

$$\mathsf{P}_t\cdot\mathsf{L}\cdot\hat{\boldsymbol{\xi}}_t = 0, \tag{37}$$

where

$$\mathsf{L} \equiv \rho\omega^2\mathsf{I} + \mathsf{F} + [\hat{\mathsf{F}}_L + \epsilon(\rho\omega^2\mathsf{I} + \mathsf{F})]\cdot\mathbf{e}_k G\mathbf{e}_k\cdot[\hat{\mathsf{F}}_R + \epsilon(\rho\omega^2\mathsf{I} + \mathsf{F})]. \tag{38}$$

The expansion of L is facilitated by using the identity

$$\mathsf{L} = (\mathsf{I} - \epsilon\hat{\mathsf{F}}_L\cdot\mathbf{e}_k G_0\mathbf{e}_k)^{-1}\cdot(\mathsf{L}_0 + \epsilon^2\mathsf{L}_0\cdot\mathbf{e}_k G\mathbf{e}_k\cdot\mathsf{L}_0)\cdot(\mathsf{I} - \epsilon\mathbf{e}_k G_0\mathbf{e}_k\cdot\hat{\mathsf{F}}_R)^{-1}, \quad (39)$$

which eliminates F in favor of the lowest order term of L,

$$\mathsf{L}_0 \equiv \rho\omega^2\mathsf{I} + \mathsf{F} + \hat{\mathsf{F}}_L\cdot\mathbf{e}_k G_0\mathbf{e}_k\cdot\hat{\mathsf{F}}_R. \quad (40)$$

On calculating L_0 explicitly in (α,ψ,θ) coordinates

$$\begin{aligned}\mathsf{L}_0 = \omega^2\rho\mathsf{I} &+ \left(\frac{\mathbf{B}}{JB^2}\partial_\theta + 2\boldsymbol{\kappa}\right)\frac{\gamma p B^2}{(B^2+\gamma p)}\left(\frac{1}{J}\partial_\theta\frac{\mathbf{B}}{B^2} - 2\boldsymbol{\kappa}\right)\\ &- J^{-1}(\nabla\psi\,\partial_\theta\,\nabla\alpha - \nabla\alpha\,\partial_\theta\,\nabla\psi)(JB^2)^{-1}\\ &\cdot(\nabla\psi\,\partial_\theta\,\nabla\alpha - \nabla\alpha\,\partial_\theta\nabla\psi)\\ &+ \sigma J^{-1}(\nabla\psi\,\partial_\theta\,\nabla\alpha - \nabla\alpha\,\partial_\theta\,\nabla\psi) + 2p'\boldsymbol{\kappa}\nabla\psi, \end{aligned} \quad (41)$$

we notice the crucial fact that the operators ∂_α and ∂_ψ in the ∇'s occurring in Eq. (40) cancel, leaving only $\partial_\theta \equiv J\,\mathbf{B}\cdot\nabla$. That is, L_0 *is an operator only along a magnetic line of force*. From Eq. (39) we see that the order in ∂_α or ∂_ψ of a term of the expansion of L is given by its order in ϵ. This is the fundamental reason why ballooning modes can be treated by a form of WKB theory projected onto the α-ψ plane.

Two more important observations are that L_0 does not contain $\mathbf{k}$, and that $\mathsf{P}_t\cdot\mathsf{L}_0\cdot\mathsf{P}_t$ does not contain the kinking term, proportional to σ in Eq. (41).

Finally, we observe that L_0 is Hermitian not only under the inner product defined by Eq. (16), but also under an inner product restricted to a line of force (α = const, ψ = const),

$$\langle\,\mathbf{f},\mathbf{g}\rangle = \int_{-\infty}^{\infty} ds\,\mathbf{f}^*\cdot\mathbf{g}, \quad (42)$$

where $ds \equiv J\,d\theta = dl/B$, where dl is the length of a field line element.

VI. LOWEST ORDER: EIGENVALUE EQUATION

Setting

$$\hat{\boldsymbol{\xi}} = \hat{\boldsymbol{\xi}}^{(0)} + \epsilon\hat{\boldsymbol{\xi}}^{(1)} + \epsilon^2\hat{\boldsymbol{\xi}}^{(2)} + \ldots, \quad (43)$$

and expanding Eq. (37) to $O(\epsilon^0)$, we have

$$\mathsf{P}_t\cdot\mathsf{L}_0\cdot\hat{\boldsymbol{\xi}}_t^{(0)} = 0. \quad (44)$$

Owing to the properties of L_0 discussed in the previous section, Eq. (44) is an ordinary differential equation in s, giving rise to a dispersion relation

$$\omega^2 = \lambda\,(\alpha,\psi,k_q/k_\alpha), \quad (45)$$

where $\lambda < 0$ is an eigenvalue such that the norm of $\hat{\boldsymbol{\xi}}_t^{(0)}$, defined by the inner product Eq. (42), is finite. Marginal modes ($\lambda = 0$), being at the edge of the continuous spectrum existing for $\lambda > 0$, are not necessarily normalized, and they need to be treated as a special case.[2,9]

The special form of the $\mathbf{k}$ dependence of λ arises from the fact that $\mathbf{k}$ enters into Eq. (44) only through the unit vector in the $\mathbf{k}$ direction

$$\mathbf{e}_k = (|\nabla\alpha|^2 + 2\theta_k\nabla\alpha\cdot\nabla q + \theta_k^2|\nabla q|^2)^{-1/2}(\nabla\alpha + \theta_k\nabla q), \quad (46)$$

where $\theta_k(\alpha,\psi)$ is the ratio of the two components of $\mathbf{k}$,

$$\theta_k \equiv k_q/k_\alpha. \quad (47)$$

To obtain a more practical form for the eigenvalue equation, we define a line-averaged Lagrangian

$$\mathscr{L}(\alpha,\psi) \equiv \tfrac{1}{2}\langle\hat{\boldsymbol{\xi}}^{(0)*},\mathsf{P}_t\cdot\mathsf{L}_0\cdot\mathsf{P}_t\cdot\hat{\boldsymbol{\xi}}^{(0)}\rangle. \quad (48)$$

Equation (44) is the Euler–Lagrange equation which extremizes $\mathscr{L}$ under arbitrary variations of $\hat{\boldsymbol{\xi}}^{(0)}$. The Lagrangian is defined with $\hat{\boldsymbol{\xi}}^{(0)*}$ as the first member of the inner product in order that it be valid in an evanescent region, where $\mathbf{k}$ is complex. In this paper, however, we are concerned only with real $\mathbf{k}$.

We now write $\hat{\boldsymbol{\xi}}_t^{(0)}$ ($= \hat{\boldsymbol{\xi}}^{(0)}$, since $\hat{\xi}_l^{(0)} = 0$) in the representation

$$\hat{\boldsymbol{\xi}}_t^{(0)} = \xi\,\mathbf{B}\times\mathbf{k}/B^2k_\alpha + \eta\mathbf{B}, \quad (49)$$

so that $\xi = \hat{\boldsymbol{\xi}}^{(0)}\cdot\nabla\psi$, $\eta = \hat{\boldsymbol{\xi}}^{(0)}\cdot\mathbf{B}/B^2$. Substituting into Eq. (48), we find

$$\begin{aligned}\mathscr{L} = -\frac{1}{2}\int_{-\infty}^{\infty} ds\,\Bigg[&\frac{\gamma p B^2}{(B^2+\gamma p)}\left(\dot\eta - \frac{2\boldsymbol{\kappa}\cdot\mathbf{B}\times\mathbf{k}}{B^2k_\alpha}\xi\right)^2\\ &+ \frac{k^2\dot\xi^2}{k_\alpha^2B^2} - 2p'\frac{\boldsymbol{\kappa}\cdot\mathbf{B}\times\mathbf{k}}{B^2k_\alpha}\xi^2\\ &- \omega^2\rho\left(\frac{k^2\xi^2}{k_\alpha^2B^2} + B^2\eta^2\right)\Bigg], \end{aligned} \quad (50)$$

where $\dot\xi \equiv d\xi/dx$, $\dot\eta \equiv d\eta/ds$. As remarked in Sec. V, the kinking term does not contribute at this order. Varying ξ and η, we find the Euler–Lagrange equations

$$\begin{aligned}\frac{d}{ds}\left(\frac{k^2\dot\xi}{k_\alpha^2B^2}\right) &+ \frac{2\boldsymbol{\kappa}\cdot\mathbf{B}\times\mathbf{k}}{B^2k_\alpha}\\ &\times\left[\frac{\gamma pB^2}{(B^2+\gamma p)}\left(\dot\eta - \frac{2\boldsymbol{\kappa}\cdot\mathbf{B}\times\mathbf{k}\xi}{B^2k_\alpha}\right) + p'\xi\right]\\ &+ \frac{\omega^2\rho k^2}{B^2k_\alpha^2}\xi = 0, \end{aligned} \quad (51)$$

$$\frac{d}{ds}\left[\frac{\gamma pB^2}{(B^2+\gamma p)}\left(\dot\eta - \frac{2\boldsymbol{\kappa}\cdot\mathbf{B}\times\mathbf{k}\xi}{B^2k_\alpha}\right)\right] + \omega^2\rho B^2\eta = 0. \quad (52)$$

This is a coupled pair of *ordinary* differential equations, and can easily be solved numerically, given an equilibrium. It is a fourth-order system because the usual approximation of incompressibility has not been made.

