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BANACH SPACE REPRESENTATIONS AND IWASAWA THEORY

BY

P. SCHNEIDER

Mathematisches Institut, Westfälische Wilhelms-Universität Münster
Einsteinstr. 62, D-48149 Münster, Germany
e-mail: pschnei@math.uni-muenster.de
http://www.uni-muenster.de/math/u/schneider

AND

J. TEITELBAUM

Department of Mathematics, Statistics, and Computer Science (M/C 249)
University of Illinois at Chicago, 851 S. Morgan St., Chicago, IL 60607, USA
e-mail: jeremy@uic.edu
http://raphael.math.uic.edu/~jeremy

ABSTRACT

We develop a duality theory between the continuous representations of a compact p-adic Lie group G in Banach spaces over a given p-adic field K and certain compact modules over the completed group ring $o_K[[G]]$. We then introduce a "finiteness" condition for Banach space representations called admissibility. It will be shown that under this duality admissibility corresponds to finite generation over the ring $K[[G]] := K \otimes o_K[[G]]$. Since this latter ring is noetherian it follows that the admissible representations of G form an abelian category. We conclude by analyzing the irreducibility properties of the continuous principal series of the group $G := \mathrm{GL}_2(\mathbb{Z}_p)$.

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Introduction

The lack of a p-adic Haar measure causes many methods of traditional representation theory to break down when applied to continuous representations of a compact p-adic Lie group G in Banach spaces over a given p-adic field K. For example, the abelian group $G = \mathbb{Z}_p$ has an enormous wealth of infinite dimensional, topologically irreducible Banach space representations, as may be seen in the paper by Diarra [Dia]. We therefore address the problem of finding an additional "finiteness" condition on such representations that will lead to a reasonable theory. We introduce such a condition that we call "admissibility". We show that the category of all admissible G-representations is reasonable — in fact, it is abelian and of a purely algebraic nature — by showing that it is anti-equivalent to the category of all finitely generated modules over a certain kind of completed group ring $K[[G]]$.

In the first part of our paper we deal with the general functional-analytic aspects of the problem. We first consider the relationship between K-Banach spaces and compact, linearly topologized o-modules where o is the ring of integers in K. As a special case of ideas of Schikhof [Sch], we recall that there is an anti-equivalence between the category of K-Banach spaces and the category of torsionfree, linearly compact o-modules, provided one tensors the Hom-spaces in the latter category with $\mathbb{Q}$. In addition we have to investigate how this functor relates certain locally convex topologies on the Hom-spaces in the two categories. This will enable us then to derive a version of this anti-equivalence with an action of a profinite group G on both sides relating K-Banach space representations of G and certain topological modules for the ring $K[[G]] := K \otimes_o o[[G]]$.

Having established these topological results we assume that G is a compact p-adic Lie group and focus our attention on the Banach representations of G that correspond under the anti-equivalence to finitely generated modules over the ring $K[[G]]$. We characterize such Banach space representations intrinsically. We then show that the theory of such "admissible" representations is purely algebraic — one may "forget" about topology and instead study finitely generated modules over the noetherian ring $K[[G]]$.

As an application of our methods we determine the topological irreducibility as well as the intertwining maps for representations of $GL_2(\mathbb{Z}_p)$ obtained by induction of a continuous character from the subgroup of lower triangular matrices. Let us stress the fact that topological irreducibility for an admissible Banach space representation corresponds to the algebraic simplicity of the dual $K[[G]]$-module. It is indeed the latter which we will analyze. These results are a complement to the treatment of the locally analytic principal series representations studied in [ST1].

Throughout this paper K is a finite extension of $\mathbb{Q}_p$ with ring of integers $o \subseteq K$ and absolute value $|\ |$. A topological o-module is called linear-topological if the zero element has a fundamental system of open neighbourhoods consisting of o-submodules. We let

$\mathrm{Mod}_{\mathrm{top}}(o) :=$ category of all Hausdorff linear-topological o-modules with morphisms being all continuous o-linear maps.

1. A duality for Banach spaces

In this section we will recall a certain duality theory for K-Banach spaces due to Schikhof ([Sch]). Because of the fundamental role it will play in our later considerations and since it is quite easy over locally compact fields we include the proofs. We set

$\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o) :=$ the full subcategory in $\mathrm{Mod}_{\mathrm{top}}(o)$ of all torsionfree and compact linear-topological o-modules.

REMARK 1.1:
(i) *An o-module is torsionfree if and only if it is flat;*
(ii) *a compact linear-topological o-module M is flat if and only if $M \cong \prod_{i \in I} o$ for some set I.*

Proof: (i) [B-CA] Chap. I §2.4 Prop. 3(ii). (ii) [SGA3] Exp. VII_B (0.3.8).

For later purposes let us note that any o-module M in $\mathrm{Mod}_{\mathrm{top}}(o)$ has a unique largest quotient module M_{cot} which is Hausdorff and torsionfree: If $(M_j)_{j \in J}$ is the family of all torsionfree Hausdorff quotient modules of M then M_{cot} is the coimage of the natural map $M \longrightarrow \prod_{j \in J} M_j$.

For any o-module M in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)$ we can construct the K-Banach space

$$M^d := \mathrm{Hom}_o^{\mathrm{cont}}(M, K) \text{ with norm } \|\ell\| := \max_{m \in M} |\ell(m)|.$$

This defines a contravariant additive functor

$$\begin{array}{ccc} \mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o) & \longrightarrow & \mathrm{Ban}(K) \\ M & \longmapsto & M^d \end{array}$$

into the category $\mathrm{Ban}(K)$ of all K-Banach spaces with morphisms being all continuous K-linear maps. Actually all maps in the image of this functor are norm decreasing. The groups of homomorphisms in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)$ are o-modules whereas in $\mathrm{Ban}(K)$ they are K-vector spaces. The above functor therefore extends naturally to a contravariant additive functor

$$\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)_{\mathbb{Q}} \longrightarrow \mathrm{Ban}(K).$$

Here $\mathfrak{A}_{\mathbb{Q}}$, for any additive category $\mathfrak{A}$, denotes the additive category with the same objects as $\mathfrak{A}$ and such that

$$\mathrm{Hom}_{\mathfrak{A}_{\mathbb{Q}}}(A,B) := \mathrm{Hom}_{\mathfrak{A}}(A,B) \otimes \mathbb{Q}$$

for any two objects A, B in $\mathfrak{A}$ with the composition of morphisms in $\mathfrak{A}_{\mathbb{Q}}$ being the $\mathbb{Q}$-linear extension of the composition in $\mathfrak{A}$.

THEOREM 1.2: *The functor*

$$\begin{array}{ccc} \mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)_{\mathbb{Q}} & \xrightarrow{\sim} & \mathrm{Ban}(K) \\ M & \longmapsto & M^d \end{array}$$

is an anti-equivalence of categories.

