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Banach fixed point theorem for digital images

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Abstract

In this paper, we prove Banach fixed point theorem for digital images. We also give the proof of a theorem which is a generalization of the Banach contraction principle. Finally, we deal with an application of Banach fixed point theorem to image processing.

Keywords: Digital image, fixed point, Banach contraction principle, digital contraction.
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1. Introduction

Digital topology is a developing area which is related to features of 2D and 3D digital images using topological properties of objects. In this field, our aim is to obtain some significant results for image processing and fixed point theory.

Fixed point theory consists of many fields of mathematics such as mathematical analysis, general topology and functional analysis. In metric spaces, this theory begins with the Banach contraction principle. There are various applications of fixed point theory in mathematics, computer science, engineering, game theory, image processing, etc. Banach fixed point theorem is the most significant test for solution of some problems in mathematics and engineering. The Banach Contraction Mapping Principle was firstly given in 1922 [1]. Its structure is so simple and useful, so it is used in existence problems in various fields of mathematical analysis. In recent times, for important studies using the Banach contraction principle, see [16, 17, 22, 23].

Up to now, several developments have been occured in this area. Digital topology was first studied by Rosenfeld [21]. Then Kong [19] introduce the digital fundamental group of a discrete object. Boxer [4] gives the digital versions of several notions from topology and [6] studies a variety of digital continuous functions.

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Some results and characteristic properties on the digital homology groups of $2D$ digital images are given in [9] and [18]. Ege and Karaca [10] construct Lefschetz fixed point theory for digital images and study the fixed point properties of digital images. Ege and Karaca [11] give relative and reduced Lefschetz fixed point theorem for digital images. They also calculate degree of the antipodal map for sphere-like digital images using fixed point properties.

This paper is organized as follows. In the first part, we give the required background about the digital topology and fixed point theory. In the next section, we state and prove main results on Banach fixed point theorem for digital images. Finally, we give an important application of Banach fixed point theorem to digital images. Lastly, we make some conclusions.

2. Preliminaries

Let X be a subset of $\mathbb{Z}^n$ for a positive integer n where $\mathbb{Z}^n$ is the set of lattice points in the n-dimensional Euclidean space and κ be represent an adjacency relation for the members of X. A digital image consists of (X, κ).

Definition 2.1. [5] Let l, n be positive integers, $1 \leq l \leq n$ and two distinct points

$$p = (p_1, p_2, \ldots, p_n),\ q = (q_1, q_2, \ldots, q_n) \in \mathbb{Z}^n$$

p and q are k_l-adjacent if there are at most l indices i such that $|p_i - q_i| = 1$ and for all other indices j such that $|p_j - q_j| \neq 1$, $p_j = q_j$.

There are some statements which can be obtained from Definition 2.1:

• Two points p and q in $\mathbb{Z}$ are 2-adjacent if $|p - q| = 1$ (see Figure 1).

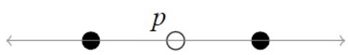

Figure 1: 2-adjacent

• Two points p and q in $\mathbb{Z}^2$ are 8-adjacent if they are distinct and differ by at most 1 in each coordinate.
• Two points p and q in $\mathbb{Z}^2$ are 4-adjacent if they are 8-adjacent and differ in exactly one coordinate (see Figure 2).

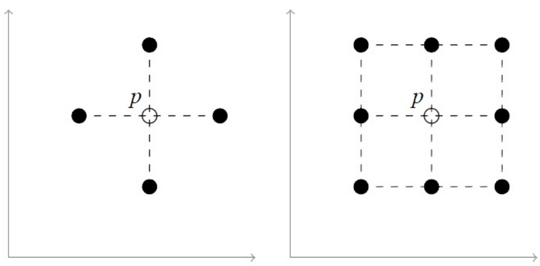

Figure 2: 4-adjacent and 8-adjacent

• Two points p and q in $\mathbb{Z}^3$ are 26-adjacent if they are distinct and differ by at most 1 in each coordinate.
• Two points p and q in $\mathbb{Z}^3$ are 18-adjacent if they are 26-adjacent and differ at most two coordinates.
• Two points p and q in $\mathbb{Z}^3$ are 6-adjacent if they are 18-adjacent and differ in exactly one coordinate (see Figure 3).