Again we observe that the $\mathbf{k}$ dependence occurs only through the ratio $\theta_k = k_q/k_\alpha$, since

$$\mathbf{k}/k_\alpha = \nabla\alpha + \theta_k\nabla q = (\nabla\zeta - q\nabla\theta) - (\theta - \theta_k)\nabla q. \quad (53)$$

Because ∇q, $\nabla\theta$, and $\nabla\zeta$ are quantities periodic in θ, the only aperiodic term in Eqs. (51) and (52) comes from the secular term $-(\theta - \theta_k)\nabla q$ in Eq. (53). This term arises from the interplay between the flutedness requirement, Eq. (23), and magnetic shear,[23] which means that the wave fronts (S = const) must twist around as we follow a field line (Fig. 3). Thus, although there is a channel around $\theta \approx \theta_k$, where the wave is approximately in phase on different surfaces ($\nabla\psi\cdot\nabla S \approx 0$), the phase must change more and more rapidly ($\nabla\psi\cdot\nabla S \to \pm\infty$) as $\theta \to \pm\infty$.

We argue in Appendix B that the asymptotic behavior of $\hat{\boldsymbol{\xi}}^{(0)}$ as $|\theta| \to \infty$, for $\omega^2 < 0$, is quasiexponential growth or decay. The requirement that $\hat{\boldsymbol{\xi}}^{(0)}$ be within the Hilbert space defined by Eq. (42) restricts us to the decaying solutions and imposes a sufficient number of constraints (4) that we expect to be able to find discrete eigenvalues as assumed in Eq. (45).

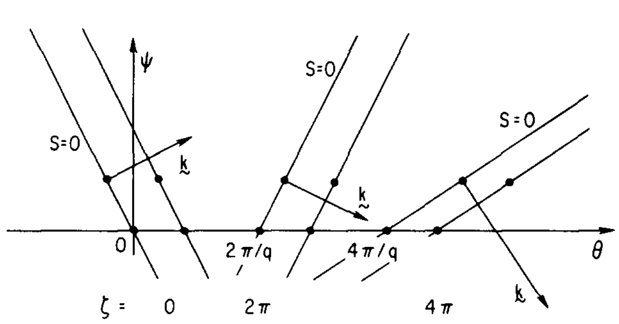

FIG. 3. Poloidal section plot for a group of four field lines contained in two phase fronts $S=$ const. As the field lines wind around the torus those on different surfaces get farther apart owing to magnetic shear. Thus the phase fronts are pulled closer together and $\mathbf{k}\equiv\nabla S$ becomes larger in the direction normal to a magnetic surface.

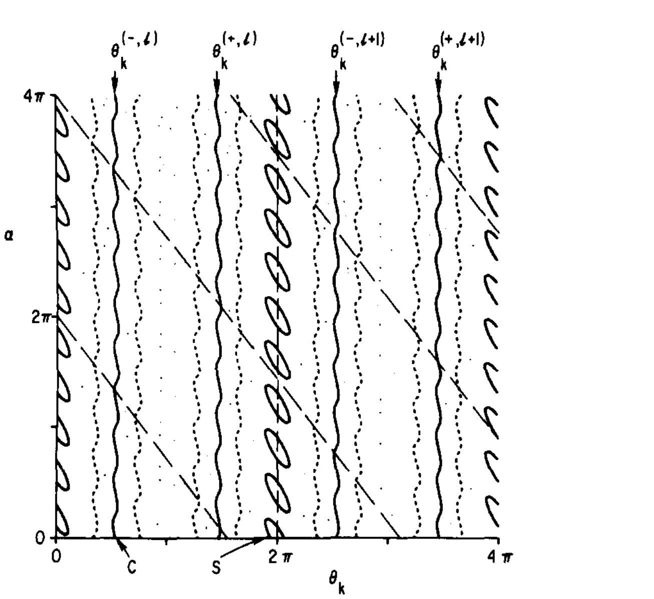

FIG. 4. Intersections of the level surfaces of $\lambda(\alpha,\psi,\theta_k)$ with a surface $\psi=$ const. The (b,l) labeling of the different branches of the inverse function $\theta_k^{(b,l)}(\alpha,\psi;\lambda=\omega^2)$ is also illustrated. The most unstable surfaces (S) are topological spherical but localized. Less unstable but less localized surfaces (C) are topologically cylindrical. The model Eq. (74) was used to produce Fig. 4 with $q=1.3$, $\nu=0.06$.

VII. PERIODICITY

The parameter θ_k has been defined so that it occurs with θ in the secular term through the combination $\theta-\theta_k$. Connor *et al.*[9] have introduced a similar quantity y_0 as the arbitrary end point of a poloidal angle integral, but here θ_k is seen as having a fundamental significance in determining the variation of the phase *across* surfaces. Owing to its universal occurrence in the secular term $\theta-\theta_k$, it does nevertheless have angle-like symmetry properties: The coefficients in the Lagrangian are invariant under the transformation P, defined by Eq. (7) or (11), *provided* $\theta_k\mapsto\theta_k+2\pi$ as well. The periodicity properties of the coefficients in the Lagrangian give rise to the following symmetry of the eigenvalues λ and associated eigenfunctions $\hat{\xi}_\lambda$. Invariance under the T operation leads to

$$\lambda(\alpha+2\pi,\psi,\theta_k)=\lambda(\alpha,\psi,\theta_k),$$
$$\hat{\xi}_\lambda(\theta\,|\alpha+2\pi,\psi,\theta_k)=a_T\hat{\xi}_\lambda(\theta\,|\alpha,\psi,\theta_k), \tag{54}$$

and invariance under the P operation gives

$$\lambda(\alpha-2\pi q,\psi,\theta_k+2\pi)=\lambda(\alpha,\psi,\theta_k),$$
$$\hat{\xi}_\lambda(\theta+2\pi|\alpha-2\pi q,\psi,\theta_k+2\pi)=a_P\hat{\xi}_\lambda(\theta\,|\alpha,\psi,\theta_k), \tag{55}$$

where a_T and a_p are arbitrary constants, which may conveniently be set to unity. Thus, λ has the same periodicity properties in (α,ψ,θ_k) space as equilibrium quantities have in (α,ψ,θ) space. This is sketched in Fig. 4 for a tokamak with rippled toroidal field (Sec. VIII).

To construct a normal mode of the system, we can superimpose any traveling waves with the same frequency. That is, for given ω^2, we need to invert Eq. (45) to find all solutions $k_q/k_\alpha=\theta_k(\alpha,\psi;\omega^2)$. As we shall always be considering normal modes, the ω^2 argument will henceforth be suppressed.

In a typical WKB problem, the inversion of the dispersion relation gives rise to two branches, corresponding to an incident and reflected wave. The extraordinary feature of the ballooning problem is that the periodicity properties of $\lambda(\alpha,\psi,\theta_k)$ mean that there is an infinite number of branches, since the line $\alpha=$ const, $\psi=$ const must intersect the level surfaces $\lambda=$ const an infinite number of times as it traverses the lattice depicted in Fig. 4. This is an infinite degeneracy of the equation of motion, Eq. (12), within the WKB approximation, and this explains how we can find normalizable local solutions in apparent contradiction with the Bloch theorem.[24] We label the different branches with two indices, b and l. The index b distinguishes branches within a "unit cell" of the lattice in Fig. 4, whereas l determines the column in which the unit cell lies (for topologically cylindrical λ surfaces):

$$-\pi+2\pi l<\theta_k^{(b,l)}\leqslant\pi+2\pi l. \tag{56}$$

Equation (56) must be modified in magnetic confinement geometries in which a helical field is dominant, or for topologically spherical λ surfaces, but a similar classification can be devised.

Introducing unknown coefficients $a^{(b,l)}$, we have the *general local solution* of Eq. (12)

$$\xi^{(0)}=\sum_{b,l}a^{(b,l)}(\alpha,\psi)\,\hat{\xi}_\lambda^{(b,l)}(\theta\,|\alpha,\psi)\times\exp[i\epsilon^{-1}S^{(b,l)}(\alpha,\psi)-i\omega t], \tag{57}$$

where

$$\hat{\xi}_\lambda^{(b,l)}(\theta\,|\alpha,\psi)\equiv\hat{\xi}_\lambda\,[\theta\,|\alpha,\psi,\theta_k^{(b,l)}(\alpha,\psi)], \tag{58}$$

and $S^{(b,l)}$ is such that $\mathbf{k}\equiv\nabla S^{(b,l)}$ has components obeying Eq. (47) with $\theta_k=\theta_k^{(b,l)}$.

The symmetry relations (54) and (55) imply

$$\theta_k^{(b,l)}(\alpha,\psi)=\theta_k^{(b,l)}(\alpha+2\pi,\psi)=\theta_k^{(b,l+1)}(\alpha-2\pi q,\psi)-2\pi, \tag{59}$$

and (with $a_T=a_P\equiv1$)

$$\hat{\xi}_\lambda^{(b,l)}(\theta\,|\alpha,\psi)=\hat{\xi}_\lambda^{(b,l)}(\theta\,|\alpha+2\pi,\psi)=\hat{\xi}_\lambda^{(b,l+1)}(\theta+2\pi|\alpha-2\pi q,\psi). \tag{60}$$

Periodicity of $\xi^{(0)}$ in the toroidal direction (i.e., invariance under the T operation) is thus ensured if

$$a^{(b,l)}(\alpha+2\pi,\psi)=a^{(b,l)}(\alpha,\psi), \tag{61}$$

and there is an integer n (the toroidal mode number) such that

$$S^{(b,l)}(\alpha+2\pi,\psi)=S^{(b,l)}(\alpha,\psi)+2\pi\epsilon n. \tag{62}$$

Poloidal periodicity (invariance under the P operation) follows if

$$a^{(b,l+1)}(\alpha-2\pi q,\psi)=a^{(b,l)}(\alpha,\psi), \tag{63}$$

and

$$S^{(b,l+1)}(\alpha-2\pi q,\psi)=S^{(b,l)}(\alpha,\psi)+2\pi\epsilon n_l, \tag{64}$$

where n_l is an integer. Assuming the eigenfunctions to be linearly independent, these conditions are also necessary for periodicity. In the following sections we show that some of these conditions can be satisfied trivially, while others give rise to "quantization" conditions which pick out the allowed values of ω^2.