Proof: Let $\mathrm{Ban}(K)^{\leq 1}$ denote the category of all K-Banach spaces $(E, \|\ \|)$ such that $\|E\| \subseteq |K|$ with morphisms being all norm decreasing K-linear maps. Clearly our functor factorizes into

$$\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o) \xrightarrow{(.)^d} \mathrm{Ban}(K)^{\leq 1} \xrightarrow{\text{forget}} \mathrm{Ban}(K).$$

For any K-Banach space $(E, \|\ \|)$ we may define by $\|v\|' := \inf\{r \in |K| : r \geq \|v\|\}$ another norm $\|\ \|'$ on E satisfying $\|E\|' \subseteq |K|$. Because of $|\pi| \leq \|v\|/\|v\|' \leq 1$ for $v \neq 0$, where π is a prime element of K, the two norms $\|\ \|$ and $\|\ \|'$ are equivalent. It follows that the right hand functor above induces an equivalence of categories

$$(\mathrm{Ban}(K)^{\leq 1})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Ban}(K).$$

We therefore are reduced to show that

$$\begin{array}{ccc} \mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o) & \longrightarrow & \mathrm{Ban}(K)^{\leq 1} \\ M & \longmapsto & M^d \end{array}$$

is an anti-equivalence of categories. Let $(E, \|\ \|)$ be a K-Banach space and denote by $E^\circ := \{v \in E : \|v\| \leq 1\}$ its unit ball. Then

$$E^d := \mathrm{Hom}_o(E^\circ, o) \text{ with the topology of pointwise convergence}$$

is a linear-topological o-module which is torsionfree and complete. In fact, E^d is the unit ball of the dual Banach space E' but equipped with the weak topology. Since

$$\begin{array}{rcl} E^d & \hookrightarrow & \prod_{v\in E^\circ} o \\ \lambda & \longmapsto & (\lambda(v))_v \end{array}$$

is a topological embedding we see that E^d is compact. This defines a functor

$$\begin{array}{rcl} \mathrm{Ban}(K)^{\leq 1} & \longrightarrow & \mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o) \\ (E, \|\ \|) & \longmapsto & E^d. \end{array}$$

It is an immediate consequence of Remark 1.1 that, for an o-module M in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)$, the o-linear map

$$\begin{array}{rcl} \iota_M\colon M & \longrightarrow & (M^d)'_s \\ m & \longmapsto & [\ell \longmapsto \ell(m)] \end{array}$$

into the weak dual $(M^d)'_s$ of the Banach space M^d is injective. Since it is easily seen to be continuous the compactness of M implies that ι_M is a closed embedding. By definition the image of ι_M is contained in M^{dd}. Assume now that there is a $\lambda \in M^{dd} \setminus \mathrm{im}(\iota_M)$. Since $\mathrm{im}(\iota_M)$ is closed in $(M^d)'_s$ there is, by Hahn-Banach ([Mon] V.1.2 Thm. 5(ii) or [NFA] 13.3), a continuous linear form on $(M^d)'_s$ which in absolute value is ≥ 1 on λ and is < 1 on $\mathrm{im}(\iota_M)$. But, as another consequence of Hahn-Banach ([NFA] 9.7), any continuous linear form on $(M^d)'_s$ is given by evaluation in a vector in M^d. Hence we find an $\ell \in M^d$ such that $|\lambda(\ell)| \geq 1$ and $|\ell(M)| < 1$. The latter implies $\|\ell\| < 1$ so that $\|\lambda(\ell)\| \leq \|\lambda\| \cdot \|\ell\|| < 1$, which is a contradiction. We obtain that $\iota_M\colon M \xrightarrow{\cong} M^{dd}$, in fact, is a topological isomorphism. This means that the ι_M constitute a natural isomorphism between the identity functor and the functor $(.)^{dd}$ on $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)$. On the other hand, any $(E, \|\ \|)$ in $\mathrm{Ban}(K)^{\leq 1}$ is isometric to a Banach space $c_0(I)$ for some set I ([Mon] IV.3 Cor. 1 or [NFA] 10.1). A straightforward explicit computation shows that $c_0(I)^{dd} = c_0(I)$. The functor $M \longrightarrow M^d$ therefore is fully faithful as well as essentially surjective and consequently an equivalence. ∎

The exactness properties of this functor are as follows.

Proposition 1.3: *For any map $f : M \longrightarrow N$ in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)$ we have:*

(i) $\ker(f)^d = M^d/\overline{f^d(N^d)}$;

(ii) $[\mathrm{coker}(f)_{\mathrm{cot}}]^d = \ker(f^d)$;

(iii) *f is surjective if and only if f^d is an isometry.*

Proof: (i) The submodules $\ker(f)$ and $\mathrm{im}(f)$ lie again in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)$. It follows from [SGA3] Exp. VII_B (0.3.7) that the surjection $M \longrightarrow \mathrm{im}(f)$ splits, i.e., we have $M \cong \ker(f) \oplus \mathrm{im}(f)$ in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)$. It suffices therefore to consider the case where f is injective and to show that then the image of f^d is dense in M^d. If not, we find by Hahn-Banach a nonzero continuous linear form λ on M^d which vanishes on the image of f^d. Up to scaling we may assume that $\lambda \in M^{dd}$, i.e., that there is a nonzero $m \in M$ such that $\lambda(\ell) = \ell(m)$. The vanishing property of λ means of course that $f(m) = 0$, which is a contradiction.

(iii) If f is surjective then f^d is an isometry by construction. Suppose now that f^d is an isometry. Let $n \in N$; we view n as a linear form in the unit ball of the dual Banach space $(N^d)'$. By Hahn-Banach n extends (via f^d) to a linear form in the unit ball of $(M^d)'$; this means of course that we find an $m \in M$ such that $f(m) = n$.

(ii) Let E denote the kernel of f^d. Then E^d is, by (iii), a torsionfree Hausdorff quotient of $\mathrm{coker}(f)$. On the other hand $[\mathrm{coker}(f)_{\mathrm{cot}}]^d$ clearly is a subspace of $\ker(f^d)$. ∎

Let M be a module in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)$. Since M is torsionfree it is an o-submodule of the K-vector space $M_K := M \otimes_o K$. Theorem 1.2 tells us that there is a natural identification

$$M_K = \mathrm{Hom}^{\mathrm{cont}}_o(o, M) \otimes \mathbb{Q} = \mathrm{Hom}^{\mathrm{cont}}_K(M^d, K) = (M^d)'$$

between M_K and the continuous dual $(M^d)'$ of the Banach space M^d. We always equip M_K with the finest locally convex topology such that the inclusion $M \subseteq M_K$ is continuous. An o-submodule $L \subseteq M_K$ is open if and only if $\alpha L \cap M$ is open in M for any $0 \neq \alpha \in o$. By construction this topology has the property that

$$\mathrm{Hom}^{\mathrm{cont}}_o(M, V) = \mathcal{L}(M_K, V)$$

for any locally convex K-vector space V where, following a common convention, we let $\mathcal{L}(.,.)$ denote the vector space of continuous linear maps between two locally convex K-vector spaces. In particular M^d at least as a vector space is the continuous dual $(M_K)'$. Since under the identification $M_K = (M^d)'$ the topology of M is induced by the weak topology on $(M^d)'$ we also see that the identification map $M_K \xrightarrow{=} (M^d)'_s$ is continuous. This shows that M_K is Hausdorff and that M_K also induces the given topology on M.