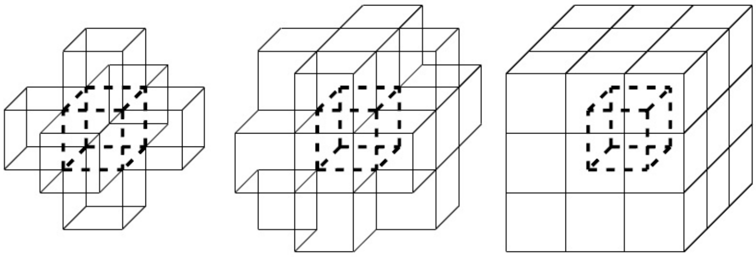

Figure 3: 6-adjacent, 18-adjacent and 26-adjacent

A κ-neighbor [5] of $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is κ-adjacent to p where $\kappa \in \{2, 4, 8, 6, 18, 26\}$ and $n \in 1, 2, 3$. The set

$$N_\kappa(p) = \{q \mid q \text{ is } \kappa - \text{adjacent to } p\}$$

is called the *κ-neighborhood* of p. *A digital interval* [4] is defined by

$$[a, b]_\mathbb{Z} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$$

where $a, b \in \mathbb{Z}$ and $a < b$.

A digital image $X \subset \mathbb{Z}^n$ is *κ-connected* [15] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i = 0, 1, \ldots, r-1$.

Definition 2.2. Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$, $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images and $f : X \longrightarrow Y$ be a function.

- If for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y, then f is said to be (κ_0, κ_1)-continuous [5].
- f is (κ_0, κ_1)-continuous [5] $\Leftrightarrow$ for every κ_0-adjacent points $\{x_0, x_1\}$ of X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are a κ_1-adjacent in Y.
- If f is (κ_0, κ_1)-continuous, bijective and f^{-1} is (κ_1, κ_0)-continuous, then f is called *(κ_0, κ_1)-isomorphism* [7] and denoted by $X \cong_{(\kappa_0,\kappa_1)} Y$.

A $(2, \kappa)$-continuous function $f : [0, m]_\mathbb{Z} \longrightarrow X$ such that $f(0) = x$ and $f(m) = y$ is called *a digital κ-path* [5] from x to y in a digital image X. In a digital image (X, κ), for every two points, if there is a κ-path, then X is called *κ-path connected. A simple closed κ-curve* of $m \geq 4$ points [8] in a digital image X is a sequence $\{f(0), f(1), \ldots, f(m-1)\}$ of images of the κ-path $f : [0, m-1]_\mathbb{Z} \longrightarrow X$ such that $f(i)$ and $f(j)$ are κ-adjacent if and only if $j = i \pm \mod m$.

A point $x \in X$ is called a κ-corner [3] if x is κ-adjacent to two and only two points $y, z \in X$ such that y and z are κ-adjacent to each other. If y, z are not κ-corners and if x is the only point κ-adjacent to both y, z, then we say that the κ-corner x is simple [2]. X is called *a generalized simple closed κ-curve* [20] if what is obtained by removing all simple κ-corners of X is a simple closed κ-curve. For a κ-connected digital image (X, κ) in $\mathbb{Z}^n$, there is a following statement [12]:

$$|X|^x = N_{3^n-1}(x) \cap X.$$

$$\kappa \in \left\{2n\ (n \geq 1), 3^n - 1\ (n \geq 2), 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1 (2 \leq r \leq n-1,\ n \geq 3)\right\}, \tag{2.1}$$

where $C_t^n = \frac{n!}{(n-t)!t!}$.

Definition 2.3. [14] Let (X, κ) be a digital image in $\mathbb{Z}^n$, $n \geq 3$ and $\overline{X} = \mathbb{Z}^n - X$. Then X is called a *closed κ-surface* if it satisfies the following.

(1) In case that $(\kappa, \overline{\kappa}) \in \{(\kappa, 2n), (2n, 3^n - 1)\}$, where the κ-adjacency is taken from (2.1) with $\kappa \neq 3^n - 2^n - 1$ and $\overline{\kappa}$ is the adjacency on $\overline{X}$, then

(a) for each point $x \in X$, $|X|^x$ has exactly one κ-component κ-adjacent to x;

(b) $|\overline{X}|^x$ has exactly two $\overline{\kappa}$-components $\overline{\kappa}$-adjacent to x; we denote by C^{xx} and D^{xx} these two components; and

(c) for any point $y \in N_\kappa(x) \cap X$, $N_{\overline{\kappa}}(y) \cap C^{xx} \neq \emptyset$ and $N_{\overline{\kappa}}(y) \cap D^{xx} \neq \emptyset$, where $N_\kappa(x)$ means the κ-neighbors of x.