Since the desired periodic solution is not normalizable with the covering space norm, the sum in Eq. (57) can converge only in the weak sense[17] (i.e., its projection onto any basis function should converge). Weak convergence is ensured by the exponential decay of the eigenfunctions.

VIII. RAY EQUATIONS

The dispersion relation, Eq. (45), may be regarded as a first-order partial differential equation for the eikonal $S^{(b,l)}(\alpha,\psi)$, whose solution will relate $S^{(b,l)}$ on different field lines. In this section and the next we shall be considering only one branch at a time, so we shall suppress the (b,l) label. We also find it convenient to work with q rather than ψ as a surface label. Thus $S=S(\alpha,q)$, and $k_q\equiv\partial_q S$.

Local solutions of Eq. (45) may be found by the method of characteristics, that is, by solving the ray equations[25]

$$\dot\alpha=\frac{\partial\lambda}{\partial k_\alpha}=-\frac{\theta_k}{k_\alpha}\frac{\partial\lambda}{\partial\theta_k}, \tag{65}$$

$$\dot q=\frac{\partial\lambda}{\partial k_q}=\frac{1}{k_\alpha}\frac{\partial\lambda}{\partial\theta_k}, \tag{66}$$

$$\dot k_\alpha=-\frac{\partial\lambda}{\partial\alpha}, \tag{67}$$

$$\dot k_q=-\frac{\partial\lambda}{\partial q}, \tag{68}$$

$$\dot S=k_\alpha\frac{\partial\lambda}{\partial k_\alpha}+k_q\frac{\partial\lambda}{\partial k_q}=0, \tag{69}$$

where dots denote derivatives with respect to a dummy "time" variable parameterizing the characteristics. Given S and ∇S on some curve in the (α,q) plane, these equations can be used to continue S into a finite region of the plane. The question of whether this continuation procedure can be continued indefinitely to produce a smooth *global* solution is more subtle, however,[26] and is the main topic of the remainder of this paper.

Equations (65)–(68) form a two-degree-of-freedom Hamiltonian system, with a four-dimensional phase space with canonical coordinates α, q, k_α, and k_q. Generically, such a system is not integrable,[13] and a phase-space point will cover at least a finite part of the constant-Hamiltonian (i.e., $\lambda=\omega^2$) manifold ergodically. Herein lies the danger: The $\lambda=\omega^2$ manifold is not compact.

This follows from the fact that λ depends on k_q and k_α only through their ratio $\theta_k=k_q/k_\alpha$. Although θ_k may be bounded, the individual components k_q and k_α will increase without bound, unless prevented by one of two circumstances: (i) Equations (65)–(68) happen to be integrable (such cases form a set of measure zero), or (ii) the path of the phase point is blocked by an invariant toroid topologically the same as one belonging to a neighboring integrable case. According to the theorem of Kolmogorov, Arnol'd, and Moser (KAM),[13] cases where (ii) occurs typically form a set of finite measure when λ is sufficiently close to an integrable Hamiltonian.

The special case of an axisymmetric system (such as an ideal tokamak) is integrable, since λ does not depend on α, and k_α is a constant of the motion, by Eq. (67). One might suppose that the KAM theorem would apply for systems close to axisymmetry, such as tokamaks with a finite number of toroidal field coils. It is the purpose of the remainder of this paper to show that, due to special properties of the ideal hydromagnetic dispersion relation, this is not strictly the case for ballooning modes.

One peculiarity of the ballooning case is already apparent in the ray equations. By Eq. (69), the eikonal is a constant on a ray trajectory. Thus, in a sense, S is an integral of the system. However, it is precisely the question of whether a global analytic solution for S can be found that we are addressing. Nevertheless, this does suggest that it may be possible to reduce the dimensionality of the phase space, and indeed, by multiplying Eqs. (65) and (66) by k_α and redefining the dummy time variable, we can write the ray equations as an autonomous system in a reduced (α,q,θ_k) phase space:

$$\dot\alpha=-\theta_k\frac{\partial\lambda}{\partial\theta_k}, \tag{70}$$

$$\dot q=\frac{\partial\lambda}{\partial\theta_k}, \tag{71}$$

$$\dot\theta_k=\theta_k\frac{\partial\lambda}{\partial\alpha}-\frac{\partial\lambda}{\partial q}. \tag{72}$$

The most general dispersion relation obeying the periodicity requirements, Eqs. (54) and (55), is of the form

$$\lambda=\sum_{m,n}a_{m,n}(q)\exp(im\theta_k+in\zeta_k), \tag{73}$$

where $\zeta_k\equiv\alpha+q\theta_k$.

In order to gain insight into possible ray trajectories, we took a choice of $n=0$ components which gave a typical[2] plot (Fig. 5) for a tokamak

$$a_{0,0}=-1+20(q-1.3)^2+10(q-1)^2,$$
$$a_{\pm 1,0}=-5(q-1)^2, \tag{74}$$

with all other components zero. We then added some toroidal ripple by taking as the only symmetry-breaking components

$$a_{0,\pm 5}=\nu/2, \tag{75}$$

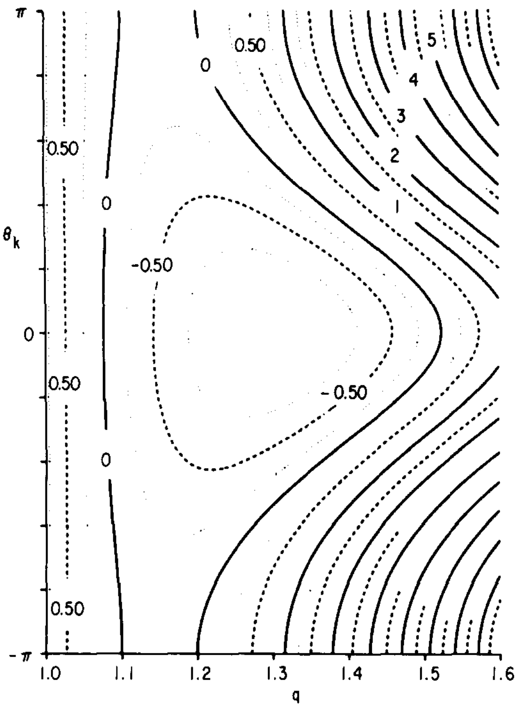

FIG. 5. Intersections of the level surfaces of $\lambda\ (\alpha,\psi,\theta_k)$ with the surface $\alpha =$ const for the model Eq. (74), with $\nu = 0$ (axisymmetric case). There are O points at $(q,\theta_k) = (1.3, 2\pi n)$, and X points at $(1.15, 2\pi n + \pi)$, $n = 0, \pm 1, \pm 2 \ldots$.

where ν was varied from 0 to 0.1. The choice of $n = 5$ ripple was influenced by a design study for a modular tokamak-torsatron hybrid [the tokatron Ref. (27)], but it is not claimed that our model is realistic for this device. In particular, we have no helical component.

From Figs. 4 and 5, we see that the level surfaces of λ fall into several categories. First, there are those which are continuously connected to surfaces in the axisymmetric case ($\nu = 0$). For $\lambda < \lambda_X$ (the value at the X point in Fig. 5) the surfaces are topologically cylindrical, infinitely extended in the α direction. We shall call these surfaces of type C. For $\lambda > \lambda_X$, there are topologically planar surfaces, which we shall not discuss further in this paper.

There are also various new types of surface which arise when $\nu \neq 0$. We shall distinguish only the surfaces associated with the minima of λ, the O point in Fig. 5. These are topologically spherical (type S), and are important as they contain the most unstable modes.

Since Eqs. (71) have λ as a constant of the motion, the rays are constrained to lie on the level surfaces of λ, and some qualitative statements about a ray are already possible once we specify the topological type of the surface. If the ray propagates on a surface of type S, it is not difficult to convince oneself that it must spiral out of an unstable fixed point and into a stable fixed point.

On surfaces of type C, a ray can also be trapped by a stable fixed point (Fig. 6), it can be periodic in α (Fig. 7), or it can cover the surface ergodically (modulo 2π in α). If such an ergodic ray is allowed physically, then there is a KAM theorem for ballooning modes, but we show in Sec. XI that this is not the case.

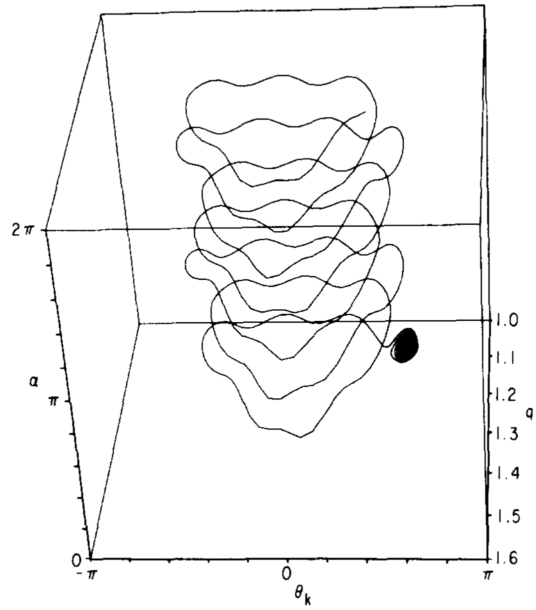

FIG. 6. A ray trajectory on a λ surface of type C. In this case ($\nu = 0.1$, $\lambda = -0.370$) there is an attracting fixed point on the surface, which ultimately traps the ray.

We now specify more precisely the types of fixed points possible. From Eqs. (70)–(72), we see that the fixed points lie in one-parameter families defined by solving

$$\frac{\partial\lambda}{\partial\theta_k} = \theta_k\,\frac{\partial\lambda}{\partial\alpha} - \frac{\partial\lambda}{\partial q} = 0. \tag{76}$$

In our model problem, the fixed points are conveniently parameterized by θ_k, since q and $\sin 5\zeta_k$ can be found analytically from Eq. (76). By scanning through all values of θ_k in the range 0 to π, we can generate all stable fixed points.