LEMMA 1.4: *The locally convex K-vector space M_K, for any M in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)$, is complete.*

Proof: Fix a prime element π of K. Let $\mathcal{F}$ be a minimal Cauchy filter on M_K. We first show that there is a $m \in \mathbb{N}$ such that

$$F \cap \pi^{-m} M \neq \emptyset \quad \text{for all } F \in \mathcal{F}.$$

Otherwise there exists for any $n \in \mathbb{N}$ a $F_n \in \mathcal{F}$ with $F_n \cap \pi^{-n} M = \emptyset$. By the minimality of $\mathcal{F}$ we may assume that

$$F_n = F_n + L_n \text{ for some open } o\text{-submodule } L_n \subseteq M_K$$

([B-GT] Chap. II §3.2 Prop. 5). We also may assume that the L_n form a decreasing sequence $L_1 \supseteq L_2 \supseteq \cdots$. The o-submodule

$$L := \sum_{n \in \mathbb{N}} (L_n \cap \pi^{-n} M)$$

is open in M_K since $L \cap \pi^{-n} M \supseteq L_n \cap \pi^{-n} M$ for all $n \in \mathbb{N}$. The L_n being decreasing and the $\pi^{-n} M$ being increasing it is clear that

$$L \subseteq L_n + \pi^{-n} M \quad \text{for all } n \in \mathbb{N}.$$

As a Cauchy filter $\mathcal{F}$ must contain a coset $v + L$ for some $v \in M_K$. If $n_0 \in \mathbb{N}$ is chosen in such a way that $v \in \pi^{-n_0} M$ we have $L_{n_0} + \pi^{-n_0} M \in \mathcal{F}$. Both sets F_{n_0} and $L_{n_0} + \pi^{-n_0} M$ belonging to the filter $\mathcal{F}$ we obtain $F_{n_0} \cap (L_{n_0} + \pi^{-n_0} M) \neq \emptyset$, i.e., $F_{n_0} \cap \pi^{-n_0} M = (F_{n_0} + L_{n_0}) \cap \pi^{-n_0} M \neq \emptyset$, which is a contradiction. We see that

$$\mathcal{F}_m := \{F \cap \pi^{-m} M : F \in \mathcal{F}\}$$

for an appropriate $m \in \mathbb{N}$ is a filter on $\pi^{-m} M$. Since $\pi^{-m} M$ is compact in M_K the filter $\mathcal{F}_m$ being also a Cauchy filter has to be convergent. By [B-GT] Chap. II §3.2 Cor. 3 then $\mathcal{F}$ is convergent, too. This proves that M_K is complete. ∎

LEMMA 1.5: *For any two o-modules M and N in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)$ we have:*

(i) *For any compact subset $C \subseteq N_K$ the closed o-submodule in N_K topologically generated by C is compact as well;*

(ii) *for any compact subset $C \subseteq N_K$ there is a $0 \neq \alpha \in o$ such that $\alpha C \subseteq N$;*

(iii) $\mathrm{Hom}^{\mathrm{cont}}_o(M, N) \otimes \mathbb{Q} = \mathcal{L}(M_K, N_K)$;

(iv) *passing to the transpose induces a K-linear isomorphism*

$$\mathcal{L}(M_K, N_K) \xrightarrow{\cong} \mathcal{L}(N^d, M^d).$$

Proof: (i) Let $< C >$ denote the o-submodule generated by C. Let $L \subseteq N_K$ be any open and therefore also closed o-submodule. Since C is compact we find finitely many $c_1, \ldots, c_n \in C$ with $C \subseteq (c_1 + L) \cup \cdots \cup (c_n + L)$. Then $< C >$ is contained in $oc_1 + \cdots + oc_n + L$. But $oc_1 + \cdots + oc_n$ is compact, too, so that we again find finitely many $a_1, \ldots, a_m \in oc_1 + \cdots + oc_n$ with

$$oc_1 + \cdots + oc_n \subseteq (a_1 + L) \cup \cdots \cup (a_m + L).$$

Together we obtain

$$< C > \subseteq \bigcup_{1 \leq i \leq m} a_i + L$$

and since the right hand side is closed the closure $\overline{< C >}$ of $< C >$ also satisfies

$$\overline{< C >} \subseteq \bigcup_{1 \leq i \leq m} a_i + L.$$

Since L was arbitrary this implies by [B-GT] Chap. II §4.2 Thm. 3 that $\overline{< C >}$ is precompact. On the other hand, as a consequence of Lemma 1.4, $\overline{< C >}$ is Hausdorff and complete. Hence $\overline{< C >}$ is compact.

(ii) By (i) we may assume that C is a compact o-submodule of N_K. Fix a prime element π of K and put $C_n := C \cap \pi^{-n} N$ for any $n \in \mathbb{N}$. These C_n form an increasing sequence $C_1 \subseteq C_2 \subseteq \cdots$ of compact o-submodules of C such that $C = \bigcup_{n \in \mathbb{N}} C_n$. We have to show that $C_m = C$ holds for some $m \in \mathbb{N}$. Being empty the subset $\bigcap_{n \in \mathbb{N}} (C \backslash C_n)$ is not dense in C. As a compact space C in particular is a Baire space ([B-GT] Chap. IX §5.3 Thm. 1) so that already some $C \backslash C_n$ is not dense in C. This means that C_n contains a non-empty open subset of C. It is then itself an open o-submodule and therefore has to be of finite index in C. Our claim obviously follows from that.

(iii) We have $\mathrm{Hom}_o^{\mathrm{cont}}(M, N) \otimes \mathbb{Q} = \mathrm{Hom}_o^{\mathrm{cont}}(M, N) \otimes_o K = \mathrm{Hom}_o^{\mathrm{cont}}(M, N_K)$ $= \mathcal{L}(M_K, N_K)$ where the second identity is a consequence of the second assertion.

(iv) This follows from (iii) and Theorem 1.2. ∎

The assertion (iii) in Lemma 1.5 in particular means that M_K and N_K are isomorphic in $\mathrm{Mod}_{\mathrm{comp}}^{\mathrm{fl}}(o)_{\mathbb{Q}}$ if and only if they are isomorphic as locally convex vector spaces.

For any two o-modules M and N in $\mathrm{Mod}_{\mathrm{comp}}^{\mathrm{fl}}(o)$ we always view $\mathrm{Hom}_o^{\mathrm{cont}}(M, N)$ as a linear-topological o-module by equipping it with the topology of compact convergence. As a consequence of Lemma 1.5 this topology is induced by the topology of compact convergence on the vector space $\mathcal{L}(M_K, N_K)$. We write $\mathcal{L}_{cc}(M_K, N_K)$ for $\mathcal{L}(M_K, N_K)$ equipped with the finest locally convex topology such that the inclusion $\mathrm{Hom}_o^{\mathrm{cont}}(M,N) \hookrightarrow \mathcal{L}_{cc}(M_K,N_K)$ is continuous. By a similar argument as before $\mathcal{L}_{cc}(M_K,N_K)$ is Hausdorff and the latter inclusion is a topological embedding. Moreover, by Lemma 1.5, $\mathcal{L}_{cc}(M_K,N_K)$ is, in both variables, a functor on $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)_{\mathbb{Q}}$.

