Banach Spaces and Descriptive Set Theory: Selected Topics

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Preface

These notes are devoted to the study of some classical problems in the geometry of Banach spaces. The novelty lies in the fact that their solution relies heavily on techniques coming from descriptive set theory.

The central theme is universality problems. In particular, the text provides an exposition of the methods developed recently in order to treat questions of the following type.

(Q) Let $\mathcal{C}$ be a class of separable Banach spaces such that every space X in the class $\mathcal{C}$ has a certain property, say property (P). When can we find a separable Banach space Y which has property (P) and contains an isomorphic copy of every member of $\mathcal{C}$?

We will consider quite classical properties of Banach spaces, such as "being reflexive", "having separable dual", "not containing an isomorphic copy of c_0", "being non-universal", etc.

It turns out that a positive answer to problem **(Q)**, for any of the above mentioned properties, is possible if (and essentially only if) the class $\mathcal{C}$ is "simple". The "simplicity" of $\mathcal{C}$ is measured in set theoretic terms. Precisely, if the class $\mathcal{C}$ is analytic in a natural "coding" of separable Banach spaces, then we can indeed find a separable space Y which is universal for the class $\mathcal{C}$ and satisfies the requirements imposed above.

The text is addressed to both functional analysts and set theorists. We have tried to follow the terminology and notation employed by these two groups of researchers. Concerning Banach space theory, we follow the conventions adopted in the monograph of Lindenstrauss and Tzafriri [LT]. Our descriptive set theoretic terminology follows the one employed in the book of Kechris [Ke]. Still, we had to make a compromise; so throughout these notes by $\mathbb{N} = \{0, 1, 2, \dots\}$ we shall denote the natural numbers.

We proceed to discuss how this work is organized. It is divided into three parts which are largely independent from each other and can be read separately.

In the first part, consisting of Chapters 1 and 2, we display the necessary background and set up the frame in which this work will be completed. The second part, consisting of Chapters 3 and 4, is devoted to the study of two "gluing" techniques for producing separable Banach spaces from given classes of Banach spaces with a Schauder basis. In the third part, consisting of Chapters 5 and 6, we present two important embedding results and their parameterized versions.

The previous material is used in Chapter 7 which is, somehow, the goal of these notes. The notion of a *strongly bounded class* of separable Banach spaces is the central concept in Chapter 7. Several natural classes of separable Banach spaces are shown to be strongly bounded. This structural information is used to answer a number of universality problems in a unified manner.

To facilitate the interested reader we have also included four appendices. The first one contains an introduction to rank theory, a basic theme in descriptive set theory which is crucial throughout this work. In the second appendix we present some basic concepts and results from Banach space theory. Beside [LT], these topics are covered in great detail in other excellent books, such as [AK] and [Di], as well as, in the two volumes of the "*Handbook of the Geometry of Banach spaces*" [JL1, JL2]. In the third appendix we give a short description of a rather technical (yet very efficient) method in descriptive set theory, known as the "Kuratowski–Tarski algorithm". The method is used to compute the complexity of sets and relations. Finally, in the fourth appendix we discuss some open problems.

A significant part of the material presented in these notes has been discovered jointly with S. A. Argyros and has been published in [AD]. Actually, this text is the natural sequel of [AD] since it is mainly focused on further discoveries contained in [D3] and in our joint papers with V. Ferenczi [DF] and with J. Lopez-Abad [DL]. Several new results are also included. Needless to say that the solutions of the main problems are based on the work of many researchers including, among others, B. Bossard, J. Bourgain, N. Ghoussoub, B. Maurey, G. Pisier, W. Schachermayer and M. Zippin. Bibliographical information on the content of each chapter is contained in its final section, named as "Comments and Remarks".

We think that this work has, mainly, two reasons of interest. The first one is that it answers some basic problems in the geometry of Banach spaces and, more important, it explains several phenomena discovered so far. The second reason is that the solutions of the relevant problems combine techniques coming from two (seemingly) unrelated disciplines; namely from Banach space theory and from descriptive set theory.

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Contents

Chapter 1

Basic concepts

1.1 Polish spaces and standard Borel spaces

Polish spaces

A *Polish* space is a separable completely metrizable topological space. There are two fundamental examples of Polish spaces. The first one is the *Baire space* $\mathbb{N}^\mathbb{N}$ consisting of all sequences of natural numbers. The second one is the *Cantor space* $2^\mathbb{N}$ consisting of all sequences of 0's and 1's.

The "definable" subsets of a Polish space X can be classified according to their complexity. At the first level we have the *Borel* subsets of X. The Borel σ-algebra $B(X)$ is further analyzed in a hierarchy of length ω_1 consisting of the open and closed sets, then the F_σ and G_δ, etc. In modern logical notation these classes are denoted by $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi$ and $\mathbf{\Delta}^0_\xi$ $(1 \leqslant \xi < \omega_1)$, where

$$\mathbf{\Sigma}^0_1 = \text{open},\ \mathbf{\Pi}^0_1 = \text{closed},\ \mathbf{\Delta}^0_1 = \text{clopen},$$

$$\mathbf{\Sigma}^0_\xi = \Big\{ \bigcup_n A_n : A_n \text{ is in } \mathbf{\Pi}^0_{\xi_n} \text{ for some } 1 \leqslant \xi_n < \xi \Big\},$$

$$\mathbf{\Pi}^0_\xi = \text{complements of } \mathbf{\Sigma}^0_\xi \text{ sets},$$

$$\mathbf{\Delta}^0_\xi = \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi.$$

Hence $\mathbf{\Sigma}^0_2 = F_\sigma$, $\mathbf{\Pi}^0_2 = G_\delta$, etc.

Beyond the class of Borel subsets of X we have the *projective* sets which are defined using the operations of projection (or continuous image) and complementation. The class of projective sets is analyzed in a hierarchy of length ω, consisting of the analytic sets (continuous images of Borel sets), the co-analytic (complements of analytic sets), etc. Again, in logical notation, we have

$$\mathbf{\Sigma}^1_1 = \text{analytic},\ \mathbf{\Pi}^1_1 = \text{co-analytic},\ \mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1,$$

$$\mathbf{\Sigma}^1_{n+1} = \text{continuous images of } \mathbf{\Pi}^1_n \text{ sets},$$

$$\mathbf{\Pi}^1_{n+1} = \text{complements of } \mathbf{\Sigma}^1_{n+1} \text{ sets},$$

$$\mathbf{\Delta}^1_{n+1} = \mathbf{\Sigma}^1_{n+1} \cap \mathbf{\Pi}^1_{n+1}.$$

A fundamental result due to Souslin (see [Ke, Theorem 14.11]) asserts that the class $\mathbf{\Delta}^1_1$ coincides with the Borel σ-algebra; that is, a subset A of a Polish space X is Borel if and only if both A and $X \setminus A$ are analytic.

Standard Borel spaces

A *measurable space* is a pair (X, S) where X is a set and S is a σ-algebra on X. A measurable space (X, S) is said to be a *standard Borel space* if there exists a Polish topology τ on X such that the Borel σ-algebra of (X, τ) coincides with the σ-algebra S.

A classical fact concerning the Borel subsets of a Polish space (X, τ) is that if $B \in B(X, \tau)$, then there exists a stronger Polish topology τ' on X with $B(X, \tau) = B(X, \tau')$ and such that B is clopen in (X, τ') (see [Ke, Theorem 13.1]). It follows that if (X, S) is a standard Borel space and $B \in S$, then B equipped with the relative σ-algebra $S_B = \{C \cap B : C \in S\}$ is also a standard Borel space.

The above fact implies that the Polish topology τ witnessing that the measurable space (X, S) is standard, is not unique. As the Borel hierarchy depends on the topology of the underlying space, we will not consider the classes $\mathbf{\Sigma}^0_\xi(X, \tau)$ $(1 \leqslant \xi \leqslant \omega)$, unless the topology τ is of particular importance. On the other hand, if τ and τ' are two Polish topologies on X both witnessing that (X, S) is standard, then $\mathbf{\Sigma}^1_n(X, \tau) = \mathbf{\Sigma}^1_n(X, \tau')$ for every $n \geqslant 1$. That is, the projective hierarchy of (X, S), which will be at the center of our focus, is independent of the topology τ.

A basic example of a standard Borel space is the *Effros–Borel structure*. Specifically, for every Polish space X by $F(X)$ we denote the set of all closed subsets of X. We endow $F(X)$ with the σ-algebra S generated by the sets

$$\{F \in F(X) : F \cap U \neq \emptyset\}$$

where U ranges over all open subsets of X. The measurable space $\big(F(X), S\big)$ is called the Effros–Borel space of $F(X)$. We have the following important result (see [Ke, Theorem 12.6]).

Theorem 1.1. *If X is Polish, then the Effros–Borel space of $F(X)$ is standard.*

A very useful tool is the following selection result due to Kuratowski and Ryll-Nardzewski (see also [Sr, Theorem 5.2.1]).

Theorem 1.2. **[KRN]** *Let X and Y be Polish spaces and let $F\colon Y \to F(X)$ be a Borel map such that $F(y) \neq \emptyset$ for every $y \in Y$. Then there exists a sequence $f_n\colon Y \to X$ $(n \in \mathbb{N})$ of Borel selectors of F (i.e., $f_n(y) \in F(y)$ for every $n \in \mathbb{N}$ and every $y \in Y$) such that the sequence $\big(f_n(y)\big)$ is dense in $F(y)$ for all $y \in Y$.*

A structure closely related to the Effros–Borel space of a Polish space X, is the hyperspace $K(X)$ of all compact subsets of X equipped with the *Vietoris topology* τ_V, that is, the topology generated by the sets

$$\{K \in K(X) : K \cap U \neq \emptyset\} \ \text{ and } \ \{K \in K(X) : K \subseteq U\}$$

where U varies over all open subsets of X. The hyperspace $\big(K(X), \tau_V\big)$ inherits most of the topological properties of the space X. In particular, we have the following theorem (see [Ke, Theorems 4.25 and 4.26]).

Theorem 1.3. *If X is Polish (respectively, compact metrizable), then $K(X)$ is Polish (respectively, compact metrizable).*

It is easy to see that for every Polish space X the Borel σ-algebra of the space $\big(K(X), \tau_V\big)$ coincides with the relative σ-algebra of the Effros–Borel structure. This observation yields the following proposition.

Proposition 1.4. *Let X and Y be Polish spaces and let $f\colon Y \to K(X)$ be a map. Then f is Borel if and only if the set $\{y \in Y : f(y) \cap U \neq \emptyset\}$ is Borel for every open subset U of X.*

1.2 Trees

The concept of a tree is a basic combinatorial tool in both Banach space theory and descriptive set theory and it will be decisive throughout these notes. We will follow the practice of descriptive set theorists and we will consider trees as sets of finite sequences.

Specifically, let Λ be a nonempty set and denote by $\Lambda^{<\mathbb{N}}$ the set of all finite sequences of elements of Λ (the empty sequence is included). We equip $\Lambda^{<\mathbb{N}}$ with the strict partial order $\sqsubset$ of extension. We will use the letters t, s, w to denote elements of $\Lambda^{<\mathbb{N}}$.

For every $\sigma \in \Lambda^{\mathbb{N}}$ and $n \in \mathbb{N}$ with $n \geqslant 1$ we set $\sigma|n = \big(\sigma(0), \dots, \sigma(n-1)\big)$, while $\sigma|0 = \emptyset$. For every $t \in \Lambda^{<\mathbb{N}}$ the *length* $|t|$ of t is defined to be the cardinality of the set $\{s : s \sqsubset t\}$. If $t, s \in \Lambda^{<\mathbb{N}}$, then by $t^\frown s$ we denote the *concatenation* of t and s. Two nodes $t, s \in \Lambda^{<\mathbb{N}}$ are said to be *comparable* if either $s \sqsubseteq t$ or $t \sqsubseteq s$; otherwise, they are said to be *incomparable*. We denote by $t \perp s$ the fact that t and s are incomparable. A subset of $\Lambda^{<\mathbb{N}}$ consisting of pairwise incomparable nodes is said to be an *antichain*, while a subset of $\Lambda^{<\mathbb{N}}$ consisting of pairwise comparable nodes is said to be a *chain*. A maximal chain is called a *branch*. Two subsets A and B of $\Lambda^{<\mathbb{N}}$ are said to be *incomparable* if for every $t \in A$ and every $s \in B$ we have $t \perp s$. Otherwise, A and B are said to be *comparable*.

A *tree* T on Λ is a subset of $\Lambda^{<\mathbb{N}}$ which is closed under taking initial segments. By $\mathrm{Tr}(\Lambda)$ we denote the set of all trees on Λ. Hence,

$$T \in \mathrm{Tr}(\Lambda) \Leftrightarrow \forall s, t \in \Lambda^{<\mathbb{N}} \ (s \sqsubset t \text{ and } t \in T \Rightarrow s \in T). \tag{1.1}$$

Notice that if $T \in \mathrm{Tr}(\Lambda)$, then $\emptyset \in T$. Also observe that if Λ is countable, then by identifying every tree T on Λ with its characteristic function (i.e., an element of $2^{\Lambda^{<\mathbb{N}}}$) we see that the set $\mathrm{Tr}(\Lambda)$ is a closed (hence compact) subspace of $2^{\Lambda^{<\mathbb{N}}}$.

A tree T on Λ is said to be *pruned* if for every $s \in T$ there exists $t \in T$ with $s \sqsubset t$. It is said to be *perfect* if for every $t \in T$ there exist two nodes t_1 and t_2 in T properly extending t and with $t_1 \perp t_2$. The *body* $[T]$ of T is the set $\{\sigma \in \Lambda^{\mathbb{N}} : \sigma|k \in T \ \forall k \in \mathbb{N}\}$.

For every subset A of $\Lambda^{\mathbb{N}}$ we set

$$T_A = \{\sigma|n : \sigma \in A \text{ and } n \in \mathbb{N}\} \in \mathrm{Tr}(\Lambda). \tag{1.2}$$

We call T_A the tree *generated* by A. Notice that T_A is pruned. Also observe that the body $[T_A]$ of T_A is equal to the closure of A in $\Lambda^{\mathbb{N}}$, where Λ is equipped with the discrete topology and $\Lambda^{\mathbb{N}}$ with the product topology. The set of all nonempty closed subsets of $\Lambda^{\mathbb{N}}$ is in one-to-one correspondence with the set of all pruned trees on Λ via the map $F \mapsto T_F$.

A tree T on Λ is said to be *well-founded* if for every $\sigma \in \Lambda^{\mathbb{N}}$ there exists $k \in \mathbb{N}$ such that $\sigma|k \notin T$, equivalently if $[T] = \emptyset$. Otherwise, it is called *ill-founded*. By $\mathrm{WF}(\Lambda)$ we denote the set of all well-founded trees on Λ. The class of ill-founded trees is denoted by $\mathrm{IF}(\Lambda)$. If $\Lambda = \mathbb{N}$, then by WF and IF we shall denote the sets of well-founded and ill-founded trees on $\mathbb{N}$ respectively.

Let Λ be an infinite set and set $\kappa = |\Lambda|$. For every well-founded tree T on Λ we define

$$T' = \{s \in T : \exists t \in T \text{ with } s \sqsubset t\} \in \mathrm{WF}(\Lambda). \tag{1.3}$$

By transfinite recursion, we define the iterated derivatives $(T^\xi : \xi < \kappa^+)$ of T by the rule

$$T^0 = T, \ T^{\xi+1} = (T^\xi)' \text{ and } T^\lambda = \bigcap_{\xi<\lambda} T^\xi \text{ if } \lambda \text{ is limit.}$$

Notice that if $T^\xi \neq \emptyset$, then $T^{\xi+1} \subsetneq T^\xi$. It follows that the transfinite sequence $(T^\xi : \xi < \kappa^+)$ is eventually empty. The *order* $o(T)$ of T is defined to be the least ordinal ξ such that $T^\xi = \emptyset$. If $T \in \mathrm{IF}(\Lambda)$, then by convention we set $o(T) = \kappa^+$. In particular, if Λ is countable, then $o(T) < \omega_1$ for every $T \in \mathrm{WF}(\Lambda)$ while $o(T) = \omega_1$ for every $T \in \mathrm{IF}(\Lambda)$.

Let S and T be trees on Λ_1 and Λ_2 respectively. A map $\phi\colon S \to T$ is called *monotone* if for every $s_1, s_2 \in S$ with $s_1 \sqsubset s_2$ we have $\phi(s_1) \sqsubset \phi(s_2)$. The following fact is quite useful.

Proposition 1.5. *Let S and T be trees on two countable sets Λ_1 and Λ_2 respectively. Then $o(S) \leqslant o(T)$ if and only if there exists a monotone map $\phi\colon S \to T$.*

Proof. First assume that there exists a monotone map $\phi\colon S \to T$. If T is ill-founded, then obviously we have $o(S) \leqslant o(T)$. So, assume that T is well-founded. The existence of the monotone map ϕ implies that S is also well-founded. By transfinite induction, we see that for every countable ordinal ξ and every $s \in S^{\xi}$ we have $\phi(s) \in T^{\xi}$. Hence, $o(S) \leqslant o(T)$.

Conversely, assume that $o(S) \leqslant o(T)$. If T is ill-founded, then we select $\sigma \in [T]$. For every $s \in S$ with $|s| = n$ we set $\phi(s) = \sigma|n$. It is easy to check that $\phi\colon S \to T$ is a monotone map.

So, it remains to treat the case where T is well-founded. The monotone map $\phi\colon S \to T$ will be constructed by recursion on the length of sequences in S, as follows. We set $\phi(\emptyset) = \emptyset$. Let $k \in \mathbb{N}$ and assume that we have defined $\phi(s) \in T$ for every $s \in S$ with $|s| \leqslant k$ so that

$$\forall \xi < \omega_1 \ (s \in S^{\xi} \Rightarrow \phi(s) \in T^{\xi}). \tag{1.4}$$

Let $w \in S$ with $|w| = k+1$. There exist $s \in S$ with $|s| = k$ and $\lambda_1 \in \Lambda_1$ such that $w = s^\frown \lambda_1$. Let $t = \phi(s)$. Invoking (1.4), we see that there exists $\lambda_2 \in \Lambda_2$ such that, setting $\phi(w) = t^\frown \lambda_2$, property (1.4) is satisfied for w and $\phi(w)$. This completes the recursive construction of the map ϕ which is easily seen to be monotone. The proof is completed. □

We will also consider trees on products of sets. In particular, if Λ_1 and Λ_2 are nonempty sets, then we identify every tree T on $\Lambda_1 \times \Lambda_2$ with the set of all pairs $(s,t) \in \Lambda_1^{<\mathbb{N}} \times \Lambda_2^{<\mathbb{N}}$ such that $|s| = |t| = k$ and either $s = t = \emptyset$ or

$$\big((s_0, t_0), \dots, (s_{k-1}, t_{k-1})\big) \in T.$$

Under the above convention the body of a tree T on $\Lambda_1 \times \Lambda_2$ is identified with the set of all $(\sigma_1, \sigma_2) \in \Lambda_1^{\mathbb{N}} \times \Lambda_2^{\mathbb{N}}$ for which $(\sigma_1|k, \sigma_2|k) \in T$ for every $k \in \mathbb{N}$. The following representation of analytic sets provides the link between trees and descriptive set theory (see [Ke]).

Theorem 1.6. *Let Λ be a countable set and let A be a subset of $\Lambda^{\mathbb{N}}$. Then A is $\mathbf{\Sigma}^1_1$ if and only if there exists a tree T on $\Lambda \times \mathbb{N}$ such that*

$$A = \mathrm{proj}[T] = \big\{\sigma \in \Lambda^{\mathbb{N}} : \exists \tau \in \mathbb{N}^{\mathbb{N}} \textit{ with } (\sigma, \tau) \in [T]\big\}.$$

We need to deal with trees which consist of *nonempty* finite sequences. We will give them a special name, as follows.

Definition 1.7. *Let* Λ *be a nonempty set. A* B-tree on Λ *is a subset* T *of* $\Lambda^{<\mathbb{N}} \setminus \{\emptyset\}$ *such that*

$$\forall t, s \in \Lambda^{<\mathbb{N}} \setminus \{\emptyset\} \ (s \sqsubset t \textit{ and } t \in T \Rightarrow s \in T). \tag{1.5}$$

Notice that T is a B-tree on Λ if and only if $T \cup \{\emptyset\} \in \mathrm{Tr}(\Lambda)$. Using this remark we can relativize to B-trees all previously presented concepts. For instance, we say that a B-tree T on Λ is pruned (respectively, well-founded, perfect) if $T \cup \{\emptyset\}$ is pruned (respectively, well-founded, perfect). The body $[T]$ of a pruned B-tree T is the body of $T \cup \{\emptyset\}$. Notice that

$$[T] = \{\sigma \in \Lambda^{\mathbb{N}} : \sigma|k \in T \ \forall k \geqslant 1\}.$$

If A is a subset of $\Lambda^{\mathbb{N}}$, then we continue to denote by T_A the B-tree generated by A; that is, $T_A = \{\sigma|n : \sigma \in A \text{ and } n \geqslant 1\}$. From the context it will be clear whether we refer to the tree or to the B-tree generated by A.

A *segment* $\mathfrak{s}$ of a tree, or of a B-tree, T on Λ is a chain of T satisfying

$$\forall s, w, t \in T \ (s, t \in T \text{ and } s \sqsubseteq w \sqsubseteq t \Rightarrow w \in T). \tag{1.6}$$

If $\sigma \in \Lambda^{\mathbb{N}}$ and $k \in \mathbb{N}$, then a segment of the form $\{t \in T : t \sqsubseteq \sigma|k\}$ is called an *initial* segment of T, while a segment of the form $\{\sigma|n : n \geqslant k \text{ and } \sigma|n \in T\}$ is called a *final* segment. More generally, a subset A of a tree (or of a B-tree) T on Λ is said to be *segment complete* if

$$\forall s, w, t \in T \ (s, t \in A \text{ and } s \sqsubseteq w \sqsubseteq t \Rightarrow w \in A). \tag{1.7}$$

Notice that a segment of T is just a segment complete chain of T.

We shall denote and name some special trees. By $\mathbb{N}^{<\mathbb{N}}$ we denote the *Baire* tree, while by $2^{<\mathbb{N}}$ we denote the *Cantor* tree. By $[\mathbb{N}]^{<\mathbb{N}}$ we denote the tree on $\mathbb{N}$ consisting of all *strictly increasing* finite sequences of natural numbers, while by Σ we denote the B-tree corresponding to $[\mathbb{N}]^{<\mathbb{N}}$ (that is, Σ consists of all nonempty strictly increasing finite sequences on $\mathbb{N}$).

Finally, let us introduce some pieces of notation closely related to trees. By $[\mathbb{N}]^\infty$ we denote the set of all infinite subsets of $\mathbb{N}$. More generally, for every infinite subset L of $\mathbb{N}$ by $[L]^\infty$ we denote the set of all infinite subsets of L. If $k \in \mathbb{N}$ with $k \geqslant 1$ and $L \in [\mathbb{N}]^\infty$, then by $[L]^k$ we denote the set of all subsets of L of cardinality k. Notice that $[L]^k$ is naturally identified with the set of all strictly increasing sequences in L of length k.

1.3 Universal spaces

There are two fundamental universality results in Banach space theory which are of particular importance in these notes. The first one is classical and asserts that the space $C(2^{\mathbb{N}})$ is *isometrically universal* for all separable Banach spaces.

Theorem 1.8. *Let X be a separable Banach space. Then there exists a closed subspace Y of $C(2^{\mathbb{N}})$ which is isometric to X.*

Proof. Let E be the closed unit ball of X^* with the weak* topology. It is a compact metrizable space. Let $f\colon 2^{\mathbb{N}} \to E$ be a continuous surjection. Define $T\colon X \to C(2^{\mathbb{N}})$ by $T(x)(\sigma) = f(\sigma)(x)$ for every $\sigma \in 2^{\mathbb{N}}$ and every $x \in X$. It is easy to see that T is a linear isometric embedding. The proof is completed. □

The second result is due to Pełczyński and provides a space U with a Schauder basis (u_n) which is universal for all basic sequences.

Theorem 1.9. **[P]** *There exists a space U with a normalized bi-monotone Schauder basis (u_n) such that for every seminormalized basic sequence (x_n) in a Banach space X there exists $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that (x_n) is equivalent to (u_{l_n}) and the natural projection P_L onto $\overline{\mathrm{span}}\{u_n : n \in L\}$ has norm one. Moreover, if U' is another space with this property, then U' is isomorphic to U.*

Proof. Let (d_n) be a countable dense subset of the sphere of $C(2^{\mathbb{N}})$. Also let (x_n) be a seminormalized basic sequence in a Banach space X. By Theorem 1.8 and Proposition B.3, there exists $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that (x_n) is equivalent to (d_{l_n}). This is the crucial observation in the construction of the space U which goes as follows.

For every $t = (n_0 < \cdots < n_k) \in \Sigma$ we set $n_t = n_k$ (by Σ we denote the B-tree on $\mathbb{N}$ consisting of all nonempty strictly increasing finite sequences; see Section 1.2). We fix a bijection $\varphi\colon \Sigma \to \mathbb{N}$ such that $\varphi(t) < \varphi(s)$ if $t \sqsubset s$. For every $t \in \Sigma$ we define $f_t \in C(2^{\mathbb{N}})$ by $f_t = d_{n_t}$. The space U is the completion of $c_{00}(\Sigma)$ under the norm

$$\|x\| = \sup\Big\{\big\|\sum_{t\in\mathfrak{s}} x(t) f_t\big\|_\infty : \mathfrak{s} \text{ is a segment of } \Sigma\Big\}.$$

Let (u_n) be the enumeration, according to φ, of the standard Hamel basis $(e_t)_{t\in\Sigma}$ of $c_{00}(\Sigma)$. The sequence (u_n) defines a normalized bi-monotone Schauder basis of U.

For every $\sigma \in [\Sigma]$ we set $L_\sigma = \{\varphi(\sigma|k) : k \geqslant 1\} \in [\mathbb{N}]^\infty$. If $\{l_0 < l_1 < \cdots\}$ is the increasing enumeration of L_σ, then we set $X_\sigma = \overline{\mathrm{span}}\{u_{l_n} : n \in \mathbb{N}\}$. Let $P_\sigma\colon U \to X_\sigma$ be the natural projection. Notice that $\|P_\sigma\| = 1$. By the remarks in the beginning of the proof, we see that for every seminormalized basic sequence (x_n) in a Banach space X, there exists $\sigma \in [\Sigma]$ such that if $L_\sigma = \{l_0 < l_1 < \cdots\}$, then (x_n) is equivalent to (u_{l_n}). Hence, the space U has the desired properties.

Finally, to see that the space U is unique (up to isomorphism) we argue as follows. Let U' be another space with the properties described in the statement of the theorem. There exist Banach spaces X and Y so that $U \cong U' \oplus X$ and $U' \cong U \oplus Y$. Moreover, there exists a space Z such that $U \cong (U \oplus U \oplus \dots)_{\ell_2} \oplus Z$. Notice that

$$U \oplus U \cong U \oplus (U \oplus U \oplus \dots)_{\ell_2} \oplus Z \cong (U \oplus U \oplus \dots)_{\ell_2} \oplus Z \cong U.$$

Similarly, we have $U' \oplus U' \cong U'$. It follows that

$$U \cong U' \oplus X \cong U' \oplus U' \oplus X \cong U' \oplus U \cong U \oplus U \oplus Y \cong U \oplus Y \cong U'.$$

The proof is completed. □

1.4 Comments and Remarks

1. The study of Borel and analytic subsets of Polish spaces was initiated with the work of Lebesgue and Souslin. The classical topological theory is presented in the monograph of Kuratowski [Ku]. The subject has been revolutionized with ideas from recursion theory leading to a powerful unified theory known as *effective descriptive set theory*. The monograph of Moschovakis [Mo] is devoted to the study of the methods and results of effective descriptive set theory as well as of the influence of strong axioms of set theory on the structure of projective sets. The book of Kechris [Ke] is an updated presentation of classical descriptive set theory and has become the standard reference on the subject. It is written under the modern point of view and with an emphasis to applications.

2. Theorem 1.8 can be traced back to the beginnings of Banach space theory and appears in the classical monograph of Stefan Banach [Ba]. As we have mentioned, Theorem 1.9 is due to Pełczyński [P]. Our presentation is based on an alternative approach to the construction of the space U due to Schechtman [Sch]. There is an unconditional version of U also due to Pełczyński.

Theorem 1.10. **[P]** *There exists a space V with an unconditional basis (v_n) such that for every unconditional basic sequence (y_n) in a Banach space Y there exists $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that (y_n) is equivalent to (v_{l_n}). Moreover, if V' is another space with this property, then V' is isomorphic to V.*

We refer to [LT, Theorem 2.d.10] for a proof of Theorem 1.10 as well as for an account of related results.

Chapter 2

The standard Borel space of all separable Banach spaces

In this chapter we will present the general framework on which the main results contained in these notes are based. This framework has been defined by Bossard in his Thesis [Bos1]. The central idea is that, while the collection of all separable Banach spaces is *not* a set, it can be naturally "coded" as a standard Borel space. This coding has been proved to be compatible with any notion, construction or operation encountered in Banach space theory. By now it has found sufficiently many applications in order to be considered as one of its internal parts.

2.1 Definitions and basic properties

Let X be a separable Banach space (not necessarily infinite-dimensional). We endow the set $F(X)$ of all closed subsets of X with the Effros–Borel structure, as this structure was described in the previous chapter. By Theorem 1.2, there exists a sequence $d_n\colon F(X) \to X$ $(n \in \mathbb{N})$ of Borel maps with $d_n(F) \in F$ for every $F \in F(X)$ and every $n \in \mathbb{N}$, and such that $\big(d_n(F)\big)$ is dense in F for every nonempty closed subset F of X.

Now let $F \in F(X)$. Then F is a linear subspace of X if and only if

$$(0 \in F) \text{ and } (\forall n, m \in \mathbb{N}\ \forall p, q \in \mathbb{Q} \text{ we have } pd_n(F) + qd_m(F) \in F). \tag{2.1}$$

It is easy to see that (2.1) defines a Borel subset of $F(X)$. By Theorem 1.8, the space $C(2^{\mathbb{N}})$ is isometrically universal for all separable Banach spaces. These observations lead to the following definition.

Definition 2.1. **[Bos1]** *For every separable Banach space X by* $\mathrm{Subs}(X)$ *we denote the subset of $F(X)$ consisting of all linear subspaces of X endowed with* *the relative Effros–Borel σ-algebra. If $X = C(2^{\mathbb{N}})$, then by* SB *we denote the space* $\mathrm{Subs}\big(C(2^{\mathbb{N}})\big)$.

We recall that a Borel subset of a standard Borel space equipped with the relative σ-algebra is a standard Borel space on its own. This fact and the above discussion yield the following basic result.

Theorem 2.2. **[Bos1]** *For every separable Banach space X the space* $\mathrm{Subs}(X)$ *is standard. In particular,* SB *is a standard Borel space.*

By Theorems 1.8 and 2.2, we view the space SB as the set of all separable Banach spaces and we call it as the **standard Borel space of all separable Banach spaces**. With this identification properties of separable Banach spaces become sets in SB. So, we define the following subsets of SB by considering classical properties of Banach spaces.

Property	Corresponding Set
being uniformly convex	UC
being reflexive	REFL
having separable dual	SD
not containing ℓ_1	NC_{ℓ_1}
being non-universal	NU

This chapter is devoted to the study of the descriptive set theoretic structure of the above defined classes. To this end, we will need some properties of the space SB which are gathered below. Most of them are rather easy consequences of the relevant definitions.

2.1.1 Properties of SB

(P1) The set $\{(Y,X) : Y \subseteq X\}$ is Borel in $\mathrm{SB}\times\mathrm{SB}$. That is, the relation "Y is a subspace of X" is Borel.

(P2) For every $X \in \mathrm{SB}$ there exists a sequence $d_n\colon \mathrm{Subs}(X) \to X$ $(n \in \mathbb{N})$ of Borel maps with $d_n(Y) \in Y$ for every $Y \in \mathrm{Subs}(X)$ and every $n \in \mathbb{N}$, and such that $\big(d_n(Y)\big)$ is norm dense in Y. This follows by Theorem 1.2.

(P3) For every $X \in \mathrm{SB}$ there exists a sequence $S_n\colon \mathrm{Subs}(X) \to X$ $(n \in \mathbb{N})$ of Borel maps with the following properties. If $Y = \{0\}$, then $S_n(Y) = 0$ for every $n \in \mathbb{N}$. If $Y \in \mathrm{Subs}(X)$ with $Y \neq \{0\}$, then $S_n(Y) \in S_Y$ for every $n \in \mathbb{N}$ and the sequence $\big(S_n(Y)\big)$ is norm dense in the sphere S_Y of Y. This follows from the fact that the sequence $\big(d_n(Y)\big)$ in (P2) above can be chosen so that $d_n(Y) \neq 0$ for every $n \in \mathbb{N}$ and every $Y \in \mathrm{Subs}(X)$ with $Y \neq \{0\}$.

(P4) For every $X \in \mathrm{SB}$ the relation $\{(y,Y) : y \in Y\}$ is Borel in $X \times \mathrm{Subs}(X)$. That is, the relation "the vector y is in the subspace Y" is Borel.

(P5) For every $X \in \mathrm{SB}$ the relation $\big\{\big((y_n),Y\big) : \overline{\mathrm{span}}\{y_n : n \in \mathbb{N}\} = Y\big\}$ is Borel in $X^{\mathbb{N}} \times \mathrm{Subs}(X)$. That is, the relation "the closed linear span of the sequence (y_n) is the space Y" is Borel. To see this notice that

$$\begin{aligned}\overline{\mathrm{span}}\{y_n : n \in \mathbb{N}\} = Y \quad \Leftrightarrow \quad & (\forall n \in \mathbb{N} \text{ we have } y_n \in Y) \text{ and} \\ & (\forall k,l \in \mathbb{N}\ \exists j \in \mathbb{N} \text{ and } \exists a_0,\dots,a_j \in \mathbb{Q} \text{ with} \\ & \Big\|d_k(Y) - \sum_{n=0}^{j} a_n y_n\Big\| \leqslant \frac{1}{l+1}).\end{aligned}$$

(P6) For every $k \in \mathbb{N}$ with $k \geqslant 1$ and every $X \in \mathrm{SB}$ the relation of k-equivalence between sequences in X is closed.

(P7) The relation $\{(X,Y) : X \cong Y\}$ is analytic in $\mathrm{SB} \times \mathrm{SB}$. That is, the relation "X is isomorphic to Y" is $\mathbf{\Sigma}^1_1$. Indeed observe that

$$\begin{aligned}X \cong Y \quad \Leftrightarrow \quad & \exists (x_n),(y_n) \in C(2^{\mathbb{N}})^{\mathbb{N}} \text{ with} \\ & (\overline{\mathrm{span}}\{x_n : n \in \mathbb{N}\} = X) \text{ and } (\overline{\mathrm{span}}\{y_n : n \in \mathbb{N}\} = Y) \text{ and} \\ & [\exists k \in \mathbb{N} \text{ with } (x_n) \text{ is } k\text{-equivalent to } (y_n)].\end{aligned}$$

The relation "X is isometric to Y" is also analytic.

(P8) For every $X \in \mathrm{SB}$ the subsets $\mathcal{NB}_X, \mathcal{SB}_X$ and $\mathcal{B}_X$ of $X^{\mathbb{N}}$ defined by

$$\begin{aligned}(x_n) \in \mathcal{NB}_X \quad &\Leftrightarrow \quad (x_n) \text{ is normalized basic,} \\ (x_n) \in \mathcal{SB}_X \quad &\Leftrightarrow \quad (x_n) \text{ is seminormalized basic, and} \\ (x_n) \in \mathcal{B}_X \quad &\Leftrightarrow \quad (x_n) \text{ is basic}\end{aligned}$$

are all Borel.

(P9) Let $X \in \mathrm{SB}$. By $\mathrm{FD}(X)$ we shall denote the subset of $\mathrm{Subs}(X)$ consisting of all finite-dimensional subspaces of X. Then $\mathrm{FD}(X)$ is a Borel subset of $\mathrm{Subs}(X)$. To see this let (d_n) be the sequence of Borel maps obtained by (P2) above and notice that

$$\begin{aligned}Y \in \mathrm{FD}(X) \quad \Leftrightarrow \quad & \exists m \in \mathbb{N}\ \forall k,l \in \mathbb{N}\ \exists a_0,\dots,a_m \in \mathbb{Q} \text{ with} \\ & \Big\|d_k(F) - \sum_{n=0}^{m} a_n d_n(F)\Big\| \leqslant \frac{1}{l+1}.\end{aligned}$$

By FD we shall denote the Borel subset of SB consisting of all finite-dimensional subspaces of $C(2^{\mathbb{N}})$.

2.1.2 Coding the dual of an $X \in \mathrm{SB}$

We will frequently deal with the dual X^* of a separable Banach space X, and so, we need to define a coding of the set $\{X^* : X \in \mathrm{SB}\}$. To this end, let (d_n) be the sequence of Borel maps described in (P2) above. Notice that for every $p, q \in \mathbb{Q}$ and every $n, m \in \mathbb{N}$ the map $pd_n + qd_m$ is Borel. Hence, we may assume that for every $X \in \mathrm{SB}$ the set $\big(d_n(X)\big)$ is dense in X and, moreover, it is closed under rational linear combinations. For every $n \in \mathbb{N}$ consider the map $r_n \colon \mathrm{SB} \to \mathbb{R}$ defined by $r_n(X) = 1/\|d_n(X)\|$ if $d_n(X) \neq 0$ and $r_n(X) = 0$ if $d_n(X) = 0$. Clearly the map r_n is Borel. Also let H be the closed unit ball of ℓ_∞ equipped with the weak* topology (equivalently, $H = [-1,1]^{\mathbb{N}}$ with the product topology).

Let $X \in \mathrm{SB}$. For every $x^* \in B_{X^*}$ consider the sequence

$$f_{x^*} = \big(r_0(X)x^*(d_0(X)), \dots, r_n(X)x^*(d_n(X)), \dots\big)$$

and notice that $f_{x^*} \in H$. We will identify the closed unit ball B_{X^*} of X^* with the set

$$K_X = \{f_{x^*} \in H : x^* \in B_{X^*}\}.$$

Define $\mathrm{D} \subseteq \mathrm{SB} \times H$ by

$$(X, f) \in \mathrm{D} \Leftrightarrow f \in K_X. \tag{2.2}$$

The basic properties of the set D are summarized below.

(P10) The set D is Borel. Indeed, notice that

$$\begin{aligned}(X,f) \in \mathrm{D} \quad \Leftrightarrow \quad & \forall n, m, k \in \mathbb{N}\ \forall p, q \in \mathbb{Q} \text{ we have} \\ & \big[pd_n(X) + qd_m(X) = d_k(X) \Rightarrow \\ & p\|d_n(X)\|f(n) + q\|d_m(X)\|f(m) = \|d_k(X)\|f(k)\big].\end{aligned}$$

(P11) For every $X \in \mathrm{SB}$ the set K_X is compact. Moreover, the map

$$(B_{X^*}, \mathrm{w}^*) \ni x^* \mapsto f_{x^*} \in K_X$$

is a homeomorphism.

(P12) Let $X \in \mathrm{SB}$ and $x_0^*, \dots, x_n^* \in$ in B_{X^*}. Then for every $a_0, \dots, a_n \in \mathbb{R}$ we have

$$\begin{aligned}\Big\|\sum_{i=0}^n a_i x_i^*\Big\| &= \sup\Big\{\Big|\sum_{i=0}^n a_i r_k(X) x_i^*\big(d_k(X)\big)\Big| : k \in \mathbb{N}\Big\} \\ &= \sup\Big\{\Big|\sum_{i=0}^n a_i f_{x_i^*}(k)\Big| : k \in \mathbb{N}\Big\} = \Big\|\sum_{i=0}^n a_i f_{x_i^*}\Big\|_\infty.\end{aligned}$$

In other words, the identification of B_{X^*} with K_X is isometric.

2.2 The class REFL

Reflexive spaces

Consider the set

$$\mathrm{REFL} = \{X \in \mathrm{SB} : X \text{ is reflexive}\}.$$

We will define a Borel map $\Phi\colon \mathrm{REFL} \to \mathrm{Tr}$ such that for every $X \in \mathrm{SB}$ we have

$$X \in \mathrm{REFL} \Leftrightarrow \Phi(X) \in \mathrm{WF}.$$

That is, the map Φ is a Borel reduction of REFL to WF (see Definition A.7). By Fact A.8, this implies that the set REFL is $\mathbf{\Pi}^1_1$ and that the map $X \mapsto o\big(\Phi(X)\big)$ is a $\mathbf{\Pi}^1_1$-rank on REFL.

Specifically, let $X \in \mathrm{SB}$. For every $\varepsilon > 0$ and every $K \geqslant 1$ we define a tree $\mathbf{T} = \mathbf{T}_{\mathrm{REFL}}(X, \varepsilon, K)$ on S_X (the sphere of X) by the rule

$$(x_n)_{n=0}^l \in \mathbf{T} \quad\Leftrightarrow\quad (x_n)_{n=0}^l \text{ is } K\text{-Schauder and } \forall a_0, \dots, a_l \in \mathbb{R}^+ \text{ with } \sum_{n=0}^l a_n = 1 \text{ we have } \Big\| \sum_{n=0}^l a_n x_n \Big\| \geqslant \varepsilon \tag{2.3}$$

where a finite sequence $(x_n)_{n=0}^l$ is said to be K-Schauder if

$$\Big\| \sum_{n=0}^m a_n x_n \Big\| \leqslant K \Big\| \sum_{n=0}^l a_n x_n \Big\|$$

for every $0 \leqslant m \leqslant l$ and every $a_0, \dots, a_n \in \mathbb{R}$. The tree $\mathbf{T}_{\mathrm{REFL}}(X, \varepsilon, K)$ describes all our attempts to build a normalized basic sequence in X having basis constant less than or equal to K and with no weakly null subsequence. Notice that if $0 < \varepsilon' \leqslant \varepsilon$ and $1 \leqslant K \leqslant K'$, then the tree $\mathbf{T}_{\mathrm{REFL}}(X, \varepsilon, K)$ is a downwards closed subtree of $\mathbf{T}_{\mathrm{REFL}}(X, \varepsilon', K')$. We have the following lemma.

Lemma 2.3. **[AD]** *Let $X \in \mathrm{SB}$. Then X is reflexive if and only if for every $\varepsilon > 0$ and every $K \geqslant 1$ the tree $\mathbf{T}_{\mathrm{REFL}}(X, \varepsilon, K)$ is well-founded.*

Proof. Let $\varepsilon > 0$ and $K \geqslant 1$ and assume, first, that the tree $\mathbf{T} = \mathbf{T}_{\mathrm{REFL}}(X, \varepsilon, K)$ is not well-founded. There exists a sequence (x_n) in X such that $(x_n)_{n=0}^l \in \mathbf{T}$ for every $l \in \mathbb{N}$. Notice that (x_n) is a normalized basic sequence. By Rosenthal's dichotomy [Ro2], either there exists $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that the sequence (x_{l_n}) is equivalent to the standard unit vector basis of ℓ_1, or there exist $M = \{m_0 < m_1 < \cdots\} \in [\mathbb{N}]^\infty$ and $x^{**} \in X^{**}$ such that the sequence (x_{m_n}) is weak* convergent to x^{**}. In the first case, we immediately obtain that X is not reflexive. In the second case, we distinguish the following subcases. If $x^{**} \in X^{**} \setminus X$, then clearly X is not reflexive. So assume that $x^{**} \in X$. Since (x_{m_n}) is basic, we see that $x^{**} = 0$. That is, the sequence (x_{m_n}) is weakly null. By Mazur's theorem, there exists a finite convex combination z of (x_{m_n}) such that $\|z\| < \varepsilon$. But this is clearly impossible by the definition of the tree $\mathbf{T}_{\mathrm{REFL}}(X, \varepsilon, K)$. Hence, X is not reflexive.

Conversely, assume that X is not reflexive. There exists $x^{**} \in X^{**} \setminus X$ with $\|x^{**}\| = 1$. If ℓ_1 embeds into X, then we can easily find ε and K such that the tree $\mathbf{T}_{\mathrm{REFL}}(X, \varepsilon, K)$ is not well-founded. If ℓ_1 does not embed into X, then, by the Odell–Rosenthal theorem [OR], there exists a sequence (z_n) in B_X which is weak* convergent to x^{**}. We may select $x^* \in X^*$ with $\|x^*\| \leqslant 1$ and $L \in [\mathbb{N}]^\infty$ such that $x^*(z_n) \geqslant 1/2$ for every $n \in L$. Notice that $1/2 \leqslant \|z_n\| \leqslant 1$ for every $n \in L$. There exists $M = \{m_0 < m_1 < \cdots\} \in [L]^\infty$ such that the sequence (z_{m_n}) is basic with basis constant, say, $K \geqslant 1$ (see [Di, page 41]). We set $x_n = z_{m_n}/\|z_{m_n}\|$ for every $n \in \mathbb{N}$. Then (x_n) is a normalized basic sequence with basis constant K. Moreover, for every $l \in \mathbb{N}$ and every $a_0, \dots, a_l \in \mathbb{R}^+$ with $\sum_{n=0}^{l} a_n = 1$ we have

$$\Big\| \sum_{n=0}^{l} a_n x_n \Big\| \geqslant \sum_{n=0}^{l} a_n \frac{x^*(z_{m_n})}{\|z_{m_n}\|} \geqslant \frac{1}{2}.$$

It follows that $(x_n)_{n=0}^{l} \in \mathbf{T}_{\mathrm{REFL}}(X, 1/2, K)$ for every $l \in \mathbb{N}$; that is, the tree $\mathbf{T}_{\mathrm{REFL}}(X, 1/2, K)$ is not well-founded. The proof is completed. □

Let $S_n \colon \mathrm{SB} \to C(2^{\mathbb{N}})$ $(n \in \mathbb{N})$ be the sequence of Borel maps described in property (P3) in Section 2.1.1. For every $X \in \mathrm{SB}$ and every $j, k \in \mathbb{N}$ with $j, k \geqslant 1$ we define a tree $T_{\mathrm{REFL}}(X, j, k)$ on $\mathbb{N}$ by the rule

$$(n_0, \dots, n_l) \in T_{\mathrm{REFL}}(X, j, k) \Leftrightarrow \big(S_{n_0}(X), \dots, S_{n_l}(X)\big) \in \mathbf{T}_{\mathrm{REFL}}(X, 1/j, k).$$

The tree $T_{\mathrm{REFL}}(X, j, k)$ is just a discrete version of the tree $\mathbf{T}_{\mathrm{REFL}}(X, 1/j, k)$. Hence, by a standard perturbation argument and Lemma 2.3, we see that

$$X \in \mathrm{REFL} \Leftrightarrow \forall j, k \in \mathbb{N} \setminus \{0\} \text{ we have } T_{\mathrm{REFL}}(X, j, k) \in \mathrm{WF}.$$

Moreover, we have the following lemma.