Further, if a closed κ-surface X does not have a simple κ-point, then X is called simple.

(2) In case that $(\kappa, \overline{\kappa}) = (3^n - 2^n - 1, 2n)$, then

(a) X is κ-connected,

(b) for each point $x \in X$, $|X|^x$ is a generalized simple closed κ-curve.

Moreover, if the image $|X|^x$ is a simple closed κ-curve, then the closed κ-surface X is called simple.

Example 2.4. [13] $MSS'_{18} = \{c_0 = (1,1,0), c_1 = (0,2,0), c_2 = (-1,1,0), c_3 = (0,0,0), c_4 = (0,1,-1), c_5 = (0,1,1)\} \subset \mathbb{Z}^3$ is a minimal simple closed 18-surface (see Figure 4).

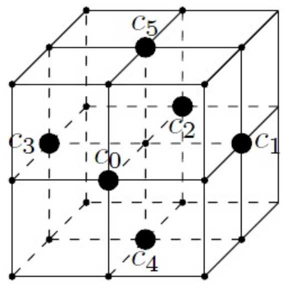

Figure 4: MSS'_{18}

Let (X, κ) be a digital image and its subset be (A, κ). (X, A) is called a digital image pair with κ-adjacency and when A is a singleton set $\{x_0\}$, then (X, x_0) is called *a pointed digital image.*

3. Banach Fixed Point Theorem for Digital Images

Let (X, κ) be a digital image and $f : (X, \kappa) \longrightarrow (X, \kappa)$ be any (κ, κ)-continuous function. We say the digital image (X, κ) has the fixed point property [10] if for every (κ, κ)-continuous map $f : X \longrightarrow X$ there exists $x \in X$ such that $f(x) = x$. The fixed point property is preserved by any digital isomorphism, i.e., it is a topological invariant. Let (X, d, κ) denote the digital metric space with κ-adjacency where d is usual Euclidean metric for $\mathbb{Z}^n$.

Definition 3.1. A sequence $\{x_n\}$ of points of a digital metric space (X, d, κ) is a Cauchy sequence if for all $\epsilon > 0$, there exists $\alpha \in \mathbb{N}$ such that for all $n, m > \alpha$, then

$$d(x_n, x_m) < \epsilon.$$

Definition 3.2. A sequence $\{x_n\}$ of points of a digital metric space (X, d, κ) converges to a limit $a \in X$ if for all $\epsilon > 0$, there exists $\alpha \in \mathbb{N}$ such that for all $n > \alpha$, then

$$d(x_n, a) < \epsilon.$$

Definition 3.3. A digital metric space (X, d, κ) is a complete digital metric space if any Cauchy sequence $\{x_n\}$ of points of (X, d, κ) converges to a point a of (X, d, κ).

Definition 3.4. Let (X, κ) be any digital image. A function $f : (X, \kappa) \to (X, \kappa)$ is called right-continuous if

$$f(a) = \lim_{x \to a^+} f(x)$$

where $a \in X$.

Definition 3.5. Let (X, d, κ) be any digital metric space and $f : (X, d, \kappa) \longrightarrow (X, d, \kappa)$ be a self digital map. If there exists $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \lambda d(x, y),$$

then f is called a digital contraction map.

Proposition 3.6. *Every digital contraction map is digitally continuous.*

Proof. Let (X, d, κ) be a digital metric space and $f : X \longrightarrow X$ be a digital contraction map. Pick $a \in X$ and let $\epsilon > 0$. Let $\delta = \epsilon$. Then if $d(a, b) < \delta$, we have

$$d(f(a), f(b)) \leq \lambda d(a, b) < \lambda\epsilon < \epsilon$$

where $\lambda \in (0, 1)$ for all $a, b \in X$. Then f is a (κ, κ)-continuous function. □

Theorem 3.7. *(Banach contraction principle)*
Let (X, d, κ) be a complete digital metric space which has a usual Euclidean metric in $\mathbb{Z}^n$. Let $f : X \longrightarrow X$ be a digital contraction map. Then f has a unique fixed point, i.e. there exists a unique $c \in X$ such that $f(c) = c$.