Given two K-Banach spaces E_1 and E_2 we write, following traditional usage, $\mathcal{L}_s(E_1,E_2)$ for the vector space $\mathcal{L}(E_1,E_2)$ equipped with the locally convex topology of pointwise convergence. We write $\mathcal{L}_{bs}(E_1,E_2)$ for $\mathcal{L}(E_1,E_2)$ equipped with the finest locally convex topology which coincides with the topology of pointwise convergence on any equicontinuous subset in $\mathcal{L}(E_1,E_2)$. Corresponding to any choice of defining norms $\|\ \|_i$ on E_i for $i=1,2$ we have the operator norm $\|\ \|$ on $\mathcal{L}(E_1,E_2)$. A subset in $\mathcal{L}(E_1,E_2)$ is equicontinuous if and only if it is bounded with respect to $\|\ \|$. Hence the topology of $\mathcal{L}_{bs}(E_1,E_2)$ can equivalently be characterized as being the finest locally convex topology which induces the topology of pointwise convergence on the unit ball with respect to $\|\ \|$ in $\mathcal{L}(E_1,E_2)$.

PROPOSITION 1.6: *Passing to the transpose induces, for any o-modules M and N in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)$, an isomorphism of locally convex K-vector spaces*

$$\mathcal{L}_{cc}(M_K,N_K) \xrightarrow{\cong} \mathcal{L}_{bs}(N^d,M^d).$$

Proof: It is clear from our preliminary discussion that it suffices to show that the o-linear isomorphism

$$\mathrm{Hom}_o^{\mathrm{cont}}(M,N) \xrightarrow{\cong} \{f \in \mathcal{L}(N^d,M^d) : \|f\| \le 1\}$$

given by the transpose is topological provided the left, resp. right, hand side carries the topology of compact, resp. pointwise convergence. We recall from the proof of Theorem 1.2 that M is the unit ball in the dual Banach space $(M^d)'$ equipped with the weak topology; we also have seen there that the closed equicontinuous subsets of the weak dual $(M^d)'_s$ are compact. By the Banach-Steinhaus theorem ([Tie] Thm. 4.3) a subset of $(M^d)'_s$ is equicontinuous if and only if it is bounded. Clearly any compact subset is bounded. It follows that for a closed subset of $(M^d)'_s$ the following properties are equivalent: Bounded, equicontinuous, bounded for the dual Banach norm, compact. This shows that the topology of compact convergence on $\mathrm{Hom}_o^{\mathrm{cont}}(M,N)$ is induced by the strong topology on $\mathcal{L}((M^d)'_s,(N^d)'_s)$. Our assertion therefore will be a consequence of the quite general fact that for any two K-Banach spaces E_1 and E_2 the transpose induces a topological isomorphism

$$\mathcal{L}_s(E_1,E_2) \xrightarrow{\cong} \mathcal{L}_b((E_2)'_s,(E_1)'_s)$$

where on the right hand side the subscript b indicates, as usual, the strong topology. This is straightforward from the definitions and the fact that

$$\begin{array}{ccc} \text{set of all open} & \xrightarrow{\sim} & \text{set of all closed equicontinuous} \\ o\text{-submodules in } E_2 & & o\text{-submodules in } (E_2)'_s \\ & & \\ L & \longmapsto & L^p := \{\ell \in (E_2)' : |\ell(v)| \leq 1 \text{ for any } v \in L\} \end{array}$$

is a bijection, which is a direct consequence of the Hahn-Banach theorem. ∎

2. Iwasawa modules and representations

From now on we let G denote a fixed profinite group. The completed group ring of G (over o) is defined to be

$$o[[G]] := \varprojlim_{H \in \mathcal{N}} o[G/H]$$

where $\mathcal{N} = \mathcal{N}(G)$ denotes the family of all open normal subgroups of G. In a natural way $o[[G]]$ is a torsionfree and compact linear-topological o-module; the ring multiplication is continuous. The surjections $o[G] \longrightarrow o[G/H]$ for $H \in \mathcal{N}$ induce in the limit a ring homomorphism

$$o[G] \longrightarrow o[[G]]$$

whose image is dense and which is injective ([Laz] II.2.2.3.1). Being the projective limit of the inclusions $G/H \subseteq o[G/H]$ the composed map

$$G \xrightarrow{\subseteq} o[G] \longrightarrow o[[G]]$$

is continuous and hence, by compactness, a homeomorphism onto its image.

Consider now a module M in $\mathrm{Mod}_{\mathrm{top}}(o)$ and let $C(G, M)$ denote the o-module of all continuous maps from G into M. It follows from the above discussion that the o-linear map

$$\begin{array}{ccc} \mathrm{Hom}_o^{\mathrm{cont}}(o[[G]], M) & \longrightarrow & C(G, M) \\ f & \longmapsto & f|G \end{array}$$

is well defined and injective.

LEMMA 2.1: *For any complete o-module M in* $\mathrm{Mod}_{\mathrm{top}}(o)$ *the map*

$$\begin{array}{ccc} \mathrm{Hom}_o^{\mathrm{cont}}(o[[G]], M) & \xrightarrow{\cong} & C(G, M) \\ f & \longmapsto & f|G \end{array}$$

is a bijection.

Proof: We extend a given $\varphi \in C(G, M)$ o-linearly to $o[G]$. By the completeness assumption and the density of $o[G]$ in $o[[G]]$ it suffices to show that this extension, which we again denote by φ, is continuous with respect to the topology induced by $o[[G]]$. Fix an open o-submodule $L \subseteq M$. By the uniform continuity of φ on G we find an $H \in \mathcal{N}$ such that

$$\varphi(g_iH) \subseteq \varphi(g_i) + L$$

for all g_i in a system of representatives for the left cosets of H in G (compare [Laz] II.2.2.5). Let $\alpha \in o$ be some element such that $\alpha \cdot \varphi(g_i) \subseteq L$ for all i. The o-submodule

$$L' := \bigoplus_i \{ \sum_{g \in g_iH} r_g g : \sum_g r_g \in \alpha o\}$$

then is open in $o[G]$ and we have

$$\varphi(L') \subseteq \sum_i (\alpha o \cdot \varphi(g_i) + L) \subseteq L. \qquad \blacksquare$$

We set $K[[G]] := o[[G]]_K$. This is a locally convex vector space as well as a K-algebra such that the multiplication is separately continuous.