Lemma 2.4. *For every $j, k \in \mathbb{N}$ with $j, k \geqslant 1$ the map $X \mapsto T_{\mathrm{REFL}}(X, j, k)$ is Borel.*

Proof. We fix $j, k \in \mathbb{N}$ with $j, k \geqslant 1$. It is enough to show that for every $t = (n_0, \dots, n_l) \in \mathbb{N}^{<\mathbb{N}}$ the set $A_t = \big\{X \in \mathrm{SB} : t \in T_{\mathrm{REFL}}(X, j, k)\big\}$ is Borel. Observe that

$$\begin{aligned} X \in A_t \quad \Leftrightarrow \quad & \big(S_{n_0}(X), \dots, S_{n_l}(X)\big) \text{ is } k\text{-Schauder and } \forall a_0, \dots, a_l \in \mathbb{Q}^+ \\ & \text{with } \sum_{i=0}^{l} a_i = 1 \text{ we have } \Big\| \sum_{i=0}^{l} a_i S_{n_i}(X) \Big\| \geqslant \frac{1}{j}. \end{aligned}$$

As the sequence (S_n) consists of Borel maps, we conclude that the set A_t is Borel. The proof is completed. □

We fix a bijection $\langle\cdot,\cdot\rangle$ between $(\mathbb{N}\setminus\{0\})^2$ and $\mathbb{N}$. For every $X\in\mathrm{SB}$ we "glue" the family $\big(T_{\mathrm{REFL}}(X,j,k) : j,k\geqslant 1\big)$ of trees and we produce a tree $T_{\mathrm{REFL}}(X)$ on $\mathbb{N}$ defined by the rule

$$p^\frown t\in T_{\mathrm{REFL}}(X)\Leftrightarrow p=\langle j,k\rangle \text{ and } t\in T_{\mathrm{REFL}}(X,j,k). \tag{2.4}$$

By Lemma 2.4, the map $\mathrm{SB}\ni X\mapsto T_{\mathrm{REFL}}(X)\in\mathrm{Tr}$ is Borel. Moreover,

$$X\in\mathrm{REFL}\Leftrightarrow T_{\mathrm{REFL}}(X)\in\mathrm{WF}.$$

This is the desired reduction. Let us summarize what we have shown so far.

Theorem 2.5. **[AD]** *The set* REFL *is* $\mathbf{\Pi}^1_1$ *and the map* $X\mapsto o\big(T_{\mathrm{REFL}}(X)\big)$ *is a* $\mathbf{\Pi}^1_1$*-rank on* REFL.

We proceed to give an estimate of the order of the tree $T_{\mathrm{REFL}}(X)$. First we notice that for every $X\in\mathrm{REFL}$ and every pair j,k of non-zero integers we have $o\big(T_{\mathrm{REFL}}(X,j,k)\big)\leqslant o\big(\mathbf{T}_{\mathrm{REFL}}(X,1/j,k)\big)$. Concerning the opposite inequality we have the following lemma.

Lemma 2.6. *Let* $X\in\mathrm{REFL}$, $K\geqslant 1$ *and* $\varepsilon>0$. *Also let* $j,k\in\mathbb{N}$ *with* $2\varepsilon^{-1}\leqslant j$ *and* $2K\leqslant k$. *Then* $o\big(\mathbf{T}_{\mathrm{REFL}}(X,\varepsilon,K)\big)\leqslant o\big(T_{\mathrm{REFL}}(X,j,k)\big)$. *In particular,*

$$o\big(T_{\mathrm{REFL}}(X)\big)=\sup\big\{o\big(\mathbf{T}_{\mathrm{REFL}}(X,\varepsilon,K)\big) : K\geqslant 1 \text{ and } \varepsilon>0\big\}+1.$$

Proof. First we observe the following consequence of Proposition B.3. Let $(x_n)_{n=0}^l$ be a normalized K-Schauder sequence in X. If $(y_n)_{n=0}^l$ is a finite sequence in X such that $\|x_n-y_n\|\leqslant(2K)^{-1}2^{-(n+2)}$ for every $n\in\{0,\dots,l\}$, then $(y_n)_{n=0}^l$ is 2-equivalent to $(x_n)_{n=0}^l$.

We continue with the proof of the lemma. For notational convenience we set $\mathbf{T}=\mathbf{T}_{\mathrm{REFL}}(X,\varepsilon,K)$ and $T=T_{\mathrm{REFL}}(X,j,k)$. If $X=\{0\}$, then $\mathbf{T}=T=\{\emptyset\}$. If $X\neq\{0\}$, then recall that the sequence $\big(S_n(X)\big)$ is dense in S_X. Using this fact and by transfinite induction, it is easy to see that for every $\xi<\omega_1$, every $(x_n)_{n=0}^l\in\mathbf{T}^\xi$ and every $(p_0,\dots,p_l)\in\mathbb{N}^{<\mathbb{N}}$ such that

$$\|x_n-S_{p_n}(X)\|\leqslant\frac{1}{2K}\cdot\frac{1}{2^{n+2}}$$

for every $n\in\{0,\dots,l\}$, we have $(p_0,\dots,p_l)\in T^\xi$. Taking into account the previous remark, we conclude that $o(\mathbf{T})\leqslant o(T)$. The proof is completed. □

Finally, we notice the following properties of the above defined trees.

Proposition 2.7. *Let* $X,Y\in\mathrm{SB}$. *Then the following hold.*

(i) *If* Y *is isomorphic to* X, *then* $o\big(T_{\mathrm{REFL}}(Y)\big)=o\big(T_{\mathrm{REFL}}(X)\big)$.

(ii) *If* Y *is a subspace of* X, *then* $o\big(T_{\mathrm{REFL}}(Y)\big)\leqslant o\big(T_{\mathrm{REFL}}(X)\big)$.

Uniformly convex spaces

A Banach space $(X, \|\cdot\|)$ is said to be *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in S_X$ with $\|x - y\| \geqslant \varepsilon$ we have that $\left\|\frac{x+y}{2}\right\| \leqslant 1 - \delta$. We consider the class

$$\mathrm{UC} = \{X \in \mathrm{SB} : X \text{ is uniformly convex}\}.$$

It is a classical result that every uniformly convex Banach space is reflexive (see [LT]) and so $\mathrm{UC} \subseteq \mathrm{REFL}$.

The class UC is Borel. To see this let $S_n \colon \mathrm{SB} \to C(2^{\mathbb{N}})$ $(n \in \mathbb{N})$ be the sequence of Borel maps described in property (P3) in Section 2.1.1. Observing that

$$X \in \mathrm{UC} \quad \Leftrightarrow \quad \forall n \in \mathbb{N} \setminus \{0\}\ \exists m \in \mathbb{N} \setminus \{0\} \text{ such that } \big[\forall k, l \in \mathbb{N} \text{ we have } \|S_k(X) - S_l(X)\| \geqslant \frac{1}{n} \Rightarrow \Big\|\frac{S_k(X) + S_l(X)}{2}\Big\| \leqslant 1 - \frac{1}{m}\big]$$

we conclude that UC is Borel.

2.3 The class SD

This section is devoted to the study of the set

$$\mathrm{SD} = \{X \in \mathrm{SB} : X^* \text{ is separable}\}.$$

To this end we will need a basic tool in Banach space theory introduced by Szlenk [Sz].

2.3.1 The Szlenk index

Let Z be a separable Banach space. Also let $\varepsilon > 0$ and let K be a weak* compact subset of B_{Z^*}. We define

$$s_\varepsilon(K) = K \setminus \bigcup \big\{V \subseteq Z^* : V \text{ is weak* open and } \|\cdot\| - \mathrm{diam}(K \cap V) \leqslant \varepsilon\big\}$$

where $\|\cdot\| - \mathrm{diam}(A) = \sup\{\|z^* - y^*\| : z^*, y^* \in A\}$ for every $A \subseteq Z^*$. That is, $s_\varepsilon(K)$ results by removing from K all relatively weak* open subsets of K which have ε norm-diameter. Notice that $s_\varepsilon(K)$ is weak* closed, $s_\varepsilon(K) \subseteq K$ and $s_\varepsilon(K_1) \subseteq s_\varepsilon(K_2)$ if $K_1 \subseteq K_2$. It follows that s_ε is a derivative on the set of all weak* compact subsets of (B_{Z^*}, w^*) (see Appendix A). Hence, by transfinite recursion, for every weak* compact subset K of B_{Z^*} we define the iterated derivatives $\big(s^\xi_\varepsilon(K) : \xi < \omega_1\big)$ of K by

$$s^0_\varepsilon(K) = K,\ s^{\xi+1}_\varepsilon(K) = s_\varepsilon\big(s^\xi_\varepsilon(K)\big) \text{ and } s^\lambda_\varepsilon(K) = \bigcap_{\xi<\lambda} s^\xi_\varepsilon(K) \text{ if } \lambda \text{ is limit.}$$

We set $\mathrm{Sz}_\varepsilon(K) = |K|_{s_\varepsilon}$ if $s_\varepsilon^\infty(K) = \emptyset$ and $\mathrm{Sz}_\varepsilon(K) = \omega_1$ otherwise. The *Szlenk index* of Z is defined by

$$\mathrm{Sz}(Z) = \sup\{\mathrm{Sz}_\varepsilon(B_{Z^*}) : \varepsilon > 0\}. \tag{2.5}$$

It is easy to see that if $0 < \varepsilon_1 < \varepsilon_2$, then $\mathrm{Sz}_{\varepsilon_1}(K) \geqslant \mathrm{Sz}_{\varepsilon_2}(K)$, and so,

$$\mathrm{Sz}(Z) = \sup\{\mathrm{Sz}_{1/n}(B_{Z^*}) : n \geqslant 1\} \tag{2.6}$$

for every separable Banach Z. The following theorem, due to Szlenk, summarizes some of the basic properties of the Szlenk index.

Theorem 2.8. [Sz] *Let Z and Y be separable Banach spaces. Then the following hold.*

(i) *If Y is isomorphic to Z, then* $\mathrm{Sz}(Y) = \mathrm{Sz}(Z)$.

(ii) *If Y is a subspace of Z, then* $\mathrm{Sz}(Y) \leqslant \mathrm{Sz}(Z)$.

(iii) *The dual Z^* of Z is separable if and only if* $\mathrm{Sz}(Z) < \omega_1$.

Parts (i) and (ii) of Theorem 2.8 follow easily from the definition of the Szlenk index. Part (iii) is essentially a consequence of the following fact.

Lemma 2.9. *Let Z be a separable Banach space and let K be a nonempty weak* compact subset of B_{Z^*}. If K is norm-separable, then for every $\varepsilon > 0$ there exists a weak* open subset V of Z^* such that $K \cap V \neq \emptyset$ and $\|\cdot\| - \mathrm{diam}(K \cap V) \leqslant \varepsilon$.*

Proof. We fix a compatible metric ρ for (B_{Z^*}, w^*) with $\rho - \mathrm{diam}(B_{Z^*}) \leqslant 1$ (notice that such a metric ρ is necessarily complete). Assume, towards a contradiction, that the lemma is false. Then we may construct a family $(V_t : t \in 2^{<\mathbb{N}})$ of relatively weak* open subsets of K such that for every $t \in 2^{<\mathbb{N}}$, setting F_t to be the weak* closure of V_t, the following are satisfied.

(a) $F_{t^\frown 0} \cap F_{t^\frown 1} = \emptyset$, $\big(F_{t^\frown 0} \cup F_{t^\frown 1}\big) \subseteq V_t$ and $\rho - \mathrm{diam}(V_t) \leqslant 2^{-|t|}$.

(b) For every $n \geqslant 1$, every $t, s \in 2^n$ with $t \neq s$ and every pair $(z^*, y^*) \in V_t \times V_s$ we have $\|z^* - y^*\| > \varepsilon$.

We set $P = \bigcup_{\sigma \in 2^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} V_{\sigma|n}$. By (a) above, we see that P is a perfect subset of K. By (b), we obtain that $\|z^* - y^*\| > \varepsilon$ for every $z^*, y^* \in P$ with $z^* \neq y^*$. That is, K is not norm-separable, a contradiction. The proof is completed. □

2.3.2 Norm-separable compact subsets of (B_{Z^*}, w^*)

Let $Z \in \mathrm{SB}$. By E we denote the compact metrizable space (B_{Z^*}, w^*). The Szlenk index is naturally extended to the set of all compact and norm-separable subsets of E and it is actually a $\mathbf{\Pi}^1_1$-rank on this set. To show this we argue as follows. We fix a basis (V_m) of the topology of E consisting of nonempty open sets. For every $n, m \in \mathbb{N}$ define the map $D_{n,m}\colon K(E) \to K(E)$ by

$$D_{n,m}(K) = \begin{cases} K \setminus V_m & : \quad K \cap V_m \neq \emptyset \text{ and } \|\cdot\| - \operatorname{diam}(K \cap V_m) \leqslant \frac{1}{n+1} \\ K & : \quad \text{otherwise.} \end{cases}$$

Notice that $D_{n,m}$ is a derivative on $K(E)$. Now define $D_n\colon K(E) \to K(E)$ by $D_n(K) = \bigcap_m D_{n,m}(K)$. Observe that

$$D_n(K) = K \setminus \bigcup\{V \subseteq E : V \text{ is open and } \|\cdot\| - \operatorname{diam}(K \cap V) \leqslant \frac{1}{n+1}\}.$$

Clearly D_n is derivative on $K(E)$ too.

Lemma 2.10. *For every $n \in \mathbb{N}$ the map D_n is Borel.*

Proof. Fix $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ be arbitrary and consider the set

$$A_m = \{K \in K(E) : K \cap V_m \neq \emptyset \text{ and } \|\cdot\| - \operatorname{diam}(K \cap V_m) \leqslant \frac{1}{n+1}\}.$$

Since the norm of Z^* is weak* lower semi-continuous, it is easy to see that A_m is a Borel subset of $K(E)$. Now observe that $D_{n,m}(K) = K$ if $K \notin A_m$ and $D_{n,m}(K) = K \setminus V_m$ if $K \in A_m$. This easily implies that the map $D_{n,m}$ is Borel for every $m \in \mathbb{N}$.

Now, consider the map $F\colon K(E)^{\mathbb{N}} \to K(E)^{\mathbb{N}}$ defined by

$$F\big((K_m)\big) = \big(D_{n,m}(K_m)\big).$$

By the above discussion, the map F is Borel. Moreover, by Lemma A.12, the map $\bigcap\colon K(E)^{\mathbb{N}} \to K(E)$ defined by $\bigcap\big((K_m)\big) = \bigcap_m K_m$ is Borel too. Finally, let $I\colon K(E) \to K(E)^{\mathbb{N}}$ be defined by $I(K) = (K_m)$ with $K_m = K$ for every m. Clearly I is continuous. Since $D_n(K) = \bigcap\big(F(I(K))\big)$, we see that D_n is a Borel map. The proof is completed. □

By Theorem A.11 and Lemma 2.10, the set

$$\Omega_Z = \{K \in K(E) : D_n^{\infty}(K) = \emptyset \ \forall n \in \mathbb{N}\}$$

is $\mathbf{\Pi}^1_1$. Notice that, by Lemma 2.9, we have

$$\Omega_Z = \{K \in K(E) : K \text{ is norm-separable}\}.$$

Invoking Theorem A.11 again, we see that the map

$$K \mapsto \sup\{|K|_{D_n} : n \in \mathbb{N}\} \tag{2.7}$$

is a $\mathbf{\Pi}^1_1$-rank on Ω_Z. Observing that $\mathrm{Sz}(K) = \sup\{|K|_{D_n} : n \in \mathbb{N}\}$ for every $K \in \Omega_Z$ we conclude that the Szlenk index is a $\mathbf{\Pi}^1_1$-rank on the set of all compact and norm-separable subsets of E.

2.3.3 The Szlenk index is a $\mathbf{\Pi}_1^1$-rank on SD

The following result, due to Bossard, shows that the Szlenk index, which is the natural index on the class of spaces with separable dual, is actually a $\mathbf{\Pi}_1^1$-rank.

Theorem 2.11. **[Bos1]** *The set* SD *is* $\mathbf{\Pi}_1^1$ *and the map* $X \mapsto \mathrm{Sz}(X)$ *is a* $\mathbf{\Pi}_1^1$*-rank on* SD.

Proof. Following the notation in Section 2.1.2, we set $H = (B_{\ell_\infty}, \mathrm{w}^*)$. By the analysis in Section 2.3.2 applied for $Z = \ell_1$, we see that the set

$$\Omega = \{K \in K(H) : K \text{ is norm-separable}\}$$

is $\mathbf{\Pi}_1^1$ and that the map $K \mapsto \sup\{|K|_{D_n} : n \in \mathbb{N}\}$ is a $\mathbf{\Pi}_1^1$-rank on Ω.

Now let D be the Borel subset of $\mathrm{SB} \times H$ defined in (2.2). For every $X \in \mathrm{SB}$ the section $D_X = \{f : (X, f) \in \mathrm{D}\}$ of D at X is compact and, by definition, equals to the set K_X. By Theorem A.14, the map $\Phi\colon \mathrm{SB} \to K(H)$ defined by $\Phi(X) = K_X$ is Borel. By property (P12) in Section 2.1.2, we see that

$$X \in \mathrm{SD} \Leftrightarrow \Phi(X) = K_X \in \Omega.$$

That is, the map Φ is a Borel reduction of the set SD to Ω. By Fact A.8, it follows that the set SD is $\mathbf{\Pi}_1^1$ and that the map

$$X \mapsto \sup\{|K_X|_{D_n} : n \in \mathbb{N}\}$$

is a $\mathbf{\Pi}_1^1$-rank on SD. Using properties (P11) and (P12) in Section 2.1.2, it is easy to see that for every $X \in \mathrm{SD}$ and every $n \in \mathbb{N}$ we have $|K_X|_{D_n} = \mathrm{Sz}_{1/n}(B_{X^*})$. Hence, invoking equality (2.6), we conclude that

$$\sup\{|K_X|_{D_n} : n \in \mathbb{N}\} = \sup\{\mathrm{Sz}_{1/n}(B_{X^*}) : n \geqslant 1\} = \mathrm{Sz}(X).$$

The proof is completed. □

2.3.4 The dual class of an analytic subset of SD

Let A be a subset of SD and consider the *dual class* A^* of A defined by

$$A^* = \{Y \in \mathrm{SB} : \exists X \in A \text{ with } Y \cong X^*\}. \tag{2.8}$$

We have the following estimate of the complexity of A^*.

Theorem 2.12. **[D1]** *Let A be an analytic subset of* SD*. Then the dual class A^* of A is analytic.*

As in Section 2.1.2, let $H = (B_{\ell_\infty}, \mathrm{w}^*)$. The proof of Theorem 2.12 is based on the following selection result. It is the analogue of property (P2) in Section 2.1.1 for the coding of the set $\{X^* : X \in \mathrm{SB}\}$.

Proposition 2.13. **[D1]** *Let S be a standard Borel space and let $A \subseteq S \times H$ be a Borel set such that for every $s \in S$ the section A_s is nonempty, compact and norm-separable. Then there exists a sequence $f_n \colon S \to H$ $(n \in \mathbb{N})$ of Borel selectors of A such that for every $s \in S$ the sequence $\big(f_n(s)\big)$ is norm dense in the section A_s.*

Let $D_n \colon K(H) \to K(H)$ $(n \in \mathbb{N})$ be the sequence of Borel derivatives defined in Section 2.3.2. Let $\varepsilon > 0$, $B \subseteq H$ and $S \subseteq B$. We say that S is *norm ε-dense* in B if for every $f \in B$ there exists $h \in S$ with $\|f - h\|_\infty \leqslant \varepsilon$.

Lemma 2.14. *Let S and A be as in Proposition 2.13. Also let $n \in \mathbb{N}$ and let $\tilde{A} \subseteq Z \times H$ be a Borel set with $\tilde{A} \subseteq A$ and such that for every $s \in S$ the section $\tilde{A}_s$ is a (possibly empty) compact set. Then there exists a sequence $h_m \colon S \to H$ $(m \in \mathbb{N})$ of Borel selectors of A such that for all $s \in S$, if the section $\tilde{A}_s$ is nonempty, then the set $\{h_m(s) : h_m(s) \in \tilde{A}_s \setminus D_n(\tilde{A}_s)\}$ is nonempty and norm ε-dense in $\tilde{A}_s \setminus D_n(\tilde{A}_s)$, where $\varepsilon = (n+1)^{-1}$.*

Proof. Let (V_m) be a countable basis of the topology of H consisting of nonempty sets. Let $m \in \mathbb{N}$ be arbitrary. By Theorem A.14, we see that the set

$$S_m = \big\{s \in S : \tilde{A}_s \cap V_m \neq \emptyset \text{ and } \|\cdot\|_\infty - \mathrm{diam}(\tilde{A}_s \cap V_m) \leqslant \frac{1}{n+1}\big\}$$

is Borel. We define $\tilde{A}_m \subseteq S \times H$ by the rule

$$\begin{aligned}(s,f) \in \tilde{A}_m \quad \Leftrightarrow \quad & \big(s \in S_m \text{ and } f \in V_m \text{ and } (s,f) \in \tilde{A}\big) \text{ or} \\ & \big(s \notin S_m \text{ and } (s,f) \in A\big).\end{aligned}$$

The set $\tilde{A}_m$ is Borel with nonempty K_σ sections. By the Arsenin–Kunugui theorem (see [Ke, Theorem 35.46]), there exists a Borel map $h_m \colon S \to H$ such that $\big(s, h_m(s)\big) \in \tilde{A}_m$ for all $s \in S$.

We claim that the sequence (h_m) is the desired one. Clearly it is a sequence of Borel selectors of A. What remains is to check that it has the desired property. So, let $s \in S$ such that $\tilde{A}_s$ is nonempty and let $f \in \tilde{A}_s \setminus D_n(\tilde{A}_s)$. By the definition of D_n, there exists $m_0 \in \mathbb{N}$ such that $s \in S_{m_0}$ and $(s,f) \in \tilde{A}_{m_0}$. Invoking the definition of $\tilde{A}_{m_0}$ we see that the set $\{g \in H : (s,g) \in \tilde{A}_{m_0}\}$ has norm-diameter less than or equal to $(n+1)^{-1}$. Since $(s, h_{m_0}(s)) \in \tilde{A}_{m_0}$, we conclude that $\|f - h_{m_0}(s)\|_\infty \leqslant (n+1)^{-1}$. The proof is completed. □

We proceed to the proof of Proposition 2.13.

Proof of Proposition 2.13. Let $A \subseteq S \times H$ be as in the statement of the proposition. By Theorem A.14, the map $\Phi_A \colon S \to K(H)$ defined by $\Phi_A(s) = A_s$ is Borel, and so, the set $\{A_s : s \in S\}$ is an analytic subset of $K(H)$.

Let $n \in \mathbb{N}$. By Theorem A.11 and Lemma 2.10, the set

$$\Omega_{D_n} = \{K \in K(H) : D_n^\infty(K) = \emptyset\}$$

is $\mathbf{\Pi}_1^1$ and the map $K \mapsto |K|_{D_n}$ is a $\mathbf{\Pi}_1^1$-rank on Ω_{D_n}. By our assumptions on the set A and Lemma 2.9, we see that for every $s \in S$ and for every $\xi < \omega_1$ if $D_n^\xi(A_s) \neq \emptyset$, then $D_n^{\xi+1}(A_s) \subsetneq D_n^\xi(A_s)$; thus, the sequence $\big(D_n^\xi(A_s) : \xi < \omega_1\big)$ of iterated derivatives of A_s must be stabilized at $\emptyset$. It follows, in particular, that $\{A_s : s \in S\} \subseteq \Omega_{D_n}$. By part (ii) of Theorem A.2, we obtain that

$$\sup\{|A_s|_{D_n} : s \in S\} = \xi_n < \omega_1.$$

Recursively, for every $\xi < \xi_n$ we define $A^\xi \subseteq S \times H$ as follows. We set $A^0 = A$. If $\xi = \zeta + 1$ is a successor ordinal, then we define A^ξ by

$$(s, f) \in A^\xi \Leftrightarrow f \in D_n\big((A^\zeta)_s\big)$$

where $(A^\zeta)_s$ is the section $\{f : (s, f) \in A^\zeta\}$ of A^ζ at s. If ξ is limit, then let

$$(s, f) \in A^\xi \Leftrightarrow (s, f) \in \bigcap_{\zeta<\xi} A^\zeta.$$

Claim 2.15. *The following hold.*

(i) *For every $\xi < \xi_n$ the set A^ξ is a Borel subset of A with compact sections.*

(ii) *For every $(s, f) \in S \times H$ with $(s, f) \in A$ there exists a unique ordinal $\xi < \xi_n$ such that $(s, f) \in A^\xi \setminus A^{\xi+1}$, equivalently $f \in (A^\xi)_s \setminus D_n\big((A^\xi)_s\big)$.*

Proof of Claim 2.15. (i) By induction on all ordinals less than ξ_n. For $\xi = 0$ it is straightforward. If $\xi = \zeta + 1$ is a successor ordinal, then, by our inductive hypothesis and Theorem A.14, the map $s \mapsto (A^\zeta)_s$ is Borel. By Lemma 2.10, the map $s \mapsto D_n\big((A^\zeta)_s\big)$ is Borel too. By the definition of $A^\xi = A^{\zeta+1}$ and invoking Theorem A.14 once more, we conclude that A^ξ is a Borel subset of A with compact sections. If ξ is limit, then the desired properties are immediate consequences of our inductive hypothesis and the definition of the set A^ξ.

(ii) For every $s \in S$ let $\xi_s = |A_s|_{D_n} \leqslant \xi_n$. Notice that A_s is partitioned into the disjoint sets $\big\{D_n^\xi(A_s) \setminus D_n^{\xi+1}(A_s) : \xi < \xi_s\big\}$. An easy induction shows that $(A^\xi)_s = D_n^\xi(A_s)$ for every $\xi < \xi_s$. It follows that $D_n^\xi(A_s) \setminus D_n^{\xi+1}(A_s) = (A^\xi)_s \setminus (A^{\xi+1})_s = (A^\xi)_s \setminus D_n\big((A^\xi)_s\big)$. The claim is proved. □

Let $\xi < \xi_n$. By part (i) of Claim 2.15, we may apply Lemma 2.14 for the set A^ξ and we obtain a sequence (h_m^ξ) of Borel selectors of A as described in Lemma 2.14. Enumerate the family $(h_m^\xi : \xi < \xi_n, m \in \mathbb{N})$ in a single sequence, say as (f_k). Clearly the sequence (f_k) is a sequence of Borel selectors of A.

Moreover, by part (ii) of the above claim and the properties of the sequence obtained by Lemma 2.14, we see that for all $s \in S$ the set $\{f_k(s) : k \in \mathbb{N}\}$ is norm ε-dense in A_s, where $\varepsilon = (n+1)^{-1}$. The result follows by applying the above procedure for every $n \in \mathbb{N}$. The proof of Proposition 2.13 is completed. $\square$

We are ready to give the proof of Theorem 2.12.

Proof of Theorem 2.12. Let A be an analytic subset of SD. By Theorem 2.11, the set SD is co-analytic. Hence, by Lusin's separation theorem (see [Ke, Theorem 28.1]), there exists $S \subseteq \mathrm{SD}$ Borel with $A \subseteq S$. Define $G \subseteq S \times H$ by

$$(X, f) \in G \Leftrightarrow (X, f) \in D.$$

It follows, by properties (P10) and (P11) in Section 2.1.2, that G is a Borel set such that for every $X \in S$ the section G_X of G at X is nonempty, compact and norm-separable. By Proposition 2.13, we obtain a sequence $f_n \colon S \to H$ $(n \in \mathbb{N})$ of Borel selectors of G such that for every $X \in S$ the sequence $\big(f_n(X)\big)$ is norm dense in the section G_X. Notice that, by property (P12) in Section 2.1.2, for every $Y \in \mathrm{SB}$ and every $X \in S$ we have

$$\begin{aligned} Y \cong X^* \quad \Leftrightarrow \quad & \exists (y_n) \in Y^{\mathbb{N}} \ \exists k \geqslant 1 \text{ with } \overline{\mathrm{span}}\{y_n : n \in \mathbb{N}\} = Y \\ & \text{and } (y_n) \text{ is } k\text{-equivalent to } \big(f_n(X)\big). \end{aligned}$$

For every $k \in \mathbb{N}$ with $k \geqslant 1$ the relation E_k in $C(2^{\mathbb{N}})^{\mathbb{N}} \times H^{\mathbb{N}}$ defined by

$$\big((y_n), (h_n)\big) \in E_k \Leftrightarrow (y_n) \overset{k}{\sim} (h_n)$$

is Borel. To see this notice that

$$\begin{aligned} (y_n) \overset{k}{\sim} (h_n) \quad \Leftrightarrow \quad & \forall m \ \forall a_0, \dots, a_m \in \mathbb{Q} \ \Big(\forall l \ \big|\sum_{n=0}^{m} a_n h_n(l)\big| \leqslant k \big\|\sum_{n=0}^{m} a_n y_n\big\|\Big) \\ & \text{and } \Big(\forall p \ \exists i \ \frac{1}{k}\big\|\sum_{n=0}^{m} a_n y_n\big\| - \frac{1}{p+1} \leqslant \big|\sum_{n=0}^{m} a_n h_n(i)\big|\Big). \end{aligned}$$

The sequence (f_n) consists of Borel functions. Therefore, the relation I_k in $C(2^{\mathbb{N}})^{\mathbb{N}} \times S$ defined by

$$\big((y_n), X\big) \in I_k \Leftrightarrow \big((y_n), (f_n(X))\big) \in E_k$$

is Borel. Finally, by property (P5) in Section 2.1.1, the relation R in $\mathrm{SB} \times C(2^{\mathbb{N}})^{\mathbb{N}}$ defined by

$$\big(Y, (y_n)\big) \in R \Leftrightarrow \overline{\mathrm{span}}\{y_n : n \in \mathbb{N}\} = Y$$

is Borel. Now let $A^* = \{Y \in \mathrm{SB} : \exists X \in A \text{ with } Y \cong X^*\}$ be the dual class of A. It follows by the above discussion that

$$\begin{aligned} Y \in A^* \quad \Leftrightarrow \quad & \exists X \in A \ \exists (y_n) \in C(2^{\mathbb{N}})^{\mathbb{N}} \ \exists k \geqslant 1 \text{ with } \big(Y, (y_n)\big) \in R \\ & \text{and } \big((y_n), X\big) \in I_k. \end{aligned}$$

Clearly the above formula gives an analytic definition of A^*. The proof of Theorem 2.12 is completed. □

2.4 The class NC_X

Throughout this section X will be a Banach space with a Schauder basis. Consider the set

$$\mathrm{NC}_X = \{Y \in \mathrm{SB} : X \text{ is not isomorphic to a subspace of } Y\}.$$

Particular cases are the classes NC_{ℓ_1} and NU consisting of all separable Banach spaces not containing ℓ_1 and of all non-universal spaces, obtained by considering $X = \ell_1$ and $X = C(2^{\mathbb{N}})$ respectively.

As in the case of the class REFL, we will define a Borel map $\Psi\colon \mathrm{NC}_X \to \mathrm{Tr}$ such that

$$Y \in \mathrm{NC}_X \Leftrightarrow \Psi(Y) \in \mathrm{WF}.$$

This will show that the set NC_X is $\mathbf{\Pi}^1_1$ and that the map $Y \mapsto o\big(\Psi(Y)\big)$ is a $\mathbf{\Pi}^1_1$-rank on NC_X. The construction of the map Ψ is based on classical work of Bourgain [Bou1].

Specifically, fix a normalized Schauder basis (e_n) of X. For every $Y \in \mathrm{SB}$ and every $\delta \geqslant 1$ we define a tree $\mathbf{T}_{\mathrm{NC}}(Y, X, (e_n), \delta)$ on Y by the rule

$$(y_n)_{n=0}^l \in \mathbf{T}_{\mathrm{NC}}(Y, X, (e_n), \delta) \Leftrightarrow (y_n)_{n=0}^l \text{ is } \delta\text{-equivalent to } (e_n)_{n=0}^l. \tag{2.9}$$

The tree $\mathbf{T}_{\mathrm{NC}}(Y, X, (e_n), \delta)$ describes all our attempts to produce a sequence (y_n) in Y which is δ-equivalent to (e_n). Notice that if $1 \leqslant \delta \leqslant \delta'$, then the tree $\mathbf{T}_{\mathrm{NC}}(Y, X, (e_n), \delta)$ is a downwards closed subtree of $\mathbf{T}_{\mathrm{NC}}(Y, X, (e_n), \delta')$. Moreover, we have the simple, though basic, fact.

Lemma 2.16. **[Bou1]** *For every $Y \in \mathrm{SB}$ we have that $Y \in \mathrm{NC}_X$ if and only if for every $\delta \geqslant 1$ the tree $\mathbf{T}_{\mathrm{NC}}(Y, X, (e_n), \delta)$ is well-founded.*

Let $d_n\colon \mathrm{SB} \to C(2^{\mathbb{N}})$ ($n \in \mathbb{N}$) be the sequence of Borel maps described in property (P2) in Section 2.1.1. For every $Y \in \mathrm{SB}$ and every $k \in \mathbb{N}$ with $k \geqslant 1$ we define a tree $T_{\mathrm{NC}}(Y, X, (e_n), k)$ on $\mathbb{N}$ by the rule

$$(n_0, \dots, n_l) \in T_{\mathrm{NC}}(Y, X, (e_n), k) \Leftrightarrow \big(d_{n_0}(Y), \dots, d_{n_l}(Y)\big) \in \mathbf{T}_{\mathrm{NC}}(Y, X, (e_n), k).$$

We "glue" the sequence $\big(T_{\mathrm{NC}}(Y, X, (e_n), k) : k \geqslant 1\big)$ of trees and we define a tree $T_{\mathrm{NC}}(Y, X, (e_n))$ on $\mathbb{N}$ by the rule

$$k^\frown t \in T_{\mathrm{NC}}(Y, X, (e_n)) \Leftrightarrow k \geqslant 1 \text{ and } t \in T_{\mathrm{NC}}(Y, X, (e_n), k). \tag{2.10}$$

Arguing as in Lemma 2.4, it is easy to see that the map

$$\mathrm{SB} \ni Y \mapsto T_{\mathrm{NC}}\big(Y, X, (e_n)\big) \in \mathrm{Tr}$$

is Borel. Using a perturbation argument, it also easily verified that

$$Y \in \mathrm{NC}_X \Leftrightarrow T_{\mathrm{NC}}\big(Y, X, (e_n)\big) \in \mathrm{WF}.$$

This is the desired reduction. The above facts are summarized below.

Theorem 2.17. **[Bos1]** *Let X be an infinite-dimensional Banach space with a Schauder basis. Let (e_n) be a normalized Schauder basis of X. Then the set NC_X is $\mathbf{\Pi}^1_1$ and the map $Y \mapsto o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big)$ is a $\mathbf{\Pi}^1_1$-rank on NC_X.*

We proceed to give an estimate of the order of the tree $T_{\mathrm{NC}}(Y, X, (e_n))$. First we notice that for every $Y \in \mathrm{NC}_X$ we have

$$o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) = \sup\big\{o\big(T_{\mathrm{NC}}(Y, X, (e_n), k)\big) : k \geqslant 1\big\} + 1.$$

Also observe that for every $k \in \mathbb{N}$ with $k \geqslant 1$ we have $o\big(T_{\mathrm{NC}}(Y, X, (e_n), k)\big) \leqslant o\big(\mathbf{T}_{\mathrm{NC}}(Y, X, (e_n), k)\big)$. Concerning the opposite inequality the following holds.

Lemma 2.18. *Let $Y \in \mathrm{NC}_X$. Also let $C \geqslant 1$ and $k \in \mathbb{N}$ such that $2C^2 \leqslant k$. Then we have $o\big(\mathbf{T}_{\mathrm{NC}}(Y, X, (e_n), C)\big) \leqslant o\big(T_{\mathrm{NC}}(Y, X, (e_n), k)\big)$. In particular,*

$$o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) = \sup\big\{o\big(\mathbf{T}_{\mathrm{NC}}(Y, X, (e_n), \delta)\big) : \delta \geqslant 1\big\} + 1.$$

Proof. Let $K \geqslant 1$ be the basis constant of (e_n) and let $(y_n)_{n=0}^l$ be a (finite) sequence in Y which is C-equivalent to $(e_n)_{n=0}^l$. Also let $(z_n)_{n=0}^l$ be a sequence in Y such that

$$\Big\| \frac{y_n}{\|y_n\|} - z_n \Big\| \leqslant \frac{1}{2CK} \cdot \frac{1}{2^{n+2}}$$

for every $n \in \{0, \dots, l\}$. We claim that $(z_n)_{n=0}^l$ is $2C^2$-equivalent to $(e_n)_{n=0}^l$. To see this, notice first $C^{-1} \leqslant \|y_n\| \leqslant C$ for every $l \in \{0, \dots, l\}$. Hence, setting $w_n = y_n/\|y_n\|$, we see that $(w_n)_{n=0}^l$ is normalized, C-equivalent to $(y_n)_{n=0}^l$ and with basis constant less than or equal to CK. By Proposition B.3, we obtain that $(z_n)_{n=0}^l$ is $2C^2$-equivalent to $(e_n)_{n=0}^l$.

Using the above observation and arguing as in the proof of Lemma 2.6, the desired estimate follows. The proof is completed. □

Finally, we notice the following stability properties of the above defined trees.

Proposition 2.19. *Let X be a Banach space with a normalized Schauder basis (e_n). Let $Y, Z \in \mathrm{SB}$. Then the following are satisfied.*

(i) *If Y is isomorphic to Z, then $o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) = o\big(T_{\mathrm{NC}}(Z, X, (e_n))\big)$.*

(ii) *If Y is a subspace of Z, then $o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) \leqslant o\big(T_{\mathrm{NC}}(Z, X, (e_n))\big)$.*

2.5 Coding basic sequences

As in the case of the class of separable Banach spaces and its natural coding SB, one can develop a similar theory for basic sequences using as universal element the canonical basis of Pełczyński's space.

In particular, let U be the universal space of Pełczyński for basic sequences and let (u_n) be its canonical Schauder basis described in Theorem 1.9. Every seminormalized basic sequence (x_n) in a Banach space X is equivalent to a subsequence (u_{l_n}) of (u_n). Hence, we may identify (x_n) with the corresponding set $\{l_0 < l_1 < \cdots\}$ of indices. Having this identification in mind, we consider the set

$$\mathcal{S} = \big\{L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty : (u_{l_n}) \text{ is shrinking}\big\}. \tag{2.11}$$

For every $L \in [\mathbb{N}]^\infty$ we set $U_L = \overline{\mathrm{span}}\{u_n : n \in L\}$. The main result of this section is the following theorem due to Bossard.

Theorem 2.20. **[Bos3]** *The set $\mathcal{S}$ is $\mathbf{\Pi}^1_1$ and the map $L \mapsto \mathrm{Sz}(U_L)$ is a $\mathbf{\Pi}^1_1$-rank on $\mathcal{S}$.*

Before we give the proof of Theorem 2.20 we need, first, to present some preliminary results which are of independent interest.

2.5.1 The convergence rank γ

Let E be a compact metrizable space, let (Y, ρ) be a complete metric space (not necessarily separable) and let $f_n\colon E \to Y$ $(n \in \mathbb{N})$ be a sequence of continuous functions. For every $K \in K(E)$ and every $n \in \mathbb{N}$ by $f_n|_K$ we shall denote the restriction of f_n on K. Consider the set

$$\mathcal{K} = \{K \in K(E) : (f_n|_K) \text{ is pointwise convergent}\}.$$

It is easy to see that the set $\mathcal{K}$ is $\mathbf{\Pi}^1_1$. As in [KL], we will define a canonical $\mathbf{\Pi}^1_1$-rank on $\mathcal{K}$ based on classical work of Zalcwasser [Za], and Gillespie and Hurwicz [GH]. To this end, we fix a countable basis (V_m) of E consisting of nonempty open sets. For every $n, m \in \mathbb{N}$ define $\Gamma_{n,m}\colon K(E) \to K(E)$ by

$$\Gamma_{n,m}(K) = \begin{cases} K \setminus V_m : & K \cap V_m \neq \emptyset \text{ and } \exists i \in \mathbb{N}\ \forall k > l > i\ \forall x \in K \cap V_m \\ & \text{we have } \rho\big(f_k(x), f_l(x)\big) \leqslant \frac{1}{n+1} \\ K : & \text{otherwise.} \end{cases}$$

Notice that $\Gamma_{n,m}$ is a derivative on $K(E)$. Now define $\Gamma_n\colon K(E) \to K(E)$ by $\Gamma_n(K) = \bigcap_m \Gamma_{n,m}(K)$. That is, $\Gamma_n(K)$ results by removing from K all nonempty relatively open subsets of K on which the sequence (f_n) is ε-uniformly convergent for some $0 < \varepsilon < (n+1)^{-1}$. Clearly Γ_n is derivative on $K(E)$. Moreover, arguing as in Lemma 2.10, we have the following lemma.

Lemma 2.21. *For every $n \in \mathbb{N}$ the map Γ_n is Borel.*

By Theorem A.11 and Lemma 2.21, the set

$$\Omega_\Gamma = \{K \in K(E) : \Gamma_n^\infty(K) = \emptyset \ \forall n \in \mathbb{N}\}$$

is $\mathbf{\Pi}_1^1$ and the map

$$K \mapsto \gamma(K) := \sup\{|K|_{\Gamma_n} : n \in \mathbb{N}\} \tag{2.12}$$

is a $\mathbf{\Pi}_1^1$-rank on Ω_Γ. Finally notice that $\Omega_\Gamma = \mathcal{K}$. To see this let $K \in K(E)$ be arbitrary. If $K \notin \mathcal{K}$, then clearly $K \notin \Omega_\Gamma$. Conversely, assume that $K \notin \Omega_\Gamma$. There exist $n \in \mathbb{N}$ and $P \in K(E)$ with $P \subseteq K$ and $\Gamma_n(P) = P$. For every $i \in \mathbb{N}$ let $P_i = \big\{x \in P : \exists k, l \in \mathbb{N} \text{ with } k > l > i \text{ and } \rho\big(f_k(x), f_l(x)\big) > (n+1)^{-1}\big\}$. By the fact that $\Gamma_n(P) = P$, we see that P_i is dense in P. Moreover, P_i is open as the sequence (f_n) consists of continuous functions. Hence, there exists $z \in P \subseteq K$ with $z \in P_i$ for all $i \in \mathbb{N}$. It follows that the sequence $\big(f_n(z)\big)$ is not Cauchy in (Y, ρ) and so $K \notin \mathcal{K}$.

By the above discussion, we conclude that the set $\mathcal{K}$ is $\mathbf{\Pi}_1^1$ and that the map $K \mapsto \gamma(K)$ is a $\mathbf{\Pi}_1^1$-rank on $\mathcal{K}$.

2.5.2 Subsequences spanning complemented subspaces

For the proof of Theorem 2.20 it will be convenient not to consider all subsequences of (u_n) but only those for which the natural projection has norm one. Specifically, for every $L \in [\mathbb{N}]^\infty$ let $P_L \colon U \to U_L$ be the natural projection and set

$$\mathrm{C} = \{L \in [\mathbb{N}]^\infty : P_L \text{ has norm one}\}.$$

Clearly C is a closed subset of $[\mathbb{N}]^\infty$. We set $\mathcal{S}_\mathrm{C} = \mathcal{S} \cap \mathrm{C}$. That is, $\mathcal{S}_\mathrm{C}$ consists of all $L = \{l_0 < l_1 < \cdots\}$ for which the sequence (u_{l_n}) is shrinking and the projection P_L onto U_L has norm one. Theorem 2.20 is essentially consequence of the following result.

Proposition 2.22. **[Bos3]** *The set $\mathcal{S}_\mathrm{C}$ is a $\mathbf{\Pi}_1^1$ subset of* C *and the map $L \mapsto \mathrm{Sz}(U_L)$ is a $\mathbf{\Pi}_1^1$-rank on $\mathcal{S}_\mathrm{C}$.*

Let U^* be the dual of U and let (u_n^*) be the bi-orthogonal functionals associated to (u_n). For every $L \in \mathrm{C}$ let Z_L be the weak* closure of $\mathrm{span}\{u_n^* : n \in L\}$. The spaces U_L^* and Z_L are isometric and weak* isomorphic via the operator $T \colon U_L^* \to Z_L$ defined by

$$T(x^*)(u) = x^*\big(P_L(u)\big) \text{ for every } u \in U.$$

Hence, we may identify the space U_L^* with the subspace Z_L of U^* (this is the reason why we consider sequences in C). Notice that $L \in \mathcal{S}_\mathrm{C}$ if and only if Z_L is equal to $\overline{\mathrm{span}}\{u_n^* : n \in L\}$.

For every $n \in \mathbb{N}$ let P_n denote the natural projection from U onto the space $U_n = \mathrm{span}\{u_k : k \leqslant n\}$. The operator $P_n^*: U^* \to U^*$ defined by

$$P_n^*(u^*)(u) = u^*\big(P_n(u)\big) \text{ for every } u \in U$$

is weak* continuous and satisfies $P_n^*(u_k^*) = u_k^*$ if $k \leqslant n$ and $P_n^*(u_k^*) = 0$ if $k > n$. Since the range of P_n^* is finite-dimensional, the map P_n^* is continuous from (U^*, w^*) to $(U^*, \|\cdot\|)$. Applying the analysis in Section 2.5.1 for $E = (B_{U^*}, \mathrm{w}^*)$, $(Y, \rho) = (U^*, \rho_{\|\cdot\|})$ and the sequence $(f_n) = (P_n^*)$, we see that the set

$$\mathcal{K} = \{K \in K(E) : (P_n^*|_K) \text{ is pointwise convergent}\}$$

is $\mathbf{\Pi}_1^1$ and that the map $K \mapsto \gamma(K) = \sup\{|K|_{\Gamma_n} : n \in \mathbb{N}\}$ is a $\mathbf{\Pi}_1^1$-rank on $\mathcal{K}$. We proceed to the proof of Proposition 2.22.

Proof of Proposition 2.22. In what follows let $E = (B_{U^*}, \mathrm{w}^*)$. For every $L \in \mathrm{C}$ let B_{Z_L} be the closed unit ball of the subspace Z_L of U^*. The map $\Phi: \mathrm{C} \to K(E)$ defined by $\Phi(L) = B_{Z_L}$ is easily seen to be Borel. Notice that

$$\begin{aligned} L \in \mathcal{S}_{\mathrm{C}} \quad &\Leftrightarrow \quad Z_L = \overline{\mathrm{span}}\{u_n^* : n \in L\} \\ &\Leftrightarrow \quad (P_n^*) \text{ is pointwise convergent on } B_{Z_L} \\ &\Leftrightarrow \quad \Phi(L) \in \mathcal{K}. \end{aligned}$$

That is, the map Φ is a Borel reduction of $\mathcal{S}_{\mathrm{C}}$ to $\mathcal{K}$. By Fact A.8, the set $\mathcal{S}_{\mathrm{C}}$ is $\mathbf{\Pi}_1^1$ and the map $L \mapsto \gamma(B_{Z_L})$ is a $\mathbf{\Pi}_1^1$-rank on $\mathcal{S}_{\mathrm{C}}$.