Proof. Assume that $a, b \in X$ are fixed points of f. Then we have the following:

$$\begin{aligned} d(a, b) = d(f(a), f(b)) \leq \lambda d(a, b) \quad &\Rightarrow \quad (1 - \lambda) d(a, b) \leq 0 \\ &\Rightarrow \quad a = b. \end{aligned}$$

Let x_0 be any point of X. Consider the iterate sequence $f(x_n) = x_{n+1}$. Using induction on n, we obtain

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}) \leq \ldots \leq \lambda^n d(f(x_0), x_0).$$

For natural numbers $n \in \mathbb{N}$ and $m \geq 1$, we conclude that

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, x_{n+m-1}) + \ldots + d(x_{n+1}, x_n) \\ &\leq [\lambda^{n+m} + \ldots + \lambda^n] d(f(x_0), x_0) \\ &\leq \frac{\lambda^n}{1 - \lambda} d(f(x_0), x_0). \end{aligned}$$

As a result, x_n is a Cauchy sequence. There is a limit point of x_n because (X, d, κ) is digital complete metric space. Let c be the limit of x_n. From the (κ, κ)-continuity of f, we get

$$f(c) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = c.$$

Therefore, f has a unique fixed point. □

Example 3.8. Let $X = [0, 2]_{\mathbb{Z}}$ be a digital interval with 2-adjacency. Consider the map

$$f : X \to X$$

defined by $f(x) = \dfrac{x}{2}$. It is clear that f has a fixed point.

We note that the following theorem is a generalization of Theorem 3.7.

Theorem 3.9. *(A generalization of the Banach contraction principle)*
Let (X, d, κ) be a complete digital metric space which has a usual Euclidean metric d in $\mathbb{Z}^n$ and let $f : X \to X$ be a digital selfmap. Assume that there exists a right-continuous real function

$$\gamma : [0, u] \to [0, u]$$

where u is sufficiently large real number such that

$$\gamma(a) < a \quad \text{if} \;\; a > 0, \tag{3.1}$$

and let f satisfies

$$d(f(x_1), f(x_2)) \leq \gamma(d(x_1, x_2)) \tag{3.2}$$

for all $x_1, x_2 \in (X, d, \kappa)$. Then f has a unique fixed point $c \in (X, d, \kappa)$ and the sequence $f^n(x)$ converges to c for every $x \in X$.

Proof. We first prove the uniqueness. Let u_1, u_2 be two fixed points of f. By (3.1) and (3.2), we get

$$d(u_1, u_2) = d(f(u_1), f(u_2)) \leq \gamma(d(u_1, u_2)) \quad \Rightarrow \quad u_1 = u_2.$$

Now let's prove the existence. For this purpose, we take a point $x_0 \in (X, d, \kappa)$ and define the sequence $f(x_n) = x_{n+1}$. For $n \in \mathbb{N}$, define the following sequence:

$$a_n = d(x_n, x_{n-1}).$$

Using (3.1) and (3.2), we obtain

$$\begin{aligned} a_{n+1} = d(x_{n+1}, x_n) &\leq \gamma(d(x_n, x_{n-1})) \\ &< d(x_n, x_{n-1}) = a_n \end{aligned}$$

for all $n \in \mathbb{N}$. Thus the sequence a_n is decreasing and so it has a limit a. If we assume that $a > 0$, we have

$$a_{n+1} \leq \gamma(a_n)$$

from (3.2). Since γ is right continuous, we get

$$a \leq \gamma(a)$$

but it contradicts with (3.1). As a result, $a_n \to 0$ as $n \to \infty$.

We would like to show x_n is a Cauchy sequence. Suppose that x_n is not a Cauchy sequence. Then there exists $\epsilon > 0$ and integers $m > n \geq k$ for every $k \geq 1$ such that

$$d(x_m, x_n) \geq \epsilon.$$

For a smallest m, we can suppose that $d(x_{m-1}, x_n) < \epsilon$. If we use the triangle inequality, we obtain

$$\begin{aligned} \epsilon \leq d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_n) \\ &< \epsilon + d(x_m, x_{m-1}). \end{aligned}$$

Since $d(x_n, x_{n-1}) \to 0$ as $n \to \infty$, we conclude that

$$\epsilon \leq d(x_m, x_n) < \epsilon \quad \Rightarrow \quad d(x_m, x_n) \to \epsilon \;\text{ as } n \to \infty.$$