COROLLARY 2.2: *For any quasi-complete Hausdorff locally convex K-vector space V we have the K-linear isomorphism*

$$\begin{array}{ccc} \mathcal{L}(K[[G]], V) & \xrightarrow{\cong} & C(G, V) \\ f & \longmapsto & f|G. \end{array}$$

Proof: The map is clearly well defined and injective. For the surjectivity let $\varphi \in C(G, V)$. Define M to be the closed o-submodule of V topologically generated by $\varphi(G)$. This M lies in $\mathrm{Mod}_{\mathrm{top}}(o)$. Since G is compact M is bounded in V. The quasi-completeness of V therefore ensures that M is complete. Hence we have, by Lemma 2.1, a continuous o-linear map f: $o[[G]] \longrightarrow M \subseteq V$ such that $f|G = \varphi$. The K-linear extension of f then is the preimage of φ we were looking for. $\blacksquare$

We will apply these results to obtain a G-equivariant version of the duality theorem of the previous section.

Definition: A K-Banach space representation E of G is a K-Banach space E together with a G-action by continuous linear automorphisms such that the map $G \times E \longrightarrow E$ describing the action is continuous.

We define

$\mathrm{Ban}_G(K) :=$ category of all K-Banach representations of G with morphisms being all G-equivariant continuous linear maps.

As a consequence of the Banach-Steinhaus theorem ([Tie] Thm. 4.1.1°), to give a K-Banach representation of G on the K-Banach space E is the same as to give a continuous homomorphism $G \longrightarrow \mathcal{L}_s(E, E)$. But $\mathcal{L}_s(E, E)$ is quasi-complete and Hausdorff ([B-TVS] III.27 Cor. 4 or [NFA] 7.14). Hence such a homomorphism extends, by Corollary 2.2, uniquely to a continuous K-linear map $K[[G]] \longrightarrow \mathcal{L}_s(E, E)$. By a density argument the latter map is a K-algebra homomorphism. This shows that a K-Banach space representation of G on E is the same as a separately continuous action $K[[G]] \times E \longrightarrow E$ of the algebra $K[[G]]$ on E.

Since the image of $o[[G]]$ in $\mathcal{L}_s(E, E)$ under the above homomorphism is compact and hence (by Banach-Steinhaus) equicontinuous we also have that a K-Banach space representation of G on E is the same as a continuous (unital) homomorphism of K-algebras $K[[G]] \longrightarrow \mathcal{L}_{bs}(E, E)$.

Definition: An Iwasawa G-module over o is an o-module M in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)$ together with a continuous (left) action $o[[G]] \times M \longrightarrow M$ of the compact o-algebra $o[[G]]$ on M such that the induced o-action on M is the given o-module structure.

Let

$\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o[[G]]) :=$ category of all Iwasawa G-modules over o with morphisms being all continuous $o[[G]]$-module homomorphisms.

A continuous (unital) homomorphism of K-algebras

$$(*) \qquad K[[G]] \longrightarrow \mathcal{L}_{cc}(M_K, M_K)$$

for some M_K in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)_{\mathbb{Q}}$ induces a continuous map $o[[G]] \longrightarrow \mathrm{Hom}^{\mathrm{cont}}_{o}(M, M_K)$ where the right hand side carries the topology of compact convergence. By [B-GT] Chap. X §3.4 Thm. 3 this is the same as a continuous map $o[[G]] \times M \longrightarrow M_K$. According to Lemma 1.5(ii) the image of this latter map is contained in $\alpha^{-1}M$ for some $0 \neq \alpha \in o$. If N denotes the closed o-submodule of M_K topologically generated by this image we therefore have $N_K = M_K$, and the above homomorphism of K-algebras $(*)$ is the tensor product with K of a continuous (unital) homomorphism of o-algebras

$$o[[G]] \longrightarrow \mathrm{Hom}_o^{\mathrm{cont}}(N, N).$$

Again by [B-GT] loc. cit. this is the same as a continuous action $o[[G]] \times N \longrightarrow N$ of the compact o-algebra $o[[G]]$ on the o-module N in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o)$ which extends the o-module structure.

Hence we see that to give a continuous (unital) homomorphism of K-algebras $(*)$ is the same as to give an object in the category $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o[[G]])_{\mathbb{Q}}$. By combining this discussion with Proposition 1.6 we arrive at the following equivariant version of Theorem 1.2.

THEOREM 2.3: *The functor*

$$\begin{aligned} \mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o[[G]])_{\mathbb{Q}} &\xrightarrow{\sim} \mathrm{Ban}_G(K) \\ M &\longmapsto M^d \end{aligned}$$

is an anti-equivalence of categories.

3. Admissible representations

In order to obtain a reasonable theory of Banach space representations it seems necessary to impose certain additional finiteness conditions. The first idea is to consider only those K-Banach space representations of G which correspond, under the duality of the previous section, to finitely generated $K[[G]]$-modules. As a consequence of the compactness of the ring $o[[G]]$ it will turn out that the theory of these representations in fact is completely algebraic in nature. In order to obtain an intrinsic characterization we will assume in this section that G is a compact p-adic Lie group. We then have:

- The subfamily of all topologically finitely generated pro-p-groups in $\mathcal{N} = \mathcal{N}(G)$ is cofinal ([B-GAL] Chap. III §1.1 Prop. 2(iii) and §7.3 and 4 and [Laz] III.2.2.6 and III.3.1.3).

- The ring $o[[G]]$ is left and right noetherian ([Laz] V.2.2.4).

The ring $K[[G]]$ then is left and right noetherian as well.

Definition: A K-Banach space representation E of G is called admissible if there is a G-invariant bounded open o-submodule $L \subseteq E$ such that, for any $H \in \mathcal{N}$, the o-submodule $(E/L)^H$ of H-invariant elements in the quotient E/L is of cofinite type.

We recall that an o-module N is called of cofinite type if its Pontrjagin dual $\mathrm{Hom}_o(N, K/o)$ is a finitely generated o-module. We also point out that an arbitrary open o-submodule $L \subseteq E$ contains the G-invariant open o-submodule $\bigcap_{g\in G} gL$.

Let

$$\mathrm{Ban}_G^{\mathrm{adm}}(K) := \begin{array}{l}\text{the full subcategory in } \mathrm{Ban}_G(K)\\ \text{of all admissible representations.}\end{array}$$

On the other hand we let $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{fg}}(o[[G]])$, resp. $\mathrm{Mod}_{\mathrm{fg}}(K[[G]])$, denote the category of all finitely generated and o-torsionfree (left unital) $o[[G]])$-modules, resp. of all finitely generated (left unital) $K[[G]])$-modules. It is clear that

$$\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{fg}}(o[[G]])_{\mathbb{Q}} = \mathrm{Mod}_{\mathrm{fg}}(K[[G]]).$$

Since $K[[G]]$ is noetherian the category $\mathrm{Mod}_{\mathrm{fg}}(K[[G]])$ is abelian.