We have already remarked that for every $L \in \mathrm{C}$ the spaces U_L^* and Z_L are isometric and weak* isomorphic. Hence, by the analysis in Section 2.3.2 applied for $Z = U$, we see that $\mathrm{Sz}(U_L) = \sup\{|B_{Z_L}|_{D_n} : n \in \mathbb{N}\}$. So the proof will be completed once we show that for every $L \in \mathcal{S}_{\mathrm{C}}$ we have

$$\sup\{|B_{Z_L}|_{\Gamma_n} : n \in \mathbb{N}\} = \sup\{|B_{Z_L}|_{D_n} : n \in \mathbb{N}\}.$$

The above equality is a consequence of the following claims.

Claim 2.23. *Let $L \in \mathcal{S}_{\mathrm{C}}$ and $K \in K(E)$ with $K \subseteq B_{Z_L}$. Also let $n, m \in \mathbb{N}$ with $7(m+1) \leqslant n+1$. Then $D_m(K) \subseteq \Gamma_n(K)$. In particular, $|B_{Z_L}|_{D_m} \leqslant |B_{Z_L}|_{\Gamma_n}$.*

Proof of Claim 2.23. We fix $x^* \in D_m(K)$ and we set $\varepsilon = 6^{-1} \cdot (m+1)^{-1}$. Let $i \in \mathbb{N}$ and let V be a weak* open subset of U^* with $x^* \in V$. The basic sequence $(u_n)_{n \in L}$ is shrinking as $L \in \mathcal{S}_{\mathrm{C}}$, and so, there exists $l \in \mathbb{N}$ with $l > i$ and such that $\|P_l^*(x^*) - x^*\| \leqslant \varepsilon$. The map $P_l^*: (U^*, \mathrm{w}^*) \to (U^*, \|\cdot\|)$ is continuous. Hence, there exists a weak* open subset W of V with $x^* \in W$ and such that $\|P_l^*(y^*) - x^*\| \leqslant \varepsilon$ for every $y^* \in W$. By the fact that $x^* \in D_m(K)$, we obtain that $\|\cdot\| - \mathrm{diam}(W \cap K) > (m+1)^{-1} = 6\varepsilon$. Thus, we may select $z^* \in W \cap K$

with $\|z^* - x^*\| \geqslant 3\varepsilon$. Invoking again the fact that $(u_n)_{n\in L}$ is shrinking, we find $k > l$ such that $\|P_k^*(z^*) - z^*\| \leqslant \varepsilon$. It follows that

$$\begin{aligned} 3\varepsilon &\leqslant \|x^* - z^*\| \leqslant \|x^* - P_l^*(z^*)\| + \|P_l^*(z^*) - P_k^*(z^*)\| + \|P_k^*(z^*) - z^*\| \\ &\leqslant 2\varepsilon + \|P_k^*(z^*) - P_l^*(z^*)\| \end{aligned}$$

and so $\|P_k^*(z^*) - P_l^*(z^*)\| \geqslant \varepsilon$. Summarizing, we see that for every weak* open subset V of U^* with $x^* \in V$ and every $i \in \mathbb{N}$ there exist $k > l > i$ and $z^* \in V \cap K$ such that

$$\|P_k^*(z^*) - P_l^*(z^*)\| \geqslant \varepsilon = \frac{1}{6(m+1)} > \frac{1}{n+1}.$$

This shows that $x^* \in \Gamma_n(K)$ and so $D_m(K) \subseteq \Gamma_n(K)$. Finally, the fact that $|B_{Z_L}|_{D_m} \leqslant |B_{Z_L}|_{\Gamma_n}$ follows by a straightforward transfinite induction taking into account that $D_m(K) \subseteq \Gamma_n(K)$ for every $K \subseteq B_{Z_L}$. The claim is proved. □

Claim 2.24. *Let $L \in \mathcal{S}_{\mathrm{C}}$ and $K \in K(E)$ with $K \subseteq B_{Z_L}$. Also let $n, m \in \mathbb{N}$ with $3(m+1) \leqslant n+1$. Then $\Gamma_m(K) \subseteq D_n(K)$. In particular, $|B_{Z_L}|_{\Gamma_m} \leqslant |B_{Z_L}|_{D_n}$.*

Proof of Claim 2.24. For notational convenience we set $\delta = (n+1)^{-1}$. We fix $x^* \in K \setminus D_n(K)$. It is enough to show that $x^* \notin \Gamma_m(K)$. To this end we argue as follows. By the definition of the derivative D_n, there exists a weak* open subset V of U^* such that $\|\cdot\| - \mathrm{diam}(V \cap K) \leqslant \delta$. The basic sequence $(u_n)_{n\in L}$ is shrinking since $L \in \mathcal{S}_{\mathrm{C}}$. Hence, we may find $i \in \mathbb{N}$ such that $\|P_k^*(x^*) - P_l^*(x^*)\| \leqslant \delta$ for every $k, l \in \mathbb{N}$ with $k > l > i$. The basis (u_n) of U is bi-monotone and so $\|P_n\| = \|P_n^*\| = 1$ for every $n \in \mathbb{N}$. For every $y^* \in V \cap K$ we have $\|x^* - y^*\| \leqslant \delta$. It follows that

$$\begin{aligned} \|P_k^*(y^*) - P_l^*(y^*)\| &\leqslant \|P_k^*(y^*) - P_k^*(x^*)\| + \|P_k^*(x^*) - P_l^*(x^*)\| + \\ &\quad + \|P_l^*(x^*) - P_l^*(y^*)\| \leqslant 3\delta. \end{aligned}$$

Summarizing, we see that there exist a weak* open subset V of U^* with $x^* \in V$ and $i \in \mathbb{N}$ such that for every $k, l \in \mathbb{N}$ with $k > l > i$ and every $y^* \in V \cap K$ we have

$$\|P_k^*(y^*) - P_l^*(y^*)\| \leqslant 3\delta = \frac{3}{n+1} \leqslant \frac{1}{m+1}.$$

Hence, $x^* \notin \Gamma_m(K)$. Finally, the inequality $|B_{Z_L}|_{D_n} \leqslant |B_{Z_L}|_{\Gamma_m}$ follows by transfinite induction and the previous discussion. The claim is proved. □

By Claims 2.23 and 2.24, we conclude that for every $L \in \mathcal{S}_{\mathrm{C}}$ we have $\sup\{|B_{Z_L}|_{\Gamma_n} : n \in \mathbb{N}\} = \sup\{|B_{Z_L}|_{D_n} : n \in \mathbb{N}\}$. The proof of Proposition 2.22 is completed. □

2.5.3 Proof of Theorem 2.20

We are ready to give the proof of Theorem 2.20. To this end, for every $L \in [\mathbb{N}]^\infty$ let $(u_n)_{n\in L}$ be the subsequence of (u_n) determined by L. Consider the relation $\sim$ on $[\mathbb{N}]^\infty \times [\mathbb{N}]^\infty$ defined by

$$L \sim M \Leftrightarrow (u_n)_{n\in L} \text{ is equivalent to } (u_n)_{n\in M}.$$

It is easy to see that $\sim$ is F_σ. By Theorem 1.9, for every $M \in [\mathbb{N}]^\infty$ there exists $L \in \mathrm{C}$ with $M \sim L$. Moreover, by Proposition 2.22, the set $\mathcal{S}_\mathrm{C}$ is $\mathbf{\Pi}^1_1$. Hence,

$$M \in \mathcal{S} \Leftrightarrow \forall L \in [\mathbb{N}]^\infty \text{ we have } (L \in \mathrm{C} \text{ and } L \sim M \Rightarrow L \in \mathcal{S}_\mathrm{C})$$

and so $\mathcal{S}$ is $\mathbf{\Pi}^1_1$. The map $L \mapsto \mathrm{Sz}(U_L)$ is a $\mathbf{\Pi}^1_1$-rank on $\mathcal{S}_\mathrm{C}$. By Fact A.3, there are relations $\leqslant_\Sigma, <_\Sigma \subseteq \mathrm{C} \times \mathrm{C}$ in $\mathbf{\Sigma}^1_1$ such that for every $L \in \mathcal{S}_\mathrm{C}$ we have

$$(M \in \mathcal{S}_\mathrm{C}) \text{ and } \mathrm{Sz}(U_M) \leqslant \mathrm{Sz}(U_L) \Leftrightarrow M \leqslant_\Sigma L$$

and

$$(M \in \mathcal{S}_\mathrm{C}) \text{ and } \mathrm{Sz}(U_M) < \mathrm{Sz}(U_L) \Leftrightarrow M <_\Sigma L.$$

By part (i) of Theorem 2.8, for every $M, L \in [\mathbb{N}]^\infty$ with $M \sim L$ we have $\mathrm{Sz}(U_M) = \mathrm{Sz}(U_L)$. It follows that for every $L \in \mathcal{S}$ we have

$$\begin{aligned} \mathrm{Sz}(U_M) \leqslant \mathrm{Sz}(U_L) \quad &\Leftrightarrow \quad (M \in \mathcal{S}) \text{ and } \mathrm{Sz}(U_M) \leqslant \mathrm{Sz}(U_L) \\ &\Leftrightarrow \quad \exists M', L' \in \mathrm{C}\ (M' \sim M \text{ and } L' \sim L \text{ and } M' \leqslant_\Sigma L') \end{aligned}$$

and

$$\begin{aligned} \mathrm{Sz}(U_M) < \mathrm{Sz}(U_L) \quad &\Leftrightarrow \quad (M \in \mathcal{S}) \text{ and } \mathrm{Sz}(U_M) < \mathrm{Sz}(U_L) \\ &\Leftrightarrow \quad \exists M', L' \in \mathrm{C}\ (M' \sim M \text{ and } L' \sim L \text{ and } M' <_\Sigma L'). \end{aligned}$$

Invoking Fact A.3 again, we see that the map $L \mapsto \mathrm{Sz}(U_L)$ is a $\mathbf{\Pi}^1_1$ rank on $\mathcal{S}$. The proof of Theorem 2.20 is completed.

We close this section with the following lemma. It shows that the coding of basic sequences is compatible with SB.

Lemma 2.25. *The following hold.*

(i) *If $A \subseteq [\mathbb{N}]^\infty$ is $\mathbf{\Sigma}^1_1$, then so is the set $\{X \in \mathrm{SB} : \exists L \in A \text{ with } X \cong U_L\}$.*

(ii) *If $A \subseteq \mathrm{SB}$ is $\mathbf{\Sigma}^1_1$, then so is the set $\{L \in [\mathbb{N}]^\infty : \exists X \in A \text{ with } U_L \cong X\}$.*

In particular, the subset S *of* SB *consisting of all spaces with a Schauder basis is analytic.*

Proof. We identify Pełczyński's space U with one of its isometric copies in $C(2^\mathbb{N})$. Noticing that the map $[\mathbb{N}]^\infty \ni L \mapsto U_L \in \mathrm{SB}$ is Borel and invoking property (P7) in Section 2.1.1, parts (i) and (ii) follow. The proof is completed. □

2.6 Applications

This section is devoted to applications of the machinery developed in this chapter. We will encounter later on several other applications. The ones that follow are just a sample of the power and elegance of descriptive set theoretic techniques in the study of classical problems in Banach space theory.

A. An old problem in Banach space theory asked whether there exists a space X with separable dual which is universal for all Banach spaces with separable dual. Szlenk [Sz] answered this in the negative. Bourgain considerably strengthened Szlenk's result by showing that if a separable Banach space Y is universal for all separable reflexive spaces, then Y must contain $C(2^{\mathbb{N}})$, and so, it is universal for all separable Banach spaces (see [Bou1]). Bourgain arrived to this result by an "overspill argument". In particular, for every space X with a normalized Schauder basis (e_n) he constructed a family $\{R_\xi(X) : \xi < \omega_1\}$ of reflexive spaces with a Schauder basis such that for every countable ordinal ξ it holds that

$$\sup\big\{o\big(\mathbf{T}_{\mathrm{NC}}(R_\xi(X), X, (e_n), \delta)\big) : \delta \geqslant 1\big\} \geqslant \xi. \tag{2.13}$$

Considering $X = C(2^{\mathbb{N}})$ we see that every separable space Y that contains every $R_\xi(X)$ must also contain $C(2^{\mathbb{N}})$. Bossard refined Bourgain's result as follows.

Theorem 2.26. **[Bos3]** *Let A be an analytic subset of* SB *such that for every $X \in \mathrm{REFL}$ there exists $Z \in A$ with $X \cong Z$. Then there exists $Y \in A$ which is universal.*

Bossard arrived to Theorem 2.26 by a "reduction argument". Specifically, he defined a Borel map $\Phi\colon \mathrm{Tr} \to \mathrm{SB}$ such that $\Phi(T)$ is reflexive if $T \in \mathrm{WF}$, while $\Phi(T)$ is universal if $T \in \mathrm{IF}$. In particular, if $A_{\cong}$ is the isomorphic saturation of A, then the set $\Phi^{-1}(A_{\cong})$ is an analytic subset of Tr that contains WF. Since WF is not analytic, there exist $T \in \mathrm{IF}$ and $Y \in A$ such that $\Phi(T) \cong Y$. By the properties of Φ, the space Y is universal.

We will give an alternative approach to Theorem 2.26 based on Bourgain's construction and on the machinery developed in Section 2.4.

Proof of Theorem 2.26. Let X be a separable Banach space with a normalized Schauder basis (e_n). Let $A_{\cong} = \{Y \in \mathrm{SB} : \exists X \in A \text{ with } Y \cong X\}$ be the isomorphic saturation of A. By property (P7) in Section 2.1.1, the equivalence relation $\cong$ of isomorphism is $\mathbf{\Sigma}^1_1$ in $\mathrm{SB} \times \mathrm{SB}$. Hence $A_{\cong}$ is analytic. We claim that $A_{\cong} \nsubseteq \mathrm{NC}_X$; in other words, we claim that there exists $Y \in A$ that contains an isomorphic copy of X.

Assume not. By Theorem 2.17, the map $Y \mapsto o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big)$ is a $\mathbf{\Pi}^1_1$-rank on NC_X. As $A_{\cong}$ is $\mathbf{\Sigma}^1_1$, by part (ii) of Theorem A.2 (i.e., boundedness), there exists a countable ordinal ζ such that

$$\sup\big\{o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) : Y \in A_{\cong}\big\} < \zeta.$$

Clearly $R_\zeta(X) \in A_{\cong}$. Hence, by Lemma 2.18 and (2.13), we see that

$$o\big(T_{\mathrm{NC}}(R_\zeta(X), X, (e_n))\big) \geqslant \sup\big\{o\big(\mathbf{T}_{\mathrm{NC}}(R_\zeta(X), X, (e_n), \delta)\big) : \delta \geqslant 1\big\} + 1 > \zeta$$

and we arrived to a contradiction. Therefore, there exists $Y \in A$ with $Y \notin \mathrm{NC}_X$. Applying the above for $X = C(2^{\mathbb{N}})$ the result follows. The proof of Theorem 2.26 is completed. □

B. Let X be a Banach space with a Schauder basis and let $Y \in \mathrm{NC}_X$. If (e_n) and (z_n) are two equivalent normalized Schauder bases of X, then clearly

$$o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) = o\big(T_{\mathrm{NC}}(Y, X, (z_n))\big).$$

However, by a result of Pełczyński and Singer [PS], there exist uncountable many non-equivalent normalized Schauder bases of X. Hence it is not clear whether the quantity

$$\sup\big\{o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) : (e_n) \text{ is normalized Schauder basis of } X\big\}$$

is bounded below ω_1. In other words, it is not clear whether the quantity $o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big)$ depends on the choice of the basis (e_n). We will show that it is independent of such a choice in a very strong sense.

Theorem 2.27. **[AD]** *Let X be a Banach space with a Schauder basis. Then there exists a $\mathbf{\Pi}^1_1$-rank $\phi_X : \mathrm{NC}_X \to \omega_1$ on NC_X such that for every $Y \in \mathrm{NC}_X$ and every normalized Schauder basis (e_n) of X we have*

$$\phi_X(Y) \geqslant o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big).$$

Proof. Consider the set

$$\mathbf{B} = \{(e_n) \in X^{\mathbb{N}} : (e_n) \text{ is a normalized Schauder basis of } X\}.$$

Then $\mathbf{B}$ is a Borel subset of $X^{\mathbb{N}}$. To see this notice first that, by property (P8) in Section 2.1.1, the set $\mathcal{NB}_X$ of all normalized basic sequences in X is Borel. On the other hand, the subset $\mathcal{D}$ of $X^{\mathbb{N}}$ consisting of all sequences in X with dense linear span is Borel (in fact, $F_{\sigma\delta}$) since

$$(x_n) \in \mathcal{D} \Leftrightarrow \forall k \ \forall m \ \exists a_0, \dots, a_l \in \mathbb{Q} \text{ with } \Big\| d_k - \sum_{n=0}^{l} a_n x_n \Big\| \leqslant \frac{1}{m+1}$$

where (d_k) is a fixed dense sequence in X. Observing that $\mathbf{B} = \mathcal{NB}_X \cap \mathcal{D}$ we conclude that $\mathbf{B}$ is Borel.

For every $Y \in \mathrm{SB}$ and every $(e_n) \in \mathbf{B}$ let $T_{\mathrm{NC}}(Y, X, (e_n))$ be the tree on $\mathbb{N}$ defined in (2.10). The map

$$\mathrm{SB} \times \mathbf{B} \ni \big(Y, (e_n)\big) \mapsto T_{\mathrm{NC}}\big(Y, X, (e_n)\big) \in \mathrm{Tr}$$

is easily seen to be Borel. It follows that the set $A \subseteq \mathrm{SB} \times \mathrm{Tr}$ defined by

$$(Y,T) \in A \Leftrightarrow \exists (e_n) \in \mathbf{B} \text{ with } T = T_{\mathrm{NC}}\big(Y, X, (e_n)\big)$$

is analytic. Notice that for every $Y \in \mathrm{SB}$ we have that $Y \in \mathrm{NC}_X$ if and only if the section $A_Y = \{T : (Y,T) \in A\}$ of A at Y is a subset of WF. We apply Theorem A.5 and we obtain a Borel map $f\colon \mathrm{SB} \to \mathrm{Tr}$ as described in Theorem A.5. We define $\phi_X(Y) = o\big(f(Y)\big)$. It is easy to see that ϕ_X is as desired. The proof is completed. □

C. Recall that an infinite-dimensional Banach space X is said to be *minimal* if it embeds in all of its infinite-dimensional subspaces. The classical sequence spaces c_0 and ℓ_p $(1 \leqslant p < \infty)$ as well as the dual of Tsirelson's space (see [CS] and [Ts]) are minimal spaces.

Now let X be a separable Banach space. If $Y, Z \in \mathrm{NC}_X$, then we cannot always expect that $Y \oplus Z \in \mathrm{NC}_X$. For instance, consider the case $X = \ell_1 \oplus \ell_2$, $Y = \ell_1$ and $Z = \ell_2$. On the other hand, if X is a minimal Banach space, then it is easy to see that for every pair $Y, Z \in \mathrm{NC}_X$ we have $Y \oplus Z \in \mathrm{NC}_X$. By a classical result of Rosenthal (see [Ro1] or [Ro4, Theorem 4.10]), the same is also true for the space $X = C(2^{\mathbb{N}})$. We will give a special name for the class of spaces sharing this property, as follows.

Definition 2.28. **[AD]** *Let X be a separable Banach space. We say that X has property* (S) *if for every $Y, Z \in \mathrm{NC}_X$ we have that $Y \oplus Z \in \mathrm{NC}_X$.*

It is natural to ask, as it was done in [BRS, Problem 8, page 227] for the spaces $C(2^{\mathbb{N}})$ and ℓ_2, whether for every Banach space X with property (S) and with a normalized Schauder basis (e_n), and for every pair $Y, Z \in \mathrm{NC}_X$ we can control the order of the tree $T_{\mathrm{NC}}(Y \oplus Z, X, (e_n))$ from the order of the corresponding trees $T_{\mathrm{NC}}(Y, X, (e_n))$ and $T_{\mathrm{NC}}(Z, X, (e_n))$. We have the following theorem which answers this question positively.

Theorem 2.29. *Let X be a Banach space with property* (S) *and with a normalized Schauder basis (e_n). Then there exists a map $\phi_X\colon \omega_1 \times \omega_1 \to \omega_1$ such that for every $\xi, \zeta < \omega_1$ and every $Y, Z \in \mathrm{NC}_X$ with $o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) = \xi$ and $o\big(T_{\mathrm{NC}}(Z, X, (e_n))\big) = \zeta$ we have*

$$o\big(T_{\mathrm{NC}}(Y \oplus Z, X, (e_n))\big) \leqslant \phi_X(\xi, \zeta).$$

Proof. We define the map ϕ_X as follows. Fix two countable ordinals ξ and ζ. By Theorem 2.17, the map $Y \mapsto T_{\mathrm{NC}}(Y, X, (e_n))$ is a $\mathbf{\Pi}^1_1$-rank on NC_X. By part (i) of Theorem A.2, the sets

$$A = \big\{Y \in \mathrm{NC}_X : o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) = \xi\big\}$$

and

$$B = \{Z \in \mathrm{NC}_X : o\big(T_{\mathrm{NC}}(Z, X, (e_n))\big) = \zeta\}$$

are Borel. Let $C(2^{\mathbb{N}}) \oplus_1 C(2^{\mathbb{N}})$ be the vector space $C(2^{\mathbb{N}}) \times C(2^{\mathbb{N}})$ equipped with the norm $\|(f,g)\| = \|f\| + \|g\|$. The map $S\colon \mathrm{SB} \times \mathrm{SB} \to \mathrm{Subs}\big(C(2^{\mathbb{N}}) \oplus_1 C(2^{\mathbb{N}})\big)$ defined by $S(Y,Z) = Y \oplus_1 Z$ is easily seen to be Borel. It follows that the set

$$C = \{W \in \mathrm{SB} : \exists Y \in A \ \exists Z \in B \text{ with } W \cong Y \oplus_1 Z\}$$

is an analytic subset of SB. The space X has property (S), and so, $C \subseteq \mathrm{NC}_X$. By part (ii) of Theorem A.2, there exists a countable ordinal η such that

$$\sup \big\{o\big(T_{\mathrm{NC}}(W, X, (e_n))\big) : W \in C\big\} \leqslant \eta.$$

We set $\phi_X(\xi, \zeta) = \eta$. Clearly f_X is as desired. The proof is completed. □

2.7 Comments and Remarks

1. Although descriptive set theoretic tools have been extensively used in Banach space theory, especially after the seminal work of Jean Bourgain in the 1980s, Bossard was the first to formalize the appropriate setting in his Thesis [Bos1], written under the supervision of Gilles Godefroy. As we have already mentioned, the coding SB of the class of separable Banach spaces was defined and analyzed in [Bos1]. There are other natural ways to code the class of separable Banach spaces, but they all lead to the same results. The coding we presented (that is, the coding SB) is, by now, standardized. The paper [Bos3] of Bossard contains a detailed discussion of these issues as well as a large part of the results obtained in [Bos1].

2. The list of classes REFL, UC, SD and NC_X is by no means exhaustive. Actually, almost every natural class of separable Banach spaces has been analyzed and its complexity has been calculated (see [AD] and [Bos3] for more details).

3. The $\mathbf{\Pi}^1_1$-rank on REFL presented in Section 2.2 is taken from [AD]. Previously, Bossard has shown [Bos1] that REFL is co-analytic but not Borel.

4. Szlenk defined his index in [Sz] and used it to show that if a separable Banach space X contains every separable reflexive space, then X^* is non-separable. The original definition in [Sz] is slightly different from the one given in Section 2.3.1. However, they both coincide for spaces not containing ℓ_1. The fact that the Szlenk index is a $\mathbf{\Pi}^1_1$-rank on SD is due to Bossard [Bos1]. Theorem 2.12 and Proposition 2.13 are taken from [D1]. We notice that Theorem 2.12 is not valid for the pre-dual class $A_* = \{Y \in \mathrm{SB} : \exists X \in A \text{ with } Y^* \cong X\}$ of an analytic subset A of SB. A counterexample is the set $A = \{X \in \mathrm{SB} : X \cong \ell_1\}$ (that is, the isomorphic class of ℓ_1).

5. As we have mentioned, the $\mathbf{\Pi}^1_1$-rank on NC_X presented in Section 2.4 is based on the work of Bourgain in [Bou1]. Bossard has extended this rank for the case of an arbitrary separable Banach space and not merely for a space X with a Schauder basis (see [Bos3]). Theorems 2.27 and 2.29 are still valid in this more general setting (see [AD]).

6. The coding of basic sequences as subsequences of the basis of Pełczyński's space is taken from [Bos1]. There is a version of Theorem 2.20 for boundedly complete sequences, also due to Bossard.

Proposition 2.30. **[Bos3]** *The set*

$$\mathcal{BC} = \big\{ L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty : (u_{l_n}) \text{ is boundedly complete} \big\}$$

is $\mathbf{\Pi}^1_1$ *and the map* $L \mapsto \mathrm{Sz}\big((U_L)_*\big)$ *is a* $\mathbf{\Pi}^1_1$*-rank on* $\mathcal{BC}$.

Proposition 2.30 is derived by Theorem 2.20 and duality arguments (see [Bos1] or [Bos3] for more details).

7. Beside the work Szlenk [Sz] and Bourgain [Bou1], there are several other results in the literature which can be called "anti-universality" results. For instance, Bourgain has shown in [Bou2] that if a separable Banach space X contains every $C(K)$ space with K countable compact, then X must be universal. In the same spirit again, Argyros [Ar] has strengthened Bourgain's result from [Bou1] by showing that if a separable Banach space X contains every separable reflexive hereditarily indecomposable space, then X must still be universal. There is also a version of Theorem 2.26 in the spirit of the results obtained in [Ar] (see [AD]).

We point out that isometric versions of various universality problems have been also considered. For instance, Godefroy and Kalton [GK] have shown that if a separable Banach space X contains an isometric copy of every separable strictly convex Banach space, then X is isometrically universal for all separable Banach spaces.

8. Theorem 2.27 as well as Definition 2.28 are taken from [AD]. Theorem 2.29 is new and answers a natural question concerning the, so-called, "Bourgain indices". It is open whether the map ϕ_X can be computed even for the simplest case $X = \ell_2$.

9. We notice that the coding SB turned out to be a very efficient tool in renorming theory (see, for instance, the work of Lancien [La]). An excellent survey of these developments can be found in [G].

Chapter 3

The ℓ_2 Baire sum

In this chapter we will present the notion of a *Schauder tree basis* and the construction of an ℓ_2 *Baire sum*, both introduced in [AD].

Schauder tree bases will serve as technical devices for producing universal spaces for certain classes of Banach spaces. Their critical rôle will be revealed in Chapter 7.

To every Schauder tree basis we associate its ℓ_2 Baire sum. It is a separable Banach space that contains, in a natural way, an isomorphic copy of every space in the class coded by the Schauder tree basis. The main goal achieved by this construction is that it provides us with an efficient "gluing" procedure. However, there is a price we have to pay. Namely, the ℓ_2 Baire sum contains subspaces which are "orthogonal" to all spaces in that class. Most of the material in this chapter is devoted to the study of these subspaces.

3.1 Schauder tree bases

Definition 3.1. **[AD]** *Let X be a Banach space, Λ a countable set, T a pruned B-tree on Λ and $(x_t)_{t\in T}$ a normalized sequence in X (with possible repetitions) which is indexed by the tree T. We say that $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ is a* Schauder tree basis *if the following are satisfied.*

(1) $X = \overline{\mathrm{span}}\{x_t : t \in T\}$.

(2) *For every $\sigma \in [T]$ the sequence $(x_{\sigma|n})_{n\geqslant 1}$ is a bi-monotone basic sequence.*

We recall that a B-tree T on Λ is a downwards closed subset of $\Lambda^{<\mathbb{N}}$ consisting of nonempty finite sequences (see Section 1.2); equivalently, T is a B-tree on Λ if $T \cup \{\emptyset\}$ is a tree on Λ.

For every Schauder tree basis $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ and every $\sigma \in [T]$ we set

$$X_\sigma = \overline{\mathrm{span}}\{x_{\sigma|n} : n \geqslant 1\}. \tag{3.1}$$

Notice that in Definition 3.1 we do not assume that the subspace X_σ of X is complemented. Also notice that if $\sigma, \tau \in [T]$ with $\sigma \neq \tau$, then this does not necessarily imply that $X_\sigma \neq X_\tau$. Let us give some examples of Schauder tree bases.

Example 3.1. Consider a Banach space X with a normalized bi-monotone Schauder basis (e_n). We set $\Lambda = \mathbb{N}$ and $T = \Sigma$, where by Σ we denote the B-tree on $\mathbb{N}$ consisting of all nonempty finite strictly increasing sequences in $\mathbb{N}$ (see Section 1.2). Notice that for all $t \in \Sigma$ we have $|t| \geqslant 1$. We define $x_t = e_{|t|-1}$ for every $t \in \Sigma$. Then $\mathfrak{X} = (X, \mathbb{N}, \Sigma, (x_t)_{t\in\Sigma})$ is a Schauder tree basis. Observe that $X_\sigma = X$ for every $\sigma \in [\Sigma]$.

Example 3.2. As in Example 3.1, let X be a Banach space with a normalized bi-monotone Schauder basis (e_n). For every $t \in \Sigma$ set $m_t = \max\{n : n \in t\}$ and define $x_t = e_{m_t}$. Again we see that $\mathfrak{X} = (X, \mathbb{N}, \Sigma, (x_t)_{t\in\Sigma})$ is a Schauder tree basis. Notice that for every $\sigma \in [\Sigma]$ the sequence $(x_{\sigma|n})_{n\geqslant 1}$ is just a subsequence of (e_n). Conversely, for every subsequence (e_{l_n}) of (e_n) there exists a (unique) branch $\sigma \in [\Sigma]$ such that $(x_{\sigma|n})_{n\geqslant 1}$ is the subsequence (e_{l_n}). That is, $\mathfrak{X} = (X, \mathbb{N}, \Sigma, (x_t)_{t\in\Sigma})$ is obtained by "spreading" all subsequences of (e_n) along the branches of Σ.

3.2 The ℓ_2 Baire sum of a Schauder tree basis

Definition 3.2. [AD] *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis. The* ℓ_2 Baire sum of $\mathfrak{X}$, *denoted by $T_2^{\mathfrak{X}}$, is defined to be the completion of $c_{00}(T)$ equipped with the norm*

$$\|z\|_{T_2^{\mathfrak{X}}} = \sup\Big\{\Big(\sum_{i=0}^{l} \Big\| \sum_{t\in\mathfrak{s}_i} z(t)x_t\Big\|_X^2\Big)^{1/2}\Big\} \tag{3.2}$$

where the above supremum is taken over all finite families $(\mathfrak{s}_i)_{i=0}^{l}$ of pairwise incomparable segments of T.

Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis. Let us gather some basic properties of the space $T_2^{\mathfrak{X}}$ associated to $\mathfrak{X}$.

A. We denote by $(e_t)_{t\in T}$ the standard Hamel basis of $c_{00}(T)$. We fix a bijection $h_T : T \to \mathbb{N}$ such that for every $t, s \in T$ with $t \sqsubset s$ we have $h_T(t) < h_T(s)$. We enumerate the tree T as (t_n) according to the bijection h_T. If (e_{t_n}) is the corresponding enumeration of $(e_t)_{t\in T}$, then the sequence (e_{t_n}) defines a normalized bi-monotone Schauder basis of $T_2^{\mathfrak{X}}$. For every $x \in T_2^{\mathfrak{X}}$ by supp(x) we denote the *support* of x, i.e., the set $\{t \in T : x(t) \neq 0\}$. The *range* of x, denoted by range(x), is the minimal interval I of $\mathbb{N}$ such that supp$(x) \subseteq \{t_n : n \in I\}$. We isolate, for future use, the following simple fact.

Fact 3.3. *Let I be an interval of $\mathbb{N}$, let $\mathfrak{s}$ be a segment of T and let R be a segment complete subset of T. Then the following hold.*

(i) *The set $\mathfrak{s}' = \mathfrak{s} \cap \{t_n : n \in I\}$ is a segment of T.*

(ii) *The subset $R' = R \cap \{t_n : n \in I\}$ of T is segment complete.*

B. For every $\sigma \in [T]$ set $\mathcal{X}_\sigma = \overline{\text{span}}\{e_{\sigma|n} : n \geqslant 1\}$. The space $\mathcal{X}_\sigma$ is isometric to X_σ. Let $P_\sigma \colon T_2^{\mathfrak{X}} \to \mathcal{X}_\sigma$ be the natural projection. Then P_σ is a norm-one projection. We notice the following consequence of the enumeration of T according to h_T. If (x_n) is a block sequence in $T_2^{\mathfrak{X}}$, then the sequence $\big(P_\sigma(x_n)\big)$ is also block in $\mathcal{X}_\sigma$.

C. More generally, let S be a segment complete subset of T and consider the subspace $\mathcal{X}_S = \overline{\text{span}}\{e_t : t \in S\}$ of $T_2^{\mathfrak{X}}$. Let $P_S \colon T_2^{\mathfrak{X}} \to \mathcal{X}_S$ be the natural projection. Again we see that P_S is a norm-one projection.

D. Let $\mathfrak{X}$ be a Schauder tree basis such that for every $\sigma \in [T]$ the sequence $(x_{\sigma|n})_{n\geqslant 1}$ is unconditional. Then the basis (e_{t_n}) of $T_2^{\mathfrak{X}}$ is unconditional.

Let Y be a subspace of $T_2^{\mathfrak{X}}$. Assume that there exist $\sigma \in [T]$ and a further subspace Y' of Y such that the operator $P_\sigma \colon Y' \to \mathcal{X}_\sigma$ is an isomorphic embedding. In this case, the subspace Y "contains information" about the Schauder tree basis $\mathfrak{X}$. On the other hand, there are subspaces of $T_2^{\mathfrak{X}}$ which are "orthogonal" to every $\mathcal{X}_\sigma$. These subspaces are naturally distinguished into three categories, as follows.

Definition 3.4. **[AD]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and let Y be a subspace of $T_2^{\mathfrak{X}}$.*

(1) *We say that Y is* X-compact *if for every $\sigma \in [T]$ the operator $P_\sigma \colon Y \to \mathcal{X}_\sigma$ is compact.*

(2) *We say that Y is* X-singular *if for every $\sigma \in [T]$ the operator $P_\sigma \colon Y \to \mathcal{X}_\sigma$ is strictly singular.*

(3) *We say that Y is* weakly X-singular *if for every finite $A \subseteq [T]$ the operator $P_{T_A} \colon Y \to \mathcal{X}_{T_A}$ is not an isomorphic embedding.*

We recall that, following the notation introduced in Section 1.2, for every $A \subseteq [T]$ by T_A we denote the B-tree generated by A, that is,

$$T_A = \{\sigma|n : \sigma \in A \text{ and } n \geqslant 1\}.$$

The following simple fact relates the above notions.

Proposition 3.5. *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and let Y be a subspace of $T_2^{\mathfrak{X}}$. Then the following are satisfied.*

(i) *If Y is X-compact, then Y is X-singular.*

(ii) *If Y is X-singular, then Y is weakly X-singular.*

Proof. Part (i) is straightforward. To see part (ii) let Y be an X-singular subspace of $T_2^{\mathfrak{X}}$ and let A be an arbitrary finite subset of $[T]$. Consider the B-tree T_A generated by A. Notice that there exist final segments $\mathfrak{s}_0, \dots, \mathfrak{s}_k$ of T and a finite-dimensional subspace F of $T_2^{\mathfrak{X}}$ such that $\mathcal{X}_{T_A} \cong F \oplus \sum_{n=0}^{k} \oplus \mathcal{X}_{\mathfrak{s}_n}$. Hence, by Lemma B.6, if the operator $P_{T_A} : Y \to \mathcal{X}_{T_A}$ was an isomorphic embedding, then there would existed $\sigma \in A$ and a subspace Y' of Y such that the operator $P_\sigma : Y' \to \mathcal{X}_\sigma$ is an isomorphic embedding too. This clearly contradicts our assumptions on the space Y. Hence, the operator $P_{T_A} : Y \to \mathcal{X}_{T_A}$ is not an isomorphic embedding and, therefore, Y is weakly X-singular. The proof is completed. □

The converse of part (ii) of Proposition 3.5 is far from being true. For instance, while every subspace of an X-singular subspace (respectively, of an X-compact subspace) is also X-singular (respectively, X-compact), this is not the case for weakly X-singular subspaces. We notice, however, that if Y is a weakly X-singular subspace and Y' is a finite co-dimensional subspace of Y, then Y' is also weakly X-singular.

The rest of this chapter will be devoted to the structure of $T_2^{\mathfrak{X}}$ and its subspaces. Among the basic results obtained in this direction is the fact that every X-singular subspace Y of $T_2^{\mathfrak{X}}$ contains *no* ℓ_p for $1 \leqslant p < 2$.

3.3 Weakly null sequences in $T_2^{\mathfrak{X}}$

In what follows let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ denote a Schauder tree basis. This section is devoted to the proof of the following result.

Theorem 3.6. **[AD]** *Let (x_n) be a bounded block sequence in $T_2^{\mathfrak{X}}$. Assume that $P_\sigma(x_n) \xrightarrow{w} 0$ in $\mathcal{X}_\sigma$ for every $\sigma \in [T]$. Then (x_n) is weakly null.*

We need some preliminary results which will play a decisive rôle in the analysis of the space $T_2^{\mathfrak{X}}$.

3.3.1 General lemmas

We start with the following lemma.

Lemma 3.7. **[AD]** *Let (x_n) be a bounded block sequence in $T_2^{\mathfrak{X}}$. Also let $\varepsilon > 0$ and $L \in [\mathbb{N}]^\infty$. Then there exist finite $A \subseteq [T]$ and $M \in [L]^\infty$ such that $\limsup_{n\in M} \|P_{\mathfrak{s}}(x_n)\| < \varepsilon$ for every segment $\mathfrak{s}$ of T with $\mathfrak{s} \cap T_A = \emptyset$.*

Proof. Let $\mathfrak{s}$ be a finite segment of T. As the sequence (x_n) is block we see that $\|P_{\mathfrak{s}}(x_n)\| \to 0$. Hence, if $\mathfrak{s}$ is segment of T such that $\limsup_{n\in M} \|P_{\mathfrak{s}}(x_n)\| \geqslant \varepsilon$ for some $M \in [L]^\infty$, then $\mathfrak{s}$ must be a final segment of T.

Now assume, towards a contradiction, that the lemma is false. Using the above observation we may construct, recursively, a decreasing sequence (M_i) of infinite subsets of L and a sequence $(\mathfrak{s}_i)$ of mutually different final segments of T such that

$$\|P_{\mathfrak{s}_i}(x_n)\| > \frac{\varepsilon}{2} \text{ for every } i \in \mathbb{N} \text{ and every } n \in M_i. \tag{3.3}$$

Let $C = \sup\{\|x_n\| : n \in \mathbb{N}\} < +\infty$. We select $k_0 \in \mathbb{N}$ such that $k_0 > 4C^2 \cdot \varepsilon^{-2}$. As the final segments $\mathfrak{s}_0, \dots, \mathfrak{s}_{k_0}$ are different, there exists an $l_0 \in \mathbb{N}$ such that if we set $\mathfrak{s}'_i = \{t \in \mathfrak{s}_i : |t| \geqslant l_0\}$ for every $i \in \{0, \dots, k_0\}$, then the family $\{\mathfrak{s}'_0, \dots, \mathfrak{s}'_{k_0}\}$ becomes a collection of pairwise incomparable final segments of T. We select $n_0 \in M_{k_0}$ such that for every $i \in \{0, \dots, k_0\}$ and every $t \in \mathfrak{s}_i \cap \mathrm{supp}(x_{n_0})$ we have $|t| \geqslant l_0$. This is possible since the sequence (x_n) is block. It follows that $\|P_{\mathfrak{s}_i}(x_{n_0})\| = \|P_{\mathfrak{s}'_i}(x_{n_0})\|$ for every $i \in \{0, \dots, k_0\}$. Since the segments $(\mathfrak{s}'_i)_{i=0}^{k_0}$ are pairwise incomparable, by the definition of the norm of $T_2^{\mathfrak{X}}$ and (3.3) above, we see that

$$C \geqslant \|x_{n_0}\| \geqslant \Big(\sum_{i=0}^{k_0} \|P_{\mathfrak{s}'_i}(x_{n_0})\|^2 \Big)^{1/2} = \Big(\sum_{i=0}^{k_0} \|P_{\mathfrak{s}_i}(x_{n_0})\|^2 \Big)^{1/2} > \sqrt{k_0 \frac{\varepsilon^2}{4}} > C.$$

This is clearly a contradiction. The proof is completed. □

The following lemma provides a strong quantitative refinement of the conclusion of Lemma 3.7.

Lemma 3.8. **[AD]** *Let (x_n) be a bounded block sequence in $T_2^{\mathfrak{X}}$. Also let $\varepsilon > 0$ and let A be a finite (possibly empty) subset of $[T]$ such that* $\limsup \|P_{\mathfrak{s}}(x_n)\| < \varepsilon$ *for every segment $\mathfrak{s}$ of T with $\mathfrak{s} \cap T_A = \emptyset$. Then there exists $L \in [\mathbb{N}]^\infty$ such that for every segment $\mathfrak{s}$ of T with $\mathfrak{s} \cap T_A = \emptyset$ we have $|\{n \in L : \|P_{\mathfrak{s}}(x_n)\| \geqslant \varepsilon\}| \leqslant 1$.*

Proof. Assume, towards a contradiction, that the lemma is false. Then for every $L \in [\mathbb{N}]^\infty$ there exist $(n_0, n_1) \in [L]^2$ and a segment $\mathfrak{s}$ with $\mathfrak{s} \cap T_A = \emptyset$ and $\|P_{\mathfrak{s}}(x_{n_i})\| \geqslant \varepsilon$ for $i \in \{0, 1\}$. By Ramsey's theorem [Ra], there exists $L \in [\mathbb{N}]^\infty$ such that for every $(n_0, n_1) \in [L]^2$ there exists such a segment $\mathfrak{s}$. Hence, by passing to a subsequence of (x_n) if necessary, we may assume that for every $n, k \in \mathbb{N}$ with $n < k$ there exists a segment $\mathfrak{s}_{n,k}$ of T with $\mathfrak{s}_{n,k} \cap T_A = \emptyset$ and such that $\|P_{\mathfrak{s}_{n,k}}(x_n)\| \geqslant \varepsilon$ and $\|P_{\mathfrak{s}_{n,k}}(x_k)\| \geqslant \varepsilon$.

Let $k \in \mathbb{N}$ with $k \geqslant 1$. For every $n < k$ let $o_n = \min\{|t| : t \in \mathfrak{s}_{n,k} \cap \mathrm{supp}(x_k)\}$ and set $\mathfrak{s}'_{n,k} = \{t \in \mathfrak{s}_{n,k} : |t| < o_n\}$. Clearly $\mathfrak{s}'_{n,k}$ is a segment of T with $\mathfrak{s}'_{n,k} \cap T_A = \emptyset$. The sequence (x_n) is block and so $\mathfrak{s}_{n,k} \cap \mathrm{supp}(x_n) \subseteq \mathfrak{s}'_{n,k}$ for every $n < k$. In fact, $\mathfrak{s}'_{n,k}$ is just the maximal initial subsegment of $\mathfrak{s}_{n,k}$ that does not intersect $\mathrm{supp}(x_k)$. Hence $\|P_{\mathfrak{s}'_{n,k}}(x_n)\| \geqslant \varepsilon$ for every $n < k$. Let $C = \sup\{\|x_n\| : n \in \mathbb{N}\} < +\infty$.

Claim 3.9. *For every $k \geqslant 1$ we have $|\{\mathfrak{s}'_{n,k} : n < k\}| \leqslant \lceil C^2/\varepsilon^2 \rceil$.*

Proof of Claim 3.9. Let $\mathfrak{s}_0, \dots, \mathfrak{s}_{l-1}$ be an enumeration of the set in question. For every $i \in \{0, \dots, l-1\}$ there exists $n_i < k$ such that $\mathfrak{s}_i = \mathfrak{s}'_{n_i,k}$. We set $\mathfrak{s}''_i = \{t \in \mathfrak{s}_{n_i,k} : |t| \geqslant o_{n_i}\} = \mathfrak{s}_{n_i,k} \setminus \mathfrak{s}'_{n_i,k}$. Notice that $\mathfrak{s}''_i$ is a segment of T with $\mathfrak{s}''_i \cap T_A = \emptyset$ and $\mathrm{supp}(x_k) \cap \mathfrak{s}_{n_i,k} \subseteq \mathfrak{s}''_i$ for every $i \in \{0, \dots, l-1\}$. It follows that $\|P_{\mathfrak{s}''_i}(x_k)\| \geqslant \varepsilon$. We observe the following. Since the segments $(\mathfrak{s}_i)_{i=0}^{l-1}$ are mutually different, the segments $(\mathfrak{s}''_i)_{i=0}^{l-1}$ are pairwise *incomparable*. Hence,

$$C \geqslant \|x_k\| \geqslant \Big(\sum_{i=0}^{l-1} \|P_{\mathfrak{s}''_i}(x_k)\|^2 \Big)^{1/2} \geqslant \sqrt{l} \cdot \varepsilon$$

which gives the desired estimate. The claim is proved. □

We set $M = \lceil C^2/\varepsilon^2 \rceil$. By Claim 3.9, for every $k \geqslant 1$ there exists a family $\{\mathfrak{s}_{i,k} : i = 0, \dots, M-1\}$ of segments of T with $\mathfrak{s}_{i,k} \cap T_A = \emptyset$ and such that for every $n < k$ there exists $i \in \{0, \dots, M-1\}$ with $\|P_{\mathfrak{s}_{i,k}}(x_n)\| \geqslant \varepsilon$. By passing to subsequences, we may assume that $\mathfrak{s}_{i,k} \to \mathfrak{s}_i$ in $2^{\Lambda^{<\mathbb{N}}}$ for every $i \in \{0, \dots, M-1\}$. Notice that every $\mathfrak{s}_i$, if nonempty, is a segment of T with $\mathfrak{s}_i \cap T_A = \emptyset$.