Stability is determined by linearizing around a fixed point and evaluating the eigenvalues of a 2×2 matrix. We

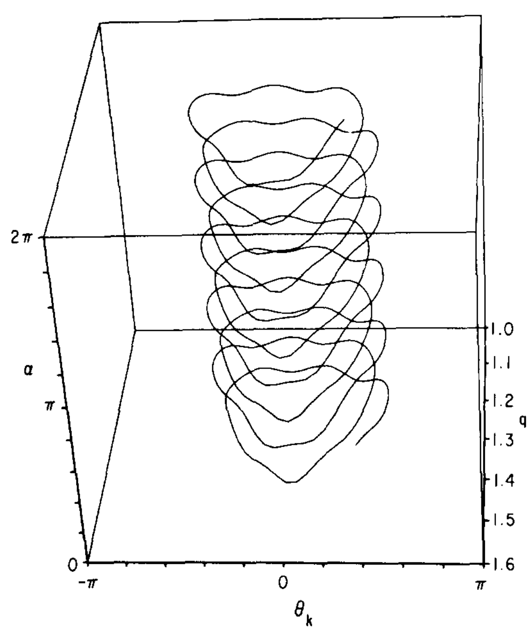

FIG. 7. A ray trajectory on a λ surface of type C, with $\nu = 0.06$, $\lambda = -0.479$. In this case there is no attracting fixed point.

find stability if and only if

$$\frac{\partial\lambda}{\partial\alpha}<0 \text{ and } D>0, \tag{77}$$

where D is the determinant

$$D=\frac{\partial^2\lambda}{\partial\theta_k^2}\left[\theta_k\frac{\partial^2\lambda}{\partial\alpha\,\partial q}-\frac{\partial^2\lambda}{\partial q^2}-\theta_k\left(\theta_k\frac{\partial^2\lambda}{\partial\alpha^2}-\frac{\partial^2\lambda}{\partial\alpha\,\partial q}\right)\right]+\frac{\partial\lambda}{\partial\alpha}\left(\frac{\partial^2\lambda}{\partial q\,\partial\theta_k}-\theta_k\frac{\partial^2\lambda}{\partial\alpha\,\partial\theta_k}\right)-\left(\frac{\partial^2\lambda}{\partial q\,\partial\theta_k}-\theta_k\frac{\partial^2\lambda}{\partial\alpha\,\partial\theta_k}\right)^2.$$

We conclude this section by pointing out that the fact that λ is a constant of the motion may be used to further reduce the phase space. By solving Eq. (45) to give $\theta_k=\theta_k^{(b,l)}(\alpha,q;\omega^2)$, we may delete Eq. (72) and obtain a *two-dimensional* autonomous set of ray equations, Eqs. (70) and (71). These equations are characteristics[25] for the linear partial differential equation

$$\partial_q S-\theta_k\,\partial_\alpha S=0. \tag{78}$$

IX. NEXT ORDER: AMPLITUDE EQUATION

In order to find an equation connecting the amplitudes $a^{(b,l)}(\alpha,\psi)$ on different field lines, we proceed to $O(\epsilon)$ in the expansion of Eq. (37),

$$\mathsf{P}_t\cdot\mathsf{L}_0\cdot\hat{\xi}_t^{(1)}+\mathsf{P}_t\cdot\mathsf{L}^{(1)}\cdot\hat{\xi}_t^{(0)}=0, \tag{79}$$

where, from Eq. (39)

$$\mathsf{L}^{(1)}=\hat{\mathsf{F}}_L^{(-1)}\cdot\mathbf{e}_k G_0\mathbf{e}_k\cdot\mathsf{L}_0+\mathsf{L}_0\cdot\mathbf{e}_k G_0\mathbf{e}_k\cdot\hat{\mathsf{F}}_R^{(-1)}. \tag{80}$$

The solution of the equation adjoint to Eq. (44) is $\hat{\xi}^{(0)*}$, so the condition for solubility of Eq. (80) is

$$\langle\hat{\xi}^{(0)*},\mathsf{P}_t\cdot\mathsf{L}^{(1)}\cdot\hat{\xi}_t^{(0)}\rangle=0, \tag{81}$$

which can be written as a conservation equation

$$\partial_\alpha\langle\hat{\xi}^{(0)*},(\nabla\alpha)k^{-2}\mathbf{k}\cdot\mathsf{L}_0\cdot\hat{\xi}_t^{(0)}\rangle+\partial_q\langle\hat{\xi}^{(0)*},(\nabla q)k^{-2}\mathbf{k}\cdot\mathsf{L}_0\cdot\hat{\xi}_t^{(0)}\rangle=0. \tag{82}$$

Comparison with Eq. (48) shows that Eq. (82) may also be written in the standard form[22] for a wave action conservation equation

$$\partial_q\Gamma^q+\partial_\alpha\Gamma^\alpha=0, \tag{83}$$

where the flux Γ^μ is defined by

$$\Gamma^\mu\equiv-\frac{\partial\bar{\mathscr{L}}}{\partial k_\mu},$$

for $\mu=q$ or α.

Observing that $\bar{\mathscr{L}}$ depends on k_μ only through the ratio θ_k [by Eq. (53)] we see that

$$\Gamma^\alpha=-\theta_k\Gamma^q, \tag{84}$$

so that Eq. (82) can be written in the form

$$\partial_q\Gamma^q-\partial_\alpha(\theta_k\Gamma^q)=0. \tag{85}$$

Comparing with Eq. (78), we see that Γ^q obeys the adjoint of the eikonal equation, and has the same characteristics.

To use Eq. (85) as an equation for the wave amplitude, we write

$$\hat{\xi}^{(0)}=a(\alpha,q)\hat{\xi}_\lambda(\theta\,|\alpha,q,\theta_k), \tag{86}$$

where $\hat{\xi}_\lambda$ is the eigenfunction defined by Eqs. (51) and (52) with normalization

$$\frac{\partial\bar{\mathscr{L}}[\hat{\xi}_\lambda]}{\partial\omega^2}=1. \tag{87}$$

Thus,

$$\Gamma^q=-\frac{a^2}{k_\alpha}\frac{\partial\bar{\mathscr{L}}[\hat{\xi}_\lambda]}{\partial\theta_k}=\frac{a^2}{k_\alpha}\frac{\partial\lambda}{\partial\theta_k}, \tag{88}$$

using the fact that $\bar{\mathscr{L}}[\hat{\xi}_\lambda]=0$ for $\omega^2=\lambda(\alpha,q,\theta_k)$ and the normalization equation (87).

Since k_α, θ_k, and $\partial\lambda/\partial\theta_k$ are real for propagating waves, the phase of a remains constant along a ray, except possibly near the edge of the domain of real values of $\theta_k^{(b,l)}$, where the ballooning ordering breaks down. This is the subject of the next section.

X. CAUSTICS

The connected parts of the boundary ∂P of the projection P of the λ surfaces on the q-α plane (Fig. 8) are called caustics, and they are crucially important in determining the global mode structure since it is along these curves that one real branch of $\theta_k^{(b,l)}$ converts into another (i.e., a left-going wave reflects into a right-going wave, or vice versa). Denote the two coalescing branches by $\theta_k^{(\pm,l)}$, with

$$\theta_k^{(+,l)}(\alpha,q)\geqslant\theta_k^{(-,l)}(\alpha,q),\quad \text{for } (\alpha,q)\in P, \tag{89}$$

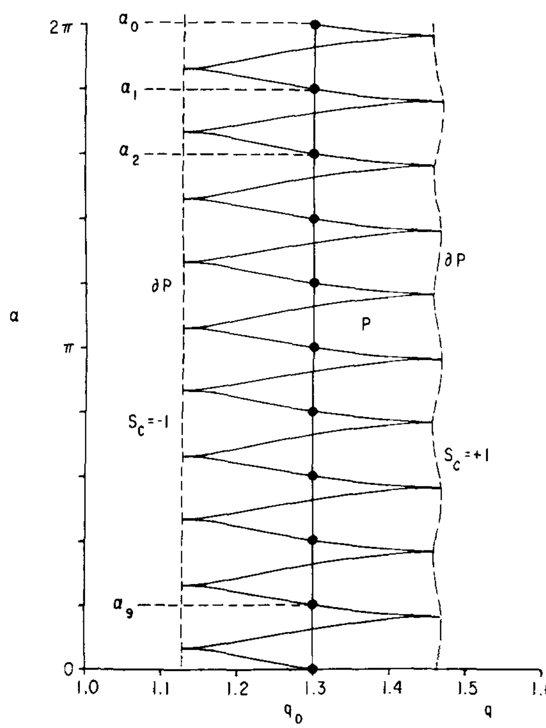

FIG. 8. Projection of a λ surface of type C. The domain P on which the rays propagate has a boundary ∂P consisting of a left-hand ($s_c=-1$) and a right-hand ($s_c=+1$) caustic, and on which $\partial\lambda/\partial\theta_k$ goes through zero. The intersections of the rays with the line of section $q=q_0$ define the mappings $T^{(i)}(\alpha_0)=\alpha_i$. The orbit is the same as that in Fig. 7.

the equality applying if and only if $(\alpha,q)\in\partial P$. We also have

$$\frac{\partial\lambda}{\partial\theta_k}=0,\quad \text{for } (\alpha,q)\in\partial P, \tag{90}$$

since the normal to the λ surface must lie in the q-α plane at the boundary of its projection. It is not necessary that θ_k be constant on ∂P, but in axisymmetric geometry it is typically a constant multiple of 2π (assuming $\theta=0$ is on the outside of the torus).