Proposition 3.1: (i) *A finitely generated $o[[G]]$-module M carries a unique Hausdorff topology — its canonical topology — such that the action $o[[G]] \times M \longrightarrow M$ is continuous;*

(ii) *any submodule of a finitely generated $o[[G]]$-module is closed in the canonical topology;*

(iii) *any $o[[G]]$-linear map between two finitely generated $o[[G]]$-modules is continuous for the canonical topologies.*

Proof: Since $o[[G]]$ is compact and noetherian this is an easy exercise. But we point out that the assertions hold for any compact ring by [AU] Cor. 1.10. ∎

It follows that equipping a module in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{fg}}(o[[G]])$ with its canonical topology induces a fully faithful embedding $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{fg}}(o[[G]]) \longrightarrow \mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o[[G]])$. This then in turn induces a fully faithful embedding $\mathrm{Mod}_{\mathrm{fg}}(K[[G]]) \longrightarrow \mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o[[G]])_{\mathbb{Q}}$. In other words we can and will view $\mathrm{Mod}_{\mathrm{fg}}(K[[G]])$ as a full subcategory of $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o[[G]])_{\mathbb{Q}}$.

For each $H \in \mathcal{N}$ let I_H denote the kernel of the projection map $o[[G]] \longrightarrow o[G/H]$. This is a family of 2-sided ideals in $o[[G]]$ which converges to zero. As a left (or right) ideal I_H is generated by the elements $h-1$ for $h \in H$. For the sake of completeness we include a proof of the following well known fact.

Lemma 3.2: *Let $H \in \mathcal{N}$ be a pro-p-group; then the ideal powers I_H^n, for $n \in \mathbb{N}$, converge to zero.*

Proof: We may assume that G is finite. Let π denote a prime element in o and $k := o/\pi o$ the residue field of o. By Clifford's theorem ([CR] (49.2)) and [Ser]

IX§1 the ideal $\ker(k[G] \longrightarrow k[G/H])$ is contained in the radical of the ring $k[G]$. Since this radical is nilpotent we have $I_H^m \subseteq \pi o[G]$ for some $m \in \mathbb{N}$. ∎

LEMMA 3.3: *Let $H \in \mathcal{N}$ be a pro-p-group; a module M in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(o[[G]])$ is finitely generated over $o[[G]]$ if and only if $M/I_H M$ is finitely generated over o.*

Proof: This is the well known Nakayama lemma; compare [BH] for a thorough discussion. ∎

LEMMA 3.4: *A K-Banach space representation E of G is admissible if and only if the dual space E' is finitely generated over $K[[G]]$.*

Proof: Let us first assume that E' is finitely generated over $K[[G]]$. There is then a finitely generated $o[[G]]$-submodule $M \subseteq E'$ such that $E' = M_K$. After equipping M with its canonical topology we have $E = M^d$. Moreover $L := \mathrm{Hom}_o^{\mathrm{cont}}(M, o)$ is a G-invariant bounded open o-submodule in E. It follows from Remark 1.1 that $E/L = \mathrm{Hom}_o^{\mathrm{cont}}(M, K/o)$ (where K/o carries the discrete topology) and hence that

$$(*) \qquad (E/L)^H = \mathrm{Hom}_o^{\mathrm{cont}}(M, K/o)^H = \mathrm{Hom}_o^{\mathrm{cont}}(M/I_H M, K/o)$$

for any $H \in \mathcal{N}$. Hence $(E/L)^H$ is of cofinite type.

On the other hand, let now $H \in \mathcal{N}$ be a pro-p-group and $L \subseteq E$ be a G-invariant bounded open o-submodule such that $(E/L)^H$ is of cofinite type. In the proof of Proposition 1.6 we had recalled that the G-invariant o-submodule $M := L^p$ in E'_s is compact. Since L is bounded we have $E' = M_K$. So the identities $(*)$ apply correspondingly and we obtain that $\mathrm{Hom}_o^{\mathrm{cont}}(M/I_H M, K/o)$ is of cofinite type. But since I_H is finitely generated as a right ideal the submodule $I_H M$ is the image of finitely many copies $M \times \cdots \times M$ under a continuous map and hence is closed in M. By Pontrjagin duality and the Nakayama lemma over o applied to the compact o-module $M/I_H M$ the latter therefore is finitely generated over o. Lemma 3.3 then implies that M is finitely generated over $o[[G]]$ and hence that E' is finitely generated over $K[[G]]$. ∎

The above proof shows that the defining condition for admissibility only needs to be tested for a single pro-p-group $H \in \mathcal{N}$. On the other hand, assume E to be an admissible representation of G and let $L \subseteq E$ be as in the above definition. Consider an $H \in \mathcal{N}$ and an *arbitrary* G-invariant open o-submodule $L_o \subseteq E$. We claim that $(E/L_o)^H$ is of cofinite type. Replacing L by αL for some appropriate $0 \neq \alpha \in o$ we may assume that $L \subseteq L_o$. As we have seen in the above proof

$M := L^p$ is a finitely generated $o[[G]]$-module. Since $o[[G]]$ is noetherian the $o[[G]]$-submodule $M_o := L_o^p$ of M also is finitely generated. As we have seen this implies that $(E/L_o)^H$ is of cofinite type.

THEOREM 3.5: *The functor*

$$\begin{aligned} \mathrm{Mod}_{\mathrm{fg}}(K[[G]]) &\xrightarrow{\sim} \mathrm{Ban}_G^{\mathrm{adm}}(K) \\ M &\longmapsto M^d \end{aligned}$$

is an anti-equivalence of categories.

Proof: Since $\mathrm{Mod}_{\mathrm{fg}}(K[[G]])$ is a full subcategory of $\mathrm{Mod}_{\mathrm{comp}}^{\mathrm{fl}}(o[[G]])_{\mathbb{Q}}$ by Proposition 3.1 this follows from Theorem 2.3 and Lemma 3.4. ∎

As an immediate consequence we obtain that the category $\mathrm{Ban}_G^{\mathrm{adm}}(K)$ is abelian.

COROLLARY 3.6: *The functor $E \longmapsto E'$ induces a bijection*

set of isomorphism classes of topologically irreducible admissible K-Banach space representations of G	$\xrightarrow{\sim}$	*set of isomorphism classes of simple $K[[G]]$-modules.*

Proof: For any proper closed G-invariant subspace $\{0\} \neq E_o \subsetneqq E$ we have, by Hahn-Banach, the exact sequence of dual vector spaces $0 \to (E/E_o)' \to E' \to E_o' \to 0$ in which all three terms are nonzero. If the $K[[G]]$-module E' is simple the representation E therefore must be topologically irreducible. On the other hand, write $E' = M_K$ for some module M in $\mathrm{Mod}_{\mathrm{fg}}^{\mathrm{fl}}(o[[G]])$ and let $\{0\} \neq V \subsetneqq M_K$ be a proper $K[[G]]$-submodule. By Proposition 3.1(ii) the nonzero $o[[G]]$-submodule $N := V \cap M$ is closed in M and hence lies in $\mathrm{Mod}_{\mathrm{comp}}^{\mathrm{fl}}(o[[G]])$. Since the quotient $(M/N)_{\mathrm{cot}} = M/N$ is nonzero as well it follows from Proposition 1.3 that the kernel of the dual map $E = M^d \longrightarrow N^d$ is a nonzero proper closed G-invariant subspace of E. ∎

One of the typical pathologies of general Banach space representations of G is avoided by the admissibility requirement as the following result shows.