Let $n, k \in \mathbb{N}$ with $n < k$ and $i \in \{0, \dots, M-1\}$. We say that k is *i-good for n* if $\|P_{\mathfrak{s}_{i,k}}(x_n)\| \geqslant \varepsilon$. Observe that for every $n \in \mathbb{N}$ there exists $i \in \{0, \dots, M-1\}$ such that the set $H^i_n = \{k > n : k \text{ is } i\text{-good for } n\}$ is infinite. Hence there exist $i_0 \in \{0, \dots, M-1\}$ and $L \in [\mathbb{N}]^\infty$ such that $H^{i_0}_n$ is infinite for every $n \in L$. Since $\mathfrak{s}_{i_0,k} \to \mathfrak{s}_{i_0}$ in $2^{\Lambda^{<\mathbb{N}}}$, we see that $\|P_{\mathfrak{s}_{i_0}}(x_n)\| \geqslant \varepsilon$ for every $n \in L$. Therefore,

$$\limsup \|P_{\mathfrak{s}_{i_0}}(x_n)\| \geqslant \varepsilon.$$

Moreover, we have $\mathfrak{s}_{i_0} \cap T_A = \emptyset$, and so, we have arrived to a contradiction. The proof of Lemma 3.8 is completed. □

By Lemma 3.8, we have the following lemma.

Lemma 3.10. **[AD]** *Let (x_n) be a bounded block sequence in $T_2^{\mathfrak{X}}$. Also let $\varepsilon > 0$ and let A be a finite (possibly empty) subset of $[T]$ such that $\limsup \|P_{\mathfrak{s}}(x_n)\| < \varepsilon$ for every segment $\mathfrak{s}$ of T with $\mathfrak{s} \cap T_A = \emptyset$. Then for every $L \in [\mathbb{N}]^\infty$ there exists a vector w which is a finite convex combination of $\{x_n : n \in L\}$ such that $\|P_{\mathfrak{s}}(w)\| \leqslant 2\varepsilon$ for every segment $\mathfrak{s}$ of T with $\mathfrak{s} \cap T_A = \emptyset$.*

Proof. Fix $L \in [\mathbb{N}]^\infty$. Applying Lemma 3.8, we obtain an infinite subset M of L such that $|\{n \in M : \|P_{\mathfrak{s}}(x_n)\| \geqslant \varepsilon\}| \leqslant 1$ for every segment $\mathfrak{s}$ of T with $\mathfrak{s} \cap T_A = \emptyset$. Let $M = \{m_0 < m_1 < \cdots\}$ and $C = \sup\{\|x_n\| : n \in \mathbb{N}\} < +\infty$. We select $k_0 \in \mathbb{N}$ such that $C \cdot (k_0+1)^{-1} < \varepsilon$ and we define

$$w = \frac{x_{m_0} + \cdots + x_{m_{k_0}}}{k_0+1}.$$

Then for every segment $\mathfrak{s}$ of T with $\mathfrak{s} \cap T_A = \emptyset$ we have

$$\|P_{\mathfrak{s}}(w)\| \leqslant \frac{1}{k_0+1} \cdot \sum_{i=0}^{k_0} \|P_{\mathfrak{s}}(x_{m_i})\| \leqslant \frac{C + \varepsilon k_0}{k_0+1} \leqslant 2\varepsilon.$$

The proof is completed. □

We isolate, for future use, the following corollaries of Lemmas 3.7, 3.8 and 3.10 respectively.

Corollary 3.11. *Let (x_n) be a bounded block sequence in $T_2^{\mathfrak{X}}$. Also let $\varepsilon > 0$ and $L \in [\mathbb{N}]^\infty$. Then there exist finite $A \subseteq [T]$ and $M \in [L]^\infty$ such that $\limsup_{n \in M} \|P_\sigma(x_n)\| < \varepsilon$ for every $\sigma \in [T] \setminus A$.*

Corollary 3.12. *Let (x_n) be a bounded block sequence in $T_2^{\mathfrak{X}}$. Also let $\varepsilon > 0$ such that $\limsup \|P_\sigma(x_n)\| < \varepsilon$ for every $\sigma \in [T]$. Then there exists $L \in [\mathbb{N}]^\infty$ such that $|\{n \in L : \|P_\sigma(x_n)\| \geqslant \varepsilon\}| \leqslant 1$ for every $\sigma \in [T]$.*

Corollary 3.13. *Let (x_n) be a bounded block sequence in $T_2^{\mathfrak{X}}$. Also let $\varepsilon > 0$ such that $\limsup \|P_\sigma(x_n)\| < \varepsilon$ for every $\sigma \in [T]$. Then for every $L \in [\mathbb{N}]^\infty$ there exists a vector w which is a finite convex combination of $\{x_n : n \in L\}$ such that $\|P_\sigma(w)\| \leqslant 2\varepsilon$ for every $\sigma \in [T]$.*

3.3.2 Sequences satisfying an upper ℓ_2 estimate

The final ingredient of the proof of Theorem 3.6 is the following lemma.

Lemma 3.14. [AD] *Let (w_n) be a bounded block sequence in $T_2^{\mathfrak{X}}$. Assume that for every $n \in \mathbb{N}$ with $n \geqslant 1$ and every $\sigma \in [T]$ we have*

$$\|P_\sigma(w_n)\| \leqslant \frac{1}{\sum_{i=0}^{n-1} |\mathrm{supp}(w_i)|^{1/2}} \cdot \frac{1}{2^{2n}}. \tag{3.4}$$

Then the sequence (w_n) satisfies an upper ℓ_2 estimate. That is, there exists a constant $M \geqslant 1$ such that for every $k \in \mathbb{N}$ and every $a_0, \dots, a_k \in \mathbb{R}$ we have

$$\Big\| \sum_{n=0}^{k} a_n w_n \Big\|_{T_2^{\mathfrak{X}}} \leqslant M \Big(\sum_{n=0}^{k} a_n^2 \Big)^{1/2}.$$

In particular, the sequence (w_n) is weakly null.

Proof. Let $C = \sup\{\|w_n\| : n \in \mathbb{N}\} < +\infty$. Let $k \in \mathbb{N}$ and $a_0, \dots, a_k \in \mathbb{R}$ with $\sum_{n=0}^{k} a_n^2 = 1$. We will show that $\big\| \sum_{n=0}^{k} a_n w_n \big\| \leqslant \sqrt{2C^2+4}$. This will finish the proof.

To this end, let $(\mathfrak{s}_j)_{j=0}^{l}$ be an arbitrary collection of pairwise incomparable segments of T. We may assume that for every $j \in \{0,\dots,l\}$ there exists $n \in \{0,\dots,k\}$ such that $\mathfrak{s}_j \cap \mathrm{supp}(w_n) \neq \emptyset$. Recursively, we define a partition $(I_n)_{n=0}^{k}$ of $\{0,\dots,l\}$ by the rule

$$\begin{aligned} I_0 &= \big\{ j \in \{0,\dots,l\} : \mathfrak{s}_j \cap \mathrm{supp}(w_0) \neq \emptyset \big\} \\ I_1 &= \big\{ j \in \{0,\dots,l\} \setminus I_0 : \mathfrak{s}_j \cap \mathrm{supp}(w_1) \neq \emptyset \big\} \\ &\vdots \\ I_k &= \Big\{ j \in \{0,\dots,l\} \setminus \Big(\bigcup_{n=0}^{k-1} I_n \Big) : \mathfrak{s}_j \cap \mathrm{supp}(w_k) \neq \emptyset \Big\}. \end{aligned}$$

The segments $(\mathfrak{s}_j)_{j=0}^{l}$ are pairwise incomparable and a fortiori disjoint. It follows that

$$|I_n| \leqslant |\mathrm{supp}(w_n)| \text{ for every } n \in \{0,\dots,k\}. \tag{3.5}$$

Also notice that for every $0 \leqslant n < m \leqslant k$ we have

$$\sum_{j \in I_m} \|P_{\mathfrak{s}_j}(w_n)\| = 0. \tag{3.6}$$

Let $n \in \{0,\dots,k\}$ and $j \in I_n$. We shall estimate the quantity

$$\begin{aligned} \|P_{\mathfrak{s}_j}(a_0 w_0 + \dots + a_k w_k)\| &\overset{(3.6)}{=} \|P_{\mathfrak{s}_j}(a_n w_n + \dots + a_k w_k)\| \\ &\leqslant |a_n| \cdot \|P_{\mathfrak{s}_j}(w_n)\| + \sum_{i=n+1}^{k} |a_i| \cdot \|P_{\mathfrak{s}_j}(w_i)\|. \end{aligned}$$

Since the Schauder basis $(e_t)_{t\in T}$ of $T_2^{\mathfrak{X}}$ is bi-monotone, by (3.4), we see that for every $i \in \{n+1,\dots,k\}$ we have $\|P_{\mathfrak{s}_j}(w_i)\| \leqslant |\mathrm{supp}(w_n)|^{-1/2} \cdot 2^{-2i}$, and moreover, $|a_i| \leqslant 1$. Hence,

$$\begin{aligned} \|P_{\mathfrak{s}_j}(a_0 w_0 + \dots + a_k w_k)\| &\leqslant |a_n| \cdot \|P_{\mathfrak{s}_j}(w_n)\| + \frac{1}{|\mathrm{supp}(w_n)|^{1/2}} \cdot \sum_{i=n+1}^{k} \frac{1}{2^{2i}} \\ &\overset{(3.5)}{\leqslant} |a_n| \cdot \|P_{\mathfrak{s}_j}(w_n)\| + \frac{1}{|I_n|^{1/2}} \cdot \frac{1}{2^n}. \end{aligned}$$

Notice that for every $n \in \{0,\dots,k\}$ we have $\sum_{j\in I_n} \|P_{\mathfrak{s}_j}(w_n)\|^2 \leqslant \|w_n\|^2 \leqslant C^2$

as the family $(\mathfrak{s}_j)_{j\in I_n}$ consists of pairwise incomparable segments. Therefore,

$$\begin{aligned}\sum_{j\in I_n} \|P_{\mathfrak{s}_j}(a_0w_0+\cdots+a_kw_k)\|^2 &\leqslant \sum_{j\in I_n}\Big(|a_n|\cdot\|P_{\mathfrak{s}_j}(w_n)\| + \frac{1}{|I_n|^{1/2}}\cdot\frac{1}{2^n}\Big)^2 \\ &\leqslant 2a_n^2\sum_{j\in I_n}\|P_{\mathfrak{s}_j}(w_n)\|^2 + 2\sum_{j\in I_n}\frac{1}{|I_n|}\cdot\frac{1}{2^n} \\ &\leqslant 2a_n^2C^2+\frac{2}{2^n}.\end{aligned}$$

It follows that

$$\sum_{n=0}^{k}\sum_{j\in I_n}\|P_{\mathfrak{s}_j}(a_0w_0+\cdots+a_kw_k)\|^2 \leqslant 2C^2\sum_{n=0}^{k}a_n^2+\sum_{n=0}^{k}\frac{2}{2^n}\leqslant 2C^2+4.$$

The family $(\mathfrak{s}_j)_{j=0}^{l}$ was arbitrary and so $\big\|\sum_{n=0}^{k}a_nw_n\big\|^2\leqslant 2C^2+4$. The proof is completed. □

3.3.3 Proof of Theorem 3.6

We are ready to give to the proof of Theorem 3.6. We will argue by contradiction. So, assume that there exists a bounded block sequence (x_n) in $T_2^{\mathfrak{X}}$ such that $\big(P_\sigma(x_n)\big)$ is weakly null in $\mathcal{X}_\sigma$ for every $\sigma\in[T]$, while (x_n) is not weakly null in $T_2^{\mathfrak{X}}$. In this case, there exist $x^*\in(T_2^{\mathfrak{X}})^*$, $\varepsilon>0$ and $L\in[\mathbb{N}]^\infty$ such that $x^*(x_n)\geqslant\varepsilon$ for every $n\in L$. By repeated applications of Corollary 3.11, we may construct, recursively, a decreasing sequence (M_k) of infinite subsets of L and an increasing sequence (A_k) of finite subsets of $[T]$ such that for every $k\in\mathbb{N}$ and every $\sigma\in[T]\setminus A_k$ we have $\limsup_{n\in M_k}\|P_\sigma(x_n)\|\leqslant 2^{-k}$. We set $A=\bigcup_k A_k$. Clearly A is countable. Let $M\in[L]^\infty$ be such that $M\setminus M_k$ is finite for every $k\in\mathbb{N}$. Notice that $\lim_{n\in M}\|P_\sigma(x_n)\|=0$ for every $\sigma\in[T]\setminus A$. Also observe that for every convex block sequence (y_n) of $(x_n)_{n\in M}$ and every $\sigma\in[T]\setminus A$ we have that $\limsup\|P_\sigma(y_n)\|=0$. Using this observation, our assumptions, Mazur's theorem and a diagonal argument, we may construct a block sequence (y_n) of (x_n) such that

(a) each vector y_n is a finite convex combination of $(x_n)_{n\in M}$ and

(b) $\lim\|P_\sigma(y_n)\|=0$ for every $\sigma\in[T]$.

Notice that, by property (a) above, we have

(c) $x^*(y_n)\geqslant\varepsilon$ for every $n\in\mathbb{N}$.

By repeated applications of Corollary 3.13, we may construct, recursively, a block sequence (w_n) of finite convex combinations of (y_n) such that for every

$n \in \mathbb{N}$ with $n \geqslant 1$ and every $\sigma \in [T]$ we have

$$\|P_\sigma(w_n)\| \leqslant \frac{1}{\sum_{i=0}^{n-1} |\text{supp}(w_i)|^{1/2}} \cdot \frac{1}{2^{2n}}.$$

Observe that the sequence (w_n) is bounded. Hence, by Lemma 3.14, the sequence (w_n) is weakly null. On the other hand, by (c) above, we have that $x^*(w_n) \geqslant \varepsilon$ for every $n \in \mathbb{N}$. This is clearly a contradiction. The proof of Theorem 3.6 is completed.

3.4 Weakly X-singular subspaces

This section is devoted to the study of the weakly X-singular subspaces of an ℓ_2 Baire sum. We start by noticing the following fact. It is an immediate consequence of Definition 3.4.

Fact 3.15. *Let Y be a block subspace of $T_2^{\mathfrak{X}}$. Assume that Y is weakly X-singular. Then for every finite $A \subseteq [T]$ there exists a normalized block sequence (y_n) in Y such that $\|P_\sigma(y_n)\| \to 0$ for every $\sigma \in A$.*

The basic property of a weakly X-singular subspace Y of $T_2^{\mathfrak{X}}$ is given in the following proposition.

Proposition 3.16. **[AD]** *Let Y be a block subspace of $T_2^{\mathfrak{X}}$. Assume that Y is weakly X-singular. Then for every $\varepsilon > 0$ there exists a normalized block sequence (y_n) in Y such that* $\limsup \|P_\sigma(y_n)\| < \varepsilon$ *for every $\sigma \in [T]$.*

Proof. The proof is a quest of a contradiction. So, suppose that there exist a weakly X-singular block subspace Y of $T_2^{\mathfrak{X}}$ and $\varepsilon > 0$ such that for every normalized block sequence (y_n) in Y there exists $\sigma \in [T]$ such that $\limsup \|P_\sigma(y_n)\| \geqslant \varepsilon$. Let $p \in \mathbb{N}$ and $r > 0$ to be determined later.

We start with a normalized block sequence (y_n^0) in Y. By our assumptions, there exist $\sigma_0 \in [T]$ and $L_0 \in [\mathbb{N}]^\infty$ such that $\|P_{\sigma_0}(y_n^0)\| > \varepsilon/2$ for every $n \in L_0$. By Lemma 3.7, there exist $M_0 \in [L_0]^\infty$ and finite $A_0 \subseteq [T]$ such that for every segment $\mathfrak{s}$ of T with $\mathfrak{s} \cap T_{A_0} = \emptyset$ we have $\limsup_{n\in M_0} \|P_{\mathfrak{s}}(y_n^0)\| < r$. By Lemma 3.8, there exists $N_0 \in [M_0]^\infty$ such that for every segment $\mathfrak{s}$ of T with $\mathfrak{s} \cap T_{A_0} = \emptyset$ we have $|\{n \in N_0 : \|P_{\mathfrak{s}}(y_n^0)\| \geqslant r\}| \leqslant 1$. Summing up, we obtain $\sigma_0 \in [T]$, finite $A_0 \subseteq [T]$ and $N_0 \in [\mathbb{N}]^\infty$ such that

(a) $\|P_{\sigma_0}(y_n^0)\| > \varepsilon/2$ for every $n \in N_0$, and

(b) if $\mathfrak{s}$ is a segment with $\mathfrak{s} \cap T_{A_0} = \emptyset$, then $|\{n \in N_0 : \|P_{\mathfrak{s}}(y_n^0)\| \geqslant r\}| \leqslant 1$.

By Fact 3.15, there exists a normalized block sequence (y_n^1) in Y such that $\|P_\sigma(y_n^1)\| \to 0$ for every $\sigma \in A_0 \cup \{\sigma_0\}$. By our assumption that the proposition is false, there exists $\sigma_1 \in [T]$ such that $\limsup \|P_{\sigma_1}(y_n^1)\| \geqslant \varepsilon$. Arguing as above, we may select finite $A_1 \subseteq [T]$ and $N_1 \in [\mathbb{N}]^\infty$ such that for every segment $\mathfrak{s}$ of T with $\mathfrak{s} \cap T_{A_1} = \emptyset$ we have $|\{n \in N_1 : \|P_{\mathfrak{s}}(y_n^1)\| \geqslant r\}| \leqslant 1$. Since $\|P_\sigma(y_n^1)\| \to 0$ for every $\sigma \in A_0 \cup \{\sigma_0\}$, we can select A_1 and σ_1 so that $(A_1 \cup \{\sigma_1\}) \cap (A_0 \cup \{\sigma_0\}) = \emptyset$. We proceed recursively up to p.

For every $i \in \{0, \dots, p\}$ we enumerate the sequence $(y_n^i)_{n \in N_i}$ as (z_n^i) according to the increasing enumeration of N_i. Also let $G_i = A_i \cup \{\sigma_i\}$. By the properties of the above construction, we have $G_i \cap G_j = \emptyset$ for every $i, j \in \{0, \dots, p\}$ with $i \neq j$. Every G_i is finite. Hence, there exists $l_0 \in \mathbb{N}$ such that if we restrict every $\tau \in \bigcup_{i=0}^p G_i$ after the l_0-level of T, then this collection of final segments becomes a collection of *pairwise incomparable* final segments. Thus, setting $T_i = \{\tau | n : n > l_0 \text{ and } \tau \in G_i\}$ for all $i \in \{0, \dots, p\}$, we see that for every $i, j \in \{0, \dots, p\}$ with $i \neq j$ and every $t \in T_i$ and $s \in T_j$, the nodes t and s are incomparable.

The sequence (z_n^i) is block. Hence, for every $i \in \{0, \dots, p\}$ we may select a subsequence (u_n^i) of (z_n^i) such that the following are satisfied.

(c) For every $n \in \mathbb{N}$ and every pair $i, j \in \{0, \dots, p\}$ with $i < j$ we have that $\max\{l : l \in \mathrm{range}(u_n^i)\} < \min\{l : l \in \mathrm{range}(u_n^j)\}$.

(d) If $n < n'$, then $\max\{l : l \in \mathrm{range}(u_n^i)\} < \min\{l : l \in \mathrm{range}(u_{n'}^j)\}$ for every $i, j \in \{0, \dots, p\}$.

(e) For every $\tau \in G_i$ and every $t \in \mathrm{supp}(u_n^i)$ with $t \sqsubset \tau$ we have $|t| > l_0$.

For every $n \in \mathbb{N}$ we define

$$w_n = u_n^0 + \cdots + u_n^p. \tag{3.7}$$

Notice that, by property (d), the sequence (w_n) is block.

Let $n \in \mathbb{N}$ be arbitrary. By (a), (c) and (e) above, for every $i \in \{0, \dots, p\}$ we may select a segment $\mathfrak{s}_n^i$ of T such that

(f) $\mathfrak{s}_n^i \subseteq \{\sigma_i | k : k > l_0\} \cap \{t_k : k \in \mathrm{range}(u_n^i)\}$, and

(g) $\|P_{\mathfrak{s}_n^i}(u_n^i)\| > \varepsilon/2$.

Using (c), (e), (f) and (g), it is easy to see that

$$\|w_n\| \geqslant \Big(\sum_{i=0}^p \|P_{\mathfrak{s}_n^i}(u_n^i)\|^2 \Big)^{1/2} \geqslant \frac{\varepsilon}{2} \sqrt{p+1}. \tag{3.8}$$

Finally for every $n \in \mathbb{N}$ we define

$$y_n = \frac{w_n}{\|w_n\|} \tag{3.9}$$

Clearly (y_n) is a normalized block sequence in Y. We will show that for an appropriate choice of p and r we have $\limsup \|P_\sigma(y_n)\| \leqslant \varepsilon/2$ for every $\sigma \in [T]$. This is clearly a contradiction.

To this end, let $\sigma \in [T]$ be arbitrary. Notice that there exists at most one $j_\sigma \in \{0, \dots, p\}$ with the property that there exists $t \in T_{j_\sigma}$ with $t \sqsubset \sigma$. For this particular j_σ we have the trivial estimate $\|P_\sigma(u_n^{j_\sigma})\| \leqslant 1$ for every $n \in \mathbb{N}$.

Now fix $i \in \{0, \dots, p\}$ with $i \neq j_\sigma$. Then every node t in T_i is not an initial segment of σ. We set

$$\mathfrak{s}^i = \{\sigma|k : k \geqslant 1\} \setminus T_{G_i} = \{\sigma|k : k \geqslant 1\} \setminus \{\tau|n : n \geqslant 1 \text{ and } \tau \in G_i\}.$$

Notice, first, that $\mathfrak{s}^i$ is a final segment of σ. Also notice that $\mathfrak{s}^i$ is nonempty. Indeed, our assumption that every node t in T_i is not an initial segment of σ, simply reduces to the fact that $\mathfrak{s}^i$ contains the final segment $\{\sigma|k : k > l_0\}$. The sequence (u_n^i) is block, and so, we may select $k_i \in \mathbb{N}$ such that $\|P_\sigma(u_n^i)\| = \|P_{\mathfrak{s}^i}(u_n^i)\|$ for every $n \geqslant k_i$. Since $A_i \subseteq G_i$, the definition of $\mathfrak{s}^i$ ensures that $\mathfrak{s}^i \cap T_{A_i} = \emptyset$. Hence, by property (b) above, we see that there exists $n_i \in \mathbb{N}$ (clearly depending on σ) such that $\|P_{\mathfrak{s}^i}(u_n^i)\| < r$ for every $n \geqslant n_i$. We set $n_\sigma = \max\big\{k_i + n_i : i \in \{0, \dots, p\} \text{ and } i \neq j_\sigma\big\}$. By the above discussion, for every $n \geqslant n_\sigma$ we have

$$\|P_\sigma(w_n)\| = \|P_\sigma(u_n^0 + \dots + u_n^p)\| \leqslant 1 + pr. \tag{3.10}$$

Combining inequalities (3.8) and (3.10), it follows that for every $n \geqslant n_\sigma$

$$\|P_\sigma(y_n)\| = \Big\|P_\sigma\Big(\frac{w_n}{\|w_n\|}\Big)\Big\| \leqslant 2\frac{1 + pr}{\varepsilon\sqrt{p+1}}. \tag{3.11}$$

If we select p and r satisfying $p + 1 > 36 \cdot \varepsilon^{-4}$ and $r < (2p)^{-1}$, then we see that $\limsup \|P_\sigma(y_n)\| \leqslant \varepsilon/2$ for every $\sigma \in [T]$ and, therefore, we have reached the desired contradiction. The proof is completed. □

The main result of this section is a functional analytic characterization of weakly X-singular block subspaces of $T_2^{\mathfrak{X}}$. To state it, we need first to introduce the following variant of Definition 3.2.

Definition 3.17. **[DL]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis. The* c_0 Baire sum *of $\mathfrak{X}$, denoted by $T_0^{\mathfrak{X}}$, is defined to be the completion of $c_{00}(T)$ equipped with the norm*

$$\|z\|_{T_0^{\mathfrak{X}}} = \sup\Big\{\Big\|\sum_{t\in\mathfrak{s}} z(t)x_t\Big\|_X : \mathfrak{s} \text{ is a segment of } T\Big\}. \tag{3.12}$$

By $\mathrm{I}\colon T_2^{\mathfrak{X}} \to T_0^{\mathfrak{X}}$ *we shall denote the natural inclusion operator.*

We have the following theorem.

Theorem 3.18. *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and let Y be a block subspace of $T_2^{\mathfrak{X}}$. Then the following are equivalent.*

(i) *The subspace Y is weakly X-singular.*

(ii) *The operator $\mathrm{I}\colon Y \to T_0^{\mathfrak{X}}$ is not an isomorphic embedding.*

(iii) *There exists a block subspace Z of Y such that the operator $\mathrm{I}\colon Z \to T_0^{\mathfrak{X}}$ is compact.*

Proof. The equivalence between (ii) and (iii) is relatively easy. Indeed, the implication (iii)⇒(ii) is straightforward. Conversely, observe that if the operator $\mathrm{I}\colon Y \to T_0^{\mathfrak{X}}$ is not an isomorphic embedding, then for every finite co-dimensional subspace Y' of Y the operator $\mathrm{I}\colon Y' \to T_0^{\mathfrak{X}}$ is not an isomorphic embedding. By Proposition B.5, we see that (ii) implies (iii).

The implication (ii)⇒(i) is also easy. To see this let Y be a block subspace of $T_2^{\mathfrak{X}}$ which is *not* weakly X-singular. By definition, there exists finite $A \subseteq [T]$ such that the operator $P_{T_A}\colon Y \to \mathcal{X}_{T_A}$ is an isomorphic embedding. Noticing that the operator $\mathrm{I}\colon \mathcal{X}_{T_A} \to T_0^{\mathfrak{X}}$ is also an isomorphic embedding, our claim follows.

We work now to prove that (i) implies (ii). We will argue by contradiction. So, assume that Y is a weakly X-singular block subspace of $T_2^{\mathfrak{X}}$ such that the operator $\mathrm{I}\colon Y \to T_0^{\mathfrak{X}}$ is an isomorphic embedding. There exists a constant $C > 0$ such that for every $y \in Y$ we have

$$C \cdot \|y\|_{T_2^{\mathfrak{X}}} \leqslant \|y\|_{T_0^{\mathfrak{X}}} \leqslant \|y\|_{T_2^{\mathfrak{X}}}. \tag{3.13}$$

We fix $k_0 \in \mathbb{N}$ and $\varepsilon > 0$ satisfying

$$k_0 > \frac{64}{C^4} \quad \text{and} \quad \varepsilon < \min\Big\{\frac{C}{2}, \frac{1}{k_0}\Big\}. \tag{3.14}$$

By our assumptions, we may apply Proposition 3.16 to the block subspace Y of $T_2^{\mathfrak{X}}$ and the chosen ε. It follows that there exists a normalized block sequence (y_n) in Y such that $\limsup \|P_\sigma(y_n)\| < \varepsilon$ for every $\sigma \in [T]$. By Corollary 3.12 and by passing to a subsequence of (y_n) if necessary, we may additionally assume that for every $\sigma \in [T]$ we have $|\{n \in \mathbb{N} : \|P_\sigma(y_n)\| \geqslant \varepsilon\}| \leqslant 1$. Since the basis of $T_2^{\mathfrak{X}}$ is bi-monotone, we may strengthen this property to the following one.

(a) For every segment $\mathfrak{s}$ of T we have $|\{n \in \mathbb{N} : \|P_{\mathfrak{s}}(y_n)\| \geqslant \varepsilon\}| \leqslant 1$.

By Fact 3.3 and (3.13), for every $n \in \mathbb{N}$ we may select a segment $\mathfrak{s}_n$ of T such that

(b) $\|P_{\mathfrak{s}_n}(y_n)\| \geqslant C$ and $\mathfrak{s}_n \subseteq \{t_k : k \in \mathrm{range}(y_n)\}$.

As the sequence (y_n) is block, we see that such a selection guarantees that

(c) $\|P_{\mathfrak{s}_n}(y_m)\| = 0$ for every $n, m \in \mathbb{N}$ with $n \neq m$.

For every $n \in \mathbb{N}$ let t_n be the $\sqsubseteq$-minimum node of $\mathfrak{s}_n$. Applying the classical Ramsey theorem, we find an infinite subset $L = \{l_0 < l_1 < l_2 < \cdots\}$ of $\mathbb{N}$ such that one of the following (mutually exclusive) cases must occur.

CASE 1: *The set $\{t_n : n \in L\}$ is an antichain.* Our hypothesis in this case implies that for every $n, m \in L$ with $n \neq m$ the segments $\mathfrak{s}_n$ and $\mathfrak{s}_m$ are incomparable. We define $z = y_{l_0} + \cdots + y_{l_{k_0}}$. Since the family $(\mathfrak{s}_{l_i})_{i=0}^{k_0}$ consists of pairwise incomparable segments of T, we obtain that

$$\|z\| \geqslant \Big(\sum_{i=0}^{k_0} \|P_{\mathfrak{s}_{l_i}}(z)\|^2\Big)^{1/2} \overset{\text{(c)}}{=} \Big(\sum_{i=0}^{k_0} \|P_{\mathfrak{s}_{l_i}}(y_{l_i})\|^2\Big)^{1/2} \overset{\text{(b)}}{\geqslant} C\sqrt{k_0+1}. \tag{3.15}$$

Now set $w = z/\|z\| \in Y$. Invoking property (a) above, inequality (3.15) and the choice of k_0 and ε made in (3.14), for every segment $\mathfrak{s}$ of T we have

$$\|P_{\mathfrak{s}}(w)\| \leqslant \frac{1+k_0\varepsilon}{C\sqrt{k_0+1}} < \frac{C}{2}.$$

It follows that

$$\|w\|_{T_0^{\mathfrak{X}}} \leqslant \frac{C}{2}$$

which contradicts inequality (3.13). Hence this case is impossible.

CASE 2: *The set $\{t_n : n \in L\}$ is a chain.* Let $\tau \in [T]$ be the branch of T determined by the infinite chain $\{t_n : n \in L\}$. By property (a) above and by passing to an infinite subset of L if necessary, we may assume that $\|P_\tau(y_n)\| < \varepsilon$ for every $n \in L$. The basis of $T_2^{\mathfrak{X}}$ is bi-monotone, and so, we have the following property.

(d) If $\mathfrak{s}$ is a segment of T with $\mathfrak{s} \subseteq \tau$, then $\|P_{\mathfrak{s}}(y_n)\| < \varepsilon$ for every $n \in L$.

We set $\mathfrak{s}'_n = \mathfrak{s}_n \setminus \tau$. Observe that the set $\mathfrak{s}'_n$ is a sub-segment of $\mathfrak{s}_n$. Notice that $\mathfrak{s}_n$ is the disjoint union of the successive segments $\mathfrak{s}_n \cap \tau$ and $\mathfrak{s}'_n$. Hence, by properties (b) and (d) above and the choice of ε, we see that

$$\|P_{\mathfrak{s}'_n}(y_n)\| \geqslant C - \varepsilon \geqslant \frac{C}{2} \tag{3.16}$$

for every $n \in L$. Also notice that if $n, m \in L$ with $n \neq m$, then the segments $\mathfrak{s}'_n$ and $\mathfrak{s}'_m$ are incomparable. We set

$$z = y_{l_0} + \cdots + y_{l_{k_0}} \quad \text{and} \quad w = \frac{z}{\|z\|}.$$

Arguing precisely as in Case 1 and using the estimate in (3.16), we conclude that

$$\|w\|_{T_0^{\mathfrak{X}}} \leqslant \frac{C}{2}.$$

This is again a contradiction. The proof is completed. □

We close this section by recording the following straightforward consequence of Theorem 3.18.

Corollary 3.19. *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and let Y be a block subspace of $T_2^{\mathfrak{X}}$. Assume that Y is weakly X-singular. Then for every $\varepsilon > 0$ and every $k \in \mathbb{N}$ there exists a finitely supported vector y in Y with $\|y\| = 1$, $k < \min\{n : n \in \mathrm{range}(y)\}$ and such that $\|P_{\mathfrak{s}}(y)\| \leqslant \varepsilon$ for every segment $\mathfrak{s}$ of T.*

3.5 X-singular subspaces

In this section we continue our analysis of the subspaces of an ℓ_2 Baire sum by focusing on the class of its X-singular subspaces. We start with the following observation.

Fact 3.20. *Let (t_n) be an infinite antichain of T. Then the sequence (e_{t_n}) is 1-equivalent to the standard basis of ℓ_2 and the subspace $\overline{\mathrm{span}}\{e_{t_n} : n \in \mathbb{N}\}$ spanned by the sequence (e_{t_n}) is X-singular.*

This fact is generalized as follows.

Proposition 3.21. **[Ar]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and let S be an infinite, well-founded, downwards closed B-subtree of T. Then the space $\mathcal{X}_S = \overline{\mathrm{span}}\{e_t : t \in S\}$ is hereditarily ℓ_2 (that is, every subspace Y of $\mathcal{X}_S$ contains an isomorphic copy of ℓ_2).*

Proof. The proof proceeds by transfinite induction on the order $o(S)$ of S. If $o(S) = 1$, then the result is trivial since the space $\mathcal{X}_S$ is isometric to ℓ_2. Let $\xi < \omega_1$ and assume that the result has been proved for every B-subtree S' of T with $o(S') < \xi$. Let S be a well-founded B-subtree of T with $o(S) = \xi$. Set $A = \{\lambda \in \Lambda : (\lambda) \in S\}$ and for every $\lambda \in A$ set $S_\lambda = \{t \in S : (\lambda) \sqsubseteq t\}$. Also let Y be an arbitrary subspace of $\mathcal{X}_S$. If for every $\lambda \in A$ the operator $P_{S_\lambda} : Y \to \mathcal{X}_{S_\lambda}$ is strictly singular, then using a standard sliding hump argument, we may find a subspace Y' of Y which is isomorphic to ℓ_2. If not, then there exist $\lambda \in A$ and a subspace Y' of Y such that the map $P_{S_\lambda} : Y' \to \mathcal{X}_{S_\lambda}$ is an isomorphic embedding. Using our inductive hypothesis, we may easily find a further subspace Y'' of Y which is isomorphic to ℓ_2. The proof is completed. □

The structure of an arbitrary X-singular subspace of $T_2^{\mathfrak{X}}$ (and not merely of those spanned by well-founded subtrees of the basis) is more complicated. In particular we have the following theorem.

Theorem 3.22. **[AD]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis. Assume that the B-tree T contains a perfect subtree. Then there exists an X-singular subspace of $T_2^{\mathfrak{X}}$ which is isomorphic to c_0.*

Proof. By passing to a subtree of T (not necessarily downwards closed), we may assume that T is the tree $4^{<\mathbb{N}}$; that is, every $t \in T$ has exactly four immediate successors in T. So, for every $n \in \mathbb{N}$ the n-level $T(n) = \{t \in T : |t| = n\}$ of T has 4^n nodes. We define

$$y_n = \sum_{t \in T(n)} \frac{1}{2^n} e_t.$$

It is easy to see that (y_n) is a normalized bi-monotone basic sequence. Let $Y = \overline{\text{span}}\{y_n : n \in \mathbb{N}\}$. Observe that Y is an X-singular subspace of $T_2^{\mathfrak{X}}$. We claim that (y_n) is equivalent to the standard basis of c_0. This will finish the proof.

To this end, let $k \in \mathbb{N}$ and $a_0, \ldots, a_k \in \mathbb{R}$ with $\max\{|a_n| : n = 0, \ldots, k\} = 1$. We will show that $\big\| \sum_{n=0}^{k} a_n y_n \big\| \leqslant 2$ which implies that (y_n) is 2-equivalent to the standard unit vector basis of c_0. We start with the following observation. Let $\mathfrak{s}$ be a segment of T and set $o_{\mathfrak{s}} = \min\{|t| : t \in \mathfrak{s}\}$. Notice that

$$\Big\| P_{\mathfrak{s}}\Big(\sum_{n=0}^{k} a_n y_n \Big) \Big\| \leqslant \sum_{n=o_{\mathfrak{s}}}^{k} |a_n| \cdot \|P_{\mathfrak{s}}(y_n)\| \leqslant \sum_{n=o_{\mathfrak{s}}}^{k} \frac{1}{2^n} \leqslant 2 \frac{1}{2^{o_{\mathfrak{s}}}}.$$

Now let $(\mathfrak{s}_j)_{j=0}^{l}$ be an arbitrary collection of pairwise incomparable segments of T. For every $j \in \{0, \ldots, l\}$ set $o_j = \min\{|t| : t \in \mathfrak{s}_j\}$. Write the set $\{o_j : j = 0, \ldots, l\}$ in increasing order as $i_0 < i_1 < \cdots < i_m$ (notice that $m \leqslant l$). For every $p \in \{0, \ldots, m\}$ let $I_p = \big\{ j \in \{0, \ldots, l\} : o_j = i_p \big\}$. The family $(I_p)_{p=0}^{m}$ forms a partition of $\{0, \ldots, l\}$. We claim that

$$\frac{|I_0|}{4^{i_0}} + \frac{|I_1|}{4^{i_1}} + \cdots + \frac{|I_m|}{4^{i_m}} \leqslant 1. \tag{3.17}$$

Indeed, notice that every node t in $T(i_p)$ has exactly $4^{i_m - i_p}$ successors in $T(i_m)$. Since the family consists of pairwise incomparable segments, we see that

$$4^{i_m - i_0}|I_0| + 4^{i_m - i_1}|I_1| + \cdots + 4^{i_m - i_{m-1}}|I_{m-1}| + |I_m| \leqslant 4^{i_m}$$

which gives the desired estimate. Now observe that

$$\Big(\sum_{j=0}^{l} \Big\| P_{\mathfrak{s}_j}\Big(\sum_{n=0}^{k} a_n y_n \Big) \Big\|^2 \Big)^{1/2} \leqslant 2 \cdot \Big(\sum_{j=0}^{l} \frac{1}{4^{o_j}} \Big)^{1/2} = 2 \cdot \Big(\sum_{p=0}^{m} \frac{|I_p|}{4^{i_p}} \Big)^{1/2} \leqslant 2$$

where the last inequality follows by (3.17). The proof is completed. □

Theorem 3.22 essentially shows that we cannot control the structure of an arbitrary X-singular subspace of $T_2^{\mathfrak{X}}$. Actually, it can be shown that there exists a Schauder tree basis $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t \in T})$ such that for every $p \in [2, +\infty)$ there exists an X-singular subspace Y of $T_2^{\mathfrak{X}}$ which is isomorphic to ℓ_p. However, we have the following result.

Theorem 3.23. **[DL]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and let Y be an X-singular subspace of $T_2^{\mathfrak{X}}$. Then for every normalized basic sequence (x_n) in Y there exists a normalized block sequence (y_n) of (x_n) satisfying an upper ℓ_2 estimate.*

In particular, every X-singular subspace Y of $T_2^{\mathfrak{X}}$ contains no ℓ_p for any $1 \leqslant p < 2$.

Proof. Let Y be an X-singular subspace of $T_2^{\mathfrak{X}}$ and let (x_n) be a normalized basic sequence in Y. A standard sliding hump argument allows us to construct a normalized block sequence (v_n) of (x_n) and a block sequence (z_n) in $T_2^{\mathfrak{X}}$ such that, setting $Z = \overline{\text{span}}\{z_n : n \in \mathbb{N}\}$, the following are satisfied.

(a) The sequences (v_n) and (z_n) are equivalent.

(b) The subspace Z of $T_2^{\mathfrak{X}}$ is block and X-singular.

By part (ii) of Proposition 3.5 and (b) above, we see that Z is a weakly X-singular subspace. Hence, using Corollary 3.19, we may construct, recursively, a normalized block sequence (w_n) of (z_n) such that for every $n \in \mathbb{N}$ with $n \geqslant 1$ and every $\sigma \in [T]$ we have

$$\|P_\sigma(w_n)\| \leqslant \frac{1}{\sum_{i=0}^{n-1} |\text{supp}(w_i)|^{1/2}} \cdot \frac{1}{2^{2n}}.$$

By Lemma 3.14, the sequence (w_n) satisfies an upper ℓ_2 estimate. Let (b_n) be the block sequence of (v_n) corresponding to (w_n). Observe that, by (a) above, the sequence (b_n) is seminormalized and satisfies an upper ℓ_2 estimate. The property of being a block sequence is transitive, and so, (b_n) is a block sequence of (x_n) as well. Hence, setting $y_n = b_n/\|b_n\|$ for every $n \in \mathbb{N}$, we see that the sequence (y_n) is the desired one.

Finally, to see that every X-singular subspace of $T_2^{\mathfrak{X}}$ can contain no ℓ_p for any $1 \leqslant p < 2$ we argue by contradiction. So, assume that Y is an X-singular subspace of $T_2^{\mathfrak{X}}$ containing an isomorphic copy of ℓ_{p_0} for some $1 \leqslant p_0 < 2$. There exists, in this case, a normalized basic sequence (x_n) in Y which is equivalent to the standard unit vector basis (e_n) of ℓ_{p_0}. Let (y_n) be a normalized block sequence of (x_n) satisfying an upper ℓ_2 estimate. Since every normalized block sequence of (e_n) is equivalent to (e_n) (see [LT, Proposition 2.a.1]), we see that there exist constants $C \geqslant c > 0$ such that for every $k \in \mathbb{N}$ and every $a_0, \dots, a_k \in \mathbb{R}$ we have

$$c \cdot \Big(\sum_{n=0}^{k} |a_n|^{p_0} \Big)^{1/p_0} \leqslant \Big\| \sum_{n=0}^{k} a_n y_n \Big\|_{T_2^{\mathfrak{X}}} \leqslant C \cdot \Big(\sum_{n=0}^{k} |a_n|^2 \Big)^{1/2}.$$

This is clearly a contradiction. The proof is completed. □

We close this section by presenting a characterization of X-singular subspaces. It is the analogue of Theorem 3.18 for this class of subspaces of $T_2^{\mathfrak{X}}$.

Theorem 3.24. **[DL]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and let Y be a subspace of $T_2^{\mathfrak{X}}$. Then the following are equivalent.*

(i) *Y is an X-singular subspace of $T_2^{\mathfrak{X}}$.*

(ii) *The operator $\mathrm{I}\colon Y \to T_0^{\mathfrak{X}}$ is strictly singular.*

Proof. It is clear that (ii) implies (i). Hence we only need to show the converse implication. We argue by contradiction. So, assume that Y is an X-singular subspace of $T_2^{\mathfrak{X}}$ such that the operator $\mathrm{I}\colon Y \to T_0^{\mathfrak{X}}$ is not strictly singular. Then there exists a further subspace Y' of Y such that the operator $\mathrm{I}\colon Y' \to T_0^{\mathfrak{X}}$ is an isomorphic embedding. Using a standard sliding hump argument, we may select, recursively, a normalized basic sequence (y_n) in Y' and a normalized block sequence (z_n) in $T_2^{\mathfrak{X}}$ such that, setting $Z = \overline{\mathrm{span}}\{z_n : n \in \mathbb{N}\}$, the following are satisfied.

(a) The sequence (z_n) is equivalent to (y_n).

(b) The subspace Z of $T_2^{\mathfrak{X}}$ is X-singular.

(c) The operator $\mathrm{I}\colon Z \to T_0^{\mathfrak{X}}$ is an isomorphic embedding.

By part (ii) of Proposition 3.5 and property (b) above, we see that Z is a block and weakly X-singular subspace of $T_2^{\mathfrak{X}}$. By Theorem 3.18, the operator $\mathrm{I}\colon Z \to T_0^{\mathfrak{X}}$ is not an isomorphic embedding, in contradiction with (c) above. The proof is completed. □

Corollary 3.25. **[DL]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and let Y be an infinite-dimensional subspace of $T_2^{\mathfrak{X}}$. Assume that Y is X-singular. Then there exists an infinite-dimensional subspace Y' of Y which is X-compact.*

Proof. By Theorem 3.24, the operator $\mathrm{I}\colon Y \to T_0^{\mathfrak{X}}$ is strictly singular. By Proposition B.5, there exists an infinite-dimensional subspace Y' of Y such that the operator $\mathrm{I}\colon Y' \to T_0^{\mathfrak{X}}$ is compact. It is easy to see that Y' must be an X-compact subspace of $T_2^{\mathfrak{X}}$ in the sense of Definition 3.4. The proof is completed. □

3.6 Schauder tree bases not containing ℓ_1

We introduce the following definition.

Definition 3.26. *We say that a Schauder tree basis $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$* does not contain ℓ_1 *if for every $\sigma \in [T]$ the space $X_\sigma = \overline{\mathrm{span}}\{x_{\sigma|n} : n \geqslant 1\}$ does not contain an isomorphic copy of ℓ_1.*

If $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ is a Schauder tree basis not containing ℓ_1, then this property is inherited to the corresponding ℓ_2 Baire sum $T_2^{\mathfrak{X}}$ of $\mathfrak{X}$. In particular, we have the following theorem.

Theorem 3.27. **[AD]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis not containing ℓ_1. Then the ℓ_2 Baire sum $T_2^{\mathfrak{X}}$ associated to $\mathfrak{X}$ does not contain an isomorphic copy of ℓ_1.*

Proof. If not, then there would existed a subspace Y of $T_2^{\mathfrak{X}}$ isomorphic to ℓ_1. By our assumptions, Y must be X-singular. By Theorem 3.23, we derive a contradiction. The proof is completed. $\square$

Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and let $(e_t)_{t\in T}$ be the canonical basis of $T_2^{\mathfrak{X}}$. Also let $(e_t^*)_{t\in T}$ be the bi-orthogonal functionals associated to $(e_t)_{t\in T}$. For every $\sigma \in [T]$ we define $\mathcal{Z}_\sigma$ to be the weak* closure of $\mathrm{span}\{e^*_{\sigma|n} : n \geqslant 1\}$. The spaces $\mathcal{X}_\sigma^*$ and $\mathcal{Z}_\sigma$ are isometric and weak* isomorphic via the operator $T\colon \mathcal{X}_\sigma^* \to \mathcal{Z}_\sigma$ defined by

$$T(x^*)(x) = x^*\big(P_\sigma(x)\big) \text{ for every } x \in T_2^{\mathfrak{X}}.$$

We have the following description of the dual $(T_2^{\mathfrak{X}})^*$ of $T_2^{\mathfrak{X}}$ for a Schauder tree basis $\mathfrak{X}$ not containing ℓ_1.