From the fact that

$$m > n \quad \Rightarrow \quad d(x_{m+1}, x_m) \leq d(x_{n+1}, x_n)$$

and (3.2), we have

$$\begin{aligned} \epsilon \leq d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_n) \\ &\leq 2d(x_{n+1}, x_n) + \gamma(d(x_m, x_n)). \end{aligned}$$

Taking the limit as $n \to \infty$, from these inequalities we get $\epsilon \leq \gamma(\epsilon)$ but this contradicts with (3.1) because $\epsilon > 0$. As a result, x_n is a Cauchy sequence and since (X, d, κ) is a complete digital metric space, $f^n(x)$ converges in (X, d, κ). □

4. An Application of Banach Fixed Point Theorem to Digital Images

In this section, we give an application of Banach fixed point theorem to image compression. The aim of image compression is to reduce redundant image information in the digital image. There are some problems in the storing an image. Memory data is usually too large and sometimes stored image has not more information than original image. It's known that the quality of compressed image can be poor. For this reason, we must pay attention to compress a digital image. Fixed point theorem can be used to image compression of a digital image. Let's show this process by an example.

Example 4.1. Let F_0 be a digital image as in the figure 5.

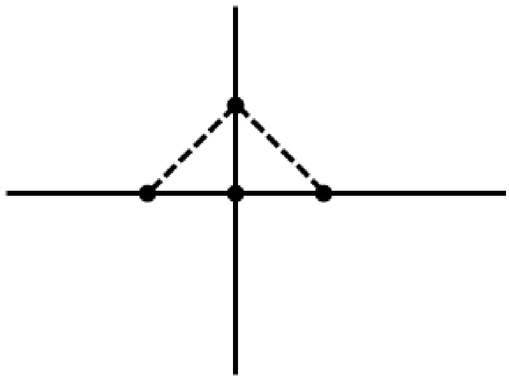

Figure 5: F_0

Starting from the digital image F_0, we can construct the following procedure:

(i) We make a copy of F_0 and glue it on the lower left vertex.

(ii) We create a copy of F_0 and glue it on lower right vertex.

(iii) So we have a new digital image which is denoted by F_1 (see figure 6).

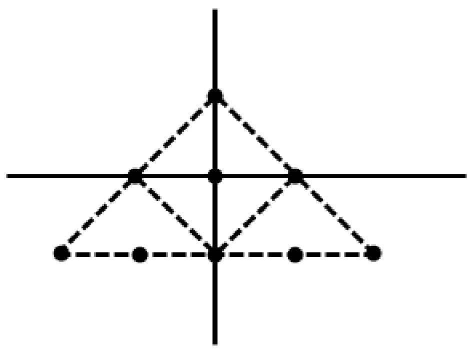

Figure 6: F_1

If the same procedure is applied to F_1, we have again a new digital image F_2 which is identical to F_1 (see figure 7).

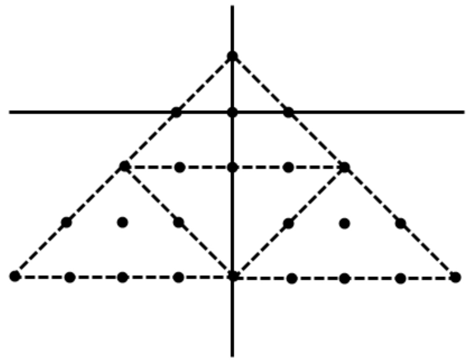

Figure 7: F_2

As a result, F_2 is the fixed point for this process. We would like to give the upper procedure in the mathematical sense. Let V be the function which takes F_i to $V(F_i)$. So we observe that $V(F_2) = F_2$, i.e. F_2 is a fixed point of this function. If we continue the process indefinitely, we obtain an infinite sequence of sets $\{F_n\}$. The sequence $\{F_n\}$ converges to F_2. It cannot be distinguished F_5 from F_2. As a result, the computer programme use F_5 instead of F_2 to better resolution. At the same time, the programme could use F_2 in place of F_5 to determine easily some properties of digital image.

Example 4.1 shows that the fixed point theory can be used for some digital imaging applications.

5. Conclusion

Our aim is to give the digital version of Banach fixed point theorem. We hope that this work will be useful for digital topology and fixed point theory. All results in this paper will help us to understand better the structure of digital images. We give an important application and use the fixed point theory to solve some problems in digital imaging. In the future, we will research other fixed point properties of digital images.

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