COROLLARY 3.7: *Any nonzero G-equivariant continuous linear map between two topologically irreducible admissible K-Banach space representations of G is an isomorphism.*

Proof: This is immediate from Theorem 3.5 and Cororollary 3.6. ∎

The simplest group to which the results of this section apply is the group $G = \mathbb{Z}_p$ of p-adic integers. As is shown in [Dia] already this group has an extreme wealth of topologically irreducible K-Banach space representations. On the other hand, for a commutative group all "reasonable" topologically irreducible K-Banach space representations should be finite dimensional. This is achieved by the admissibility requirement. The ring $o[[\mathbb{Z}_p]]$ is the ring considered in classical Iwasawa theory; it is isomorphic to the power series ring $o[[T]]$ in one variable over o ([Was] 7.1). It follows ([Was] §13.2) that $K[[G]]$ is a principal ideal domain in which every maximal ideal is of finite codimension.

Remark: In [ST2] we have introduced the notion of an analytic module over the algebra $D(G,K)$ of K-valued distributions on G and we have advocated the conjecture that any $D(G,K)$-module of finite presentation is analytic. Since $K[[G]]$ is naturally a subalgebra of $D(G,K)$ base change would (assuming this conjecture) induce a functor from $\mathrm{Mod}_{\mathrm{fg}}(K[[G]])$ into the category of analytic $D(G,K)$-modules. Since the latter are dual to a certain class of locally analytic G-representations this functor should correspond to the passage from a K-Banach space representation to the subspace of locally analytic vectors. The next basic question in this context then would be whether the ring extension $K[[G]] \longrightarrow D(G,K)$ is faithfully flat. This is in the spirit of whether every admissible K-Banach space representation of G contains a locally analytic vector.

4. The group $G = GL_2(\mathbb{Z}_p)$

In this section we will analyze a certain infinite series of Iwasawa modules for the group $G := GL_2(\mathbb{Z}_p)$. Let $B \subseteq G$ denote the Iwahori subgroup of all matrices which are lower triangular modulo p. In B we consider the subgroups P, P^-, and T of lower triangular, upper triangular, and diagonal matrices, respectively. We also need the subgroups U and U^- of unipotent matrices in P and P^-, respectively. We fix a continuous character $\chi\colon T \longrightarrow o^\times$. By Corollary 2.2 it extends uniquely to a continuous homomorphism of K-algebras $\chi\colon K[[T]] \longrightarrow K$. The inclusions $P \subseteq B \subseteq G$, resp. the projection $P \longrightarrow T$, induce continuous algebra monomorphisms $K[[P]] \subseteq K[[B]] \subseteq K[[G]]$, resp. a continuous algebra epimorphism $K[[P]] \longrightarrow K[[T]]$. We denote by $K^{(\chi)}$ the one dimensional $K[[P]]$-module given by the composed homomorphism $K[[P]] \longrightarrow K[[T]] \xrightarrow{\chi} K$. Our aim is to study the finitely generated $K[[G]]$- and $K[[B]]$-modules

$$M_\chi := K[[G]] \underset{K[[P]]}{\otimes} K^{(\chi)} \quad \text{and} \quad N_\chi := K[[B]] \underset{K[[P]]}{\otimes} K^{(\chi)},$$

respectively. In a similar way (and by a slight abuse of notation) we have the finitely generated $K[[B]]$-module

$$N_\chi^- := K[[B]] \underset{K[[P^-]]}{\otimes} K^{(\chi)}.$$

Put

$$w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G \quad \text{and} \quad w\chi(t) := \chi(w^{-1}tw).$$

As a consequence of the Bruhat decomposition $G = B\dot{\cup}BwP$ the module M_χ, as a $K[[B]]$-module, decomposes into

$$M_\chi \cong N_\chi \oplus N_{w\chi}^-.$$

For later use we note that this decomposition is *not* $K[[G]]$-equivariant since obviously $wN_\chi \subseteq N_{w\chi}^-$.

The module theoretic properties of the series of modules N_χ and M_χ are governed by one numerical invariant $c(\chi) \in K$ of the character χ which is defined by the expansion

$$\chi(\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}) = \exp(c(\chi)\log(a))$$

for a sufficiently close to 1 (the existence follows from the topological cyclicity of the group $1 + p\mathbb{Z}_p$).

In order to investigate the module N_χ we use the Iwahori decomposition, which says that multiplication induces a homeomorphism $U^- \times P \xrightarrow{\sim} B$. It implies that $o[[B]] = o[[U^-]]\widehat{\otimes}o[[P]]$ where $\widehat{\otimes}$ is the completed tensor product for linear-topological o-modules ([SGA3] Exp. VII_B (0.3)). The inclusion $K[[U^-]] \subseteq K[[B]]$ therefore induces an isomorphism of $K[[U^-]]$-modules

$$(*) \qquad K[[U^-]] \xrightarrow{\cong} N_\chi.$$

In particular, any $K[[B]]$-submodule of N_χ corresponds to a certain ideal in the ring $K[[U^-]]$. Since the matrix

$$\gamma := \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

is a topological generator of U^-, the ring $K[[U^-]]$ is the ring of formal power series in $\gamma - 1$ whose coefficients are bounded. As already recalled earlier this is a principal ideal domain and each ideal is generated by a polynomial in $\gamma - 1$ with all its zeros lying in the open unit disk ([Was] §7.1).

PROPOSITION 4.1: *If $c(\chi) \notin \mathbb{N}_0$ then N_χ is a simple $K[[B]]$-module.*

Proof: Let $N \subseteq N_\chi$ be a nonzero $K[[B]]$-submodule, $I \subseteq K[[U^-]]$ be the ideal which corresponds to N under the above isomorphism $(*)$, and $F_I(\gamma - 1)$ be a polynomial which generates I and has all its zeros in the open unit disk. The action of the element

$$t_a := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in T$$

on N_χ is given on the left hand side of $(*)$ by

$$F(\gamma - 1) \longmapsto \chi(t_a) \cdot F(\gamma^a - 1)$$

for any bounded power series $F(x)$. Using the bounded power series

$$\omega_a(x) := (x+1)^a - 1 = \sum_{n \in \mathbb{N}} \binom{a}{n} x^n$$

this can be rewritten as

$$F(\gamma - 1) \longmapsto \chi(t_a) \cdot F(\omega_a(\gamma - 1)).$$

Since this action preserves the ideal I it follows that with z every $\omega_a(z)$, for $a \in \mathbb{Z}_p{}^\times$, is a zero of the polynomial $F_I(x)$. This is only possible if $z+1$ is a p^m-th root of unity for some $m \in \mathbb{N}$. We therefore see that there are natural numbers k_{o} and ℓ such that $F_I(x)$ divides $\omega_{p^{k_{\mathrm{o}}}}(x)^\ell$. In particular, for any natural number $k \geq k_{\mathrm{o}}$, the polynomial $\omega_{p^k}(x)^\ell$ lies in I. We now look at the action of the element $u := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ on N_χ. It is straightforward to check that on the left hand side of $(*)$ we have

$$u(\gamma^n) = \chi(\begin{pmatrix} (1+np)^{-1} & 0 \\ 0 & 1+np \end{pmatrix}) \cdot \gamma^{n/(1+np)} \quad \text{for any } n \in \mathbb{N}_0.$$