Proposition 3.28. **[AD]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis not containing ℓ_1. Then*

$$(T_2^{\mathfrak{X}})^* = \overline{\mathrm{span}}\Big\{ \bigcup_{\sigma\in[T]} B_{\mathcal{Z}_\sigma} \Big\}.$$

Proof. We set $Z^* = \overline{\mathrm{span}}\big\{ \bigcup_{\sigma\in[T]} B_{\mathcal{Z}_\sigma} \big\}$. Assume, towards a contradiction, that Z^* is a proper subspace of the dual $(T_2^{\mathfrak{X}})^*$ of $T_2^{\mathfrak{X}}$. Hence, by the Hahn–Banach theorem, there exists $x^{**} \in (T_2^{\mathfrak{X}})^{**}$ with $\|x^{**}\| = 1$ and such that $x^{**}(z^*) = 0$ for every $z^* \in Z^*$. We select $x^* \in (T_2^{\mathfrak{X}})^*$ with $\|x^*\| = 1$ and $x^{**}(x^*) = 1$. By Theorem 3.27 and our assumptions, we see that the space $T_2^{\mathfrak{X}}$ does not contain ℓ_1. Since $x^{**}(e_t^*) = 0$ for every $t \in T$, by the Odell–Rosenthal theorem [OR], we may select a bounded block sequence (x_n) in $T_2^{\mathfrak{X}}$ which is weak* convergent to x^{**}. We may also assume that

$$x^*(x_n) > 1/2 \;\text{ for every } n \in \mathbb{N}. \tag{3.18}$$

The fact that $x^{**}|_{Z^*} = 0$ implies that $P_\sigma(x_n) \xrightarrow{\text{w}} 0$ in $\mathcal{X}_\sigma$ for every $\sigma \in [T]$. By Theorem 3.6, we see that the sequence (x_n) is weakly null, contradicting (3.18) above. The proof is completed. $\square$

Notice that for every Schauder tree basis $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ and every $\sigma \in [T]$ we have

$$\begin{aligned}(x_{\sigma|n})_{n\geqslant 1} \text{ is shrinking} \quad &\Leftrightarrow \quad (e_{\sigma|n})_{n\geqslant 1} \text{ is shrinking}\\ &\Leftrightarrow \quad \mathcal{Z}_\sigma = \overline{\text{span}}\{e^*_{\sigma|n} : n \geqslant 1\}.\end{aligned}$$

Hence, by Proposition 3.28, we obtain the following corollary.

Corollary 3.29. **[AD]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis such that for every $\sigma \in [T]$ the sequence $(x_{\sigma|n})_{n\geqslant 1}$ is shrinking. Let (t_n) be the enumeration of T according to the bijection $h_T : T \to \mathbb{N}$ described in Section 3.2. If (e_{t_n}) is the corresponding enumeration of $(e_t)_{t\in T}$, then the basis (e_{t_n}) of $T_2^{\mathfrak{X}}$ is shrinking.*

It should be noted that, by Theorem 3.22, the analogue of Corollary 3.29 for boundedly complete sequences is not valid.

3.7 Comments and Remarks

1. Almost all the material in this chapter, including the basic notions, is taken form [AD]. As the reader might have already observed, the space $T_2^{\mathfrak{X}}$ is a variant of James' fundamental example [Ja]. We also notice that norms similar to the ones given in (3.2) were previously defined by Bourgain [Bou1] and Bossard [Bos3] (but of course not at this level of generality).

2. Theorem 3.18 and Corollary 3.19 are new.

3. Proposition 3.21 was essentially noticed by Argyros in [Ar]. We should point out that all subspaces of $T_2^{\mathfrak{X}}$ spanned by well-founded subtrees of the basis are reflexive (see [Bou1] and [Bos3]).

4. Theorems 3.23 and 3.24 and Corollary 3.25 are taken from [DL]. Another result contained in that paper and concerning the structure of an X-singular subspace is the following.

Theorem 3.30. **[DL]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and let Y be an infinite-dimensional X-singular subspace of $T_2^{\mathfrak{X}}$. Then every infinite-dimensional subspace Z of Y contains an unconditional basic sequence.*

5. Proposition 3.28 is valid without the assumption that the Schauder tree basis $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ does not contain ℓ_1. In particular, the following holds.

Theorem 3.31. **[AD]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis. Then*

$$(T_2^{\mathfrak{X}})^* = \overline{\text{span}}\Big\{ \bigcup_{\sigma\in[T]} B_{\mathcal{Z}_\sigma} \Big\}.$$

Chapter 4

Amalgamated spaces

Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and consider the ℓ_2 Baire sum $T_2^{\mathfrak{X}}$ associated to $\mathfrak{X}$. The space $T_2^{\mathfrak{X}}$ contains, naturally, a complemented copy of every space in the class coded by the Schauder tree basis. Moreover, by Theorem 3.23, there is information on the kind of subspaces present in $T_2^{\mathfrak{X}}$. This is enough for a large number of applications. However, by Theorem 3.22, for every interesting Schauder tree basis $\mathfrak{X}$ the space $T_2^{\mathfrak{X}}$ contains an isomorphic copy of c_0.

The aim this chapter is to present a refinement of the previous construction, also introduced in [AD]. The refinement will lead to a new space for which we have significant control over the isomorphic types of its subspaces. The method is to use the Davis–Fiegel–Johnson–Pełczyński [DFJP] interpolation scheme in similar spirit as in the work of Argyros and Felouzis [AF].

4.1 Definitions and basic properties

Definition 4.1. **[AD]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis. We define*

$$W^0_{\mathfrak{X}} = \mathrm{conv}\Big\{ \bigcup_{\sigma\in[T]} B_{\mathcal{X}_\sigma} \Big\} \ \ \textit{and} \ \ W_{\mathfrak{X}} = \overline{\mathrm{conv}}\Big\{ \bigcup_{\sigma\in[T]} B_{\mathcal{X}_\sigma} \Big\}. \tag{4.1}$$

Notice that for every Schauder tree basis $\mathfrak{X}$ the set $W_{\mathfrak{X}}$ is closed, convex, bounded and symmetric. This observation permits us to define the following space which is the central object of study in this chapter.

Definition 4.2. **[AD]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and $1 < p < +\infty$. The* p-amalgamation space *of $\mathfrak{X}$, denoted by $A_p^{\mathfrak{X}}$, is defined to be the p-interpolation space of the pair $(T_2^{\mathfrak{X}}, W_{\mathfrak{X}})$ (see Definition* B.7*). By $J\colon A_p^{\mathfrak{X}} \to T_2^{\mathfrak{X}}$ we shall denote the natural inclusion operator.*

Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and $1 < p < +\infty$. As in Section 3.2, we fix a bijection $h_T : T \to \mathbb{N}$ such that for every $t, s \in T$ with $t \sqsubset s$ we have $h_T(t) < h_T(s)$. In what follows, by (t_n) we shall denote the enumeration of the tree T according to the bijection h_T, while by (e_{t_n}) we shall denote the canonical Schauder basis of $T_2^{\mathfrak{X}}$ described in Section 3.2. Finally, for every $n \in \mathbb{N}$ with $n \geqslant 1$ let $\|\cdot\|_n$ be the Minkowski gauge of the set $2^n W_{\mathfrak{X}} + 2^{-n} B_{T_2^{\mathfrak{X}}}$. Clearly $\|\cdot\|_n$ is an equivalent norm on $T_2^{\mathfrak{X}}$.

A. For every $n \in \mathbb{N}$, denoting by $P_n : T_2^{\mathfrak{X}} \to \mathrm{span}\{e_{t_k} : k \leqslant n\}$ the natural onto projection, we see that $P_n(W_{\mathfrak{X}}) \subseteq W_{\mathfrak{X}}$ and $P_n(W^0_{\mathfrak{X}}) \subseteq W^0_{\mathfrak{X}}$. Moreover, setting $\bar{e}_t = J^{-1}(e_t)$ for every $t \in T$, it is easy to see that $2^{-1} \leqslant \|\bar{e}_t\|_{A_p^{\mathfrak{X}}} \leqslant 1$. By Proposition B.9, we obtain that the sequence $(\bar{e}_{t_n})$ defines a seminormalized Schauder basis of the space $A_p^{\mathfrak{X}}$. In addition, if (e_{t_n}) is a shrinking basis of $T_2^{\mathfrak{X}}$, then $(\bar{e}_{t_n})$ is a shrinking basis of $A_p^{\mathfrak{X}}$.

B. Let $x \in W_{\mathfrak{X}}$ with $\|x\| = 1$. Then for every $n \in \mathbb{N}$ with $n \geqslant 1$ we have $2^{-(n+1)} \leqslant \|x\|_n \leqslant 2^{-n}$. Let $\bar{x} = J^{-1}(x)$ and notice that the previous remark implies that $2^{-1} \leqslant \|\bar{x}\|_{A_p^{\mathfrak{X}}} \leqslant 1$. For every $\sigma \in [T]$ we set

$$\bar{\mathcal{X}}_\sigma = \overline{\mathrm{span}}\{\bar{e}_{\sigma|n} : n \geqslant 1\}. \tag{4.2}$$

It follows that the operator $J : \bar{\mathcal{X}}_\sigma \to T_2^{\mathfrak{X}}$ is an isomorphic embedding onto the subspace $\mathcal{X}_\sigma$ of $T_2^{\mathfrak{X}}$. In particular, for every Schauder tree basis $\mathfrak{X}$ and every $1 < p < +\infty$ the subspace $\bar{\mathcal{X}}_\sigma$ of $A_p^{\mathfrak{X}}$, the subspace $\mathcal{X}_\sigma$ of $T_2^{\mathfrak{X}}$ and the space X_σ defined in (3.1) are all mutually isomorphic. Moreover, the Banach–Mazur distance between $\bar{\mathcal{X}}_\sigma$ and X_σ is at most 2.

C. For every finite subset A of $[T]$ let $T_A = \{\sigma|n : \sigma \in A \text{ and } n \geqslant 1\}$ be the B-tree generated by A and set $\bar{\mathcal{X}}_{T_A} = \overline{\mathrm{span}}\{\bar{e}_t : t \in T_A\}$. It is easy to see that the operator $J : \bar{\mathcal{X}}_{T_A} \to T_2^{\mathfrak{X}}$ is an isomorphic embedding onto the subspace $\mathcal{X}_{T_A}$ of $T_2^{\mathfrak{X}}$ (notice, however, that the isomorphic constant is not uniform and depends on the cardinality of A).

D. More generally, let S be a segment complete subset of T and set

$$\bar{\mathcal{X}}_S = \overline{\mathrm{span}}\{\bar{e}_t : t \in S\}. \tag{4.3}$$

Consider the natural projection $\bar{P}_S : A_p^{\mathfrak{X}} \to \bar{\mathcal{X}}_S$.

Fact 4.3. *For every segment complete subset S of T we have $\|\bar{P}_S\| = 1$.*

Fact 4.3 implies, in particular, that $\bar{\mathcal{X}}_S$ is complemented in $A_p^{\mathfrak{X}}$. We notice that this property of the space $A_p^{\mathfrak{X}}$ is essentially a consequence of the fact that the external norm used in the interpolation scheme—that is, the norm in (B.1)—is an unconditional one. This property is *not* valid for other variants.

Proof of Fact 4.3. Let $\bar{x}$ be a finitely supported vector in $A_p^{\mathfrak{X}}$; that is, $\bar{x} \in \text{span}\{\bar{e}_t : t \in T\}$. We set $x = J(\bar{x})$. Notice that x is finitely supported in $T_2^{\mathfrak{X}}$ and $J\big(\bar{P}_S(\bar{x})\big) = P_S(x)$.

Fix $n \in \mathbb{N}$ with $n \geqslant 1$ and set $\lambda_n = \|x\|_n$. Then for every $\lambda > \lambda_n$ we have

$$\frac{x}{\lambda} \in 2^n W_{\mathfrak{X}} + 2^{-n} B_{T_2^{\mathfrak{X}}}.$$

Since $P_S(W_{\mathfrak{X}}) \subseteq W_{\mathfrak{X}}$ and $P_S(B_{T_2^{\mathfrak{X}}}) \subseteq B_{T_2^{\mathfrak{X}}}$, we obtain that

$$\frac{P_S(x)}{\lambda} \in 2^n W_{\mathfrak{X}} + 2^{-n} B_{T_2^{\mathfrak{X}}}.$$

In particular, $\|P_S(x)\|_n \leqslant \lambda_n = \|x\|_n$. Therefore,

$$\|\bar{P}_S(\bar{x})\|_{A_p^{\mathfrak{X}}} = \Big(\sum_{n=1}^{\infty} \|P_S(x)\|_n^p\Big)^{1/p} \leqslant \Big(\sum_{n=1}^{\infty} \|x\|_n^p\Big)^{1/p} = \|\bar{x}\|_{A_p^{\mathfrak{X}}}.$$

The proof of Fact 4.3 is completed. □

Up to this point we have seen certain analogies between the spaces $A_p^{\mathfrak{X}}$ and $T_2^{\mathfrak{X}}$. The results that follow reveal the structural differences between $A_p^{\mathfrak{X}}$ and $T_2^{\mathfrak{X}}$ and show, in particular, that the p-amalgamation space $A_p^{\mathfrak{X}}$ is a much "smaller" space than $T_2^{\mathfrak{X}}$.

Theorem 4.4. **[AD]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis such that for every $\sigma \in [T]$ the space X_σ is reflexive. Let $1 < p < +\infty$. Then the p-amalgamation space $A_p^{\mathfrak{X}}$ of $\mathfrak{X}$ is reflexive.*

Proof. By part (iv) of Proposition B.8, it is enough to show that the set $W_{\mathfrak{X}}$ is weakly compact. To this end, let

$$C_{\mathfrak{X}} = \bigcup_{\sigma\in[T]} B_{\mathcal{X}_\sigma}.$$

Notice that $W_{\mathfrak{X}} = \overline{\text{conv}}\{C_{\mathfrak{X}}\}$.

Claim 4.5. *The set $C_{\mathfrak{X}}$ is weakly compact.*

Proof of Claim 4.5. Let (x_n) be a sequence in $C_{\mathfrak{X}}$. We need to find a vector $z \in C_{\mathfrak{X}}$ and a subsequence of (x_n) which is weakly convergent to z. Clearly we may assume that every vector x_n is finitely supported. By the definition of the set $C_{\mathfrak{X}}$, for every $n \in \mathbb{N}$ the support of x_n is a chain. Let $\mathfrak{s}_n$ be the minimal initial segment of T that contains $\text{supp}(x_n)$. By identifying every initial segment $\mathfrak{s}_n$ with its characteristic function (that is, an element of 2^T), we find $L \in [\mathbb{N}]^\infty$ and a (possible empty) initial segment $\mathfrak{s}$ of T such that the sequence $(\mathfrak{s}_n)_{n\in L}$ converges to $\mathfrak{s}$ in 2^T. We select a branch $\sigma \in [T]$ such that $\mathfrak{s} \subseteq \{\sigma|n : n \geqslant 1\}$. For every $n \in L$, set $z_n = P_\sigma(x_n)$ and $w_n = x_n - z_n = x_n - P_\sigma(x_n)$.

By our assumptions, the space $B_{\mathcal{X}_\sigma}$ is reflexive. Noticing that $z_n \in B_{\mathcal{X}_\sigma}$ for every $n \in L$, we see that there exist $M \in [L]^\infty$ and a vector $z \in B_{\mathcal{X}_\sigma} \subseteq C_{\mathfrak{X}}$ such that $(z_n)_{n\in M}$ is weakly convergent to z. Now for every $n \in M$ let t_n be the $\sqsubseteq$-minimum node of $\mathrm{supp}(w_n)$. Applying Ramsey's theorem, we obtain $N \in [M]^\infty$ such that the family $\{t_n : n \in N\}$ is either a chain, or an antichain. We claim that it must be an antichain. Assume not. Let $k \in N$ be such that $|t_k| \leqslant |t_n|$ for all $n \in N$. It follows that $t_k \in \mathfrak{s}_n$ for every $n \in N$, and so, $t_k \in \mathfrak{s} \subseteq \{\sigma|n : n \geqslant 1\}$. But this is clearly impossible by the definition of the vector w_k. Hence, the set $\{t_n : n \in N\}$ is an antichain. Using this remark, we see that the sequence $(w_n)_{n\in N}$ satisfies an upper ℓ_2 estimate, and therefore, it is weakly null. It follows by the above discussion that the subsequence $(x_n)_{n\in N}$ of (x_n) is weakly convergent to $z \in C_{\mathfrak{X}}$. The claim is proved. □

By Claim 4.5 and the Krein–Smulian theorem, we conclude that the set $W_{\mathfrak{X}}$ is weakly compact. The proof of Theorem 4.4 is completed. □

The following result is a basic dichotomy concerning the isomorphic types of subspaces of $A_p^{\mathfrak{X}}$.

Theorem 4.6. [**AD**] *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and $1 < p < +\infty$. Let Z be a block subspace of $A_p^{\mathfrak{X}}$. Then either*

(i) *Z contains an isomorphic copy of ℓ_p, or*

(ii) *there exists finite $A \subseteq [T]$ such that the operator $\bar{P}_{T_A} : Z \to \bar{\mathcal{X}}_{T_A}$ is an isomorphic embedding.*

Theorem 4.6 can be reformulated as follows.

Corollary 4.7. *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and $1 < p < +\infty$. Let Z be a block subspace of $A_p^{\mathfrak{X}}$. Then either*

(i) *Z contains an isomorphic copy of ℓ_p, or*

(ii) *there exists a finite subset $\{\sigma_0, \dots, \sigma_k\}$ of $[T]$ such that Z is isomorphic to a subspace of $\sum_{n=0}^k \oplus X_{\sigma_n}$.*

The rest of this chapter is devoted to the proof of Theorem 4.6. The basic tool will be a combinatorial result concerning sequences in $T_2^{\mathfrak{X}}$. This result will be presented in Section 4.2. The proof of Theorem 4.6 will be completed in Section 4.3.

4.2 Finding incomparable sets of nodes

This section contains the key result towards the proof of Theorem 4.6. We will comment, briefly, on the rôle of this result.

The main problem behind our reasoning is the following. Given a bounded block sequence (x_n) in $T_2^{\mathfrak{X}}$ when can we extract a subsequence of (x_n) which is equivalent to the standard basis of ℓ_2? This problem is, of course, reduced to the problem of finding sufficient conditions on a sequence (x_n) in order to be able to infer that the sequence satisfies an upper and, respectively, a lower ℓ_2 estimate.

Concerning the upper ℓ_2 estimate the problem is solved in Lemma 3.14 (in fact, Lemma 3.14 is nearly optimal). We stress the fact that the condition given in Lemma 3.14 does *not* imply that the sequence satisfies a lower ℓ_2 estimate. Actually, concerning the lower ℓ_2 estimate one has the following, more or less obvious, condition.

Fact 4.8. *Let (x_n) be a bounded block sequence in $T_2^{\mathfrak{X}}$ and $c > 0$. Assume that there exists a sequence (A_n) of mutually incomparable, segment complete subsets of T such that $\|P_{A_n}(x_n)\| \geqslant c$ for every $n \in \mathbb{N}$. Then for every $k \in \mathbb{N}$ and every $a_0, \dots, a_k \in \mathbb{R}$ we have*

$$c \cdot \Big(\sum_{n=0}^{k} a_n^2 \Big)^{1/2} \leqslant \big\| \sum_{n=0}^{k} a_n x_n \big\|_{T_2^{\mathfrak{X}}}.$$

In particular, the sequence (x_n) satisfies a lower ℓ_2 estimate.

The main result of this section provides sufficient conditions for the existence of a sequence of incomparable sets of nodes as described in Fact 4.8 and thereby giving sufficient conditions for checking the lower ℓ_2 estimate.

Proposition 4.9. **[AD]** *Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t\in T})$ be a Schauder tree basis and let (t_n) be the enumeration of T according to the bijection $h_T \colon T \to \mathbb{N}$ described in Section 3.2. Let (y_n) be a normalized block sequence in $T_2^{\mathfrak{X}}$ and $\lambda > 0$ such that*

$$\{y_n : n \in \mathbb{N}\} \subseteq \lambda W_{\mathfrak{X}} + \frac{1}{200} B_{T_2^{\mathfrak{X}}}.$$

Also let $r \leqslant 100^{-3} \cdot \lambda^{-1}$ and assume that

$$\limsup \|P_\sigma(y_n)\| < r \text{ for every } \sigma \in [T].$$

Then there exist $L \in [\mathbb{N}]^\infty$ and for every $n \in L$ a segment complete subset A_n of T such that the following are satisfied.

(i) *For every $n \in L$ we have $A_n \subseteq \{t_k : k \in \mathrm{range}(y_n)\}$ and $\|P_{A_n}(y_n)\| \geqslant 2/3$.*

(ii) *For every pair $n, m \in L$ with $n \neq m$ the sets A_n and A_m are incomparable.*

Proof. Let $n \in \mathbb{N}$ be arbitrary. By our assumptions, there exist $w_n \in W_{\mathfrak{X}}^0$ and $x_n \in T_2^{\mathfrak{X}}$ such that $\|x_n\| \leqslant 100^{-1}$ and $y_n = \lambda w_n + x_n$. This implies, in particular, that $\|y_n - \lambda w_n\| \leqslant 100^{-1}$. Let $I_n = \mathrm{range}(y_n)$ and $R_n = \{t_k : k \in I_n\}$. Without loss of generality we may assume that $\mathrm{supp}(w_n) \subseteq R_n$ for every $n \in \mathbb{N}$. Indeed, by Fact 3.3, the set R_n is segment complete and so $P_{R_n}(W^0_{\mathfrak{X}}) \subseteq W^0_{\mathfrak{X}}$ and $\|P_{R_n}\| = 1$ for every $n \in \mathbb{N}$. So, in what follows we will assume that $\mathrm{supp}(w_n) \subseteq R_n$. This property clearly implies that the sequence (w_n) is block as well.

For every $n \in \mathbb{N}$ we select a family $\{\mathfrak{s}^0_n, \dots, \mathfrak{s}^{d_n}_n\}$ of pairwise incomparable segments of T such that $\mathfrak{s}^i_n \subseteq R_n$ for all $i \in \{0, \dots, d_n\}$, and

$$\|\lambda w_n\| = \Big(\sum_{i=0}^{d_n} \|P_{\mathfrak{s}^i_n}(\lambda w_n)\|^2 \Big)^{1/2}. \tag{4.4}$$

Since $\|y_n\| = 1$ and $\|y_n - \lambda w_n\| \leqslant 100^{-1}$, we have $99/100 \leqslant \|\lambda w_n\| \leqslant 101/100$. We set $\theta = 8^2 \cdot 100^{-2} \cdot \lambda^{-2}$. Notice that $\lambda\sqrt{\theta} = 8/100$. We define

$$G_n = \big\{ i \in \{0, \dots, d_n\} : \|P_{\mathfrak{s}^i_n}(w_n)\| \geqslant \theta \big\}. \tag{4.5}$$

Claim 4.10. *For every $n \in \mathbb{N}$ the following hold.*

(i) $|G_n| \leqslant 4 \cdot \lambda^{-2} \cdot \theta^{-2}$.

(ii) $\Big(\sum_{i \in G_n} \|P_{\mathfrak{s}^i_n}(y_n)\|^2 \Big)^{1/2} \geqslant 9/10$.

Proof of Claim 4.10. (i) Notice that

$$2 \geqslant \|\lambda w_n\| \geqslant \Big(\sum_{i \in G_n} \|P_{\mathfrak{s}^i_n}(\lambda w_n)\|^2 \Big)^{1/2} \geqslant \lambda \Big(\sum_{i \in G_n} \theta^2 \Big)^{1/2} = \lambda \cdot \theta \cdot \sqrt{|G_n|}$$

which gives the desired estimate.

(ii) Fix $n \in \mathbb{N}$. Since $w_n \in W^0_{\mathfrak{X}}$, the vector w_n is of the form $w_n = \sum_{j=0}^{k_n} a^j_n x^j_n$ where $k_n \in \mathbb{N}$, $\sum_{j=0}^{k_n} a^j_n = 1$ with $a^j_n > 0$ and $x^j_n \in B_{\mathcal{X}_{\sigma^j_n}}$ for some $\sigma^j_n \in [T]$. For every $i \in \{0, \dots, d_n\}$ let $\beta^i_n = \|P_{\mathfrak{s}^i_n}(w_n)\|$. We claim, first, that $\sum_{i=0}^{d_n} \beta^i_n \leqslant 1$. Indeed, for every $i \in \{0, \dots, d_n\}$ let

$$H_i = \big\{ j \in \{0, \dots, k_n\} : \mathrm{supp}(x^j_n) \cap \mathfrak{s}^i_n \neq \emptyset \big\}.$$

Since $\mathrm{supp}(x^j_n)$ is a chain and the family $\{\mathfrak{s}^0_n, \dots, \mathfrak{s}^{d_n}_n\}$ consists of pairwise incomparable segments, we see that $H_{i_1} \cap H_{i_2} = \emptyset$ if $i_1 \neq i_2$. Moreover,

$$\beta^i_n = \|P_{\mathfrak{s}^i_n}(w_n)\| = \Big\| P_{\mathfrak{s}^i_n}\Big(\sum_{j=0}^{k_n} a^j_n x^j_n \Big) \Big\| = \Big\| P_{\mathfrak{s}^i_n}\Big(\sum_{j \in H_i} a^j_n x^j_n \Big) \Big\| \leqslant \sum_{j \in H_i} a^j_n$$

and so

$$\sum_{i=0}^{d_n} \beta^i_n \leqslant \sum_{i=0}^{d_n} \sum_{j \in H_i} a^j_n \leqslant \sum_{j=0}^{k_n} a^j_n = 1$$

which gives the desired estimate. By the definition of G_n, we have that if $i \notin G_n$, then $\beta_n^i < \theta$. It follows that

$$\begin{aligned}\Big(\sum_{i\notin G_n} \|P_{\mathfrak{s}_n^i}(\lambda w_n)\|^2\Big)^{1/2} &= \lambda\Big(\sum_{i\notin G_n}(\beta_n^i)^2\Big)^{1/2} < \lambda\Big(\sum_{i\notin G_n}\beta_n^i\cdot\theta\Big)^{1/2} \\ &= \lambda\cdot\sqrt{\theta}\Big(\sum_{i\notin G_n}\beta_n^i\Big)^{1/2} \leqslant \lambda\cdot\sqrt{\theta} = \frac{8}{100}\end{aligned}$$

by the choice of θ. This fact combined with equality (4.4) and the estimate $\|\lambda w_n\| \geqslant 99/100$ yields that

$$\Big(\sum_{i\in G_n} \|P_{\mathfrak{s}_n^i}(\lambda w_n)\|^2\Big)^{1/2} \geqslant \frac{91}{100}. \tag{4.6}$$

Using inequality (4.6) and the estimate $\|y_n - \lambda w_n\| \leqslant 1/100$, we conclude that

$$\Big(\sum_{i\in G_n} \|P_{s_n^i}(y_n)\|^2\Big)^{1/2} \geqslant \frac{9}{10}.$$

The claim is proved. □

By part (i) of Claim 4.10, the choice of θ and by passing to a subsequence of (y_n) if necessary, we may assume that $|G_n| = k$ for every $n \in \mathbb{N}$, where

$$k \leqslant \frac{4\cdot 100^4}{8^4}\cdot\lambda^2.$$

Of course, the exact numerical estimate of the size of the set G_n is not important. What is crucial is that the bound is *uniform.* For every $n \in \mathbb{N}$ let us re-enumerate the family $\{\mathfrak{s}_n^i : i \in G_n\}$ of incomparable segments of T, say as $\{\mathfrak{s}_n^0, \dots, \mathfrak{s}_n^{k-1}\}$.

Claim 4.11. *Let $i \in \{0, \dots, k-1\}$ and $M_i \in [\mathbb{N}]^\infty$. Then there exist $N_i \in [M_i]^\infty$ and for every $n \in N_i$ disjoint segments $\mathfrak{g}_n^i$ and $\mathfrak{b}_n^i$ of T such that the following are satisfied.*

(i) *For every $n \in N_i$ we have $\mathfrak{s}_n^i = \mathfrak{g}_n^i \cup \mathfrak{b}_n^i$ (that is, $\mathfrak{g}_n^i$ and $\mathfrak{b}_n^i$ form a partition of $\mathfrak{s}_n^i$). Moreover, for every $t \in \mathfrak{b}_n^i$ and every $s \in \mathfrak{g}_n^i$ we have $t \sqsubset s$.*

(ii) *For every $n \in N_i$ it holds that $\|P_{\mathfrak{b}_n^i}(y_n)\| < r$.*

(iii) *For every $n, m \in N_i$ with $n \neq m$ if both $\mathfrak{g}_n^i$ and $\mathfrak{g}_m^i$ are nonempty, then they are incomparable.*

Proof of Claim 4.11. For every $n \in M_i$ let t_n be the $\sqsubseteq$-minimum node of $\mathfrak{s}_n^i$. Notice that $t_n \neq t_m$ for every pair $n, m \in M_i$ with $n \neq m$. By Ramsey's theorem, there exists $I \in [M_i]^\infty$ such that one of the (mutually exclusive) possibilities must occur.

CASE 1: *The set* $\{t_n : n \in I\}$ *is an antichain.* In this case we set $N_i = I$ and for every $n \in N_i$ we define $\mathfrak{g}^i_n = \mathfrak{s}^i_n$ and $\mathfrak{b}^i_n = \emptyset$. It is easy to see that these choices satisfy the requirements of the claim.

CASE 2: *The set* $\{t_n : n \in I\}$ *is a chain.* Let $\tau \in [T]$ be the branch of T determined by the infinite chain $\{t_n : n \in I\}$. By our assumptions on the sequence (y_n), we see that

$$\limsup_{n\in I} \|P_\tau(y_n)\| \leqslant \limsup \|P_\tau(y_n)\| < r.$$

Hence, there exists $N_i \in [I]^\infty$ such that $\|P_\tau(y_n)\| < r$ for all $n \in N_i$. For every $n \in N_i$ we set $\mathfrak{g}^i_n = \mathfrak{s}^i_n \setminus \tau$ and $\mathfrak{b}^i_n = \mathfrak{s}^i_n \cap \tau$. It is also easy to check that these choices are as desired. The claim is proved. □

Applying Claim 4.11 successively, we obtain $N \in [\mathbb{N}]^\infty$ and for every $n \in N$ and every $i \in \{0, \dots, k-1\}$ disjoint segments $\mathfrak{g}^i_n$ and $\mathfrak{b}^i_n$ such that the following are satisfied.

(a) For every $n \in N$ and every $i \in \{0, \dots, k-1\}$ we have $\mathfrak{s}^i_n = \mathfrak{g}^i_n \cup \mathfrak{b}^i_n$ and, moreover, if $t \in \mathfrak{b}^i_n$ and $s \in \mathfrak{g}^i_n$, then $t \sqsubset s$.

(b) For every $n \in N$ and every $i \in \{0, \dots, k-1\}$ we have $\|P_{\mathfrak{b}^i_n}(y_n)\| < r$.

(c) For every $i \in \{0, \dots, k-1\}$ and every pair $n, m \in N$ with $n \neq m$ if the segments $\mathfrak{g}^i_n$ and $\mathfrak{g}^i_m$ are nonempty, then they are incomparable.

For every pair $i, j \in \{0, \dots, k-1\}$ let $C_{i,j}$ be the subset of $[N]^2$ consisting of all $(n, m) \in [N]^2$ for which the segments $\mathfrak{g}^i_n$ and $\mathfrak{g}^j_m$ are *nonempty and comparable.* Applying Ramsey's theorem successively, we find $L \in [N]^\infty$ such that for every $i, j \in \{0, \dots, k-1\}$ we have that either $[L]^2 \cap C_{i,j} = \emptyset$ or $[L]^2 \subseteq C_{i,j}$. We claim that for every pair i, j the first alternative must occur. Indeed, assume on the contrary that there exist $i, j \in \{0, \dots, k-1\}$ such that $[L]^2 \subseteq C_{i,j}$. Let $L = \{l_0 < l_1 < l_2 < \cdots\}$ be the increasing enumeration of L. Notice, first, that the segments $\mathfrak{g}^i_{l_0}$, $\mathfrak{g}^i_{l_1}$ and $\mathfrak{g}^j_{l_2}$ are nonempty. Moreover, both $\mathfrak{g}^i_{l_0}$ and $\mathfrak{g}^i_{l_1}$ are comparable with $\mathfrak{g}^j_{l_2}$. Let t_0, t_1 and t_2 be the $\sqsubseteq$-minimum nodes of $\mathfrak{g}^i_{l_0}$, $\mathfrak{g}^i_{l_1}$ and $\mathfrak{g}^j_{l_2}$ respectively. Since $l_0 < l_1 < l_2$ and the sequence (y_n) is block, we see that $t_0 \sqsubset t_2$ and $t_1 \sqsubset t_2$. But this implies that the segments $\mathfrak{g}^i_{l_0}$ and $\mathfrak{g}^i_{l_1}$ are comparable, in contradiction with property (c) above. Hence $[L]^2 \cap C_{i,j} = \emptyset$ for every $i, j \in \{0, \dots, k-1\}$.

For every $n \in L$ we set

$$A_n = \big\{t : t \in \mathfrak{g}^i_n \text{ for some } i \in \{0, \dots, k-1\}\big\}. \tag{4.7}$$

We claim that the set L and the family $\{A_n : n \in L\}$ satisfy the requirements of the proposition. Indeed, notice first that, by property (a) above and the remarks preceding (4.4), for every $n \in L$ we have $A_n \subseteq R_n = \{t_k : k \in \mathrm{range}(y_n)\}$. Moreover, the discussion in the previous paragraph implies that if $n, m \in L$ with $n \neq m$, then A_n is incomparable with A_m. What remains is to estimate the quantity $\|P_{A_n}(y_n)\|$ for every $n \in L$. To this end fix $n \in L$. By our hypotheses on r and the estimate on k, we have

$$r\sqrt{k} \leqslant \frac{1}{100^3 \cdot \lambda} \cdot \frac{2 \cdot 100^2 \cdot \lambda}{8^2} \leqslant \frac{1}{10}. \tag{4.8}$$

By property (a) above, we see that $\|P_{\mathfrak{s}_n^i}(y_n)\| \leqslant \|P_{\mathfrak{g}_n^i}(y_n)\| + \|P_{\mathfrak{b}_n^i}(y_n)\|$ for every $i \in \{0, \dots, k-1\}$. Hence, by property (b) and the estimate (4.8), we obtain

$$\begin{aligned} \Big(\sum_{i=0}^{k-1} \|P_{\mathfrak{s}_n^i}(y_n)\|^2\Big)^{1/2} &\leqslant \Big(\sum_{i=0}^{k-1} \|P_{\mathfrak{g}_n^i}(y_n)\|^2\Big)^{1/2} + r\sqrt{k} \\ &\leqslant \Big(\sum_{i=0}^{k-1} \|P_{\mathfrak{g}_n^i}(y_n)\|^2\Big)^{1/2} + \frac{1}{10}. \end{aligned}$$

So, by part (ii) of Claim 4.10, we conclude that

$$\|P_{A_n}(y_n)\| \geqslant \Big(\sum_{i=0}^{k-1} \|P_{\mathfrak{g}_n^i}(y_n)\|^2\Big)^{1/2} \geqslant \Big(\sum_{i=0}^{k-1} \|P_{\mathfrak{s}_n^i}(y_n)\|^2\Big)^{1/2} - \frac{1}{10} \geqslant \frac{2}{3}.$$

The proof of Proposition 4.9 is completed. □

4.3 Proof of Theorem 4.6

Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t \in T})$ be a Schauder tree basis and $1 < p < +\infty$. Let $J \colon A_p^{\mathfrak{X}} \to T_2^{\mathfrak{X}}$ be the natural inclusion operator. Fix a block subspace Z of $A_p^{\mathfrak{X}}$. We consider the following (mutually exclusive) cases.

CASE 1: *The operator $J \colon Z \to T_2^{\mathfrak{X}}$ is not an isomorphic embedding.* We will show that this case implies part (i) of the theorem; that is, the subspace Z must contain an isomorphic copy of ℓ_p. To see this notice first that the operator $J \colon Z \to T_2^{\mathfrak{X}}$ is one-to-one. Hence, for every finite co-dimensional subspace Z' of Z the operator $J \colon Z' \to T_2^{\mathfrak{X}}$ is not an isomorphic embedding. By Proposition B.5, there exists a block subspace Z'' of Z such that the operator $J \colon Z'' \to T_2^{\mathfrak{X}}$ is compact.

We claim that Z'' is hereditarily ℓ_p and therefore that Z contains an isomorphic copy of ℓ_p. This is a consequence of the following general fact.

Proposition 4.12. *Let $(X, \|\cdot\|)$ be a Banach space, let W be a closed, convex, bounded and symmetric subset of X and let $1 < p < +\infty$. Consider the p-interpolation space $\Delta_p(X, W)$ of the pair (X, W) and let $J \colon \Delta_p(X, W) \to X$*

be the natural inclusion map. Let Y be an infinite-dimensional subspace of $\Delta_p(X,W)$. Assume that the operator $J\colon Y \to X$ is strictly singular. Then Y is hereditarily ℓ_p.

Proof. Notice that it is enough to show that Y contains an isomorphic copy of ℓ_p. To this end, for every $n \in \mathbb{N}$ with $n \geqslant 1$ let $\|\cdot\|_n$ be the equivalent norm on X induced by the Minkowski gauge of the set $2^n W + 2^{-n} B_X$. Consider the space $E = \big(\sum_{n=1}^{\infty} \oplus (X, \|\cdot\|_n)\big)_{\ell_p}$. For every $k \in \mathbb{N}$ with $k \geqslant 1$ let P_k be the natural (bounded) projection from E onto the subspace $E_k = \big(\sum_{n=1}^{k} \oplus (X, \|\cdot\|_n)\big)_{\ell_p}$ of E. By the discussion in Appendix B.3, the interpolation space $\Delta_p(X,W)$ is identified with the "diagonal" subspace $\Delta = \{(x,x,\dots) \in E : x \in X\}$ of E. Moreover, our assumption that the operator $J\colon Y \to X$ is strictly singular reduces to the fact that for every $k \in \mathbb{N}$ with $k \geqslant 1$ the operator $P_k\colon Y \to E_k$ is strictly singular. Using these observations the result follows by a standard sliding hump argument. The proof is completed. □

Case 2: *The operator $J\colon Z \to T_2^{\mathfrak{X}}$ is an isomorphic embedding.* We set $Y = J(Z)$. Notice, first, that Y is a (closed) block subspace of $T_2^{\mathfrak{X}}$. Our main goal is to show that there exists a finite subset A of $[T]$ such that the operator $P_{T_A}\colon Y \to \mathcal{X}_{T_A}$ is an isomorphic embedding. This property easily yields that the operator $\bar{P}_{T_A}\colon Z \to \bar{\mathcal{X}}_{T_A}$ is also an isomorphic embedding; that is, part (ii) of the theorem is valid.

We will argue by contradiction. So, assume that for every finite subset A of $[T]$ the operator $P_{T_A}\colon Y \to \mathcal{X}_{T_A}$ is not an isomorphic embedding. In other words and according to Definition 3.4, Y is a weakly X-singular block subspace of $T_2^{\mathfrak{X}}$. Hence, the structural results obtained in Section 3.4 can be applied to the subspace Y. In particular, using Corollary 3.19, we may select, recursively, a normalized block sequence (y_n) in Y such that for every $n \in \mathbb{N}$ with $n \geqslant 1$ and every $\sigma \in [T]$ we have

$$\|P_\sigma(y_n)\| \leqslant \frac{1}{\sum_{i=0}^{n-1} |\mathrm{supp}(y_i)|^{1/2}} \cdot \frac{1}{2^{2n}}.$$

Such a selection guarantees the following.

(a) By Lemma 3.14, the sequence (y_n) satisfies an upper ℓ_2 estimate.

(b) For every $\sigma \in [T]$ we have $\|P_\sigma(y_n)\| \to 0$.

We wish to apply Proposition 4.9 to the sequence (y_n). The fact that it can indeed be applied is the content of the following claim.

Claim 4.13. *For every $\varepsilon > 0$ there exists a constant $\lambda = \lambda(\varepsilon) > 0$ such that $B_Y \subseteq \lambda W_{\mathfrak{X}} + \varepsilon B_{T_2^{\mathfrak{X}}}$.*

Proof of Claim 4.13. The desired property is essentially a consequence of the fact that the operator $J\colon Z \to Y$ is an isomorphism. Specifically, let $C \geqslant 1$ be such that

$$\|z\|_{A_p^{\mathfrak{X}}} \leqslant C \cdot \|J(z)\|_{T_2^{\mathfrak{X}}} \tag{4.9}$$

for every $z \in Z$. Fix $\varepsilon > 0$ and select $k \in \mathbb{N}$ with $k \geqslant 1$ and such that $C \cdot 2^{-k+1} < \varepsilon$. We set $\lambda = C \cdot 2^{k+1}$ and we claim that λ is as desired. Indeed, let $y \in B_Y$ be arbitrary and set $z = J^{-1}(y) \in Z$. By (4.9), we have

$$\Big(\sum_{n=1}^{\infty} \|y\|_n^p\Big)^{1/p} = \|z\|_{A_p^{\mathfrak{X}}} \leqslant C.$$

Hence $\|y\|_k \leqslant C < 2C$ and, therefore,

$$y \in (C \cdot 2^{k+1}) W_{\mathfrak{X}} + (C \cdot 2^{-k+1}) B_{T_2^{\mathfrak{X}}} \subseteq \lambda W_{\mathfrak{X}} + \varepsilon B_{T_2^{\mathfrak{X}}}.$$

The claim is proved. □

By Claim 4.13, we see that $\{y_n : n \in \mathbb{N}\} \subseteq \lambda W_{\mathfrak{X}} + 200^{-1} B_{T_2^{\mathfrak{X}}}$ for some $\lambda > 0$. Applying Proposition 4.9 and passing to a subsequence of (y_n) if necessary, we see that there exists a sequence (A_n) of mutually incomparable, segment complete subsets of T such that

(c) for every $n \in \mathbb{N}$ we have $A_n \subseteq \{t_k : k \in \text{range}(y_n)\}$ and $\|P_{A_n}(y_n)\| \geqslant 2/3$.

By Fact 4.8 and properties (a) and (c) above, we see, in particular, that

(d) there exists a constant $\delta \geqslant 1$ such that the sequence (y_n) is δ-equivalent to the standard unit vector basis of ℓ_2.

For every $t \in T$ let $e_t^* \in (T_2^{\mathfrak{X}})^*$ be the bi-orthogonal functional of e_t. We select a sequence (y_n^*) in $(T_2^{\mathfrak{X}})^*$ such that for every $n \in \mathbb{N}$ we have

(e) $\|y_n^*\| = 1$, $y_n^* \in \text{span}\{e_t^* : t \in A_n\}$ and $y_n^*(y_n) \geqslant 2/3$.

The above choice yields that for every $x \in T_2^{\mathfrak{X}}$ we have

$$\Big(\sum_{n\in\mathbb{N}} y_n^*(x)^2\Big)^{1/2} \leqslant \|x\|. \tag{4.10}$$

Indeed, observe that $|y_n^*(x)| \leqslant \|P_{A_n}(x)\|$ for every $n \in \mathbb{N}$. Since the sequence (A_n) consists of pairwise incomparable segment complete subsets of T, inequality (4.10) follows.

We set $E = \overline{\text{span}}\{y_n : n \in \mathbb{N}\}$. We define $P\colon T_2^{\mathfrak{X}} \to E$ by

$$P(x) = \sum_{n\in\mathbb{N}} \frac{y_n^*(x)}{y_n^*(y_n)}\, y_n. \tag{4.11}$$

We isolate below two basic properties of the operator P.

Claim 4.14. *The operator P is a well-defined bounded projection. Moreover, if $\delta \geqslant 1$ is as in* (d) *above, then $\|P\| \leqslant 2\delta$.*

Proof of Claim 4.14. Clearly P is a projection. We shall estimate the norm of P. To this end, fix a finitely supported vector x in $T_2^{\mathfrak{X}}$. The sequence (y_n) is δ-equivalent to the standard ℓ_2 basis, and so

$$\|P(x)\| \leqslant \delta \cdot \Big(\sum_{n\in\mathbb{N}} y_n^*(x)^2 \cdot y_n^*(y_n)^{-2}\Big)^{1/2} \overset{\text{(e)}}{\leqslant} 2\delta \cdot \Big(\sum_{n\in\mathbb{N}} y_n^*(x)^2\Big)^{1/2}.$$

By (4.10), we conclude that $\|P\| \leqslant 2\delta$. The claim is proved. □

Claim 4.15. *We have $P(W_{\mathfrak{X}}) \subseteq \overline{\mathrm{conv}}\{\pm 2y_n : n \in \mathbb{N}\}$.*

Proof of Claim 4.15. Fix $w \in W_{\mathfrak{X}}^0$. By definition, the vector w is of the form $w = \sum_{i=0}^k a_i x_i$ where $k \in \mathbb{N}$, $\sum_{i=0}^k a_i = 1$ with $a_i > 0$ and $x_i \in B_{\mathcal{X}_{\sigma_i}}$ for some $\sigma_i \in [T]$. For every $i \in \{0, \dots, k\}$ let $\mathfrak{s}_i$ be the minimal segment of T that contains $\mathrm{supp}(x_i)$ (this segment exists because the support of x_i is a chain). For every $n \in \mathbb{N}$ we set $F_n = \big\{i \in \{0, \dots, k\} : \mathfrak{s}_i \cap A_n \neq \emptyset\big\}$. Since the sequence (A_n) consists of pairwise incomparable sets of nodes, we see that $F_n \cap F_m = \emptyset$ if $n \neq m$. Moreover,

$$|y_n^*(w)| \leqslant \sum_{i\in F_n} a_i \cdot |y_n^*(x_i)| \leqslant \sum_{i\in F_n} a_i.$$

Hence, setting $\theta_n = |y_n^*(w) \cdot y_n^*(y_n)^{-1}|$ for every $n \in \mathbb{N}$ and invoking property (e), we obtain that

$$\sum_{n\in\mathbb{N}} \theta_n \overset{\text{(e)}}{\leqslant} 2 \cdot \sum_{n\in\mathbb{N}} |y_n^*(w)| \leqslant 2 \cdot \sum_{n\in\mathbb{N}} \sum_{i\in F_n} a_i \leqslant 2 \cdot \sum_{i=0}^{k} a_i = 2.$$

The above inequality clearly implies that $P(w) \in \mathrm{conv}\{\pm 2y_n : n \in \mathbb{N}\}$. The claim is proved. □

We are ready for the last step of the argument. To this end, we need the following lemma.