Implicit in the WKB expansion is the assumption that $\mathbf{k}$ is analytic and slowly varing, so that its α and q derivatives are $O(1)$ quantities. Near a caustic, however, this ordering is violated. To see this, we expand λ near a particular caustic $C\in\partial P$. On the caustic, let

$$\lambda=\omega^2,\quad q=q_c(\alpha),\quad \theta_k^{(\pm,l)}=\theta_c(\alpha). \tag{91}$$

Then, to lowest nontrivial order near the caustic

$$\lambda=\omega^2+(q-q_c)\left(\frac{\partial\lambda}{\partial q}\right)_c+\frac{1}{2}(\theta_k-\theta_c)^2\left(\frac{\partial^2\lambda}{\partial\theta_k^2}\right)_c+\ldots, \tag{92}$$

where subscript c means evaluation on the caustic. Setting $\lambda=\omega^2$ in P [so that $\operatorname{sgn}(q-q_c)=-\operatorname{sgn}(\partial_q\lambda)_c$],

$$\theta_k^{(b,l)}=\theta_c(\alpha)+b\left|\frac{2\,\partial\lambda/\partial q}{\partial^2\lambda/\partial\theta_k^2}\right|_c^{1/2}|q-q_c|^{1/2}+O(q-q_c), \tag{93}$$

where the branch index $b=\operatorname{sgn}(\partial\lambda/\partial\theta_k)=\pm 1$. By solving Eq. (78) in P, we find the eikonal near the caustic

$$S^{(b,l)}=S_c^{(b,l)}(\alpha)+(k_q)_c(q-q_c)-s_c b\left(\frac{2}{3}\right)\times\left|\frac{2\,\partial\lambda/\partial q}{\partial^2\lambda/\partial\theta_k^2}\right|_c^{1/2}\frac{(k_\alpha)_c|q-q_k|^{3/2}}{(1+\theta_c\partial_\alpha q_c)}+\ldots, \tag{94}$$

where $S_c^{(b,l)}(\alpha)\equiv S^{(b,l)}[\alpha,q_c(\alpha)]$, and $s_c=+/-$ on a right/left caustic,

$$s_c\equiv\operatorname{sgn}(\partial_q\lambda)_c. \tag{95}$$

From Eq. (88)

$$|a^{(b,l)}|=\left|\frac{(k_\alpha\Gamma^q)^2/2}{(\partial_q\lambda)(\partial^2\lambda/\partial\theta_k^2)}\right|_c^{1/4}\frac{1}{|q-q_c|^{1/4}}+\ldots. \tag{96}$$

Thus, not only is θ_k nonanalytic on ∂P, but $|a|$ diverges there, according to the WKB ordering.

In order to match the two branches $\theta_k^{(\pm,l)}$ across a caustic, we must introduce a boundary layer ordering. To estimate the width of the boundary, we determine the distance beyond which the nonanalytic contribution to the phase S/ϵ becomes small. That is, we set the last term of Eq. (94) to be $O(\epsilon)$, whence we find that the boundary layer is in the region $|q-q_c|\lesssim O(\epsilon^{2/3})$. To find the boundary layer equation, we define a stretched coordinate $x=O(1)$ such that

$$q-q_c(\alpha)=\epsilon^{2/3}x, \tag{97}$$

and assume $\boldsymbol{\xi}$ to vary on the x scale. We expand $\boldsymbol{\xi}$ in powers of $\epsilon^{1/3}$

$$\boldsymbol{\xi}(\alpha,x,\theta,t)=\sum_{n=0}^{\infty}\epsilon^{n/3}\,\hat{\boldsymbol{\xi}}^{(n/3)}(\alpha,x,\theta)\exp(i\epsilon^{-1}S-i\omega t). \tag{98}$$

Since both branches are coupled in the boundary layer, we *define* S to be given exactly by the first two terms of Eq. (94) and allow $\hat{\boldsymbol{\xi}}$ to carry the residual phase information. From Eq. (97) we see that

$$\boldsymbol{\nabla}=\epsilon^{-2/3}[\boldsymbol{\nabla}q-(\partial_\alpha q_c)\boldsymbol{\nabla}\alpha]\partial_x+\boldsymbol{\nabla}^{(0)}, \tag{99}$$

where $\boldsymbol{\nabla}^{(0)}$ acts only on equilibrium quantities and on $\mathbf{k}$. Since the formal elimination of $\hat{\xi}_l$ goes through with arbitrary $S(\alpha,q)$, we may use Eq. (37) with S as defined above. At $O(\epsilon^0)$, we find

$$(\mathsf{P}_t\cdot\mathsf{L}_0\cdot\mathsf{P}_t)_c\cdot\hat{\boldsymbol{\xi}}^{(0)}=0, \tag{100}$$

which is satisfied by

$$\hat{\boldsymbol{\xi}}^{(0)}=a(\alpha,x)\hat{\boldsymbol{\xi}}_c(\theta\,|\alpha), \tag{101}$$

with $\hat{\boldsymbol{\xi}}_c\equiv(\hat{\boldsymbol{\xi}}_\lambda)_c$ normalized as in Eq. (87).

At $O(\epsilon^{1/3})$, Eq. (37) gives

$$(\mathsf{P}_t\cdot\mathsf{L}_0\cdot\mathsf{P}_t)_c\cdot\hat{\boldsymbol{\xi}}^{(1/3)}+(\mathsf{P}_t\cdot\mathsf{L}\cdot\mathsf{P}_t)^{(1/3)}\cdot\hat{\boldsymbol{\xi}}^{(0)}=0, \tag{102}$$

where

$$(\mathsf{P}_t\cdot\mathsf{L}\cdot\mathsf{P}_t)^{1/3}=[\mathsf{P}_t\cdot(\mathsf{L}_0\cdot\mathbf{e}_k\mathbf{u}+\mathbf{u}\mathbf{e}_k\cdot\mathsf{L}_0)\cdot\mathsf{P}_t]_c\,\partial_x, \tag{103}$$

with

$$\mathbf{u}\equiv k^{-1}[\boldsymbol{\nabla}q-(\partial_\alpha q_c)\boldsymbol{\nabla}\alpha]. \tag{104}$$

The solubility condition for Eq. (102) is

$$\langle\hat{\boldsymbol{\xi}}_c^*,(\mathsf{P}_t\cdot\mathsf{L}\cdot\mathsf{P}_t)^{(1/3)}\cdot\hat{\boldsymbol{\xi}}^{(0)}\rangle=0. \tag{105}$$

Using the identity

$$\frac{\partial}{\partial k_\alpha}\mathsf{P}_t=-k^{-1}[\mathbf{e}_k(\boldsymbol{\nabla}_t\alpha)+(\boldsymbol{\nabla}_t\alpha)\mathbf{e}_k] \tag{106}$$

(true also with α replaced by q), and the normalization Eq. (87) we can show that the left-hand side of Eq. (105) is

$$\text{lhs}=i\left(\frac{\partial\lambda}{\partial k_q}-(\partial_\alpha q_c)\frac{\partial\lambda}{\partial k_\alpha}\right)_c\partial_x a. \tag{107}$$

This vanishes automatically, by Eq. (90).

At $O(\epsilon^{2/3})$, Eq. (37) gives

$$(\mathsf{P}_t\cdot\mathsf{L}_0\cdot\mathsf{P}_t)_c\cdot\hat{\boldsymbol{\xi}}^{(2/3)}+(\mathsf{P}_t\cdot\mathsf{L}\cdot\mathsf{P}_t)^{(1/3)}\cdot\hat{\boldsymbol{\xi}}^{(1/3)}+(\mathsf{P}_t\cdot\mathsf{L}\cdot\mathsf{P}_t)^{(2/3)}\cdot\hat{\boldsymbol{\xi}}^{(0)}=0. \tag{108}$$

After a considerable amount of algebra, the solubility condition for Eq. (108) may be written

$$\left(\frac{(1+\theta_k\partial_\alpha q_c)^2}{2k_\alpha^2}\frac{\partial^2\lambda}{\partial\theta_k^2}\right)_c\partial_x^2 a-x(\partial_q\lambda)_c\,a=0. \tag{109}$$

The solution of Eq. (109) evanescent in the region outside P [i.e., $s_c\,x>0$, see Eq. (95)] is

$$a=A_c\,\mathrm{Ai}(\mu x), \tag{110}$$

where Ai is an Airy function of the first kind,[28]

$$\mu\equiv s_c\left|\frac{2k_\alpha^2\,\partial\lambda/\partial q}{(1+\theta_k\partial_\alpha q_k)^2\,\partial^2\lambda/\partial\theta_k^2}\right|_c^{1/3}, \tag{111}$$

and A_c is an arbitrary constant.

The asymptotic behavior of Eq. (110) as we go into the interior of P, $s_c x\to-\infty$ is[28]

$$a\sim\frac{A_c}{2\pi^{1/2}|\mu x|^{1/4}}\left[\exp\left(\frac{2i|\mu x|^{3/2}}{3}-\frac{i\pi}{4}\right)+\exp\left(\frac{-2i|\mu x|^{3/2}}{3}+\frac{i\pi}{4}\right)\right]. \tag{112}$$

Equations (97), (101), (111), and (112) match the outer solutions $a^{(b,l)}\hat{\xi}_\lambda^{(b,l)} \exp(i\epsilon^{-1}S^{(b,l)})$, with $S^{(b,l)}$ and $|a^{(b,l)}|$ given by Eqs. (94) and (96), provided we take all outer quantities to be continuous as we move from the $(-)$ branch to the $(+)$ branch at the caustic, except for the action flux Γ^q, which changes sign, and the phase $\epsilon^{-1}S$, which suffers a discontinuity given by

$$\epsilon^{-1}S_c^{(+,l)} = \epsilon^{-1}S_c^{(-,l)} + (s_c\pi/2) + 2\pi N_c, \tag{113}$$

where N_c is an arbitrary integer.