It follows that, for $k \geq k_{\mathrm{o}}$, with $\omega_{p^k}(x)^\ell$ also

$$u((\gamma^{p^k} - 1)^\ell) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \chi(\begin{pmatrix} (1+jp^{k+1})^{-1} & 0 \\ 0 & 1+jp^{k+1} \end{pmatrix}) \cdot \gamma^{jp^k/(1+jp^{k+1})}$$

lies in the ideal I. If $\omega_{p^k}(x)^\ell$ and its image under u, for some $k \geq k_{\mathrm{o}}$, have no zero in common, then I has to be the unit ideal which means that $N = N_\chi$. In the opposite case we obtain

$$\begin{aligned} 0 &= \sum_{j=0}^{\ell}(-1)^j \binom{\ell}{j} \chi\left(\begin{pmatrix} (1+jp^{k+1})^{-1} & 0 \\ 0 & 1+jp^{k+1} \end{pmatrix}\right) \\ &= \sum_{j=0}^{\ell}(-1)^j \binom{\ell}{j} \exp(c(\chi)\log(1+jp^{k+1})) \end{aligned}$$

for any sufficiently big $k \geq k_o$. This implies that the function

$$\sum_{j=0}^{\ell}(-1)^j \binom{\ell}{j} \exp(c(\chi)\log(1+jy))$$

of the variable y which is analytic in a sufficiently small open disk around zero has infinitely many zeros and hence vanishes identically. In order to prove our assertion we therefore have to show that this is only possible if $c(\chi) \in \mathbb{N}_0$. But if $c(\chi) \notin \mathbb{N}_0$, then evaluating all higher derivatives of the above function in zero would lead to the identities

$$\sum_{j=1}^{\ell}(-1)^j \binom{\ell}{j} j^m = 0 \quad \text{for any } m \in \mathbb{N}.$$

This is clearly impossible. ∎

The proof of the following companion result is completely analogous and is therefore omitted.

Proposition 4.2: *If $c(\chi) \notin -\mathbb{N}_0$ then N_χ^- is a simple $K[[B]]$-module.*

Lemma 4.3: $\mathrm{Hom}_{K[[B]]}(N_{\chi'}, N_\chi^-) = \mathrm{Hom}_{K[[B]]}(N_\chi^-, N_{\chi'}) = 0$ *for any two continuous characters χ and $\chi'\colon T \longrightarrow o^\times$.*

Proof: We compute

$$\begin{aligned} &\mathrm{Hom}_{K[[B]]}(N_{\chi'}, N_\chi^-) = \mathrm{Hom}_{K[[P]]}(K^{(\chi')}, N_\chi^-) \\ &\subseteq \mathrm{Hom}_{K[[U]]}(K, N_\chi^-) = \mathrm{Hom}_{K[[U]]}(K, K[[U]]) = 0. \end{aligned}$$

The other vanishing follows by a completely symmetric computation. ∎

Theorem 4.4: *If $c(\chi) \notin \mathbb{N}_0$ then M_χ is a simple $K[[G]]$-module.*

Proof: By our above results the decomposition $M_\chi \cong N_\chi \oplus N_{w\chi}^-$ is a $K[[B]]$-invariant decomposition into two nonisomorphic simple $K[[B]]$-modules. But as noted already at the beginning it is not $K[[G]]$-invariant. Hence M_χ must be a simple $K[[G]]$-module. ∎

The simple $K[[G]]$-modules which we have exhibited above are all non-isomorphic as the following result implies.

PROPOSITION 4.5: *We have* $\mathrm{Hom}_{K[[G]]}(M_{\chi'}, M_\chi) = 0$ *for any two continuous characters* $\chi \neq \chi' \colon T \longrightarrow o^\times$.

Proof: Because of Lemma 4.3 it is sufficient to show that $\mathrm{Hom}_{K[[B]]}(N_{\chi'}, N_\chi) = \mathrm{Hom}_{K[[B]]}(N^-_{\chi'}, N^-_\chi) = 0$. Since the arguments are completely symmetric we only discuss the vanishing of the first space. Making, as usual, the identification $(*)$ we have

$$\begin{aligned}\mathrm{Hom}_{K[[B]]}(N_{\chi'}, N_\chi) &= \mathrm{Hom}_{K[[P]]}(K^{(\chi')}, N_\chi)\\ &= \{F \in K[[U^-]] : g(F) = \chi'(g)\cdot F \text{ for any } g \in P\}.\end{aligned}$$

Assume now that there is a nonzero F in this latter space. Since any central matrix $g = \begin{pmatrix} b & 0 \\ 0 & b\end{pmatrix}$ in T acts by multiplication with $\chi(g)$ on N_χ it follows immediately that χ and χ' have to coincide on those matrices. On the other hand, the action of an element $t_a \in T$ as described in the proof of Proposition 4.1 gives rise to the equation

$$\chi(t_a)\cdot F((1+x)^a - 1) = \chi'(t_a)\cdot F(x)$$

between bounded power series over K. It was shown in the proof of [ST1] Prop. 5.5 that this implies $c(\chi') - c(\chi) \in 2\mathbb{N}_0$ and $F(x) = [\log(1+x)]^{(c(\chi')-c(\chi))/2}$. Since the power series $\log(1+x)$ is not bounded we in fact obtain $c(\chi') = c(\chi)$ and $F(x) = 1$. Going back to the above equation it follows that $\chi(t_a) = \chi'(t_a)$. Hence we have shown that the existence of a nonzero F forces the characters χ and χ' to coincide. ∎

To finish we briefly explain the dual picture. In the Banach space $C(G,K)$ of all K-valued continuous functions on G we have the closed subspace $\mathrm{Ind}_P^G(\chi) := \{f \in C(G,K) : f(gp) = \chi(p^{-1})f(g) \text{ for any } g \in G, p \in P\}$.

Via the left translation action this is a K-Banach space representation of G (a "principal series" representation). By the interpretation of $K[[G]]$ as the space of bounded K-valued measures on G we have that $K[[G]]$ is the continuous dual of $C(G,K)$. It easily follows that $\mathrm{Ind}_P^G(\chi)' = M_{\chi^{-1}}$.

In particular, by Lemma 3.4, $\mathrm{Ind}_P^G(\chi)$ is an admissible G-representation. As a consequence of Corollary 3.6 and Theorem 4.4 we see that $\mathrm{Ind}_P^G(\chi)$ is topologically irreducible if $c(\chi) \notin -\mathbb{N}_0$. This latter fact (for a slightly restricted class of χ) was proved in a direct and completely different way in [Tru].

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