Lemma 4.16. [N] *Let E be a Banach space and let W be a convex subset of E. Let $\lambda > 0$ and $0 < \varepsilon < 1$, and assume that $B_E \subseteq \lambda W + \varepsilon B_E$. Then $B_E \subseteq \frac{\lambda}{1-\varepsilon}\overline{W}$.*

Proof. Fix $e \in E$. By our assumptions, there exist $w_0 \in W$ and $z_0 \in B_E$ such that $e = \lambda w_0 + \varepsilon z_0$. We set $e_1 = e - \lambda w_0$. Then $\|e_1\| \leqslant \varepsilon\|z_0\| \leqslant \varepsilon$. Hence, there exist $w_1 \in W$ and $z_1 \in B_E$ such that $e_1 = \lambda\varepsilon w_1 + \varepsilon^2 z_1$. We set $e_2 = e_1 - \lambda\varepsilon w_1 = e - (\lambda w_0 + \lambda\varepsilon w_1)$ and we notice that $\|e_2\| \leqslant \varepsilon^2\|z_1\| \leqslant \varepsilon^2$. Continuing recursively, we construct a sequence (w_n) in W and a sequence

(e_n) in E such that for every $k \in \mathbb{N}$ with $k \geqslant 1$ we have $\|e_k\| \leqslant \varepsilon^k$ and $e_k = e - \big(\lambda \sum_{n=0}^{k-1} \varepsilon^n w_n\big)$. These properties and the convexity of W yield that

$$w = (1-\varepsilon) \sum_{n=0}^{\infty} \varepsilon^n w_n \in \overline{W}.$$

Moreover, $e = \big(\lambda \cdot (1-\varepsilon)^{-1}\big) w$. The proof is completed. □

By Claim 4.13, there exists $\lambda > 0$ such that $B_Y \subseteq \lambda W_{\mathfrak{X}} + (4\delta)^{-1} B_{T_2^{\mathfrak{X}}}$. Notice that E is a subspace of Y, and so, $B_E \subseteq B_Y$. Hence

$$B_E \subseteq \lambda W_{\mathfrak{X}} + \frac{1}{4\delta} B_{T_2^{\mathfrak{X}}}.$$

Applying P in the above inclusion and taking into account that P is a projection on E with $\|P\| \leqslant 2\delta$, we see that

$$B_E \subseteq \lambda P(W_{\mathfrak{X}}) + \frac{1}{2} B_E.$$

Clearly $\lambda P(W_{\mathfrak{X}})$ is a convex subset of E. Therefore, by Lemma 4.16, we obtain that

$$B_E \subseteq 2\lambda \overline{P(W_{\mathfrak{X}})}.$$

Invoking Claim 4.15, we conclude that

$$B_E \subseteq 2\lambda \overline{\mathrm{conv}}\{\pm 2y_n : n \in \mathbb{N}\}. \tag{4.12}$$

We have reached the contradiction. Indeed, by (d) above, the sequence (y_n) is equivalent to the standard unit vector basis of ℓ_2. On the other hand, inclusion (4.12) implies that the basic sequence (y_n) is equivalent to the standard unit vector basis of ℓ_1. This is clearly a contradiction. The proof of Theorem 4.6 is completed.

4.4 Comments and Remarks

1. We stress the fact that Theorem 4.6 is valid for an *arbitrary* subspace of $A_p^{\mathfrak{X}}$.

Theorem 4.17. **[AD]** *Let $\mathfrak{X}$ be a Schauder tree basis and $1 < p < +\infty$. Let Z be a subspace of $A_p^{\mathfrak{X}}$. Then either*

(i) *Z contains an isomorphic copy of ℓ_p, or*

(ii) *there exists finite $A \subseteq [T]$ such that the operator $\bar{P}_{T_A} \colon Z \to \bar{\mathcal{X}}_{T_A}$ is an isomorphic embedding.*

The proof of Theorem 4.17 follows exactly the same steps of the proof of Theorem 4.6. The only difference is that one has to replace the standard notion of a block sequence with the more general, but less conventional, notion of a "pointwise null sequence" (see [AD, Definition 31]). Actually, once the basic arguments of the proof of Theorem 4.6 have been understood, then one easily realizes that it takes a small step to go from block subspaces to arbitrary ones.

Concerning the proof of Theorem 4.6 we point out that it follows some of the arguments in the work of Argyros and Felouzis [AF]. There is, however, a notable exception: the proof of Proposition 4.9. The corresponding result in [AF] is based on a probabilistic argument (see also [M] for an exposition). This argument is not suitable for the proof of Proposition 4.9 due, mainly, to the fact that we have to deal with infinitely splitting trees. The proof we presented, taken from [AD], is of combinatorial nature and relies heavily on the description of the norm of $T_2^{\mathfrak{X}}$.

2. We notice that there are many variants of the p-amalgamation space. For instance, let U be a Banach space with an unconditional basis (u_n) and let $\mathfrak{X}$ be a Schauder tree basis. The *U-amalgamation space* $A_U^{\mathfrak{X}}$ of $\mathfrak{X}$ is the interpolation space of the pair $(T_2^{\mathfrak{X}}, W_{\mathfrak{X}})$ which is obtained using the norm of the space U in (B.1) instead of the ℓ_p norm. It is straightforward to adapt the machinery developed in this chapter in this setting. In particular, Theorem 4.17 has the following reformulation.

Theorem 4.18. **[AD]** *Let $\mathfrak{X}$ be a Schauder tree basis and let U be a Banach space with an unconditional basis. Let Z be a subspace of $A_U^{\mathfrak{X}}$. Then either*

(i) *Z contains a subspace isomorphic to a subspace of U, or*

(ii) *there exists finite $A \subseteq [T]$ such that the operator $\bar{P}_{T_A}\colon Z \to \bar{\mathcal{X}}_{T_A}$ is an isomorphic embedding.*

More involved is the construction of the *HI amalgamation space* $A_{\mathrm{HI}}^{\mathfrak{X}}$ of a Schauder tree basis $\mathfrak{X}$, a variant also introduced in [AD]. It is obtained using the method of "HI Schauder sums" developed by Argyros and Felouzis [AF]. This method allows one to use an HI norm as an external norm in (B.1) yielding the following result.

Theorem 4.19. **[AD]** *Let $\mathfrak{X}$ be a Schauder tree basis. Let Z be a subspace of $A_{\mathrm{HI}}^{\mathfrak{X}}$. Then either*

(i) *Z contains an HI subspace, or*

(ii) *there exists finite $A \subseteq [T]$ such that the operator $\bar{P}_{T_A}\colon Z \to \bar{\mathcal{X}}_{T_A}$ is an isomorphic embedding.*

Chapter 5

Zippin's embedding theorem

A deep result of Per Enflo [E] asserts that there exists a separable Banach space without a Schauder basis. On the other hand, by Theorem 1.8, every separable Banach space embeds into a Banach space with a Schauder basis. An old problem in Banach space theory (see [LT, Problem 1.b.16]) asked whether every space X with separable dual is isomorphic to a subspace of a space Y with a shrinking Schauder basis.

More generally, for every separable Banach space X with a certain property, say property (P), one is looking for a space Y with a Schauder basis and with property (P) and such that Y contains an isomorphic copy of X. This "embedding problem" is one of the central problems in Banach space theory and, by now, there are several strong results obtained in this direction.

This chapter is devoted to the proof of the following theorem of Zippin (as well as to a parameterized version of it) which answers affirmatively this "embedding problem" for two of the most important properties of Banach spaces.

Theorem 5.1. **[Z]** *The following hold.*

(i) *Every Banach space X with separable dual embeds into a space Y with a shrinking Schauder basis.*

(ii) *Every separable reflexive space X embeds into a reflexive space Y with a Schauder basis.*

Our presentation is based on an alternative proof of Theorem 5.1 due to Ghoussoub, Maurey and Schachermayer [GMS]. The approach of [GMS] is based on a selection result which has other applications beside its use in the proof of Theorem 5.1. The parameterized version of Theorem 5.1, due to Bossard [Bos2], asserts that the embedding of X into Y can be done "uniformly" in X.

5.1 Fragmentation, slicing and selection

Throughout this section by $E = (E, \tau)$ we denote a fixed compact metrizable space. We also fix a countable basis (V_m) of the topology τ of E consisting of nonempty open sets.

5.1.1 Fragmentation

A *fragmentation* $\Delta\colon E \times E \to \mathbb{R}$ *on* E is a metric on E such that for every nonempty closed subset K of E and every $\varepsilon > 0$ there exists an open subset V of E with $K \cap V \neq \emptyset$ and such that $\Delta - \mathrm{diam}(K \cap V) \leqslant \varepsilon$ where, as usual, we set $\Delta - \mathrm{diam}(A) = \sup\{\Delta(x, y) : x, y \in A\}$ for every $A \subseteq E$.

Notice that if Δ is a fragmentation on E, then the metric space (E, Δ) must be separable. For if not, then we would be able to find $\varepsilon > 0$ and a family $\Gamma = \{x_\xi : \xi < \omega_1\} \subseteq E$ such that $\Delta(x_\xi, x_\zeta) \geqslant \varepsilon$ for every $\xi \neq \zeta$. Refining if necessary, we may assume that for every $\xi < \omega_1$ the point x_ξ is a condensation point of Γ in (E, τ). Setting K to be the closure of Γ in (E, τ), we see that for every relatively open subset V of K there exist $x, y \in V$ with $\Delta(x, y) \geqslant \varepsilon$, contradicting the fact that Δ is a fragmentation. Using this observation and Baire's classical characterization of Baire-1 functions we obtain the following proposition.

Proposition 5.2. *A metric Δ on E is a fragmentation if and only if the identity map* $\mathrm{Id}\colon (E, \tau) \to (E, \Delta)$ *is Baire-1.*

Proof. A classical result in the theory of Borel functions asserts that if X is a Polish space, Y is a metrizable space (not necessarily separable) and $f\colon X \to Y$ is Borel, then $f(X)$ is separable (see [Sr, Theorem 4.3.8]). Hence, if Id is Baire-1, then the metric space (E, Δ) is separable. By Baire's classical characterization of Baire-1 functions (see [Ke, Theorem 24.15]), we see that for every $K \in K(E)$ the map $\mathrm{Id}|_K$ has a point of continuity. This implies that Δ is a fragmentation.

Conversely, let Δ be a fragmentation and $K \in K(E)$. By our assumptions, we see that for every $\varepsilon > 0$ the set $O_\varepsilon = \big\{x \in K : \mathrm{osc}(\mathrm{Id}|_K)(x) < \varepsilon\big\}$ contains a dense open subset of K, where by $\mathrm{osc}(\mathrm{Id}|_K)$ we denote the oscillation map of the function $\mathrm{Id}|_K$. It follows that $\mathrm{Id}|_K$ has a point of continuity. Invoking Baire's characterization again, we conclude that $\mathrm{Id}\colon (E, \tau) \to (E, \Delta)$ is Baire-1. The proof is completed. □

As a consequence we have the following corollary.

Corollary 5.3. *Let Δ be a fragmentation on E. Then the map* $\Delta\colon (E, \tau) \times (E, \tau) \to \mathbb{R}$ *is Baire-1.*

Proof. The metric space (E, Δ) is separable. Hence, by Proposition 5.2, we see that the identity map $\mathrm{Id}\colon (E,\tau)\times(E,\tau)\to(E,\Delta)\times(E,\Delta)$ is Baire-1. The proof is completed. $\square$

We should point out that the converse implication in Corollary 5.3 is not valid. That is, if Δ is a metric on E such that the map $\Delta\colon (E,\tau)\times(E,\tau)\to\mathbb{R}$ is Baire-1, then Δ is not necessarily a fragmentation (consider, for instance, the discrete metric on E).

5.1.2 Slicing associated to a fragmentation

Let Δ be a fragmentation on E. For every $\varepsilon>0$ we define the *slicing function* $f_{\Delta,\varepsilon}\colon K(E)\to K(E)$ associated to Δ and ε as follows. We set $f_{\Delta,\varepsilon}(\emptyset)=\emptyset$. For every nonempty $K\in K(E)$ let

$$m_K=\min\{m\in\mathbb{N}: K\cap V_m\neq\emptyset \text{ and } \Delta-\mathrm{diam}(K\cap V_m)\leqslant\varepsilon\}.$$

Notice that m_K is well-defined since Δ is a fragmentation and (V_m) is a basis of E. We set

$$f_{\Delta,\varepsilon}(K)=K\setminus V_{m_K}.$$

That is, $f_{\Delta,\varepsilon}(K)$ results by removing from K the *first* nonempty relatively basic open subset of K with Δ-diameter less than or equal to ε. Clearly $f_{\Delta,\varepsilon}(K)$ is closed and $f_{\Delta,\varepsilon}(K)\subseteq K$ for every $K\in K(E)$ and every $\varepsilon>0$.

The *slicing* of E associated to Δ and $\varepsilon>0$ is a decreasing transfinite sequence $(E^{\xi}_{\Delta,\varepsilon}:\xi<\omega_1)$ of closed subsets of E defined recursively by

$$E^0_{\Delta,\varepsilon}=E,\ E^{\xi+1}_{\Delta,\varepsilon}=f_{\Delta,\varepsilon}(E^{\xi}_{\Delta,\varepsilon}) \text{ and } E^{\lambda}_{\Delta,\varepsilon}=\bigcap_{\xi<\lambda}E^{\xi}_{\Delta,\varepsilon} \text{ if } \lambda \text{ is limit.}$$

There exists a countable ordinal ζ such that $E^{\zeta}_{\Delta,\varepsilon}=\emptyset$. The *index* $\mathrm{Ind}(\Delta,\varepsilon,E)$ of this slicing is defined to be the least ordinal $\xi<\omega_1$ such that $E^{\xi+1}_{\Delta,\varepsilon}=\emptyset$. Notice that if $\mathrm{Ind}(\Delta,\varepsilon,E)=\xi$, then $E^{\xi}_{\Delta,\varepsilon}\neq\emptyset$. We also observe that for every $\xi\leqslant\mathrm{Ind}(\Delta,\varepsilon,E)$ the set $E^{\xi}_{\Delta,\varepsilon}\setminus E^{\xi+1}_{\Delta,\varepsilon}$ has Δ-diameter less than or equal to ε.

5.1.3 Derivative associated to a fragmentation

As in Section 5.1.2, let Δ be a fragmentation of E. For every $\varepsilon>0$ we define the *derivative* $D_{\Delta,\varepsilon}\colon K(E)\to K(E)$ associated to Δ and ε by

$$\begin{aligned} D_{\Delta,\varepsilon}(K) &= K\setminus\bigcup\{V_m: K\cap V_m\neq\emptyset \text{ and } \Delta-\mathrm{diam}(K\cap V_m)\leqslant\varepsilon\}\\ &= K\setminus\bigcup\{V\subseteq E: V \text{ is open and } \Delta-\mathrm{diam}(K\cap V)\leqslant\varepsilon\}.\end{aligned}$$

That is, $D_{\Delta,\varepsilon}(K)$ results by removing from K *all* relatively basic open subsets of K with Δ-diameter less than or equal to ε. Observe that $D_{\Delta,\varepsilon}$ is a derivative on $K(E)$ according to the terminology in Appendix A.

Notice, however, that the slicing function $f_{\Delta,\varepsilon}$ is *not* a derivative on $K(E)$. Actually, $f_{\Delta,\varepsilon}$ is just a version of $D_{\Delta,\varepsilon}$ with delay. Precisely, for every $K \in K(E)$ let $(K^{\xi}_{\Delta,\varepsilon} : \xi < \omega_1)$ be the *slicing* of K according to Δ and $\varepsilon > 0$ defined recursively by

$$K^0_{\Delta,\varepsilon} = K,\ K^{\xi+1}_{\Delta,\varepsilon} = f_{\Delta,\varepsilon}(K^{\xi}_{\Delta,\varepsilon}) \text{ and } K^{\lambda}_{\Delta,\varepsilon} = \bigcap_{\xi<\lambda} K^{\xi}_{\Delta,\varepsilon} \text{ if } \lambda \text{ is limit.}$$

We have the following proposition.

Proposition 5.4. *Let Δ be a fragmentation on E and $\varepsilon > 0$. Then for every $K \in K(E)$ we have $K^{\omega}_{\Delta,\varepsilon} \subseteq D_{\Delta,\varepsilon}(K)$. Moreover,*

$$\mathrm{Ind}(\Delta, \varepsilon, E) \leqslant \omega \cdot |E|_{D_{\Delta,\varepsilon}}.$$

Proof. Fix $\varepsilon > 0$ and $K \in K(E)$ nonempty. For notational simplicity for every $n < \omega$ we set $K_n = K^n_{\Delta,\varepsilon}$. Let $m_n \in \mathbb{N}$ be defined by

$$m_n = \min\{m \in \mathbb{N} : K_n \cap V_m \neq \emptyset \text{ and } \Delta - \mathrm{diam}(K_n \cap V_m) \leqslant \varepsilon\}.$$

Observe that for every $n \in \mathbb{N}$ we have $K_{n+1} \stackrel{\text{def}}{=} K_n \setminus V_{m_n} = K \setminus (V_{m_0} \cup \cdots \cup V_{m_n})$. Hence $m_n \neq m_l$ if $n \neq l$.

Let $k \in \mathbb{N}$ be such that $K \cap V_k \neq \emptyset$ and $\Delta - \mathrm{diam}(K \cap V_k) \leqslant \varepsilon$. We claim that there exists $n \in \mathbb{N}$ with $K_n \cap V_k = \emptyset$. Assume not. The sequence (K_n) is decreasing. It follows that for every $n \in \mathbb{N}$ we have

$$k \in \{m \in \mathbb{N} : K_n \cap V_m \neq \emptyset \text{ and } \Delta - \mathrm{diam}(K_n \cap V_m) \leqslant \varepsilon\},$$

and so, $m_n \leqslant k$. But this is clearly impossible and our claim is proved. The above discussion implies that

$$K^{\omega}_{\Delta,\varepsilon}(K) \stackrel{\text{def}}{=} \bigcap_n K_n \subseteq D_{\Delta,\varepsilon}(K)$$

as desired.

Now, taking into account the fact that $K^{\omega}_{\Delta,\varepsilon} \subseteq D_{\Delta,\varepsilon}(K)$ for every $K \in K(E)$, we will show that for every countable ordinal ξ it holds that

$$E^{\omega\cdot\xi}_{\Delta,\varepsilon} \subseteq D^{\xi}_{\Delta,\varepsilon}(E). \tag{5.1}$$

We proceed by transfinite induction. For $\xi = 0$ the above inclusion is trivially valid. If $\xi = \zeta + 1$ is a successor ordinal, then using the fact that $D_{\Delta,\varepsilon}$ is a derivative and our inductive hypothesis we see that

$$E^{\omega\cdot(\zeta+1)}_{\Delta,\varepsilon} = \big(E^{\omega\cdot\zeta}_{\Delta,\varepsilon}\big)^{\omega}_{\Delta,\varepsilon} \subseteq D_{\Delta,\varepsilon}(E^{\omega\cdot\zeta}_{\Delta,\varepsilon}) \subseteq D_{\Delta,\varepsilon}\big(D^{\zeta}_{\Delta,\varepsilon}(E)\big) = D^{\zeta+1}_{\Delta,\varepsilon}(E).$$

If ξ is a limit ordinal, then

$$E^{\omega\cdot\xi}_{\Delta,\varepsilon} = \bigcap_{\zeta<\xi} E^{\omega\cdot\zeta}_{\Delta,\varepsilon} \subseteq \bigcap_{\zeta<\xi} D^{\zeta}_{\Delta,\varepsilon}(E) = D^{\xi}_{\Delta,\varepsilon}(E).$$

Finally, observe that inclusion (5.1) yields the desired estimate for $\mathrm{Ind}(\Delta,\varepsilon,E)$. Indeed, set $\eta = |E|_{D_{\Delta,\varepsilon}}$ and notice that $D^{\eta}_{\Delta,\varepsilon}(E)=\emptyset$. Therefore, by (5.1), we obtain that $E^{\omega\cdot\eta}_{\Delta,\varepsilon}=\emptyset$ and so $\mathrm{Ind}(\Delta,\varepsilon,E)\leqslant \omega\cdot\eta = \omega\cdot |E|_{D_{\Delta,\varepsilon}}$. The proof is completed. □

5.1.4 The "last bite" of a slicing

Let Δ be a fragmentation on E and $\varepsilon>0$. Consider the slicing $(E^{\xi}_{\Delta,\varepsilon}:\xi<\omega_1)$ associated to Δ and ε defined in Section 5.1.2. For every nonempty $K\in K(E)$ there is an ordinal $\eta\leqslant \mathrm{Ind}(\Delta,\varepsilon,E)$ such that $K\cap E^{\eta}_{\Delta,\varepsilon}\neq\emptyset$ and $K\cap E^{\eta+1}_{\Delta,\varepsilon}=\emptyset$. We set

$$\mathrm{Ind}(\Delta,\varepsilon,K)=\eta \ \text{ and } \ L_{\Delta,\varepsilon}(K)=K\cap E^{\eta}_{\Delta,\varepsilon}$$

and we call η the *index* of K with respect to the slicing $(E^{\xi}_{\Delta,\varepsilon}:\xi<\omega_1)$. Also let $L_{\Delta,\varepsilon}(\emptyset)=\emptyset$. The map $L_{\Delta,\varepsilon}\colon K(E)\to K(E)$ is called the "*last bite*" of the slicing. Let us summarize the basic properties of the "last bite" map.

Proposition 5.5. **[GMS]** *Let Δ be a fragmentation on E and $\varepsilon>0$. Then the following are satisfied.*

- **(i)** *For every $K\in K(E)$ we have $\Delta-\mathrm{diam}\big(L_{\Delta,\varepsilon}(K)\big)\leqslant\varepsilon$.*
- **(ii)** *If $K\subseteq C$ are in $K(E)$ and nonempty, then* $\mathrm{Ind}(\Delta,\varepsilon,K)\leqslant\mathrm{Ind}(\Delta,\varepsilon,C)$.
- **(iii)** *If $K\subseteq C$ are in $K(E)$ and $x\in K\cap L_{\Delta,\varepsilon}(C)$, then* $\mathrm{Ind}(\Delta,\varepsilon,K)=\mathrm{Ind}(\Delta,\varepsilon,C)$ *and $x\in L_{\Delta,\varepsilon}(K)\subseteq L_{\Delta,\varepsilon}(C)$.*
- **(iv)** *If (K_n) is a decreasing sequence in $K(E)$ and $K=\bigcap_n K_n$, then the sequence of ordinals $\big(\mathrm{Ind}(\Delta,\varepsilon,K_n)\big)$ is eventually constant. It follows, in particular, that there exists $n_0\in\mathbb{N}$ such that $L_{\Delta,\varepsilon}(K)=\bigcap_{n\geqslant n_0}L_{\Delta,\varepsilon}(K_n)$.*

Proof. Parts (i), (ii) and (iii) are straightforward consequences of the relevant definitions. For part (iv) notice that $\cdots\leqslant\mathrm{Ind}(\Delta,\varepsilon,K_1)\leqslant\mathrm{Ind}(\Delta,\varepsilon,K_0)$ since the sequence (K_n) is decreasing. Hence, the sequence $\big(\mathrm{Ind}(\Delta,\varepsilon,K_n)\big)$ must be eventually constant. The proof is completed. □

5.1.5 The "dessert" selection of a fragmentation

Let Δ be a fragmentation on E. For every $n\in\mathbb{N}$ let $L_n\colon K(E)\to K(E)$ be the "last bite" map associated to Δ and $\varepsilon=2^{-n}$ as it was defined in Section 5.1.4. We define, recursively, a sequence of maps $S_n\colon K(E)\to K(E)$ $(n\in\mathbb{N})$ by

$$S_0(K)=K \text{ and } S_{n+1}(K)=L_{n+1}\big(S_n(K)\big).$$

By part (i) of Proposition 5.5, for every nonempty $K \in K(E)$ and every $n \in \mathbb{N}$ with $n \geqslant 1$ the set $S_n(K)$ is nonempty and has Δ-diameter less than or equal to 2^{-n}. Moreover, the sequence $\big(S_n(K)\big)$ is decreasing. If follows that $\bigcap_n S_n(K)$ is a singleton. The "*dessert*" *selection* associated to Δ is the map $s_\Delta\colon K(E) \to E$ defined by

$$s_\Delta(K) = \bigcap_n S_n(K).$$

The basic properties of the "dessert" selection are summarized in the following theorem due to Ghoussoub, Maurey and Schachermayer.

Theorem 5.6. **[GMS]** *Let E be a compact metrizable space and let Δ be a fragmentation on E. Then the "dessert" selection $s_\Delta\colon K(E) \to E$ satisfies the following.*

(i) *For every nonempty $K \in K(E)$ we have $s_\Delta(K) \in K$.*

(ii) *If $K \subseteq C$ are in $K(E)$ and $s_\Delta(C) \in K$, then $s_\Delta(K) = s_\Delta(C)$.*

(iii) *If (K_m) is a decreasing sequence in $K(E)$ and $K = \bigcap_m K_m$, then*

$$\lim_m \Delta\big(s_\Delta(K_m), s_\Delta(K)\big) = 0.$$

Proof. Part (i) is clear. For part (ii) notice first that $s_\Delta(C) \in K \cap L_1(C)$. By part (iii) of Proposition 5.5, we have $s_\Delta(C) \in L_1(K) \subseteq L_1(C)$. Hence, according to our notation, $s_\Delta(C) \in S_1(K) \subseteq S_1(C)$. Again we see that $s_\Delta(C) \in S_1(K) \cap L_2\big(S_1(C)\big)$. Continuing inductively, we obtain that $s_\Delta(C) \in S_n(K) \subseteq S_n(C)$ for every $n \in \mathbb{N}$. It follows that $s_\Delta(C) = s_\Delta(K)$ as desired.

For part (iii) we argue as follows. Inductively and using part (iv) of Proposition 5.5, we select a sequence (m_n) in $\mathbb{N}$ and a sequence (ξ_n) of countable ordinals such that for every $n \in \mathbb{N}$ with $n \geqslant 1$ the following hold.

(a) For every $m \in \mathbb{N}$ with $m \geqslant m_n$ we have $\mathrm{Ind}(\Delta, 2^{-n}, K_m) = \xi_n$.

(b) The sequence $\{S_n(K_m) : m \geqslant m_n\}$ is decreasing.

(c) We have $\bigcap_{m \geqslant m_n} S_n(K_m) = S_n(K)$.

It follows that for every $m \geqslant m_n$ the points $s_\Delta(K_m)$ and $s_\Delta(K)$ belong to the set $E^{\xi_n}_{\Delta,2^{-n}} \setminus E^{\xi_n+1}_{\Delta,2^{-n}}$ which has (by definition) Δ-diameter less than or equal to 2^{-n}. The proof is completed. □

5.2 Parameterized fragmentation

As in the previous section, in what follows let $E = (E, \tau)$ denote a compact metrizable space and let (V_m) be a countable basis of the topology on E consisting of nonempty sets. We wish to parameterize the construction presented in Section 5.1. To this end we give the following definition.

Definition 5.7. *Let Z be a standard Borel space. A* parameterized Borel fragmentation on E *is a map $\mathcal{D}\colon Z \times E \times E \to \mathbb{R}$ such that, setting $\mathcal{D}_z(x,y) = \mathcal{D}(z,x,y)$ for every $z \in Z$ and every $x, y \in E$, the following are satisfied.*

(1) *For every $z \in Z$ the map $\mathcal{D}_z\colon E \times E \to \mathbb{R}$ is a fragmentation on E.*

(2) *The map $\mathcal{D}$ is Borel.*

Let $\mathcal{D}$ be a parameterized Borel fragmentation on E. By condition (1) in Definition 5.7, for every $z \in Z$ the map $\mathcal{D}_z$ is a fragmentation on E. Hence, fixing $z \in Z$ we can define for every $\varepsilon > 0$ the slicing map $f_{\mathcal{D}_z,\varepsilon}$, the derivative $D_{\mathcal{D}_z,\varepsilon}$ and the "last bite" map $L_{\mathcal{D}_z,\varepsilon}$ associated to $\mathcal{D}_z$ and ε. The parameterized versions of these maps are the *parameterized slicing* map $f_{\mathcal{D},\varepsilon}\colon Z \times K(E) \to K(E)$, the *parameterized derivative* $D_{\mathcal{D},\varepsilon}\colon Z \times K(E) \to K(E)$ and the *parameterized "last bite"* map $L_{\mathcal{D},\varepsilon}\colon Z \to K(E) \to K(E)$ associated to $\mathcal{D}$ and $\varepsilon > 0$, defined by $f_{\mathcal{D},\varepsilon}(z,K) = f_{\mathcal{D}_z,\varepsilon}(K)$, $D_{\mathcal{D},\varepsilon}(z,K) = D_{\mathcal{D}_z,\varepsilon}(K)$ and $L_{\mathcal{D},\varepsilon}(z,K) = L_{\mathcal{D}_z,\varepsilon}(K)$ respectively. Finally, the *parameterized "dessert" selection* associated to $\mathcal{D}$ is the map $s_{\mathcal{D}}\colon Z \times K(E) \to E$ defined by $s_{\mathcal{D}}(z,K) = s_{\mathcal{D}_z}(K)$, where $s_{\mathcal{D}_z}$ is the "dessert" selection associated to the fragmentation $\mathcal{D}_z$. The main result in this section is the following theorem.

Theorem 5.8. *Let E be a compact metrizable space and let Z be a standard Borel space. Let $\mathcal{D}\colon Z \times E \times E \to E$ be a parameterized Borel fragmentation on E. Then the parameterized "dessert" selection $s_{\mathcal{D}}\colon Z \times K(E) \to E$ associated to $\mathcal{D}$ is Borel.*

For the proof of Theorem 5.8 we need the following lemma.

Lemma 5.9. *Let E, Z and $\mathcal{D}$ be as in Theorem* 5.8*. Then for every $\varepsilon > 0$ the parameterized slicing map $f_{\mathcal{D},\varepsilon}$ and the parameterized derivative $D_{\mathcal{D},\varepsilon}$ associated to $\mathcal{D}$ and ε are both Borel.*

Proof. Fix $\varepsilon > 0$. For every $m \in \mathbb{N}$ let $A_m \subseteq Z \times K(E)$ be defined by

$$(z,K) \in A_m \Leftrightarrow (K \cap V_m \neq \emptyset) \text{ and } (\forall x, y \in K \cap V_m \text{ we have } \mathcal{D}(z,x,y) \leqslant \varepsilon).$$

Claim 5.10. *For every $m \in \mathbb{N}$ the set A_m is Borel.*

Proof of Claim 5.10. By condition (2) in Definition 5.7, for every $p \in \mathbb{N}$ the set $B_{m,p} \subseteq Z \times K(E) \times E \times E$ defined by

$$(z,K,x,y) \in B_{m,p} \Leftrightarrow (x, y \in K \cap V_m) \text{ and } \mathcal{D}(z,x,y) > \varepsilon + \frac{1}{p+1}$$

is Borel. Let $z \in Z$. By Corollary 5.3 and our assumptions on $\mathcal{D}$, we see that the map $\mathcal{D}_z\colon (E,\tau) \times (E,\tau) \to \mathbb{R}$ is Baire-1. It follows that for every $(z,K) \in Z \times K(E)$ the section $\{(x,y) : (z,K,x,y) \in B_{m,p}\}$ of $B_{m,p}$ at (z,K)

is K_σ. By a classical result of Arsenin and Kunugui (see [Ke, Theorem 35.46]), the set

$$C_{m,p} = \mathrm{proj}_{Z\times K(E)} B_{m,p} = \big\{(z,K) : \exists x,y \in E \text{ with } (z,K,x,y) \in B_{m,p}\big\}$$

is Borel too. Noticing that

$$(z,K) \in A_m \Leftrightarrow (K \cap V_m \neq \emptyset) \text{ and } (\forall p \in \mathbb{N} \text{ we have } (z,K) \notin C_{m,p})$$

we conclude that A_m is Borel. The claim is proved. □

The above claim easily implies that the maps $f_{\mathcal{D},\varepsilon}$ and $D_{\mathcal{D},\varepsilon}$ are Borel. Indeed, notice that $f_{\mathcal{D},\varepsilon}(z,K) = K \setminus V_m$ if $(z,K) \in A_m$ and $(z,K) \notin A_i$ for every $i < m$ (the argument for $D_{\mathcal{D},\varepsilon}$ is similar). The proof of Lemma 5.9 is completed. □

We are ready to proceed to the proof of Theorem 5.8.

Proof of Theorem 5.8. By Lemma A.12, it is enough to show that for every $\varepsilon > 0$ the parameterized "last bite" map $L_{\mathcal{D},\varepsilon}$ associated to $\mathcal{D}$ and $\varepsilon > 0$ is Borel. So, let us fix $\varepsilon > 0$. For every $z \in Z$ let $\xi_z = \mathrm{Ind}(\mathcal{D}_z, \varepsilon, E)$. Also let $(E^{\xi}_{\mathcal{D}_z,\varepsilon} : \xi < \omega_1)$ be the slicing of E associated to $\mathcal{D}_z$ and ε.

Claim 5.11. *We have* $\sup\{\mathrm{Ind}(\mathcal{D}_z, \varepsilon, E) : z \in Z\} = \xi_\varepsilon < \omega_1$.

Proof of Claim 5.11. By Lemma 5.9 and according to the terminology in Appendix A, the map $D_{\mathcal{D},\varepsilon} \colon Z \times K(E) \to K(E)$ is a parameterized Borel derivative on $K(E)$. By Theorem A.10, the set

$$\Omega_{D_{\mathcal{D},\varepsilon}} = \{(z,K) \in Z \times K(E) : D^{\infty}_{\mathcal{D}_z,\varepsilon}(K) = \emptyset\}$$

is $\mathbf{\Pi}^1_1$ and the map $(z,K) \mapsto |K|_{D_{\mathcal{D}_z,\varepsilon}}$ is a $\mathbf{\Pi}^1_1$-rank on $\Omega_{D_{\mathcal{D},\varepsilon}}$. Fix $z \in Z$. Since the map $\mathcal{D}_z$ is a fragmentation on E we see that the iterated derivatives of E according to $D_{\mathcal{D}_z,\varepsilon}$ must be stabilized at $\emptyset$, and so, the set $\mathcal{A} = \{(z,E) : z \in Z\}$ is a subset of $\Omega_{D_{\mathcal{D},\varepsilon}}$. The set $\mathcal{A}$ is Borel. Hence, by part (ii) of Theorem A.2, there exists a countable ordinal ζ such that $\sup\{|E|_{D_{\mathcal{D}_z,\varepsilon}} : z \in Z\} = \zeta$. By Proposition 5.4, for every $z \in Z$ we have

$$\mathrm{Ind}(\mathcal{D}_z, \varepsilon, E) \leqslant \omega \cdot |E|_{D_{\mathcal{D}_z,\varepsilon}} \leqslant \omega \cdot \zeta.$$

The claim is proved. □

Let ξ_ε be the countable ordinal obtained by Claim 5.11. By transfinite recursion, for every $\xi \leqslant \xi_\varepsilon$ we define $A^\xi \subseteq Z \times E$ as follows. We set $A^0 = Z \times E$. If $\xi = \zeta + 1$ is a successor ordinal, then we set

$$(z,x) \in A^\xi \Leftrightarrow x \in f_{\mathcal{D},\varepsilon}\big(z, (A^\zeta)_z\big)$$

where $(A^\zeta)_z = \{x \in E : (z,x) \in A^\zeta\}$ is the section of A^ζ at z. If ξ is a limit ordinal, then we set

$$(z,x) \in A^\xi \Leftrightarrow (z,x) \in \bigcap_{\zeta<\xi} A^\zeta.$$

Claim 5.12. *The following hold.*

(i) *For every $\xi \leqslant \xi_\varepsilon$ the set A^ξ is Borel and has compact sections.*

(ii) *For every $\xi \leqslant \xi_\varepsilon$ and every $z \in Z$ the section $(A^\xi)_z$ of A^ξ at z coincides with the set $E^\xi_{\mathcal{D}_z,\varepsilon}$.*

Proof of Claim 5.12. (i) The proof is similar to the proof of Claim 2.15 and proceeds by transfinite induction on all ordinals less than or equal to ξ_ε. For $\xi = 0$ it is straightforward. If $\xi = \zeta + 1$ is a successor ordinal, then by our inductive assumption and Theorem A.14, we see that the map $z \mapsto (A^\zeta)_z$ is Borel. By Lemma 5.9, the map $z \mapsto f_{\mathcal{D},\varepsilon}\big(z, (A^\zeta)_z\big)$ is Borel too. Hence, by the definition of $A^\xi = A^{\zeta+1}$ and invoking Theorem A.14 once more, we conclude that A^ξ is Borel. If ξ is limit, then the result follows immediately by the definition of the set A^ξ and our inductive assumption.

(ii) It follows by straightforward transfinite induction. The claim is proved. □

For every $\xi \leqslant \xi_\varepsilon$ we define the "hitting" set $H_\xi \subseteq Z \times K(E)$ by

$$(z,K) \in H_\xi \Leftrightarrow \exists x \in E \text{ with } (x \in K \text{ and } (z,x) \in A^\xi).$$

Using part (i) of Claim 5.12 and arguing as in the proof of Lemma 5.9, we see that H_ξ is Borel for every $\xi \leqslant \xi_\varepsilon$. By part (i) of Claim 5.12 and the definition of the map $L_{\mathcal{D},\varepsilon}$, we obtain that

$$L_{\mathcal{D},\varepsilon}(z,K) = K \cap (A^\xi)_z \text{ if and only if } (z,K) \in H_\xi \text{ and } (z,K) \notin H_{\xi+1}.$$

This clearly implies that $L_{\mathcal{D},\varepsilon}$ is a Borel map. The proof of Theorem 5.8 is completed. □

5.3 The embedding

This section is devoted to the proof of Theorem 5.1. We will give the proof of part (i) and we will explain, later on, how the construction yields part (ii) as well. We need, first, to introduce some pieces of notation.

Notation 5.1. By $\phi\colon 2^{<\mathbb{N}} \to \mathbb{N}$ we denote the unique bijection satisfying $\phi(s) < \phi(t)$ if either $|s| < |t|$, or $|s| = |t| = n$ and $s <_{\mathrm{lex}} t$. (Here, by $<_{\mathrm{lex}}$ we denote the usual lexicographical order on 2^n.) Moreover, for every $t \in 2^{<\mathbb{N}}$ we set $V_t = \{\sigma \in 2^{\mathbb{N}} : t \sqsubset \sigma\}$.

We also isolate, for future use, the following elementary facts. Let $n, k \in \mathbb{N}$ and $i \in \{0,1\}$. Set $A_k = \{\phi^{-1}(m) : k \leqslant m \leqslant 2k\}$ and $\pi_k = \{V_t : t \in A_k\}$. We also set $t_n = \emptyset$ if $n = 0$ and $t_n = \phi^{-1}(n-1)^\frown i$ if $n \geqslant 1$. Then the following hold.

(F1) The family π_k forms a partition of $2^{\mathbb{N}}$ into clopen pieces. Moreover, π_{k+1} is obtained from π_k by splitting the clopen set $V_{\phi^{-1}(k)}$ (which belongs to π_k) into the clopen sets $V_{\phi^{-1}(k)^\frown 0}$ and $V_{\phi^{-1}(k)^\frown 1}$.

(F2) If $n \leqslant k$, then for every $t \in A_k$ compatible with t_n we have $t_n \sqsubseteq t$. Moreover, the family $\{V_t : t \in A_k \text{ and } t_n \sqsubseteq t\}$ forms a partition of V_{t_n}.

(F3) If $n > k$, then there exists a unique $t \in A_k$ with $t \sqsubset t_n$.

Now let X be a Banach space with separable dual. By Theorem 1.8, we may assume that X is a subspace of $C(2^{\mathbb{N}})$. Let $\mathbf{1} \in C(2^{\mathbb{N}})$ be the constant function equal to 1. We fix a function $g_0 \in C(2^{\mathbb{N}})$ that separates the points of $2^{\mathbb{N}}$ and with $\|g_0\|_\infty = 1$. By replacing X with $\overline{\mathrm{span}}\{X \cup g_0 \cup \mathbf{1}\}$ if necessary, we may assume that g_0 and $\mathbf{1}$ belong to X. We define $\Delta_X \colon 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to \mathbb{R}$ by

$$\Delta_X(\sigma, \tau) = \sup\big\{|f(\sigma) - f(\tau)| : f \in B_X\big\}. \tag{5.2}$$

Clearly, Δ_X is a metric on $2^{\mathbb{N}}$. Also observe that if (f_n) is a dense sequence in S_X, then $\Delta_X(\sigma,\tau) = \sup\{|f_n(\sigma) - f_n(\tau)| : n \in \mathbb{N}\}$ for every $\sigma, \tau \in 2^{\mathbb{N}}$. Arguing as in Lemma 2.9, it is easy to verify that if Δ_X is *not* a fragmentation, then we would be able to find $\varepsilon > 0$ and a perfect subset P of $2^{\mathbb{N}}$ such that $\Delta_X(\sigma, \sigma') \geqslant \varepsilon$ for every $\sigma, \sigma' \in P$ with $\sigma \neq \sigma'$. Hence, our assumption that X^* is separable reduces to the fact that Δ_X is a fragmentation of $2^{\mathbb{N}}$. We apply the analysis presented in Section 5.1 and we obtain the "dessert" selection $s_X \colon K(2^{\mathbb{N}}) \to 2^{\mathbb{N}}$ associated to the fragmentation Δ_X.

We define a sequence (t_n^X) in $2^{<\mathbb{N}}$ as follows. We set $t_0^X = \emptyset$. For every $n \in \mathbb{N}$ with $n \geqslant 1$ let $t = \phi^{-1}(n-1)$ where ϕ is the bijection between $2^{<\mathbb{N}}$ and $\mathbb{N}$ described in Notation 5.1. Consider the element $s_X(V_t)$ of $2^{\mathbb{N}}$. By part (i) of Theorem 5.6, there exists a unique $i \in \{0,1\}$ such that $t^\frown i \sqsubset s_X(V_t)$. We set $t_n^X = t^\frown j$ where $j = i + 1 \mod 2$. Having defined the sequence (t_n^X), we define a normalized sequence (e_n^X) in $C(2^{\mathbb{N}})$ by the rule

$$e_n^X = \mathbf{1}_{V_{t_n^X}} \tag{5.3}$$

for every $n \in \mathbb{N}$. Before we proceed to our discussion on the properties of the sequence (e_n^X) we make the following simple observation. Let $k, m \in \mathbb{N}$ with $t_k^X \sqsubset t_m^X$ (by the properties of ϕ this implies that $k < m$). Then there exists a node $s \in 2^{<\mathbb{N}}$ with $t_k^X \sqsubset s$, $|s| = |t_m^X|$ and such that $e_m^X(\sigma) = 0$ for every $\sigma \in V_s$.

Claim 5.13. *The sequence (e_n^X) is a normalized monotone basis of $C(2^{\mathbb{N}})$.*

Proof of Claim 5.13. First we observe that $\mathbf{1}_{V_t} \in \mathrm{span}\{e_n^X : n \in \mathbb{N}\}$ for every $t \in 2^{<\mathbb{N}}$. Hence $\overline{\mathrm{span}}\{e_n^X : n \in \mathbb{N}\} = C(2^{\mathbb{N}})$. Thus, it is enough to show that (e_n^X) is a monotone basic sequence. To see this let $k, m \in \mathbb{N}$ with $k < m$ and $a_0, \dots, a_m \in \mathbb{R}$. There exists $\sigma \in 2^{\mathbb{N}}$ such that

$$\Big\| \sum_{n=0}^{k} a_n e_n^X \Big\|_\infty = \Big| \sum_{n=0}^{k} a_n e_n^X(\sigma) \Big|.$$

By the remarks before the statement of the claim, we may find $\tau \in 2^{\mathbb{N}}$ such that $e_n^X(\tau) = e_n^X(\sigma)$ if $0 \leqslant n \leqslant k$ while $e_n^X(\tau) = 0$ if $k < n \leqslant m$. Hence,

$$\Big\| \sum_{n=0}^{k} a_n e_n^X \Big\|_\infty = \Big| \sum_{n=0}^{k} a_n e_n^X(\sigma) \Big| = \Big| \sum_{n=0}^{m} a_n e_n^X(\tau) \Big| \leqslant \Big\| \sum_{n=0}^{m} a_n e_n^X \Big\|_\infty.$$

The claim is proved. □

For every $k \in \mathbb{N}$ let $P_k \colon C(2^{\mathbb{N}}) \to \mathrm{span}\{e_n^X : n \leqslant k\}$ be the natural projection. We will give a representation of P_k which will be very useful in the argument below. As in (F1) above, let $A_k = \{\phi^{-1}(m) : k \leqslant m \leqslant 2k\}$ and $\pi_k = \{V_t : t \in A_k\}$.

Claim 5.14. *For every $k \in \mathbb{N}$ and every $f \in C(2^{\mathbb{N}})$ we have*

$$P_k(f) = \sum_{V \in \pi_k} f\big(s_X(V)\big)\mathbf{1}_V.$$

Proof of Claim 5.14. Fix $k \in \mathbb{N}$. Let $R_k \colon C(2^{\mathbb{N}}) \to C(2^{\mathbb{N}})$ denote the operator defined by

$$R_k(f) = \sum_{V \in \pi_k} f\big(s_X(V)\big)\mathbf{1}_V = \sum_{t \in A_k} f\big(s_X(V_t)\big)\mathbf{1}_{V_t}.$$

Since $\|R_k\| \leqslant 1$, by Claim 5.13, it is enough to show that $R_k(e_n^X) = e_n^X$ if $n \leqslant k$ while $R_k(e_n^X) = 0$ if $n > k$. By (F2) and part (i) of Theorem 5.6, we see that $R_k(e_n^X) = e_n^X$ for every $n \leqslant k$. Now let $n > k$ and consider the unique node t in A_k with $t \sqsubset t_n^X$ obtained by (F3) above. We claim that $s_X(V_t) \notin V_{t_n^X}$; clearly, this implies that $R_k(e_n^X) = 0$. To see this let w be the immediate predecessor of t_n^X in $2^{<\mathbb{N}}$. Notice that $t \sqsubseteq w \sqsubset t_n^X$ and so $V_{t_n^X} \subseteq V_w \subseteq V_t$. By the definition of the sequence (t_n^X), we have

$$s_X(V_w) \notin V_{t_n^X}. \tag{5.4}$$

This implies that $s_X(V_t) \notin V_{t_n^X}$. For if not, by part (ii) of Theorem 5.6, we would have that $s_X(V_w) \in V_{t_n^X}$ in contradiction with (5.4) above. The claim is proved. □

For every $\mu \in M(2^{\mathbb{N}}) = C(2^{\mathbb{N}})^*$ and every $f \in C(2^{\mathbb{N}})$ we set $\mu(f) = \int f\, d\mu$. We define

$$W_0 = \bigcup_{k\in\mathbb{N}} P_k(B_X). \tag{5.5}$$

Notice that W_0 is bounded, convex and symmetric. The following lemma is the key step towards the proof of Theorem 5.1.