This is the expected result[26] for typical wave equations, but we feel that it was necessary to check that nothing pathological happens in the case of ballooning modes, in view of the fact that the conclusions of the next section depend critically on Eq. (113) being precisely correct (in the limit $\epsilon \to 0$).

XI. THE SPECTRUM

The results of Secs. VIII–X allow us to find linear superpositions of local solutions which are also global solutions on the covering space; that is, which are valid not only within the propagating region, but at the caustics as well. However, we expect that only for certain values of ω^2 will these global solutions satisfy the periodicity conditions of Eqs. (61) and (62).

Actually, since the assumed configuration (Fig. 4) of the λ surfaces is such that branches with different l values are disjoint (no rays propagate between them), and since we ignore tunneling (it is exponentially small), Eqs. (63) and (64) can be satisfied trivially by suitable choice of arbitrary constants. Thus we restrict attention to the single projection shown in Fig. 8 and consider the implications of Eqs. (61) and (62).

The ray trajectories are sketched in Fig. 8, and are seen to bounce back and forth between the caustics, with a mean drift in the negative α direction. By considering the intersection of a ray with a line of section, $q = q_0$ (Fig. 8), we define a mapping T from a typical initial value $\alpha = \alpha_1$, to the final value $\alpha = \alpha_2$ after one complete bounce. That is,

$$\alpha_2 = T(\alpha_1, \omega^2), \tag{114}$$

where T is a 2π-periodic function of α_1 due to the periodicity of λ. Thus, T can be regarded as a mapping of the circle onto itself. By iterating the mapping an infinite number of times, one can define a mean α drift per bounce, which we denote by $-\Omega(\omega^2)$, the sign being chosen for agreement with conventions used previously in the axisymmetric case.[2]

First, we express the ray tracing results and periodicity constraint in terms of T. Since S is constant on a ray, by Eq. (69), except for the jumps given by Eq. (113), the ray tracing for the phase increment over one bounce yields

$$\epsilon^{-1}\{S[T(\alpha)] - S(\alpha)\} = (2N+1)\pi, \tag{115}$$

where the radial mode number N is any integer $O(\epsilon^{-1})$, and the argument $q = q_0$ is suppressed.

The equation for the amplitude is found by applying Gauss' theorem to the action conservation equation, Eq. (85), on a narrow strip bounded by two rays. We get

$$\Gamma^q[T(\alpha)] = \Gamma^q(\alpha)/T'(\alpha). \tag{116}$$

Thus, Γ^q is proportional to the probability density of an infinitely iterated initial point (taking the iterates modulo 2π).

The periodicity conditions, Eqs. (61) and (62), imply

$$\Gamma^q(\alpha + 2\pi) = \Gamma^q(\alpha), \tag{117}$$

and

$$\epsilon^{-1}[S(\alpha + 2\pi) - S(\alpha)] = -2\pi n, \tag{118}$$

where the toroidal mode number n is any integer $O(\epsilon^{-1})$. The requirement that Eqs. (115) and (118) be simultaneously satisfied is to provide our quantization condition.

A great deal is known on the general properties of circle mappings owing to the investigations of Poincaré and Denjoy,[29] Arnol'd,[30] and Herman.[31] Herman proved a conjecture of Arnol'd that, provided T is a smooth invertible function of α, there is for almost all Ω a smooth, invertible change of variable

$$\alpha = h(\beta), \tag{119}$$

to a new angle variable in which the mapping appears as a rigid rotation through the angle $-\Omega$

$$\beta_2 = \beta_1 - \Omega(\omega^2). \tag{120}$$

However, if Ω is a rational multiple of 2π, the mapping or an iterate (say $T^{(m)}$) has fixed points (modulo 2π), called m cycles, and the theorem breaks down[29] because the m cycles act as *attractors* for successive iterates, either under forward iteration of T (stable cycles) or backward iteration of T (unstable cycles). Since the rationals have measure zero on the unit interval, this would appear to be a nongeneric case. However, there is a "lock in" phenomenon[30] which causes $\Omega/2\pi$ to assume rational values over a finite (perhaps small) band of values of any physical variable of which it is a function. In our case this is ω^2.

Herman's theorem is much stronger than the KAM theorem for Hamiltonian systems, since the measure of exceptional values of Ω remains strictly zero even for mappings far from the rigid rotation. This indicates that the lock in "widths" of the m cycles must become small very rapidly as m increases. This has been confirmed numerically (see later).

We proceed tentatively by first supposing that $\Omega/2\pi$ is irrational, so that, by Herman's theorem, the transformation defined by Eq. (119) almost always exists. Then Eq. (116) has the smooth solution

$$\Gamma^q(\alpha) = \Gamma_0/h'[h^{-1}(\alpha)], \tag{121}$$

where Γ_0 is a constant. This means that the iterates cover the circle ergodically with a smoothly varying probability density. Equation (117) is satisfied automatically. We express the wave phase $\phi \equiv S/\epsilon$ as a function of β,

$$\phi(\beta) = \epsilon^{-1}S[h(\beta)]. \tag{122}$$

Then Eqs. (115) and (118) become

$$\phi(\beta - \Omega) - \phi(\beta) = (2N+1)\pi, \tag{123}$$

$$\phi(\beta + 2\pi) - \phi(\beta) = -2\pi n. \tag{124}$$

The smooth solution of these equations is

$$\phi(\beta) = -n\beta = -2\pi[N + (1/2)]\beta/\Omega, \tag{125}$$

whence we have the quantization condition

$$\Omega(\omega^2)/2\pi = (2N+1)/2n. \tag{126}$$

We now observe that $\Omega/2\pi$ has been selected by the physics to be a rational fraction, which contradicts our original assumption. Thus we now assume that $\Omega/2\pi$ is a rational fraction. Indeed, as we shall now show, it is the number given by Eq. (126).

Suppose $(2N+1)/2n = P/Q$, where P and Q are relatively prime integers. Then, one can verify that Eqs. (115)–(118) are satisfied by the singular solutions

$$\epsilon^{-1}S(\alpha) = -\frac{2\pi n}{Q}\left(\sum_{i=1}^{\infty}\theta(\alpha-\alpha_i) - \sum_{i=-\infty}^{0}\theta(\alpha_i-\alpha)\right), \tag{127}$$

$$\Gamma^q(\alpha) = \sum_{i=-\infty}^{\infty}\delta(\alpha-\alpha_i), \tag{128}$$

where $\theta(\cdot)$ and $\delta(\cdot)$ are, respectively, the Heaviside step function and the Dirac delta function. The set $\{\alpha_i\}$ is a Q cycle obeying the conditions

$$T(\alpha_i) = \alpha_{i-P} \tag{129}$$

and

$$\alpha_i + 2\pi = \alpha_{i+Q}\,. \tag{130}$$

Equation (127) represents a descending staircase with riser height $2\pi n/Q$. The general solution is the superposition of all such solutions from disjoint Q cycles, of which there may be many (or even an infinite number, in the axisymmetric case).

Since the rays we have been using are the projection on the q-α plane of the Hamiltonian system Eqs. (65)–(68) on the *unbounded* constant-λ manifold, in four-dimensional phase space, we have no reason to expect a smooth probability density Γ^q to exist in general; phase space elements can be arbitrarily elongated in a direction transverse to the q-α plane.

Since Ω_0 is a function of ω^2, the bands in Ω_0 for which Eq. (126) is satisfied correspond to continua in the eigenmode spectrum. Previous workers[14,15] have not found this unstable continuum in systems with magnetic shear because they assumed that continuum eigenfunctions must have a singularity localized to a magnetic surface (cf. Appendix B), whereas the singularity for ballooning modes is localized to the ray passing through the fixed points α_i, and spans a range of magnetic surfaces. Spies[32] has recognized the possibility of such a continuum in systems with everywhere-closed field lines.

There is one case where a smooth h exists (trivially), viz. the ideal axisymmetric tokamak for which α is an ignorable coordinate. In this case, we can write[3]

$$\Omega(\omega^2) = \oint \theta_k\, dq, \tag{131}$$

where the integral is over a complete bounce of a ray trajectory. The quantization condition Eq. (126) is then the same as that used in a numerical comparison with a finite element code,[2] where it was found that the WKB method gave good results for $n \gtrsim 5$. What was not previously noticed, however, is that Eq. (126) predicts an infinite degeneracy in the ballooning mode spectrum. If Eq. (126) is satisfied for the pair (n_0, N_0), then it is satisfied, with the same value of ω^2, for the pairs $(3n_0, 3N_0+1)$, $(5n_0, 5N_0+2)$.... Thus, it is not surprising that even the tiniest amount of symmetry breaking couples energy up to indefinitely high harmonics of n_0, and gives rise to a singular mode.

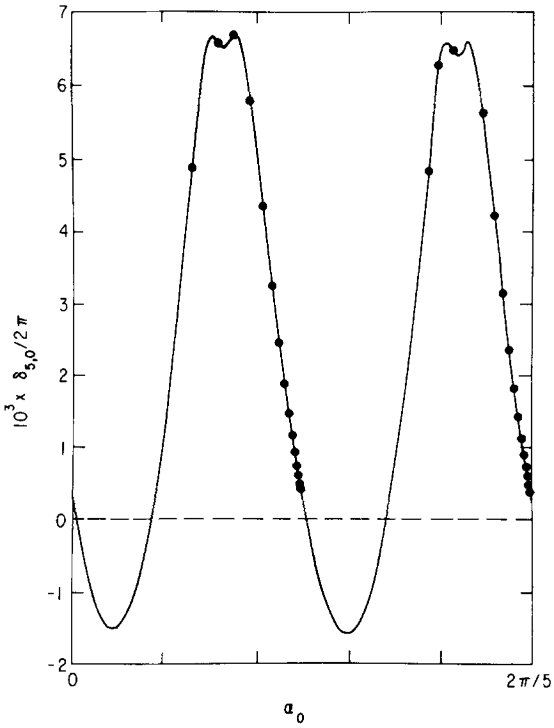

FIG. 9. Plot of the departure from rigid rotation $\delta_{n,N} \equiv \alpha_{2n} - \alpha_0 + (2N+1)2\pi$ in the case $n=5$, $N=0$, $\nu=0.06$, $\lambda = -0.479$ (cf. Figs. 7 and 8). The circles show successive passages of a ray trajectory. Note the accumulation near the stable zeros of $\delta_{n,N}(\alpha)$.