Lemma 5.15. **[GMS]** *For every $\mu \in M(2^{\mathbb{N}})$ we have*

$$\lim_k \sup_{w\in W_0} \mu\big(w - P_k(w)\big) = 0.$$

Proof. Let $r > 0$ be given. For every $k \in \mathbb{N}$ and every $\sigma \in 2^{\mathbb{N}}$ there exists a unique clopen set $V \in \pi_k$ such that $\sigma \in V$. We shall denote it by $V_k(\sigma)$. Notice that $V_k(\sigma) \subseteq V_n(\sigma)$ for every $\sigma \in 2^{\mathbb{N}}$ and every $k, n \in \mathbb{N}$ with $n < k$. Let $f \in B_X$ be arbitrary. By Claim 5.14, for every $k \in \mathbb{N}$ we have

$$\begin{aligned} |\mu\big(f - P_k(f)\big)| &= \Big|\sum_{V\in\pi_k} \int_V f(\sigma) - f\big(s_X(V)\big)\, d\mu(\sigma)\Big| \\ &= \Big|\int f(\sigma) - f\big(s_X(V_k(\sigma))\big)\, d\mu(\sigma)\Big| \\ &\leqslant \int \Delta_X\big(s_X(\{\sigma\}), s_X(V_k(\sigma))\big)\, d\mu(\sigma). \end{aligned}$$

For every $\sigma \in 2^{\mathbb{N}}$ it holds $\{\sigma\} = \bigcap_k V_k(\sigma)$ and the sequence $\big(V_k(\sigma)\big)$ is decreasing. Hence, by part (iii) of Theorem 5.6, we have $\Delta_X\big(s_X(\{\sigma\}), s_X(V_k(\sigma))\big) \to 0$ for every $\sigma \in 2^{\mathbb{N}}$. Notice that $0 \leqslant \Delta_X(\sigma, \tau) \leqslant 2$ for every pair $\sigma, \tau \in 2^{\mathbb{N}}$. By Lebesgue's dominated convergence theorem, there exists $l \in \mathbb{N}$ such that for every $k \geqslant l$ and every $f \in B_X$ we have

$$|\mu(f - P_k(f))| \leqslant \frac{r}{2}. \tag{5.6}$$

Now let $w \in W_0$ be arbitrary and $k \geqslant l$. There exist $f \in B_X$ and $n \in \mathbb{N}$ with $w = P_n(f)$. If $n \leqslant k$, then $w - P_k(w) = 0$. If $n > k$, then

$$w - P_k(w) = \big(f - P_k(f)\big) - \big(f - P_n(f)\big).$$

Invoking (5.6), we see that $|\mu\big(w - P_k(w)\big)| \leqslant r$ for every $w \in W_0$ and every $k \geqslant l$. The proof is completed. $\square$

We are ready for the last step of the proof of part (i) of Theorem 5.1. Let

$$W_X = \overline{W}_0 = \overline{\bigcup_{k\in\mathbb{N}} P_k(B_X)}. \tag{5.7}$$

We notice the following properties of W_X.

(P1) W_X is closed, convex, bounded and symmetric.

(P2) $B_X \subseteq W_X$.

(P3) $P_k(W_X) \subseteq W_X$ for every $k \in \mathbb{N}$.

By (P1), the Davis–Fiegel–Johnson–Pełczyński interpolation scheme can be applied to the pair $\big(C(2^{\mathbb{N}}), W_X\big)$ and $p = 2$ (see Appendix B.3). Denote by Z the interpolation space and by $J\colon Z \to C(2^{\mathbb{N}})$ the natural inclusion map. By (P2), we see that X is contained in Z. We observe that $e_n^X \in \mathrm{span}\{W_X\}$ for every $n \in \mathbb{N}$. For $n = 0$ this follows form the fact that $\mathbf{1} \in B_X$ and $P_0(\mathbf{1}) = e_0^X$. Now let $n \geqslant 1$. Also let $g_0 \in B_X$ be the fixed function that separates the points of $2^{\mathbb{N}}$ and write $g_0 = \sum_{k\in\mathbb{N}} a_k e_k^X$. Notice that $a_k \neq 0$ for every $k \in \mathbb{N}$. Hence $a_n e_n^X = P_n(g_0) - P_{n-1}(g_0) \in W_X - W_X$.

We set $z_n^X = J^{-1}(e_n^X)$ for every $n \in \mathbb{N}$. It follows, by (P3), the above discussion and Proposition B.9, that (z_n^X) is a monotone Schauder basis (not normalized) of Z. For every $k \in \mathbb{N}$ let $Q_k\colon Z \to \mathrm{span}\{z_n^X : n \leqslant k\}$ be the natural onto projection. It is easy to see that $J\big(Q_k(z)\big) = P_k\big(J(z)\big)$ for every $z \in Z$ and every $k \in \mathbb{N}$. We claim that (z_n^X) is shrinking. This is equivalent to saying that

$$\lim_k \sup_{z\in B_Z} z^*\big(z - Q_k(z)\big) = 0 \text{ for every } z^* \in Z^*. \tag{5.8}$$

Notice that we need to check (5.8) only for a norm dense subset of Z^*. By part (vi) of Proposition B.8, the dual operator $J^*\colon M(2^{\mathbb{N}}) \to Z^*$ has norm dense range. Hence, it is enough to show that

$$\lim_k \sup_{z\in B_Z} \mu\big(J(z) - P_k(J(z))\big) = 0 \text{ for every } \mu \in M(2^{\mathbb{N}}). \tag{5.9}$$

By the definition of Z, for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that every $z \in B_Z$ can be written as $z = z_1 + z_2$ where $J(z_1) \in 2^n W_0$ and $\|J(z_2)\| \leqslant \varepsilon$. Combining this fact with Lemma 5.15, we see that (5.9) is valid. This shows that (z_n^X) is shrinking. The proof of part (i) of Theorem 5.1 is completed.

As we have already mentioned in the beginning of the section we shall explain how the proof of part (i) of Theorem 5.1 yields part (ii) as well. But before that let us isolate in a definition the above described construction.

Definition 5.16. *Let X be a subspace of $C(2^{\mathbb{N}})$ with separable dual and such that $\mathbf{1}, g_0 \in X$. By $Z(X)$ we shall denote the space constructed following the procedure described above. We shall call the space $Z(X)$ as the* Ghoussoub–Maurey–Schachermayer space associated to *X*.

Let us gather what we have shown so far concerning the Ghoussoub–Maurey–Schachermayer construction.

Theorem 5.17. **[GMS]** *Let X be a subspace of $C(2^{\mathbb{N}})$ with separable dual and such that $\mathbf{1}, g_0 \in X$. Then the space $Z(X)$ associated to X has a shrinking Schauder basis and contains an isomorphic copy of X.*

By Theorem 5.17, part (ii) of Theorem 5.1 is an immediate consequence of the following lemma.

Lemma 5.18. *Let X be a subspace of $C(2^{\mathbb{N}})$ such that $\mathbf{1}, g_0 \in X$. Assume that X is reflexive. Then the following hold.*

(i) *The set W_X defined in (5.7) is weakly compact.*

(ii) *The space $Z(X)$ associated to X is reflexive.*

Proof. By Proposition B.8, it is enough to show that the set W_X is weakly compact. To this end, let $J\colon Z(X) \to C(2^{\mathbb{N}})$ be the inclusion map. We have already mentioned that X is contained in $Z(X)$. We set $K = J^{-1}(B_X)$. Notice that K is a weakly compact subset of $Z(X)$. Recall that for every $k \in \mathbb{N}$ by $Q_k\colon Z(X) \to \mathrm{span}\{z_n^X : n \leqslant k\}$ we denote the natural onto projection. As the basis (z_n^X) of $Z(X)$ is shrinking, by Lemma B.10, we see that the set

$$K' = K \cup \bigcup_{k \in \mathbb{N}} Q_k(K)$$

is a weakly compact subset of $Z(X)$. Hence, so is the set $J(K')$. Finally notice that $J(K') = W_X$. The proof is completed. $\square$

5.4 Parameterizing Zippin's theorem

This section is devoted to the proof of a result, due to Bossard, asserting that the Ghoussoub–Maurey–Schachermayer construction (as presented in Section 5.3) can be done "uniformly" in X. Before we give the precise statement, we introduce the following notation. For every $X \in \mathrm{SB}$ we set

$$E_X = \overline{\mathrm{span}}\{X \cup \mathbf{1} \cup g_0\}$$

where, as in Section 5.3, by $\mathbf{1}$ we denote the constant function on $2^{\mathbb{N}}$ equal to 1 while by $g_0 \in C(2^{\mathbb{N}})$ we denote a fixed function that separates the points of $2^{\mathbb{N}}$ and with $\|g_0\|_\infty = 1$. Clearly X is a closed subspace of E_X. Also notice that E_X has separable dual (respectively, E_X is reflexive) if and only if X^* is separable (respectively, X is reflexive). We are ready to state and prove the main result of this section.

Theorem 5.19. **[Bos2]** *Let B be a Borel subset of* SD. *Then the set $\mathcal{Z} \subseteq B \times \mathrm{SB}$ defined by*

$$(X, Y) \in \mathcal{Z} \Leftrightarrow Y \text{ is isometric to } Z(E_X)$$

is analytic.

Proof. Let $d_n\colon \mathrm{SB}\to C(2^{\mathbb{N}})$ and $S_n\colon \mathrm{SB}\to C(2^{\mathbb{N}})$ ($n\in\mathbb{N}$) be the sequences of Borel maps described in properties (P2) and (P3) in Section 2.1.1. Recall that the sequence $\big(d_n(X)\big)$ is norm dense in X for every $X\in\mathrm{SB}$, while the sequence $\big(S_n(X)\big)$ is norm dense in the sphere S_X of X for every $X\in\mathrm{SB}$ with $X\neq\{0\}$. Notice that $E_X\neq\{0\}$ for all $X\in\mathrm{SB}$.

Claim 5.20. *The map* $\mathrm{SB}\ni X\mapsto E_X\in\mathrm{SB}$ *is Borel.*

Proof of Claim 5.20. Let U be a nonempty open subset of $C(2^{\mathbb{N}})$. Noticing that

$$E_X\cap U\neq\emptyset \Leftrightarrow \exists n\in\mathbb{N}\ \exists p,q\in\mathbb{Q} \text{ with } d_n(X)+p\mathbf{1}+qg_0\in U$$

we see that the set $\{X : E_X\cap U\neq\emptyset\}$ is Borel. The claim is proved. □

Let B be as in the statement of the theorem. Define $\mathcal{D}\colon B\times 2^{\mathbb{N}}\times 2^{\mathbb{N}}\to\mathbb{R}$ by

$$\mathcal{D}(X,\sigma,\tau)=\sup\big\{|S_n(E_X)(\sigma)-S_n(E_X)(\tau)| : n\in\mathbb{N}\big\}. \tag{5.10}$$

By Claim 5.20, we see that the map $\mathcal{D}$ is Borel. We observe that for every $X\in B$ and every $\sigma,\tau\in 2^{\mathbb{N}}$ we have $\mathcal{D}(X,\sigma,\tau)=\Delta_{E_X}(\sigma,\tau)$, where Δ_{E_X} is the fragmentation on $2^{\mathbb{N}}$ associated to the space E_X and defined in (5.2). It follows that $\mathcal{D}$ is a parameterized Borel fragmentation according to Definition 5.7. We apply Theorem 5.8 and we obtain a Borel map $s\colon B\times K(2^{\mathbb{N}})\to 2^{\mathbb{N}}$ such that for every $(X,K)\in B\times K(2^{\mathbb{N}})$ the point $s(X,K)$ coincides with the point $s_{E_X}(K)$, where s_{E_X} is the "dessert" selection associated to the fragmentation Δ_{E_X}. For every $X\in B$ let $(e_n^{E_X})$ be the monotone basis of $C(2^{\mathbb{N}})$ defined in (5.3). We have the following claim.

Claim 5.21. *The map* $B\ni X\mapsto (e_n^{E_X})\in C(2^{\mathbb{N}})^{\mathbb{N}}$ *is Borel.*

Proof of Claim 5.21. It is enough to show that for every $n\in\mathbb{N}$ the map $B\ni X\mapsto e_n^{E_X}$ is Borel. So fix $n\in\mathbb{N}$. Let $t=\emptyset$ if $n=0$; otherwise, set $t=\phi^{-1}(n-1)$ where ϕ is the bijection between $2^{<\mathbb{N}}$ and $\mathbb{N}$ described in Notation 5.1. We define $B_0=\{X\in B : t^\frown 1\sqsubset s(E_X,V_t)\}$ and $B_1=B\setminus B_0$. Invoking the fact that s is Borel and Claim 5.20, we see that B_0 is Borel. Notice that $e_n^{E_X}=\mathbf{1}_{V_{t^\frown 0}}$ if $X\in B_0$ and $e_n^{E_X}=\mathbf{1}_{V_{t^\frown 1}}$ if $X\in B_1$. Therefore, the map $B\ni X\mapsto e_n^{E_X}$ is Borel. The claim is proved. □

We need the following elementary observation. Let Z be a standard Borel space, let Y be a Polish space and let $f_n\colon Z\to Y$ ($n\in\mathbb{N}$) be a sequence of Borel maps. Then the map $\Phi\colon Z\to F(Y)$ defined by $\Phi(z)=\overline{\{f_n(z) : n\in\mathbb{N}\}}$ for every $z\in Z$, is Borel. Now let W_{E_X} be the closed subset of $C(2^{\mathbb{N}})$ associated to the space E_X and defined in (5.7).

Claim 5.22. *The map* $B\ni X\mapsto W_{E_X}\in F\big(C(2^{\mathbb{N}})\big)$ *is Borel.*

Proof of Claim 5.22. In light of the previous observation, it is enough to show that for every $n, k \in \mathbb{N}$ and every rational q in $[-1,1]$ the map $f_{n,k,q}\colon B \to C(2^{\mathbb{N}})$ defined by $f_{n,k,q}(X) = P_k\big(qS_n(E_X)\big)$, is Borel. By Claims 5.14 and 5.20 and taking into account the fact that the parameterized selection s is Borel, we see that the map $B \ni X \mapsto W_{E_X}$ is also Borel. The claim is proved. □

Let $m \in \mathbb{N}$ with $m \geqslant 1$. For every $X \in B$ we set

$$W^m_{E_X} = \overline{2^m W_{E_X} + 2^{-m} B_{C(2^{\mathbb{N}})}}. \tag{5.11}$$

Using Claim 5.22, it is easy to see that the map $B \ni X \mapsto W^m_{E_X} \in F\big(C(2^{\mathbb{N}})\big)$ is Borel. This fact has, in turn, the following consequence.

Claim 5.23. *If $\|\cdot\|_{m,X}$ denotes the Minkowski gauge of the closed, convex and symmetric set $W^m_{E_X}$, then the function $B \times C(2^{\mathbb{N}}) \ni (X,f) \mapsto \|f\|_{m,X}$ is Borel.*

Proof of Claim 5.23. Let $r \in \mathbb{R}$ with $r > 0$ and notice that

$$\|f\|_{m,X} < r \Leftrightarrow \exists q \in \mathbb{Q} \text{ with } 0 < q < r \text{ and } f \in qW^m_{E_X}.$$

The claim is proved. □

We define $\mathcal{B} \subseteq B \times C(2^{\mathbb{N}})^{\mathbb{N}}$ by

$$\big(X,(y_n)\big) \in \mathcal{B} \Leftrightarrow (z^{E_X}_n) \text{ is 1-equivalent to } (y_n)$$

where $(z^{E_X}_n)$ is the sequence $(e^{E_X}_n)$ regarded as a basis of the space $Z(E_X)$. In order to finish the proof of Theorem 5.19, it is enough to show that the set $\mathcal{B}$ is Borel. Indeed,

$$\begin{aligned}(X,Y) \in \mathcal{Z} \quad \Leftrightarrow \quad & \exists (y_n) \in C(2^{\mathbb{N}}) \text{ with } \big[(\forall n\ y_n \in Y) \text{ and} \\ & (\overline{\mathrm{span}}\{y_n : n \in \mathbb{N}\} = Y) \text{ and } \big(X,(y_n)\big) \in \mathcal{B}\big].\end{aligned}$$

By property (P5) in Section 2.1.1 and invoking the Borelness of the set $\mathcal{B}$, we see that $\mathcal{Z}$ is analytic, as desired. So, it remains to show that $\mathcal{B}$ is Borel. Observe that

$$\begin{aligned}\big(X,(y_n)\big) \in \mathcal{B} \quad \Leftrightarrow \quad & \forall k \in \mathbb{N}\ \forall a_0,\dots,a_k \in \mathbb{Q}\ \Big(\forall N \geqslant 1 \text{ we have} \\ & \sum_{1\leqslant m\leqslant N} \Big\|\sum_{n=0}^{k} a_n e^{E_X}_n\Big\|^2_{m,X} \leqslant \Big\|\sum_{n=0}^{k} a_n y_n\Big\|^2\Big) \\ & \text{and } \Big(\forall p \geqslant 1\ \exists N \geqslant 1 \text{ with} \\ & \Big\|\sum_{n=0}^{k} a_n y_n\Big\|^2 - \frac{1}{p} \leqslant \sum_{1\leqslant m\leqslant N} \Big\|\sum_{n=0}^{k} a_n e^{E_X}_n\Big\|^2_{m,X}\Big).\end{aligned}$$

By Claims 5.21 and 5.23, we conclude that $\mathcal{B}$ is Borel. The proof of Theorem 5.19 is completed. □

We close this section by mentioning two consequences of Theorem 5.19. They will be of particular importance later on.

Corollary 5.24. **[DF]** *Let A be an analytic subset of* SD*. Then there exists an analytic subset A' of* SD *with the following properties.*

(i) *Every $Y \in A'$ has a shrinking Schauder basis.*

(ii) *For every $X \in A$ there exists $Y \in A'$ containing an isomorphic copy of X.*

Proof. By Theorem 2.11, the set SD is $\mathbf{\Pi}^1_1$. Hence, by Lusin's separation theorem, there exists a Borel set $B \subseteq \mathrm{SD}$ with $A \subseteq B$. We apply Theorem 5.19 for the Borel set B and we obtain the analytic set $\mathcal{Z}$ as described in Theorem 5.19. We define $A' \subseteq \mathrm{SB}$ by

$$Y \in A' \Leftrightarrow \exists X \in \mathrm{SB} \ \big[(X \in A) \text{ and } (X,Y) \in \mathcal{Z}\big].$$

Clearly A' is as desired. The proof is completed. □

By Theorem 2.5, the set REFL is $\mathbf{\Pi}^1_1$. Hence, by Lemma 5.18, we obtain the following analogue of Corollary 5.24 for reflexive spaces. The proof is identical to the proof of Corollary 5.24.

Corollary 5.25. **[DF]** *Let A be an analytic subset of* REFL*. Then there exists an analytic subset A' of* REFL *with the following properties.*

(i) *Every $Y \in A'$ has a Schauder basis.*

(ii) *For every $X \in A$ there exists $Y \in A'$ containing an isomorphic copy of X.*

5.5 Comments and Remarks

1. The slicing methods presented in Section 5.1 have been developed by Ghoussoub, Maurey and Schachermayer [GMS]. They can be performed in more general topological spaces. We refer the reader to [GMS, GGMS] and the references therein for a detailed presentation, as well as, for further applications of these techniques.

2. The notion of a "parameterized Borel fragmentation" given in Definition 5.7 is new. It is the analogue of the notion of a parameterized Borel derivative presented in Appendix A. Theorem 5.8 is new as well.

3. As we have already mentioned, the proof of Zippin's theorem given in Section 5.3 is due to Ghoussoub, Maurey and Schachermayer [GMS]. We notice that it was known, prior to [Z], that a reflexive subspace of a space with a shrinking basis embeds into a reflexive space with a basis (see [DFJP]).

4. Theorem 5.19 is due to Bossard [Bos2]. His approach, however, is different (he used Theorem 2.11 instead of Theorem 5.8 and worked with the Szlenk indices of the corresponding spaces). Bossard used Theorem 5.19 to derive the following.

Corollary 5.26. **[Bos2]** *There exists a map $\phi\colon \omega_1 \to \omega_1$ such that for every $\xi < \omega_1$ if X is a Banach space with $\mathrm{Sz}(X) \leqslant \xi$, then X embeds into a Banach space Y with a shrinking basis, satisfying $\mathrm{Sz}(Y) \leqslant \phi(\xi)$.*

We notice that, in this direction, there exist two sharp quantitative refinements of Theorem 5.1. The first is due to Odell, Schlumprecht and Zsák and deals with separable reflexive Banach spaces.

Theorem 5.27. **[OSZ]** *Let ξ be a countable ordinal and let X be a separable reflexive Banach space such that $\max\big\{\mathrm{Sz}(X), \mathrm{Sz}(X^*)\big\} \leqslant \omega^{\xi\cdot\omega}$. Then X embeds isomorphically into a reflexive Banach space Y with a Schauder basis also satisfying $\max\big\{\mathrm{Sz}(Y), \mathrm{Sz}(Y^*)\big\} \leqslant \omega^{\xi\cdot\omega}$.*

The second is due to Freeman, Odell, Schlumprecht and Zsák and deals with Banach spaces with separable dual.

Theorem 5.28. **[FOSZ]** *Let ξ be a countable ordinal and let X be a separable Banach space such that $\mathrm{Sz}(X) \leqslant \omega^{\xi\cdot\omega}$. Then X embeds isomorphically into a Banach space Y with a shrinking Schauder basis also satisfying $\mathrm{Sz}(Y) \leqslant \omega^{\xi\cdot\omega}$.*

5. Corollaries 5.24 and 5.25 were noticed in [DF]. We should point out that they can be also derived by Theorems 5.28 and 5.27 respectively, using the machinery developed in Sections 2.3 and 2.5.

Chapter 6

The Bourgain–Pisier construction

This chapter is devoted to the study of an embedding result due to Jean Bourgain and Gilles Pisier [BP]. The main goal in the Bourgain–Pisier construction is to embed isometrically a given separable Banach space X into a separable $\mathcal{L}_\infty$-space Y (that is, into a Banach space with prescribed local structure) in such a way that the quotient Y/X is "small".

Theorem 6.1. **[BP]** *Let X be a separable Banach space and $\lambda > 1$. Then there exists a separable $\mathcal{L}_{\infty,\lambda+}$-space, denoted by $\mathcal{L}_\lambda[X]$, which contains X isometrically and is such that the quotient $\mathcal{L}_\lambda[X]/X$ has the Radon–Nikodym and the Schur properties.*

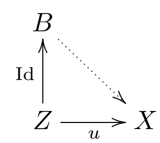
We recall that a Banach space X has the *Schur property* if every weakly null sequence in X is automatically norm convergent. It is an immediate consequence of Rosenthal's dichotomy [Ro2] that a space with the Schur property is hereditarily ℓ_1. Generalities about the Radon–Nikodym property can be found in Appendix B.5.

The building blocks of the space $\mathcal{L}_\lambda[X]$ in Theorem 6.1 are obtained using a method of extending operators due to Kisliakov [Ki]. This method has found many other remarkable applications beside its use in Theorem 6.1. It is presented in Section 6.1. The rest of the material in this chapter is devoted to the proof of Theorem 6.1 as well as to a parameterized version of it. The parameterized version is taken from [D3] and asserts that the space $\mathcal{L}_\lambda[X]$ is constructed from X in a "Borel way". This information gives further control on the resulting space $\mathcal{L}_\lambda[X]$ and it will be crucial for the results in Chapter 7.

6.1 Kisliakov's extension

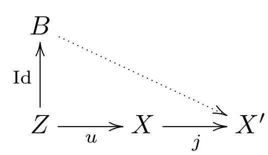
This section is devoted to the study of a method of extending operators invented by Kisliakov [Ki]. Let us start, first, with a brief motivating discussion.

Consider a Banach space X. Suppose that B is an arbitrary Banach space and that Z is a subspace of B. Suppose, further, that we are given a bounded linear operator $u\colon Z \to X$. When can we extend the operator u to a bounded linear operator $\tilde{u} : B \to X$? Pictorially, we are asking when we can close off the diagram:



By definition, this extension problem has an affirmative answer if and only if the space X is injective.

Suppose now that we are allowed to embed X isometrically into another Banach space X' via an isometric embedding $j\colon X \to X'$. By making an appropriate choice of X' and j, can we close off the following diagram?



It is easy to see that if we put no restrictions on X', then such an extension is always possible. Indeed, let X' be the space $\ell_\infty(B_{X^*})$ and let $j\colon X \to X'$ be the natural isometric embedding. Since $\ell_\infty(B_{X^*})$ is injective (see [LT]), we can extend the operator $j \circ u$. Observe, however, that this solution is uneconomical. The space X' is too large.

The method discovered by Kisliakov lies somewhere between the above extremes. It has been heavily investigated and it has been used in many remarkable ways. Let us give the main definition.

Definition 6.2. **[Ki]** *Let B and X be Banach spaces and $\eta \leqslant 1$. Let Z be a subspace of B and let $u\colon Z \to X$ be a bounded linear operator with $\|u\| \leqslant \eta$. Let $B \oplus_1 X$ be the vector space $B \times X$ equipped with the norm $\|(b,x)\| = \|b\| + \|x\|$ and consider the subspace $N = \big\{(z, -u(z)) : z \in Z\big\}$ of $B \oplus_1 X$. We define $X_1 = (B \oplus_1 X)/N$. Moreover, denoting by $Q\colon B \oplus_1 X \to X_1$ the natural quotient map, we define $\tilde{u}\colon B \to X_1$ and $j\colon X \to X_1$ by*

$$\tilde{u}(b) = Q(b,0) \quad \textit{and} \quad j(x) = Q(0,x)$$

for every $b \in B$ and every $x \in X$. We call the family $(X_1, j, \tilde{u})$ as the canonical triple *associated to (B, Z, X, u, η).*

The rest of this section is devoted to the study of the canonical triple introduced above.

6.1.1 Basic properties

Proposition 6.3. [Ki] *Let B, Z, X, u and η be as in Definition 6.2. Let $(X_1, j, \tilde{u})$ be the canonical triple associated to (B, Z, X, u, η). Then the following are satisfied.*

(i) *The operator j is an isometric embedding.*

(ii) *We have $\|\tilde{u}\| \leqslant 1$ and $\tilde{u}(z) = j\big(u(z)\big)$ for all $z \in Z$.*

(iii) *The spaces B/Z and $X_1/j(X)$ are isometric.*

Proof. (i) Fix $x \in X$. By definition, we have $j(x) = Q(0, x)$. It follows that

$$\begin{aligned} \|x\| \geqslant \|Q(0,x)\| &= \inf\{\|(0,x) + (z, -u(z))\| : z \in Z\} \\ &= \inf\{\|z\| + \|x - u(z)\| : z \in Z\} \\ &\geqslant \inf\{\|z\| + \|x\| - \|u(z)\| : z \in Z\} \\ &\geqslant \inf\{\|x\| + (1-\eta)\|z\| : z \in Z\} \geqslant \|x\|. \end{aligned}$$

Therefore j is an isometric embedding.

(ii) To see that $\|\tilde{u}\| \leqslant 1$ notice that for every $b \in B$ we have

$$\|\tilde{u}(b)\| = \|Q(b,0)\| \leqslant \|(b,0)\| = \|b\|.$$

Let $z \in Z$. Then $\|\tilde{u}(z) - j\big(u(z)\big)\| = \|Q(z,0) - Q(0,u(z))\| = \|Q(z, -u(z))\| = 0$, and so $\tilde{u}(z) = j\big(u(z)\big)$, as claimed.

(iii) We define a map $U\colon B/Z \to X_1/j(X)$ as follows. For every $\beta \in B/Z$ we select $b \in B$ such that $\beta = b + Z$ and we set $U(\beta) = \tilde{u}(b) + j(X)$. The fact that $\tilde{u}(z) = j\big(u(z)\big)$ for every $z \in Z$, already established in part (ii), implies that $U(\beta)$ is independent of the choice of b. In other words, U is a well-defined linear operator. It is easy to see that $\|U\| \leqslant 1$. We will show that U is an isometry.

We argue as follows. Fix $\alpha \in X_1/j(X)$. There exist $b \in B$ and $x \in X$ such that $\alpha = Q(b,x) + j(X)$. Hence

$$\alpha = Q(b,0) + Q(0,x) + j(X) = \tilde{u}(b) + j(x) + j(X) = U(b+Z).$$

Therefore, U is onto. Moreover,

$$\begin{aligned} \|b+Z\|_{B/Z} \geqslant \|\alpha\| &= \inf\{\|Q(b,x) + Q(0,x')\| : x' \in X\} \\ &= \inf\{\|Q(b,x'')\| : x'' \in X\} \\ &= \inf\{\|b+z\| + \|x'' - u(z)\| : x'' \in X \text{ and } z \in Z\} \\ &\geqslant \inf\{\|b+z\| : z \in Z\} = \|b+Z\|_{B/Z}. \end{aligned}$$

The proof is completed. □

6.1.2 Preservation of isomorphic embeddings

Lemma 6.4. **[BP]** *Let B, Z, X, u and η be as in Definition* 6.2*. Let $\delta > 0$ and assume that $\|u(z)\| \geqslant \delta\|z\|$ for every $z \in Z$. Consider the canonical triple $(X_1, j, \tilde{u})$ associated to (B, Z, X, u, η). Then $\|\tilde{u}(b)\| \geqslant \delta\|b\|$ for every $b \in B$.*

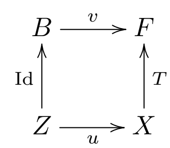
Proof. The fact that $\|u\| \leqslant \eta \leqslant 1$ implies that $\delta \leqslant 1$. Fix $b \in B$ and notice that

$$\begin{aligned} \|\tilde{u}(b)\| &= \|Q(b,0)\| = \inf\{\|b+z\| + \|u(z)\| : z \in Z\} \\ &\geqslant \inf\{\delta\|b+z\| + \delta\|z\| : z \in Z\} \geqslant \delta\|b\|. \end{aligned}$$

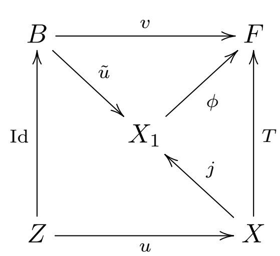
The proof is completed. □

6.1.3 Minimality

Lemma 6.5. **[BP]** *Let B, Z, X, u and η be as in Definition* 6.2 *and consider the canonical triple $(X_1, j, \tilde{u})$ associated to (B, Z, X, u, η). Let F be a Banach space. Also let $T\colon X \to F$ and $v\colon B \to F$ be bounded linear operators such that the following diagram commutes:*



Then there exists a unique bounded linear operator $\phi\colon X_1 \to F$ such that the following diagram commutes:



Moreover, $\|\phi\| \leqslant \max\{\|T\|, \|v\|\}$.

Proof. We start with the following simple observation. Fix $\alpha \in X_1$. Let $b, b' \in B$ and let $x, x' \in X$ such that $\alpha = Q(b,x) = Q(b',x')$. There exists $z_0 \in Z$ such that $b' = b + z_0$ and $x' = x - u(z_0)$. The fact that $v(z) = T\big(u(z)\big)$ for all $z \in Z$ implies that $v(b') + T(x') = v(b) + T(x)$.

We are ready to define the operator $\phi\colon X_1 \to F$. So, let $\alpha \in X_1$ and select $b \in B$ and $x \in X$ such that $\alpha = Q(b,x)$. We set $\phi(\alpha) = v(b) + T(x)$. The previous remark yields that ϕ is well-defined (that is, independent of the choice of b and x) and $\|\phi\| \leqslant \max\{\|T\|, \|v\|\}$. Finally, the operator ϕ is unique. Indeed, if $\psi\colon X_1 \to F$ is another operator such that $v = \psi \circ \tilde{u}$ and $T = \psi \circ j$, then it is easy to see that $\psi = \phi$. The proof is completed. □

6.1.4 Uniqueness

Lemma 6.6. **[BP]** *Let B, Z, X, u and η be as in Definition* 6.2 *and consider the canonical triple $(X_1, j, \tilde{u})$ associated to (B, Z, X, u, η).*

Let X' be a Banach space, let $j'\colon X \to X'$ be an isometric embedding and let $u'\colon B \to X'$ be a linear operator with $\|u'\| \leqslant 1$. Assume that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{u'} & X' \\ {\scriptstyle \mathrm{Id}}\uparrow & & \uparrow{\scriptstyle j'} \\ Z & \xrightarrow[u]{} & X \end{array}$$

Assume, moreover, that the triple (X', j', u') satisfies the minimality property described in Lemma 6.5*. That is, whenever F is a Banach space and $T\colon X \to F$ and $v\colon B \to F$ are bounded linear operators such that $v(z) = T\big(u(z)\big)$ for all $z \in Z$, then there exists a unique bounded linear operator $\phi'\colon X' \to F$ such that $v = \phi' \circ \tilde{u}$, $T = \phi' \circ j$ and $\|\phi\| \leqslant \max\{\|T\|, \|v\|\}$.*

Then there exists a linear isometry $I\colon X_1 \to X'$ such that $I \circ j = j'$.

Proof. The proof is essentially a consequence of the minimality of the canonical triple $(X_1, j, \tilde{u})$ established in Lemma 6.5 and of our assumption that the triple (X', j', u') is also minimal. Indeed, there exist two unique linear operators $\phi\colon X_1 \to X'$ and $\phi'\colon X' \to X_1$ satisfying the following properties.

(a) $\|\phi\| \leqslant 1$ and $\|\phi'\| \leqslant 1$.

(b) $j' = \phi \circ j$ and $j = \phi' \circ j'$.

(c) $u' = \phi \circ \tilde{u}$ and $\tilde{u} = \phi' \circ u'$.

We claim that $\phi' \circ \phi$ and $\phi \circ \phi'$ are the identity operators on X_1 and X' respectively. This fact and properties (a) and (b) above yield that the operator $\phi\colon X_1 \to X'$ is the desired isometry.

As the argument is symmetric it is enough to show that $\phi' \circ \phi$ is the identity on X_1. To this end we argue as follows. We set $F = X_1$. We define $T\colon X \to F$ and $v\colon B \to F$ by $T = \phi' \circ \phi \circ j$ and $v = \tilde{u}$. Property (b) above implies that $T = j$. Therefore, the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{v} & F \\ {\scriptstyle \mathrm{Id}}\uparrow & & \uparrow{\scriptstyle T} \\ Z & \xrightarrow[u]{} & X \end{array}$$

By Lemma 6.5, there exists a *unique* operator $S\colon X_1 \to F$ such that $T = S \circ j$. Denote by S' the identity operator on X_1 and by S'' the operator $\phi' \circ \phi$. Since $T = \phi' \circ \phi \circ j = j$, we see that $T = S \circ j = S' \circ j = S'' \circ j$. Invoking the uniqueness of S, we conclude that $S' = S''$. In other words, the operator $\phi' \circ \phi$ is the identity operator on X_1. The proof is completed. □

6.2 Admissible embeddings

In this section we will continue our study of the canonical triple $(X_1, j, \tilde{u})$. In particular we will focus on the properties of the isometric embedding j. Since the kind of questions addressed in this section do not depend on the whole structure of the canonical triple, one is naturally led to take an abstract approach to the study of the embedding j. This is the content of the following definition due to Bourgain and Pisier.

Definition 6.7. **[BP]** *Let X and X' be two Banach spaces and $\eta \leqslant 1$. Let $J\colon X \to X'$ be an isometric embedding. We say that the isometric embedding J is* η-admissible *if there exist a Banach space B, a subspace Z of B and an operator $u\colon Z \to X$ with $\|u\| \leqslant \eta$ such that the following is satisfied. If $(X_1, j, \tilde{u})$ is the canonical triple associated to (B, Z, X, u, η), then there exists an isometry $T\colon X_1 \to X'$ making the following diagram commutative:*

$$\begin{array}{ccccc} B & \xrightarrow{\tilde{u}} & X_1 & \xrightarrow{T} & X' \\ {\scriptstyle \mathrm{Id}}\uparrow & & \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle J} \\ Z & \xrightarrow[u]{} & X & \xrightarrow[\mathrm{Id}]{} & X \end{array}$$

We call the quadruple (B, Z, u, T) as the witness of the η-admissibility *of the embedding $J\colon X \to X'$.*

We will present a characterization of η-admissible embeddings which will be very useful in the arguments below. To state it we need to introduce the following terminology. Let X and Y be Banach spaces and let $\pi\colon X \to Y$ be a surjective operator. We say that π is a *metric surjection* if the induced isomorphism between $X/\mathrm{Ker}(\pi)$ and Y is an isometry. Notice that a linear surjection $\pi\colon X \to Y$ is a metric surjection if and only if $\|\pi\| \leqslant 1$ and for every $y \in Y$ and every $\varepsilon > 0$ there exists $x \in X$ with $\pi(x) = y$ and such that $\|x\| - \varepsilon \leqslant \|y\| \leqslant \|x\|$.

Lemma 6.8. **[BP]** *Let X and X' be Banach spaces and let $J\colon X \to X'$ be an isometric embedding. Also let $\eta \leqslant 1$. Then the following are equivalent.*

(i) *The embedding J is η-admissible.*

(ii) *There exist a Banach space E and a metric surjection $\pi\colon E \oplus_1 X \to X'$ such that for every $e \in E$ and every $x \in X$ we have*

$$\|\pi(e,x)\| \geqslant \|x\| - \eta\|e\|$$

and $\pi(0,x) = J(x)$.

Proof. Assume, first, that J is η-admissible. Let (B, Z, u, T) be the quadruple witnessing the η-admissibility of J and consider the canonical triple $(X_1, j, \tilde{u})$ associated to (B, Z, X, u, η). By Definition 6.7, we may actually assume that $X' = X_1$ and $J = j$. As in Definition 6.2, let $N = \{(z, -u(z)) : z \in Z\}$ and recall that $X_1 = (B \oplus_1 X)/N$. Also let $Q\colon B \oplus_1 X \to X_1$ be the natural quotient map. We set $E = B$ and $\pi = Q$, and we claim that these choices satisfy all properties required in part (ii). Indeed, it is clear that π is a metric surjection and $\pi(0,x) = j(x)$ for all $x \in X$. Moreover, for every $b \in B$ and every $x \in X$ we have

$$\begin{aligned} \|Q(b,x)\| &= \inf\{\|(b,x) + (z, -u(z))\| : z \in Z\} \\ &\geqslant \inf\{\eta\|z + b\| + \|x - u(z)\| : z \in Z\} \\ &\geqslant \inf\{\eta\|z\| - \eta\|b\| + \|x\| - \eta\|z\| : z \in Z\} \\ &\geqslant \|x\| - \eta\|b\| \end{aligned}$$

as desired.

Conversely, let E be the Banach space and let $\pi\colon E \oplus_1 X \to X'$ be the metric surjection described in part (ii). Notice that if $(e,x) \in \mathrm{Ker}(\pi)$, then $0 = \|\pi(e,x)\| \geqslant \|x\| - \eta\|e\|$. Therefore,

$$\|x\| \leqslant \eta\|e\| \tag{6.1}$$

for all $(e,x) \in \mathrm{Ker}(\pi)$. Let Z be the projection of $\mathrm{Ker}(\pi)$ into E. Formally,

$$Z = \{e \in E : \exists x \in X \text{ with } \pi(e,x) = 0\}.$$

Fix $z \in Z$ and let $x, y \in X$ such that $\pi(z,x) = \pi(z,y) = 0$. Then $\pi(0, x-y) = 0$ which yields, by our assumptions, that $J(x-y) = 0$. Since J is an isometry, we conclude that $x = y$. What we have just proved is that there exists a map $w\colon Z \to X$ such that $\big(z, w(z)\big) \in \mathrm{Ker}(\pi)$ for every $z \in Z$. Clearly w is linear. Invoking (6.1), we obtain that $\|w\| \leqslant \eta$. Set $u = -w$ and let $(X_1, j, \tilde{u})$ be the canonical triple associated to (E, Z, X, u, η). We define $T\colon X_1 \to X'$ as follows. Let $\alpha \in X_1$ and select $e \in E$ and $x \in X$ such that $\alpha = Q(e,x)$. We set $T(\alpha) = \pi(e,x)$. It is easy to check that T is a well-defined isometry and that the following diagram commutes:

$$\begin{array}{ccccc} E & \xrightarrow{\tilde{u}} & X_1 & \xrightarrow{T} & X' \\ \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle J} \\ Z & \xrightarrow[u]{} & X & \xrightarrow[\mathrm{Id}]{} & X \end{array}$$

Hence, the embedding $J\colon X \to X'$ is η-admissible. The proof is completed. □

We are ready to begin the analysis of η-admissible embeddings.

6.2.1 Stability under compositions

Lemma 6.9. **[BP]** *Let X, X', X'' be Banach spaces and $\eta \leqslant 1$. Let $J\colon X \to X'$ and $J'\colon X' \to X''$ be isometric embeddings and assume that both J and J' are η-admissible. Then so is $J' \circ J$.*

Proof. We will use the characterization of η-admissibility established in Lemma 6.8. Specifically, by Lemma 6.8, there exist two Banach spaces E and E' and a pair $\pi\colon E \oplus_1 X \to X'$ and $\pi'\colon E' \oplus_1 X' \to X''$ of metric surjections satisfying the properties described in part (ii) of Lemma 6.8 for the isometric embeddings J and J' respectively. We set $E'' = E \oplus_1 E'$ and we define $\pi''\colon E'' \oplus_1 X \to X''$ by the rule

$$\pi''\big((e,e'),x\big) = \pi'\big(e', \pi(e,x)\big)$$

for every $(e,e') \in E''$ and every $x \in X$. We claim that the Banach E'' and the operator π'' witness that the isometric embedding $J' \circ J$ is η-admissible. To see this notice first that π'' is a metric surjection and $\pi''(0,x) = J'\big(J(x)\big)$. What remains is to show that for every $(e,e') \in E''$ and every $x \in X$ we have

$$\|\pi''\big((e,e'),x\big)\| \geqslant \|x\| - \eta\|(e,e')\| = \|x\| - \eta(\|e\| + \|e'\|).$$

Indeed, invoking the corresponding properties of π' an π respectively, we see that

$$\begin{aligned} \|\pi''\big((e,e'),x\big)\| &= \|\pi'\big(e', \pi(e,x)\big)\| \geqslant \|\pi(e,x)\| - \eta\|e'\| \\ &\geqslant \|x\| - \eta\|e\| - \eta\|e'\| = \|x\| - \eta(\|e\| + \|e'\|). \end{aligned}$$

The proof is completed. □

6.2.2 Stability under quotients

Lemma 6.10. **[BP]** *Let X, X' be two Banach spaces, let Y be a subspace of X and let $\eta \leqslant 1$. Let $J\colon X \to X'$ be an isometric embedding and assume that J is η-admissible. Then the induced isometric embedding $\bar{J}\colon X/Y \to X'/J(Y)$ is η-admissible.*

Proof. Let (B, Z, u, T) be the quadruple witnessing that the isometric embedding J is η-admissible. Let $(X_1, j, \tilde{u})$ be the canonical triple associated to (B, Z, X, u, η). Clearly we may assume that $X' = X_1$ and $J = j$. Thus, what we have to show is that the induced isometric embedding $\overline{j} : X/Y \to X_1/j(Y)$ is η-admissible.

Let $q\colon X \to X/Y$ be the natural quotient map and define $w\colon Z \to X/Y$ by $w = q\circ u$. Notice that $\|w\| \leqslant \eta$. Let $(X_2, j_2, \tilde{w})$ be the canonical triple associated to $(B, Z, X/Y, w, \eta)$. The proof will be completed once we show that there exists an isometry $S\colon X_2 \to X_1/j(Y)$ making the following diagram commutative:

$$\begin{array}{ccccc} B & \xrightarrow{\tilde{w}} & X_2 & \xrightarrow{S} & X_1/j(Y) \\ \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle j_2} & & \uparrow{\scriptstyle \bar{j}} \\ Z & \xrightarrow[w]{} & X/Y & \xrightarrow[\mathrm{Id}]{} & X/Y \end{array}$$

We define the desired isometry S as follows. Let $Q_2\colon B \oplus_1 (X/Y) \to X_2$ and $Q_1\colon B \oplus_1 X \to X_1$ be the natural quotient maps. Let $\alpha \in X_2$ be arbitrary. We select $b \in B$ and $\gamma \in X/Y$ such that $\alpha = Q_2(b, \gamma)$. There exists $x \in X$ such that $\gamma = x + Y$. We set $S(\alpha) = Q_1(b, x) + j(Y)$. It is easy to check that S is an isometry making the above diagram commutative. The proof is completed. □

6.2.3 Metric properties

The isomorphic properties of the resulting space in the Bourgain–Pisier construction are consequences of the metric properties of Kisliakov's embedding, and in particular, of the metric properties of η-admissible embeddings. We isolate, below, the crucial inequality satisfied by all η-admissible embeddings.

Lemma 6.11. *Let X, X' be Banach spaces, let $J\colon X \to X'$ be an isometric embedding and let $\eta \leqslant 1$. Assume that J is η-admissible. Let $q\colon X' \to X'/J(X)$ be the natural quotient map. Consider a finite sequence $\alpha_0, \dots, \alpha_k$ in X' and assume that $\alpha_0 + \dots + \alpha_k \in J(X)$. Then*

$$\|\alpha_0\| + \dots + \|\alpha_k\| \geqslant \|\alpha_0 + \dots + \alpha_k\| + (1-\eta)\cdot\big[\|q(\alpha_0)\| + \dots + \|q(\alpha_k)\|\big]. \quad (6.2)$$

Inequality (6.2) appears in the work of Bourgain and Pisier in a probabilistic form. Actually, this probabilistic form is easily seen to be equivalent to inequality (6.2). The reader will find more details in Appendix B.5. We proceed to the proof of Lemma 6.11.

Proof of Lemma 6.11. As in the proof of Lemma 6.10, we may assume that J is the isometric embedding of a canonical triple. Precisely, let (B, Z, u, T) be the quadruple witnessing that the isometric embedding J is η-admissible. Let $(X_1, j, \tilde{u})$ be the canonical triple associated to (B, Z, X, u, η). In what follows we will assume $X' = X_1$ and $J = j$. Therefore, $X' = X_1 = (B \oplus_1 X)/N$ where $N = \{(z, -u(z)) : z \in Z\}$. Let $Q\colon B \oplus_1 X \to X_1$ be the natural quotient map and recall that $j(x) = Q(0, x)$ for all $x \in X$.

Let $\varepsilon > 0$ be arbitrary. We select a finite sequence $x_0, \dots, x_k$ in X and a finite sequence $b_0, \dots, b_k$ in B such that for every $n \in \{0, \dots, k\}$ the following are satisfied.

(a) $\alpha_n = Q(b_n, x_n)$.

(b) $\|x_n\| + \|b_n\| - \varepsilon/(k+1) \leqslant \|\alpha_n\|$.

By our assumptions, we have $\alpha_0 + \cdots + \alpha_k \in j(X)$. Hence, there exists a vector $x \in X$ such that $\alpha_0 + \cdots + \alpha_k = j(x) = Q(0, x)$. It follows that

$$Q(b_0 + \cdots + b_k, x_0 + \cdots + x_k - x) = 0$$

and so, there exists a vector $z \in Z$ such that

(c) $b_0 + \cdots + b_k = z$ and

(d) $x_0 + \cdots + x_k - x = -u(z)$.