We have computed $T^{(2n)}(\alpha)$ for the model dispersion relation, Eqs. (73) and (74). An example ($n=5$) is shown in Fig. 9. Since $n=5$ is resonant with the assumed toroidal ripple, the amplitude of the variation in $T^{(2n)}(\alpha)$ is relatively large. Even in this case, however, the variation is numerically quite small.

The model spectrum determined by Eq. (126) has been plotted in Fig. 10, where it is seen that the width of the continuum bands associated with values of $n_0 \lesssim 50$ can hardly be resolved. A much more dramatic effect is the broadening out of the $n_0 \to \infty$ accumulation point into a continuum band. The "eigenfunctions" in this continuum are δ functions at the attracting fixed points defined by Eqs. (76) and (77). A continuum band also develops at $\nu = 0.0068$, due to the appearance of stable fixed points near the X points in Fig. 5. The two bands merge when $\nu \gtrsim 0.08$.

The dummy time variable in Eqs. (70)–(72) is not the physical time. In fact, to get real characteristics when $\omega^2 < 0$, we must take the physical time to be imaginary. Thus the choice of stable fixed points is not dictated by causality, and we could equally well evolve our dummy time backwards and find the unstable fixed points as solutions. These two sets of solutions are completely uncoupled within the WKB approximation, and are degenerate.

Finally, we remark that although $\mathbf{k}$ goes to infinity on the fixed points and cycles, the WKB ordering breaks down there since the amplitude variation (assumed slow) becomes infinitely rapid. Thus a localized calculation should be performed with a new ordering appropriate to fixed points. However the WKB prediction of a singularity in the neighborhood of a fixed point is presumably still correct.

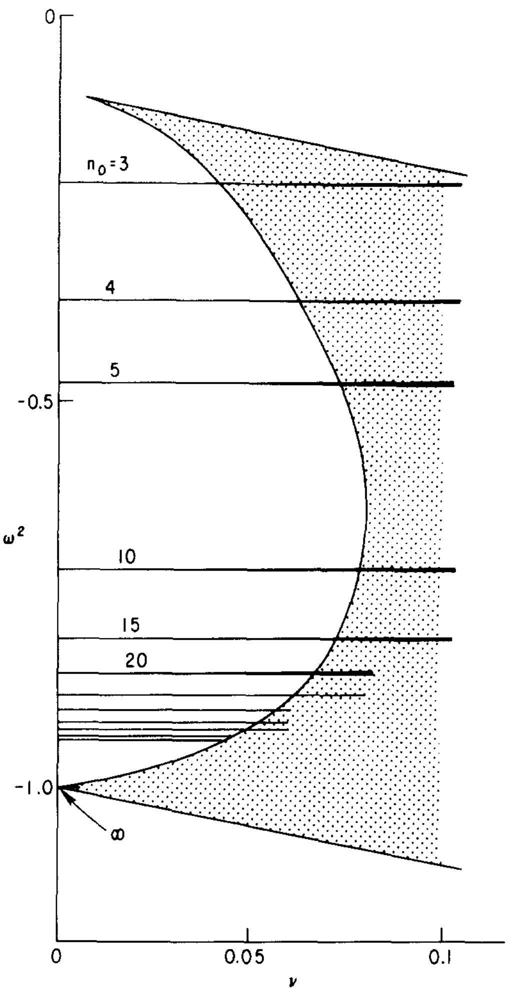

FIG. 10. Plot of the allowed eigenvalues ω^2 as a function of ripple parameter ν. Selected continuum bands associated with various toroidal mode numbers n_0 of the axisymmetric case (with $N_0 = 0$) have been plotted, as well as the much wider continuum band growing out of the $n_0 \to \infty$ accumulation point.

XII. CONCLUSION

We have shown that no strict KAM theorem exists for ideal ballooning modes in machines close to axisymmetry, and the eigenmodes are singular for all nonaxisymmetric systems. There are various ways to resolve this nonphysical conclusion by extending the model. One way is to look at the initial value problem for a smooth wave packet with the hope that nonlinear effects set in before the ray focusing effect. The degeneracy of the spectrum, however, suggests that singularities will still develop because of nonlinear coupling to infinite n modes. This is borne out by computer experiments.[33] However, the extremely narrow width of the low-n_0 spectral bands means that a wavepacket quasimode will retain its form for many exponential growth times. Thus, from a practical point of view discrete eigenvalues do persist in the linear theory of nonaxisymmetric systems. For instance, a Galerkin method calculation of normal modes might be expected to give the appearance of convergence to discrete eigenvalues.

We can stay within ideal hydromagnetics also by allowing flows in the background state. Velocity shear may be expected to damp the modes at high n. Such an effect has been predicted for internal gravity waves,[34] whose dispersion relation bears some resemblance to that of ballooning modes.

Finite-Larmor-radius effects[4,6,35] will also resolve the singularity by introducing a $|\mathbf{k}|^2$ dependence in λ and can stabilize the modes (or at least reduce their growth rate).

Presumably, only the high-n_0 quasimodes are susceptible to FLR stabilization, so that it is only if the critical β at $n_0 = 20$ (say) is much greater than that at very large n_0 that FLR effects can be an important stabilizing mechanism, just as in tokamaks.[2,4] The width of the continuum band associated with the accumulation point also suggests that the most unstable modes, corresponding to localized regions of bad curvature, must rapidly couple to very high n and thus must be easily susceptible to FLR stabilization. Thus averaging methods for ideal hydromagnetic stability may give a truer picture than might at first appear.

ACKNOWLEDGMENTS

This paper grew out of the work on the $1/n$ correction problem in axisymmetric geometry with the authors of Refs. 2 and 3, with whom useful conversations are gratefully acknowledged. One of us (R. L.D.) would also like to acknowledge useful conversations with W. M. Nevins, J. W. Connor, M. D. Kruskal, J. Mather, R. B. White, F. S. Henyey, and W. A. Coppel, and the assistance of P. Vianna and M. Selberg in providing computer visualizations of the mapping from the covering space to the torus. He would also like to acknowledge the hospitality of the Aspen Center for Physics, where the work in Secs. V and X was performed.

This work was supported by the United States Department of Energy Contract No. DE-AC02-76-CHO3073.

APPENDIX A: DOES THE BALLOONING TRANSFORMATION PROVE THE EXISTENCE OF THE PERIODIC REPRESENTATION?

In this appendix we seek to determine the restrictions placed on the Fourier analysis/transform technique[9] by the requirement that the individual components of the infinite sum each obey the same equation as the sum itself. For ease of comparison, we use the same notation as Ref. 9, where it is shown that an arbitrary periodic function $\phi(\theta) = \Sigma_m a_m \times \exp(-im\theta)$ may be mapped on a function $\hat{\phi}(\eta)$, square integrable on the domain $-\infty < \eta < \infty$, by three steps: Fourier analysis ($\phi \mapsto a_m$), interpolation between the integers [$a_m \mapsto a(s)$], and inversion as a Fourier transform [$\hat{a}(s) \mapsto \hat{\phi}(\eta)$].

The interpolation formula (A3) of Ref. 9 is a special case of the more general algorithm

$$\hat{a}(s) = \sum_m a_m F(s-m), \tag{A1}$$

where $F(s)$ is a function with zeros at $s = \pm 1, \pm 2, \ldots$, and such that $F(0) \neq 0$. In Ref. 9, $F(s) = (\sin \pi s)/\pi s$.

The third step is Fourier inversion

$$\hat{\phi}(\eta) = \int_{-\infty}^{\infty} ds \exp(-is\eta)\hat{a}(s). \tag{A2}$$

Poisson's sum formula can now be used to show $\phi(\theta) = \Sigma_m \hat{\phi}(\theta - 2\pi m)/F(0)$.

Substituting Eq. (A1) in Eq. (A2), we find

$$\hat{\phi}(\eta) = \hat{F}(\eta)\phi(\eta), \tag{A3}$$

where

$$\hat{F}(\eta) = \int_{-\infty}^{\infty} ds \exp(-is\eta)F(s). \tag{A4}$$

We shall normalize $F(s)$ so that $F(0) = 1$. As in Ref. 8, we assume that $\phi(\theta)$ obeys the equation

$$L(\partial_\theta)\phi(\theta) = \lambda\phi(\theta), \tag{A5}$$

for $0 < \theta \leqslant 2\pi$, and, by analytical continuation, for $-\infty < \theta < \infty$. Equation (A3) then implies that $\hat{\phi}(\eta)$ obeys the equation

$$\hat{F}(\eta)L(\partial_\eta)[\hat{\phi}(\eta)/\hat{F}(\eta)] = \lambda\hat{\phi}(n). \tag{A6}$$

This is to be compared with the equation

$$L(\partial_\eta)\hat{\phi}(\eta) = \lambda\hat{\phi}(\eta), \tag{A7}$$

which expresses the assumption made in Ref. 9 that $\hat{\phi}(\eta)$ obeys the *same* equation as $\phi(\theta)$. Equations (A6) and (A7) can only be satisfied simultaneously in general if $\hat{F}(\eta) = 1$ [so that $F(s) = \delta(s)$]. By Eq. (A3), this implies that the transformation is the identity transformation! We conclude that the Fourier construction, at least in its simple form, devoid of asymptotic arguments,[11] actually proves nothing regarding the existence or otherwise of square integrable solutions on the infinite domain.

APPENDIX B: LARGE $|\theta|$ BEHAVIOR

We can remove the secular terms from the coefficients in the Lagrangian, Eq. (50), by the change of variable

$$\xi = k_\alpha\, v/k. \tag{B1}$$

To leading order in $|\theta|^{-1}$, $k_\alpha/k \sim 1/|q'\theta\nabla\psi|$, all aperiodic terms cancel, and we have

$$\mathscr{L}_\pm \sim -\frac{1}{2}\int ds\left[\frac{\gamma p B^2}{(B^2+\gamma p)}\left(\dot{\eta} + v\,\mathrm{sgn}(q'\theta)\frac{\mathbf{B}\cdot\nabla\sigma}{p'|\nabla\psi|}\right)^2 + \frac{\dot{v}^2}{B^2} - \omega^2\rho\left(\frac{v^2}{B^2} + B^2\eta^2\right)\right], \tag{B2}$$

where $\mathscr{L}_\pm$ denotes the contribution $\mathscr{L}$ from θ greater/less than some large positive/negative value. We have used Eq. (17) to express the geodesic curvature in terms of $\mathbf{B}\cdot\nabla\sigma$.

At finite θ the coefficients are not quasiperiodic, but the important point is that they are now bounded for all θ.

We now show that if there is some value of $\omega^2 < 0$ for which there is a solution pair (v,η) of the Euler–Lagrange equations such that v and η are uniformly bounded as $|\theta| \to \infty$, then the solution must in fact be square integrable. That is, $\hat{\xi}^{(0)}$ lies in the Hilbert space corresponding to Eq. (42), and ω^2 is a point in the discrete spectrum of $\mathsf{P}_t\cdot\mathsf{L}_0\cdot\mathsf{P}_t$. This generalizes to the nonaxisymmetric case the result that there is no unstable continuum associated with singularities occurring on a single magnetic surface,[14,15] the relation to singularities coming through the question of the convergence of the sum in Eq. (57).

To demonstrate this result we use *reductio ad absurdum*: suppose v and η are *not* square integrable. Consider $\mathscr{L}_+$, which we take to be an integral from θ_+ to $2\theta_+$, where $\theta_+ \to \infty$. Integration by parts, using the Euler–Lagrange equations, gives (taking $q' > 0$)

$$\mathscr{L}_+ = -\frac{1}{2}\left[\frac{\gamma p B^2}{(B^2+\gamma p)}\left(\dot{\eta} + v\frac{\mathbf{B}\cdot\nabla\sigma}{p'|\nabla\psi|}\right)\eta + \frac{\dot{v}v}{B^2}\right]_{\theta_+}^{2\theta_+}. \tag{B3}$$

From the assumption that v and η are bounded as $\theta \to \infty$, we see that $\mathscr{L}_+$ must also be bounded as $\theta_+ \to \infty$. However we can also write $\mathscr{L}_+$ in the form

$$\mathscr{L}_+ = \omega^2\mathscr{K}_+ - \mathscr{W}_+, \tag{B4}$$

where $\mathscr{K}_+$ and $\mathscr{W}_+$ are positive integrals which, by hypothesis, diverge as $\theta_+ \to \infty$. Thus, as $\theta_+ \to \infty$,

$$\mathscr{W}_+/\mathscr{K}_+ \to \omega^2. \tag{B5}$$

But $\mathscr{W}_+/\mathscr{K}_+ \geqslant 0$, in contradiction to the assumption $\omega^2 < 0$.

In the axisymmetric case we can show that the large-θ behavior of v (and η) may be represented by a series in inverse powers of θ in the form

$$v(\theta) = \theta^{\mu_i} e^{\lambda_i\theta}\left[v_i^{(0)}(\theta) + \theta^{-1} v_i^{(1)}(\theta) + \ldots\right], \tag{B6}$$

where the coefficients are 2π-periodic functions of θ, and μ_i and λ_i $(i = 1,\ldots,4)$ are characteristic exponents for each of the four independent solutions. We can use Floquet theory[36] to show that $\lambda_i = \pm\lambda_1, \pm\lambda_1^*$ so that there are two large solutions as $\theta \to \infty$ and two as $\theta \to -\infty$. Suppression of the large solutions gives a well-posed eigenvalue problem.

Characterization of the precise asymptotic behavior in the nonaxisymmetric case, when the coefficients of the Euler–Lagrange equations are only quasiperiodic, is more difficult although the theory of Sacker and Sell[37] gives some insight. We can appeal to a general theorem[38] on the stability of discrete eigenvalues to small perturbations to support our assumption, in Sec. VIII, that $\lambda(\alpha,\psi,\theta_k)$ depends analytically on the parameter ν representing the departure from axisymmetry.

[1] A. M. M. Todd, J. Manickam, M. Okabayashi, M. S. Chance, R. C. Grimm, J. M. Greene, and J. L. Johnson, Nucl. Fusion **19**, 743 (1979).

[2] R. L. Dewar, J. Manickam, R. C. Grimm, and M. S. Chance, Nucl. Fusion **21**, 493 (1981); Corrigendum **22**, 307 (1982).

[3] R. L. Dewar, M. S. Chance, A. H. Glasser, J. M. Greene, and E. A. Frieman, WKB Theory for High-*n* Modes in Axisymmetric Toroidal Plasmas, Princeton Plasma Physics Laboratory Report PPPL-1587, 1979.

[4] W. M. Tang, R. L. Dewar, and J. Manickam, Nucl. Fusion **22**, 1079 (1981).

[5] D. Correa-Restrepo, Z. Naturforsch. **33a**, 789 (1978).

[6] W. M. Nevins and L. D. Pearlstein (private communication).

[7] A. H. Glasser in *Finite Beta Theory Proceedings of the Workshop*, Varenna, 1977, edited by B. Coppi and W. L. Sadowski (U. S. Dept. of Energy, Washington, D. C., 1977), CONF-7709167, p. 55.

[8] Y. C. Lee and J. W. Van Dam, Ref. 7, p. 93.

[9] J. W. Connor, R. J. Hastie, and J. B. Taylor, Proc. R. Soc. London Ser. A **365**, 1 (1979).

[10] B. Coppi, J. Filreis, and F. Pegoraro, Ann. Phys. N. Y. **121**, 1 (1979).

[11] R. D. Hazeltine, D. A. Hitchcock, and S. M. Mahajan, Phys. Fluids **24**, 180 (1981).

[12] W. S. Massey, *Algebraic Topology: An Introduction* (Harcourt, Brace, and World, New York, 1967), Chap. V.

[13] See, e.g., M. V. Berry in *Topics in Nonlinear Dynamics*, edited by S. Jorna

(American Institute of Physics, New York, 1978), p. 16.
[14]Y. P. Pao, Nucl. Fusion **18**, 621 (1975).
[15]E. Hameiri and J. H. Hammer, Phys. Fluids **22**, 1700 (1979).
[16]J. M. Greene and J. L. Johnson, Phys. Fluids **5**, 510 (1962).
[17]P. Roman, *Some Modern Mathematics for Physicists and Other Outsiders* (Pergamon, New York, 1975), Vol. 2.
[18]R. C. Grimm, R. L. Dewar, and J. Manickam, J. Comp. Phys. **49**, 94 (1983).
[19]I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. R. Soc. London, Ser. A **244**, 17 (1958).
[20]H. P. Furth, J. Killeen, M. N. Rosenbluth, and B. Coppi, *Plasma Physics and Controlled Nuclear Fusion Research* (IAEA, Vienna, 1966), Vol. 1, p. 103.
[21]J. M. Greene and J. L. Johnson, Plasma Phys. **10**, 729 (1968).
[22]R. L. Dewar, Phys. Fluids **13**, 2710 (1970).
[23]J. M. Greene and M. S. Chance, Nucl. Fusion **21**, 453 (1981).
[24]Y. C. Lee (private communication).
[25]P. R. Garabedian, *Partial Differential Equations* (Wiley, New York, 1967), Chap. 2.
[26]See, e.g., I. C. Percival, Adv. Chem. Phys. **36**, 1 (1977).
[27]G. V. Sheffield, U. R. Christensen, and H. P. Furth, Bull. Am. Phys. Soc. **26**, 860 (1981).
[28]See, e.g., *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965), pp. 446–448.
[29]See, e.g., E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw–Hill, New York, 1955), Chap. 17.
[30]V. I. Arnol'd, Izvestia Akad. Nauk. SSSR, Ser. Mat. **25**, 21 (1961); Am. Math. Soc. Trans. Ser. 2 **46**, 213 (1965).
[31]M. R. Herman, Inst. Hautes Etudes Sci., Publ. Math. **49**, 5 (1979).
[32]G. O. Spies, Nucl. Fusion **19**, 1532 (1979). E. Hameiri (private communication, 1979) has also recognized the possibility of an unstable spectrum distinct from the stable Alfvén and cusp continua, but has not given details.
[33]D. A. Monticello, W. Park, S. C. Jardin, M. S. Chance, R. L. Dewar, R. B. White, R. C. Grimm, J. Manickam, H. R. Strauss, J. L. Johnson, J. M. Greene, A. H. Glasser, P. K. Kaw, P. H. Rutherford, and E. J. Valeo, *Plasma Physics and Controlled Nuclear Fusion Research* (IAEA, Vienna, 1981), Vol. I, p. 227.
[34]W. Munk, in *Evolution of Physical Oceanography*, edited by B. A. Warren and C. Wunsch (MIT, Cambridge, MA, 1981), pp. 273–4.
[35]T. M. Antonsen, Jr. and B. Lane, Phys. Fluids **23**, 1205 (1980).
[36]V. A. Yakubovich and V. M. Starzhinskii, *Linear Differential Equations With Periodic Coefficients* (Wiley, New York, 1975).
[37]R. J. Sacker and G. R. Sell, J. Diff. Eqs. **27**, 320 (1978).
[38]T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 1966), p. 214.