The above equalities and the fact that j is an isometric embedding imply that

$$\begin{aligned} \|x_0\| + \cdots + \|x_k\| &\geqslant \|x_0 + \cdots + x_k\| = \|x - u(z)\| \geqslant \|x\| - \|u(z)\| \\ &= \|j(x)\| - \|u(z)\| = \|\alpha_0 + \cdots + \alpha_k\| - \|u(z)\| \\ &\geqslant \|\alpha_0 + \cdots + \alpha_k\| - \eta\|z\| \\ &= \|\alpha_0 + \cdots + \alpha_k\| - \eta\|b_0 + \cdots + b_k\| \\ &\geqslant \|\alpha_0 + \cdots + \alpha_k\| - \eta\big[\|b_0\| + \cdots + \|b_k\|\big]. \qquad (6.3) \end{aligned}$$

Adding in both sides of inequality (6.3) the quantity $\|b_0\| + \cdots + \|b_k\|$ and taking into account property (b) above, we obtain that

$$\varepsilon + \|\alpha_0\| + \cdots + \|\alpha_k\| \geqslant \|\alpha_0 + \cdots + \alpha_k\| + (1 - \eta) \cdot \big[\|b_0\| + \cdots + \|b_k\|\big]. \quad (6.4)$$

Now notice that for every $n \in \{0, \ldots, k\}$ we have

(e) $\|q(\alpha_n)\| \leqslant \|b_n\|$.

Indeed, fix $n \in \{0, \ldots, k\}$ and observe that

$$\begin{aligned} \|q(\alpha_n)\| &= \inf\{\|Q(b_n, x_n) + j(x'')\| : x'' \in X\} \\ &= \inf\{\|Q(b_n, x_n) + Q(0, x'')\| : x'' \in X\} \leqslant \|Q(b_n, 0)\| \leqslant \|b_n\|. \end{aligned}$$

Plugging in (6.4) the estimate in (e), we see that

$$\varepsilon + \|\alpha_0\| + \cdots + \|\alpha_k\| \geqslant \|\alpha_0 + \cdots + \alpha_k\| + (1 - \eta) \cdot \big[\|q(\alpha_0)\| + \cdots + \|q(\alpha_k)\|\big].$$

Since $\varepsilon > 0$ was arbitrary, inequality (6.2) follows. The proof is completed. □

We close this section by recording the following special case of Lemma 6.11.

Corollary 6.12. *Let X, X' be Banach spaces, let $J : X \to X'$ be an isometric embedding and let $\eta \leqslant 1$. Assume that J is η-admissible. Let $q : X' \to X'/J(X)$ be the natural quotient map. Consider two vectors α and β in X' and assume that $\alpha + \beta \in J(X)$. Then*

$$\|\alpha\| + \|\beta\| \geqslant \|\alpha + \beta\| + (1 - \eta) \cdot \big[\|q(\alpha)\| + \|q(\beta)\|\big]. \qquad (6.5)$$

6.3 Inductive limits of finite-dimensional spaces

A *system of isometric embeddings* is a sequence (X_n, j_n) where (X_n) is a sequence of Banach spaces and $j_n\colon X_n \to X_{n+1}$ is an isometric embedding for every $n \in \mathbb{N}$. The *inductive limit* of a system (X_n, j_n) of isometric embeddings is a Banach space X defined as follows. First we consider the vector subspace of $\prod_n X_n$ consisting of all sequences (x_n) such that $j_n(x_n) = x_{n+1}$ for all n large enough. We equip this subspace with the semi-norm $\|(x_n)\| = \lim \|x_n\|$. Let $\mathcal{X}$ be the vector space obtained after passing to the quotient by the kernel of that semi-norm. The inductive limit X of the system (X_n, j_n) is then defined to be the completion of $\mathcal{X}$. Notice that there exists a sequence (J_n) of isometric embeddings $J_n\colon X_n \to X$ such that $J_{n+1} \circ j_n = J_n$ for every $n \in \mathbb{N}$ and if $E_n = J_n(X_n)$, then the union $\bigcup_n E_n$ is dense in X. Hence, in practice, we may do as if the sequence (X_n) was an increasing (with respect to inclusion) sequence of subspaces of a bigger space and we may identify the space X with the closure of the vector space $\bigcup_n X_n$.

The main result in this section is the following theorem due to Bourgain and Pisier.

Theorem 6.13. **[BP]** *Let $0 < \eta < 1$. Let (F_n, j_n) be a system of isometric embeddings where the sequence (F_n) consists of finite-dimensional Banach spaces and for every $n \in \mathbb{N}$ the isometric embedding $j_n\colon F_n \to F_{n+1}$ is η-admissible. Then the inductive limit of the system (F_n, j_n) has the Radon–Nikodym and the Schur properties.*

Theorem 6.13 is the basic tool for verifying the crucial properties of the Bourgain–Pisier construction. Actually in these notes we will not use the full power of Theorem 6.13 but only the fact that the inductive limit is a Schur space. Therefore, in this section we will present only the proof of this property. A proof that the inductive limit has the Radon–Nikodym property can be found in Appendix B.5.

We will need the following "unconditional" version of Mazur's theorem.

Lemma 6.14. *Let (z_k) be a weakly null sequence in a Banach space X. Then for every $\varepsilon > 0$ there exist $m_0 < m_1 < \cdots < m_j$ in $\mathbb{N}$ and $a_0, \dots, a_j \in \mathbb{R}^+$ with $\sum_{i=0}^{j} a_i = 1$ and such that*

$$\max\Big\{\Big\|\sum_{i=0}^{j} \epsilon_i a_i z_{m_i}\Big\| : \epsilon_0, \dots, \epsilon_j \in \{-1, 1\}\Big\} < \varepsilon. \tag{6.6}$$

Proof. Clearly we may assume that $X = \overline{\operatorname{span}}\{z_k : k \in \mathbb{N}\}$. By Theorem 1.8, we may also assume that X is a subspace of $C(2^{\mathbb{N}})$, and so, each z_k is a continuous function on $2^{\mathbb{N}}$. By Lebesgue's dominated convergence theorem, a sequence (x_k) in $C(2^{\mathbb{N}})$ is weakly null if and only if (x_k) is bounded and pointwise convergent to 0. Hence, setting $y_k = |z_k|$ for every $k \in \mathbb{N}$, we see that the sequence (y_k) is also weakly null. We apply Mazur's theorem to the sequence (y_k) and the given $\varepsilon > 0$ and we find $m_0 < \cdots < m_j$ in $\mathbb{N}$ and $a_0, \dots, a_j \in \mathbb{R}^+$ with $\sum_{i=0}^j a_i = 1$ and such that $\| \sum_{i=0}^j a_i y_{m_i} \| < \varepsilon$. Finally notice that

$$\max\Big\{ \Big\| \sum_{i=0}^{j} \epsilon_i a_i z_{m_i} \Big\|_\infty : \epsilon_0, \dots, \epsilon_j \in \{-1,1\} \Big\} \leqslant \Big\| \sum_{i=0}^{j} a_i y_{m_i} \Big\|_\infty < \varepsilon.$$

The proof is completed. □

We proceed to the proof of Theorem 6.13.

Proof of Theorem 6.13*: the Schur property.* Fix $0 < \eta < 1$. Let (F_n, j_n) be a system of isometric embeddings such that each F_n is finite-dimensional and for every $n \in \mathbb{N}$ the isometric embedding $j_n \colon F_n \to F_{n+1}$ is η-admissible. Let X be the inductive limit of the system (F_n, j_n). We view the sequence (F_n) as being an increasing (with respect to inclusion) sequence of finite-dimensional subspaces of X such that $\bigcup_n F_n$ is dense in X. For every $n \in \mathbb{N}$ by $q_n \colon X \to X/F_n$ we shall denote the natural quotient map, while for every pair $n, m \in \mathbb{N}$ with $n < m$ by $I(n,m) \colon F_n \to F_m$ we shall denote the inclusion operator. By our assumptions, the isometric embedding $I(n, n+1) \colon F_n \to F_{n+1}$ is η-admissible for every $n \in \mathbb{N}$. Hence, by Lemma 6.9, we see that the isometric embedding $I(n,m)$ is also η-admissible for every pair $n, m \in \mathbb{N}$ with $n < m$. This yields the following claim.

Claim 6.15. *Let $0 < \theta \leqslant 1$ and let (w_k) be a normalized sequence in X. Also let $\{n_0 < n_1 < \cdots\}$ be an infinite subset of $\mathbb{N}$. Assume that the following hold.*

(i) *$w_k \in F_{n_k}$ for every $k \in \mathbb{N}$.*

(ii) *$\|q_{n_k}(w_{k'})\| \geqslant \theta$ for every $k, k' \in \mathbb{N}$ with $k' > k$.*

Then for every $m_0 < \cdots < m_j$ in $\mathbb{N}$ and every $a_0, \dots, a_j$ in $\mathbb{R}$ there exist $\epsilon_0, \dots, \epsilon_j \in \{-1,1\}$ such that

$$\Big\| \sum_{i=0}^{j} \epsilon_i a_i w_{m_i} \Big\| \geqslant (1-\eta) \cdot \theta \cdot \sum_{i=0}^{j} |a_i|. \tag{6.7}$$

Proof of Claim 6.15. Fix $m_0 < \cdots < m_j$ in $\mathbb{N}$ and $a_0, \dots, a_j$ in $\mathbb{R}$. Recursively, we will select the signs $\epsilon_0, \dots, \epsilon_j$ in such a way that for every $l \leqslant j$ we have

$$\Big\| \sum_{i=0}^{l} \epsilon_i a_i w_{m_i} \Big\| \geqslant (1-\eta) \cdot \theta \cdot \sum_{i=0}^{l} |a_i|. \tag{6.8}$$

We set $\epsilon_0 = 1$. Assume that for some $l < j$ we have selected $\epsilon_0, \ldots, \epsilon_l \in \{-1, 1\}$ so that (6.8) is satisfied. We set

$$z = \sum_{i=0}^{l} \epsilon_i a_i w_{m_i}, \quad x = a_{l+1} w_{m_{l+1}}, \quad \alpha = z + x \quad \text{and} \quad \beta = z - x.$$

Notice that

$$\alpha, \beta \in F_{n_{m_{l+1}}} \quad \text{and} \quad \alpha + \beta = 2z \in F_{n_{m_l}}.$$

The inclusion operator $I(n_{m_l}, n_{m_{l+1}})$ is η-admissible. Therefore, by inequality (6.5) in Corollary 6.12, property (ii) in the statement of the claim and our inductive hypothesis, we obtain that

$$\|\alpha\| + \|\beta\| \geqslant 2 \cdot (1 - \eta) \cdot \theta \cdot \sum_{i=0}^{l+1} |a_i|.$$

Hence, there is $\epsilon_{l+1} \in \{-1, 1\}$ so that $\big\| \sum_{i=0}^{l+1} \epsilon_i a_i w_{m_i} \big\| \geqslant (1 - \eta) \cdot \theta \cdot \sum_{i=0}^{l+1} |a_i|$. The recursive selection is completed. The claim is proved. □

After this preliminary discussion we are ready to prove that the inductive limit X has the Schur property. We will argue by contradiction. So assume that there exists a normalized weakly null sequence (x_k) in X. By an obvious approximation argument, we may assume that

$$\{x_k : k \in \mathbb{N}\} \subseteq \bigcup_n F_n. \tag{6.9}$$

Claim 6.16. *For every $n \in \mathbb{N}$ we have* $\limsup \|q_n(x_k)\| \geqslant 1/3$.

Proof of Claim 6.16. Suppose not. Then, by passing to a subsequence of (x_k) if necessary, we may find a sequence (y_k) in F_n such that $\|x_k + y_k\| \leqslant 1/3$ for every $k \in \mathbb{N}$. The sequence (y_k) is bounded and the space F_n is finite-dimensional. Therefore, by passing to further subsequences, we may assume that there exists a vector $y \in F_n$ such that $y_k \to y$. The sequence $(x_k + y_k)$ is weakly convergent to the vector y, and so,

$$\|y\| \leqslant \limsup \|x_k + y_k\| \leqslant 1/3. \tag{6.10}$$

On the other hand for every $k \in \mathbb{N}$ we have

$$\|y_k\| = \|(x_k + y_k) - x_k\| \geqslant \|x_k\| - \|x_k + y_k\| = 1 - \|x_k + y_k\| \geqslant 2/3.$$

Hence $\|y\| \geqslant 2/3$ in contradiction with (6.10) above. The claim is proved. □

By inclusion (6.9) and Claim 6.16, we may select a subsequence (z_k) of (x_k) and an infinite subset $\{n_0 < n_1 < \cdots\}$ of $\mathbb{N}$ such that the following are satisfied.

(a) The sequence (z_k) is normalized and weakly null.

(b) $z_k \in F_{n_k}$ for every $k \in \mathbb{N}$.

(c) $\|q_{n_k}(z_{k'})\| \geqslant 1/4$ for every $k, k' \in \mathbb{N}$ with $k' > k$.

We have reached the contradiction. Indeed, by (a) above, we may apply Lemma 6.14 to the sequence (z_k) and $\varepsilon = (1-\eta)/8$. Hence, there exist $m_0 < \cdots < m_j$ in $\mathbb{N}$ and $a_0, \ldots, a_j \in \mathbb{R}^+$ with $\sum_{i=0}^{j} a_i = 1$ and such that

$$\max\Big\{ \Big\| \sum_{i=0}^{j} \epsilon_i a_i z_{m_i} \Big\| : \epsilon_0, \ldots, \epsilon_j \in \{-1, 1\} \Big\} < \frac{1-\eta}{8}.$$

This is clearly impossible by Claim 6.15. The proof of Theorem 6.13 is completed. □

6.4 The construction

This section is devoted to the proof of Theorem 6.1. So, let $\lambda > 1$ and let X be a separable Banach space. In the argument below we shall use the following simple fact.

Fact 6.17. *Let H be a finite-dimensional Banach space and let $\varepsilon > 0$. Then there exist $m \in \mathbb{N}$, a subspace Z of ℓ_∞^m and an isomorphism $T\colon Z \to H$ satisfying $\|z\| \leqslant \|T(z)\| \leqslant (1+\varepsilon)\|z\|$ for every $z \in Z$.*

We fix $0 < \eta < 1$ such that $\frac{1}{\lambda} < \eta < 1$. We also fix $\varepsilon > 0$ with $1 + \varepsilon < \lambda\eta$. Let (F_n) be an increasing sequence of finite-dimensional subspaces of X such that $\bigcup_n F_n$ is dense in X (the sequence (F_n) is not necessarily *strictly* increasing since we are not assuming that the space X is infinite-dimensional). Recursively, we shall construct

(C1) a system (E_n, j_n) of isometric embeddings, and

(C2) a sequence (G_n) of finite-dimensional spaces

such that for every $n \in \mathbb{N}$ the following are satisfied.

(P1) $G_n \subseteq E_n$ and $G_0 = \{0\}$.

(P2) The embedding $j_n\colon E_n \to E_{n+1}$ is η-admissible and $E_0 = X$.

(P3) $(j_{n-1} \circ \cdots \circ j_0)(F_{n-1}) \cup j_{n-1}(G_{n-1}) \subseteq G_n$ for every $n \geqslant 1$.

(P4) $d(G_n, \ell_\infty^{m_n}) \leqslant \lambda$ where $\dim(G_n) = m_n \geqslant n$.

As the first step is identical to the general one, we may assume that for some $k \in \mathbb{N}$ with $k \geqslant 1$ the spaces $G_0, \dots, G_k$ and $E_0, \dots, E_k$ as well as the η-admissible isometric embeddings $j_0, \dots, j_{k-1}$ have been constructed. Let H_k be the subspace of E_k spanned by $(j_{k-1} \circ \cdots \circ j_0)(F_k) \cup G_k$. (If $k = 0$, then we set $H_0 = F_0$.) Let m_k be the least integer with $m_k \geqslant k+1$ and for which there exist a subspace Z_k of $\ell_\infty^{m_k}$ and an isomorphism $T \colon Z_k \to H_k$ satisfying $\|z\| \leqslant \|T(z)\| \leqslant (1+\varepsilon)\|z\|$ for every $z \in Z_k$. By Fact 6.17, we see that m_k is well-defined. Define $u \colon Z_k \to E_k$ by

$$u(z) = \frac{1}{\lambda} T(z) \tag{6.11}$$

and notice that $u(Z_k) = H_k$, $\|u\| \leqslant \eta$ and $\|u^{-1}|_{H_k}\| \leqslant \lambda$. Let $(Y, j, \tilde{u})$ be the canonical triple associated to $(\ell_\infty^{m_k}, Z_k, u, E_k, \eta)$. We set $E_{k+1} = Y$, $j_{k+1} = j$ and $G_{k+1} = \tilde{u}(\ell_\infty^{m_k})$. By Proposition 6.3 and Lemma 6.4, the spaces G_{k+1} and E_{k+1}, and the embedding j_{k+1} satisfy (P1)–(P4) above. The construction is completed.

Now let Z be the inductive limit of the system (E_n, j_n). As we have remarked in Section 6.3, the sequence (E_n) can be identified with an increasing sequence of subspaces of Z. Under this point of view, we let $\mathcal{L}_\lambda[X]$ be the closure of $\bigcup_n G_n$. By property (P3), we see that the space $\mathcal{L}_\lambda[X]$ contains an isometric copy of X while, by property (P4) and Fact B.11, it follows that the space $\mathcal{L}_\lambda[X]$ is $\mathcal{L}_{\infty,\lambda+}$. What remains is to analyze the quotient $\mathcal{L}_\lambda[X]/X$.

To this end notice first that $\mathcal{L}_\lambda[X]/X$ naturally embeds into Z/X. For every $n \in \mathbb{N}$ let $J_n \colon E_n/X \to E_{n+1}/X$ be the isometric embedding induced by $j_n \colon E_n \to E_{n+1}$. Observe that the space Z/X is isometric to the inductive limit of the system $(E_n/X, J_n)$. By Lemma 6.10 and property (P2) above, we see that the isometric embedding J_n is η-admissible for every $n \in \mathbb{N}$. Finally, notice that each E_n/X is finite-dimensional. Indeed, by Proposition 6.3, for every $n \in \mathbb{N}$ the space E_{n+1}/E_n is isometric to $\ell_\infty^{m_n}/Z_n$. Using this our claim follows by a straightforward induction. Therefore, Theorem 6.13 can be applied to the system $(E_n/X, J_n)$ yielding, in particular, that the space Z/X has the Radon–Nikodym and the Schur properties. As these properties are inherited to subspaces and $\mathcal{L}_\lambda[X]/X$ is isometric to a subspace of Z/X, we conclude that the quotient $\mathcal{L}_\lambda[X]/X$ has the Radon–Nikodym and the Schur properties. The proof of Theorem 6.1 is completed.

6.5 Parameterizing the construction

This section is devoted to the proof of the following parameterized version of Theorem 6.1.

Theorem 6.18. **[D3]** *For every $\lambda > 1$ the set $\mathcal{L}_\lambda \subseteq \mathrm{SB} \times \mathrm{SB}$ defined by*

$$(X, Y) \in \mathcal{L}_\lambda \Leftrightarrow Y \textit{ is isometric to } \mathcal{L}_\lambda[X]$$

is analytic.

Proof. Let $\lambda > 1$ be given and fix $\eta > 0$ and $\varepsilon > 0$ such that $\frac{1}{\lambda} < \eta < 1$ and $1 + \varepsilon < \lambda\eta$. Below we will adopt the following notational conventions. By Ω we shall denote the Borel subset of $\mathrm{SB} \times \mathrm{SB} \times C(2^{\mathbb{N}})^{\mathbb{N}} \times C(2^{\mathbb{N}})^{\mathbb{N}}$ defined by

$$\big(X, Y, (x_n), (y_n)\big) \in \Omega \quad \Leftrightarrow \quad \begin{array}{l} \forall n \in \mathbb{N}\ (x_n \in X \text{ and } y_n \in Y) \text{ and} \\ (x_n) \text{ is dense in } X,\ (y_n) \text{ is dense in } Y, \\ Y \subseteq X \text{ and } \forall n \in \mathbb{N}\ \exists m \in \mathbb{N} \text{ with } y_n = x_m. \end{array}$$

That is, an element $\big(X, Y, (x_n), (y_n)\big) \in \Omega$ codes a separable Banach space X, a dense sequence (x_n) in X, a subspace Y of X and a subsequence (y_n) of (x_n) which is dense in Y. Given $\omega = \big(X, Y, (x_n), (y_n)\big) \in \Omega$ we set $p_0(\omega) = X$ and $p_1(\omega) = Y$. We will reserve the letter t to denote elements of $\Omega^{<\mathbb{N}}$. The letter α shall be used to denote elements of $\Omega^{\mathbb{N}}$. For every nonempty $t \in \Omega^{<\mathbb{N}}$ and every $i < |t|$ we set $X_i^t = p_0\big(t(i)\big)$ and $Y_i^t = p_1\big(t(i)\big)$. Respectively, for every $\alpha \in \Omega^{\mathbb{N}}$ and every $i \in \mathbb{N}$ we set $X_i^\alpha = p_0\big(\alpha(i)\big)$ and $Y_i^\alpha = p_1\big(\alpha(i)\big)$. If X, Y and Z are nonempty sets and $f\colon X \times Y \to Z$ is a map, then for every $x \in X$ by f^x we shall denote the function $f^x\colon Y \to Z$ defined by $f^x(y) = f(x, y)$ for every $y \in Y$. Finally, by $d_m\colon \mathrm{SB} \to C(2^{\mathbb{N}})$ $(m \in \mathbb{N})$ we denote the sequence of Borel maps described in property (P2) in Section 2.1.1.

The proof of Theorem 6.18 is based on the fact that we can appropriately encode the Bourgain–Pisier construction so that it can be performed "uniformly" in X. To this end we introduce the following terminology.

A. Let $k \in \mathbb{N}$ with $k \geqslant 2$. A *code of length* k is a pair (C, ϕ) where C is a Borel subset of Ω^k and $\phi\colon C \times C(2^{\mathbb{N}}) \to C(2^{\mathbb{N}})$ is a Borel map such that for every $t \in C$ the following are satisfied.

(C1) For every $i < k$ the space Y_i^t is finite-dimensional and $Y_0^t = \{0\}$.

(C2) The map $\phi^t\colon X_{k-2}^t \to C(2^{\mathbb{N}})$ is a linear isometric embedding satisfying $\phi^t(X_{k-2}^t) \subseteq X_{k-1}^t$ and $\phi^t(Y_{k-2}^t) \subseteq Y_{k-1}^t$.

The *code of length* 1 is the pair (C_1, ϕ_1) where $C_1 \subseteq \Omega$ and $\phi_1\colon C_1 \times C(2^{\mathbb{N}}) \to C(2^{\mathbb{N}})$ are defined by

$$t = \big(X, Y, (x_n), (y_n)\big) \in C_1 \Leftrightarrow Y = \{0\}$$

and $\phi_1(t, x) = x$ for every $t \in C_1$ and every $x \in C(2^{\mathbb{N}})$. Clearly C_1 is Borel and ϕ_1 is a Borel map. Notice that for every $X \in \mathrm{SB}$ there exists $t \in C_1$ with $X = X_0^t$.

B. Let $\{(C_k,\phi_k) : k \geqslant 1\}$ be a sequence such that for every $k \geqslant 1$ the pair (C_k,ϕ_k) is a code of length k. We say that the sequence $\{(C_k,\phi_k) : k \geqslant 1\}$ is a *tree-code* if for every $k,m \in \mathbb{N}$ with $1 \leqslant k \leqslant m$ we have $C_k = \big\{t|k : t \in C_m\big\}$. The *body* $\mathcal{C}$ of a tree-code $\{(C_k,\phi_k) : k \geqslant 1\}$ is defined by

$$\mathcal{C} = \big\{\alpha \in \Omega^{\mathbb{N}} : \alpha|k \in C_k \ \forall k \geqslant 1\big\}.$$

Clearly $\mathcal{C}$ is a Borel subset of $\Omega^{\mathbb{N}}$.

Let $\{(C_k,\phi_k) : k \geqslant 1\}$ be a tree-code and let $\mathcal{C}$ be its body. For every $k \geqslant 1$ the map ϕ_k induces a map $\Phi_k \colon \mathcal{C} \times C(2^{\mathbb{N}}) \to C(2^{\mathbb{N}})$ defined by $\Phi_k(\alpha,x) = \phi_k(\alpha|k,x)$ for every $\alpha \in \mathcal{C}$ and every $x \in C(2^{\mathbb{N}})$. We need to introduce two more maps. First, for every $n,m \in \mathbb{N}$ with $n < m$ we define $\Phi_{n,m} \colon \mathcal{C} \times C(2^{\mathbb{N}}) \to C(2^{\mathbb{N}})$ recursively by the rule $\Phi_{n,n+1}(\alpha,x) = \Phi_{n+2}(\alpha,x)$ and $\Phi_{n,m+1}(\alpha,x) = \Phi_{m+2}\big(\alpha,\Phi_{n,m}(\alpha,x)\big)$. Also we set $J_n = \Phi_{n+2}$ for every $n \in \mathbb{N}$. We isolate, for future use, the following fact concerning these maps. Its proof is a straightforward consequence of the relevant definitions and of condition (C2) above.

Fact 6.19. *Let $\{(C_k,\phi_k) : k \geqslant 1\}$ be a tree-code and let $\mathcal{C}$ be its body. Then the following are satisfied.*

(i) *For every $n,k,m \in \mathbb{N}$ with $k \geqslant 1$ and $n < m$ the maps Φ_k and $\Phi_{n,m}$ are Borel. Moreover, for every $\alpha \in \mathcal{C}$ we have $\Phi^{\alpha}_{n,m}(X^{\alpha}_n) \subseteq X^{\alpha}_m$ and $\Phi^{\alpha}_{n,m}(Y^{\alpha}_n) \subseteq Y^{\alpha}_m$.*

(ii) *Let $\alpha \in \mathcal{C}$. Then for every $n \in \mathbb{N}$ the map $J^{\alpha}_n|_{X^{\alpha}_n}$ is a linear isometric embedding satisfying $J^{\alpha}_n(X^{\alpha}_n) \subseteq X^{\alpha}_{n+1}$ and $J^{\alpha}_n(Y^{\alpha}_n) \subseteq Y^{\alpha}_{n+1}$.*

C. Let $\{(C_k,\phi_k) : k \geqslant 1\}$ be a tree-code and let $\mathcal{C}$ be its body. Also let $\alpha \in \mathcal{C}$. Consider the sequence $(X^{\alpha}_0, Y^{\alpha}_0, X^{\alpha}_1, Y^{\alpha}_1, \dots)$ and notice that Y^{α}_n is a finite-dimensional subspace of X^{α}_n for every $n \in \mathbb{N}$. In the coding we are developing the sequences (X^{α}_n) and (Y^{α}_n) will correspond to the sequences (E_n) and (G_n) obtained by the Bourgain–Pisier construction performed to the space $X = X^{\alpha}_0$. This is made precise using the auxiliary concept of λ-coherence which we are about to introduce.

Let $\alpha \in \mathcal{C}$ be arbitrary. For every $n \in \mathbb{N}$ let $F_n(X^{\alpha}_0) = \overline{\mathrm{span}}\{d_i(X^{\alpha}_0) : i \leqslant n\}$. Clearly $\big(F_n(X^{\alpha}_0)\big)$ is an increasing sequence of finite-dimensional subspaces of X^{α}_0 with $\bigcup_n F_n(X^{\alpha}_0)$ dense in X^{α}_0. Let $(E^{\alpha}_n, j^{\alpha}_n)$ be the system of isometric embeddings and let (G^{α}_n) be the sequence of finite-dimensional spaces obtained by performing the construction described in Section 6.4 to the space X^{α}_0, the sequence $\big(F_n(X^{\alpha}_0)\big)$ and the numerical parameters λ, η and ε. We say that the tree-code $\{(C_k,\phi_k) : k \geqslant 1\}$ is λ-*coherent* if for every $\alpha \in \mathcal{C}$ there exists a sequence $T^{\alpha}_n \colon X^{\alpha}_n \to E^{\alpha}_n$ $(n \geqslant 1)$ of isometries such that $G^{\alpha}_n = T^{\alpha}_n(Y^{\alpha}_n)$ for every

$n \geqslant 1$ and making the following diagram commutative:

$$\begin{array}{ccccccccc}
E_0^\alpha & \xrightarrow{j_0^\alpha} & E_1^\alpha & \xrightarrow{j_1^\alpha} & E_2^\alpha & \xrightarrow{j_2^\alpha} & E_3^\alpha & \xrightarrow{j_3^\alpha} & \cdots \\
\uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle T_1^\alpha} & & \uparrow{\scriptstyle T_2^\alpha} & & \uparrow{\scriptstyle T_3^\alpha} & & \\
X_0^\alpha & \xrightarrow[J_0^\alpha|_{X_0^\alpha}]{} & X_1^\alpha & \xrightarrow[J_1^\alpha|_{X_1^\alpha}]{} & X_2^\alpha & \xrightarrow[J_2^\alpha|_{X_2^\alpha}]{} & X_3^\alpha & \xrightarrow[J_3^\alpha|_{X_3^\alpha}]{} & \cdots
\end{array}$$

The basic property guaranteed by the above requirements is isolated in the following fact (the proof is straightforward).

Fact 6.20. *Let $\{(C_k, \phi_k) : k \geqslant 1\}$ be a λ-coherent tree-code and let $\mathcal{C}$ be its body. Also let $\alpha \in \mathcal{C}$. Then the inductive limit of the system of embeddings $(Y_n^\alpha, J_n^\alpha|_{Y_n^\alpha})$ is isometric to the space $\mathcal{L}_\lambda[X_0^\alpha]$.*

We are ready to state the main technical step in the proof of Theorem 6.18.

Lemma 6.21. **[D3]** *There exists a λ-coherent tree-code $\{(C_k, \phi_k) : k \geqslant 1\}$.*

Granting Lemma 6.21 the proof of Theorem 6.18 is completed as follows. Let $\{(C_k, \phi_k) : k \geqslant 1\}$ be the λ-coherent tree-code obtained above. Denote by $\mathcal{C}$ its body. By Fact 6.20, we have

$$\begin{aligned}(X, Y) \in \mathcal{L}_\lambda \quad \Leftrightarrow \quad & \exists \alpha \in \Omega^{\mathbb{N}} \text{ with } \alpha \in \mathcal{C},\ X = X_0^\alpha \text{ and such that } Y \text{ is isometric} \\ & \text{to the inductive limit of the system } (Y_n^\alpha, J_n^\alpha|_{Y_n^\alpha}).\end{aligned}$$

Let $\alpha \in \mathcal{C}$. There is a canonical dense sequence in the inductive limit Z^α of the system $(Y_n^\alpha, J_n^\alpha|_{Y_n^\alpha})$. Indeed, by the discussion in Section 6.3, the sequence of spaces (Y_n^α) can be identified with an increasing sequence of subspaces of Z^α. Under this point of view, the sequence $\big(d_m(Y_n^\alpha) : n, m \in \mathbb{N}\big)$ is a dense sequence in Z^α. Let $\{(n_i, m_i) : i \in \mathbb{N}\}$ be an enumeration of the set $\mathbb{N} \times \mathbb{N}$ such that $\max\{n_i, m_i\} \leqslant i$ for every $i \in \mathbb{N}$. It follows that

$$\begin{aligned}(X, Y) \in \mathcal{L}_\lambda \quad \Leftrightarrow \quad & \exists (y_i) \in C(2^{\mathbb{N}})^{\mathbb{N}}\ \exists \alpha \in \Omega^{\mathbb{N}} \text{ with } \alpha \in \mathcal{C},\ X = X_0^\alpha \text{ and} \\ & Y = \overline{\mathrm{span}}\{y_i : i \in \mathbb{N}\} \text{ and } \forall l \in \mathbb{N}\ \forall b_0, \dots, b_l \in \mathbb{Q} \\ & \Big\| \sum_{i=0}^{l} b_i y_i \Big\| = \Big\| \sum_{i=0}^{l} b_i \Phi_{n_i, l+1}\big(\alpha, d_{m_i}(Y_{n_i}^\alpha)\big) \Big\|.\end{aligned}$$

Invoking part (i) of Fact 6.19, we see that the above formula gives an analytic definition of the set $\mathcal{L}_\lambda$, as desired.

It remains to prove Lemma 6.21. To this end we need the following easy fact.

Fact 6.22. *Let S be a standard Borel space, let X be a Polish space and let $f_n : S \to X$ $(n \in \mathbb{N})$ be a sequence of Borel maps. Then the map $F : S \to F(X)$, defined by $F(s) = \overline{\{f_n(s) : n \in \mathbb{N}\}}$ for every $s \in S$, is Borel.*

We are ready to proceed to the proof of Lemma 6.21.

Proof of Lemma 6.21. The λ-coherent tree-code $\{(C_k, \phi_k) : k \geqslant 1\}$ will be constructed by recursion. For $k = 1$ let (C_1, ϕ_1) be the code of length 1 defined in the beginning of the proof of Theorem 6.18. Assume that for some $k \geqslant 1$ and every $l \leqslant k$ we have constructed the code (C_l, ϕ_l) of length l. We will construct the code (C_{k+1}, ϕ_{k+1}) of length $k + 1$.

First, we define recursively a family of Borel functions $f_l \colon C_k \times C(2^{\mathbb{N}}) \to C(2^{\mathbb{N}})$ $(1 \leqslant l \leqslant k)$ by the rule $f_1(t, x) = x$ and $f_{l+1}(t, x) = \phi_{l+1}\big(t|l+1, f_l(t, x)\big)$. Notice that $f^t_k(X^t_0) \subseteq X^t_{k-1}$ for every $t \in C_k$. Also let $F_{k-1} \colon \mathrm{SB} \to \mathrm{SB}$ and $H_k \colon C_k \to \mathrm{SB}$ be defined by $F_{k-1}(X) = \overline{\mathrm{span}}\{d_i(X) : i \leqslant k-1\}$ and

$$H_k(t) = \overline{\mathrm{span}}\big\{Y^t_{k-1} \cup f^t_k\big(F_{k-1}(X^t_0)\big)\big\}$$

respectively. Observe that for every $X \in \mathrm{SB}$ and every $t \in C_k$ the spaces $F_{k-1}(X)$ and $H_k(t)$ are both finite-dimensional subspaces of X and X^t_{k-1} respectively.

Claim 6.23. *The maps F_{k-1} and H_k are Borel.*

Proof of Claim 6.23. For every $s \in \mathbb{Q}^k$ consider the map $f_s \colon \mathrm{SB} \to C(2^{\mathbb{N}})$ defined by $f_s(X) = \sum_{i=0}^{k-1} s(i) d_i(X)$. Clearly f_s is Borel. Notice that $F_{k-1}(X)$ is equal to the closure of the set $\{f_s(X) : s \in \mathbb{Q}^k\}$. Invoking Fact 6.22, the Borelness of the map F_{k-1} follows. The proof that H_k is also Borel proceeds similarly. The claim is proved. □

We fix a dense sequence (σ_i) in $2^{\mathbb{N}}$. For every $d \in \mathbb{N}$ with $d \geqslant k$ we define an operator $v_d \colon C(2^{\mathbb{N}}) \to \ell^d_\infty$ by

$$v_d(f) = \big(f(\sigma_0), \dots, f(\sigma_{d-1})\big).$$

Notice that $\|v_d(f)\| \leqslant \|f\|$. Moreover, observe that the map $C(2^{\mathbb{N}}) \ni f \mapsto \|v_d(f)\|$ is continuous. For every $d \geqslant k$ let B_d be the subset of C_k defined by

$$\begin{aligned} t \in B_d \quad &\Leftrightarrow \quad \forall f \in H_k(t) \text{ we have } \|f\| \leqslant (1+\varepsilon)\|v_d(f)\| \\ &\Leftrightarrow \quad \forall n \in \mathbb{N} \text{ we have } \|d_n(H_k(t))\| \leqslant (1+\varepsilon)\big\|v_d\big(d_n(H_k(t))\big)\big\|. \end{aligned}$$

By the above formula, we see that B_d is Borel. Also observe that $C_k = \bigcup_{d \geqslant k} B_d$. We define recursively a partition $\{P_d : d \geqslant k\}$ of C_k by the rule $P_k = B_k$ and $P_{d+1} = B_{d+1} \setminus (P_k \cup \dots \cup P_d)$. Notice that P_d is a Borel subset of B_d.

Let $d \geqslant k$ be arbitrary. Let E be the Banach space $C(2^{\mathbb{N}}) \oplus_1 \ell^d_\infty$. Consider the map $N \colon P_d \to \mathrm{Subs}(E)$ defined by

$$N(t) = \big\{(-f, \lambda v_d(f)) : f \in H_k(t)\big\}.$$

Arguing as in Claim 6.23, it is easy to see that N is Borel. Let $\mathbf{d}_m\colon \mathrm{Subs}(E)\to E$ ($m\in\mathbb{N}$) be the sequence of Borel maps obtained by property (P2) in Section 2.1.1 applied for $X=E$. We recall that by $d_m\colon C(2^{\mathbb{N}})\to C(2^{\mathbb{N}})$ ($m\in\mathbb{N}$) we denote the corresponding sequence obtained for $X=C(2^{\mathbb{N}})$. We may, and we will, assume that $d_0(X)=0$ and $\mathbf{d}_0(E')=0$ for every $X\in\mathrm{SB}$ and every $E'\in\mathrm{Subs}(E)$. We fix a countable dense subset (r_m) of ℓ_∞^d such that $r_0=0$. Let $Q\colon P_d\times E\to\mathbb{R}$ be the map

$$Q(t,e)=\inf\big\{\|e+\mathbf{d}_m\big(N(t)\big)\| : m\in\mathbb{N}\big\}.$$

Clearly Q is Borel. We fix a bijection $\langle\cdot,\cdot\rangle\colon\mathbb{N}\times\mathbb{N}\to\mathbb{N}$. For every $n\in\mathbb{N}$ by m_n^0 and m_n^1 we shall denote the unique integers satisfying $n=\langle m_n^0,m_n^1\rangle$. We define $C_d\subseteq\Omega^{k+1}$ by

$$\begin{aligned} t'\in C_d \quad\Leftrightarrow\quad & t'|k\in P_d \text{ and if } t'(k)=\big(X,Y,(x_n),(y_n)\big) \text{ and } t=t'|k,\\ & \text{then } \forall i\in\mathbb{N}\ \forall b_0,\dots,b_i\in\mathbb{Q} \text{ we have}\\ & \Big\|\sum_{n=0}^{i} b_n x_n\Big\| = Q\Big(t,\sum_{n=0}^{i} b_n\big(d_{m_n^0}(X_{k-1}^t),r_{m_n^1}\big)\Big) \text{ and}\\ & \forall n\in\mathbb{N} \text{ we have } y_n=x_{\langle 0,n\rangle}. \end{aligned}$$

Clearly the above formula defines a Borel subset of Ω^{k+1}. Also observe that $C_d\cap C_{d'}=\emptyset$ if $d\neq d'$.

Let us comment on some properties of the set C_d. Fix $t'\in C_d$ and set $t=t'|k$. By definition, we have $t\in P_d\subseteq B_d$. It follows that the operator $v_d\colon H_k(t)\to\ell_\infty^d$ is an isomorphic embedding satisfying $\|v_d(x)\|\leqslant\|x\|\leqslant(1+\varepsilon)\|v_d(x)\|$ for every $x\in H_k(t)$. We set $Z_t=v_d\big(H_k(t)\big)$ and we define $u\colon Z_t\to X_{k-1}^t$ by

$$u(z)=\frac{1}{\lambda}\big(v_d|_{H_k(t)}\big)^{-1}(z).$$

Notice that $\|u\|\leqslant\eta$. As in Definition 6.2, let $(X_1,j,\tilde{u})$ be the canonical triple associated to $(\ell_\infty^d,Z_t,u,X_{k-1}^t,\eta)$. There is a natural way to select a dense sequence in X_1. Indeed, let $Q_t\colon\ell_\infty^d\oplus_1 X_{k-1}^t\to X_1$ be the natural quotient map. Setting $\alpha_n=Q_t\big((r_{m_n^1},d_{m_n^0}(X_{k-1}^t))\big)$ for every $n\in\mathbb{N}$, we see that the sequence (α_n) is dense in X_1. Let $t'(k)=\big(X_k^{t'},Y_k^{t'},(x_n),(y_n)\big)$. By the definition of the set C_d, it follows that the map

$$X_1\ni\alpha_n\mapsto x_n\in X_k^{t'}$$

can be extended to a linear isometry $T_{t'}\colon X_1\to X_k^{t'}$. In other words, we have the following commutative diagram:

$$\begin{array}{ccccc} \ell_\infty^d & \xrightarrow{\ \tilde{u}\ } & X_1 & \xrightarrow{\ T_{t'}\ } & X_k^{t'}\\ \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle j} & & \\ Z_t & \xrightarrow[\ u\ ]{} & X_{k-1}^t & & \end{array}$$

It is clear from what we have said that the map

$$X^t_{k-1} \ni d_n(X^t_{k-1}) \mapsto x_{\langle n,0\rangle} \in X^{t'}_k$$

can be also extended to a linear isometric embedding $J^{t'}\colon X^t_{k-1} \to X^{t'}_k$ satisfying $J^{t'}(Y^t_{k-1}) \subseteq Y^{t'}_k$. Moreover, it easy to see that this extension can be done "uniformly" in t'. Precisely, there exists a Borel map $\phi_d\colon C_d \times C(2^{\mathbb{N}}) \to C(2^{\mathbb{N}})$ such that $\phi_d(t',x) = J^{t'}(x)$ for every $t' \in C_d$ and every $x \in X^{t'|k}_{k-1} = X^{t'}_{k-1}$.

We are finally in the position to construct the code (C_{k+1}, ϕ_{k+1}) of length $k+1$. First we set

$$C_{k+1} = \bigcup_{d\geqslant k} C_d.$$

As we have already remarked the sets $(C_d)_{d\geqslant k}$ are pairwise disjoint. We define $\phi_{k+1}\colon C_{k+1} \times C(2^{\mathbb{N}}) \to C(2^{\mathbb{N}})$ as follows. Let $(t',x) \in C_{k+1} \times C(2^{\mathbb{N}})$ and let d be the unique integer with $t' \in C_d$. We set $\phi_{k+1}(t',x) = \phi_d(t',x)$. Clearly the pair (C_{k+1}, ϕ_{k+1}) is a code of length $k+1$.

This completes the recursive construction of the family $\{(C_k,\phi_k) : k \geqslant 1\}$. The fact that the family $\{(C_k,\phi_k) : k \geqslant 1\}$ is a tree-code follows immediately by the definition of the set C_d above. Moreover, as one can easily realize, the tree-code $\{(C_k,\phi_k) : k \geqslant 1\}$ is in addition λ-coherent. The proof of Lemma 6.21 is completed. □

As we have already indicated above, having completed the proof of Lemma 6.21 the proof of Theorem 6.18 is also completed. □

6.6 Consequences

This section is devoted to applications of Theorems 6.1 and 6.18. To this end we will need a result on quotient spaces which is of independent interest.

6.6.1 A result on quotient spaces

We begin by introducing some pieces of notation and some terminology. Let $2^{<\mathbb{N}}$ be the Cantor tree. For every $s,t \in 2^{<\mathbb{N}}$ let $s \wedge t$ denote the $\sqsubset$-maximal node w of $2^{<\mathbb{N}}$ with $w \sqsubseteq s$ and $w \sqsubseteq t$. If $s,t \in 2^{<\mathbb{N}}$ are incomparable with respect to $\sqsubseteq$, then we write $s \prec t$ provided that $(s\wedge t)^\frown 0 \sqsubseteq s$ and $(s\wedge t)^\frown 1 \sqsubseteq t$. We say that a subset D of $2^{<\mathbb{N}}$ is a *dyadic subtree* of $2^{<\mathbb{N}}$ if D can be written in the form $D = \{s_t : t \in 2^{<\mathbb{N}}\}$ so that for every $t_1,t_2 \in 2^{<\mathbb{N}}$ we have $t_1 \sqsubset t_2$ (respectively, $t_1 \prec t_2$) if and only if $s_{t_1} \sqsubset s_{t_2}$ (respectively, $s_{t_1} \prec s_{t_2}$). It is easy to see that such a representation of D as $\{s_t : t \in 2^{<\mathbb{N}}\}$ is unique. In the sequel when we write $D = \{s_t : t \in 2^{<\mathbb{N}}\}$, where D is a dyadic subtree, we will assume that this is the canonical representation of D described above.

Let $\mathcal{O} = \{\emptyset\} \cup \{t^\frown 0 : t \in 2^{<\mathbb{N}}\}$. Namely, $\mathcal{O}$ is the subset of the Cantor tree consisting of all sequences ending with 0. If $D = \{s_t : t \in 2^{<\mathbb{N}}\}$ is a dyadic subtree of $2^{<\mathbb{N}}$, then we set $\mathcal{O}_D = \{s_t : t \in \mathcal{O}\}$. Let $h_D \colon \mathcal{O}_D \to \mathbb{N}$ be the unique bijection satisfying $h_D(s_{t_1}) < h_D(s_{t_2})$ if either $|t_1| < |t_2|$, or $|t_1| = |t_2|$ and $t_1 \prec t_2$. By $h \colon \mathcal{O} \to \mathbb{N}$ we shall denote the bijection corresponding to the Cantor tree itself.

Proposition 6.24. **[D3]** *Let E be a minimal Banach space not containing ℓ_1. Also let X be a Banach space and let Y be a subspace of X. Assume that the quotient X/Y has the Schur property. Then the following are satisfied.*

(i) *If Y is non-universal, then so is X.*

(ii) *If Y does not contain an isomorphic copy of E, then neither X does.*

Proof. (i) This part is essentially a consequence of a result due to Lindenstrauss and Pełczyński [LP2] asserting that the property of not containing an isomorphic copy of $C(2^{\mathbb{N}})$ is a three-space property. However, we shall give a proof for this special case which is more direct.

For every $t \in 2^{<\mathbb{N}}$ let $V_t = \{\sigma \in 2^{\mathbb{N}} : t \sqsubset \sigma\}$; that is, V_t is the clopen subset of $2^{\mathbb{N}}$ determined by the node t. We set $f_t = \mathbf{1}_{V_t}$. Clearly $f_t \in C(2^{\mathbb{N}})$ and $\|f_t\|_\infty = 1$. Let (t_n) be the enumeration of the set $\mathcal{O}$ according to the bijection h introduced above, and consider the corresponding sequence (f_{t_n}). The main properties of the sequence (f_{t_n}) are summarized in the following claim.

Claim 6.25. *The following hold.*

(i) *The sequence (f_{t_n}) is a normalized monotone Schauder basis of $C(2^{\mathbb{N}})$.*

(ii) *Let $D = \{s_t : t \in 2^{<\mathbb{N}}\}$ be a dyadic subtree of $2^{<\mathbb{N}}$ and let (s_n) be the enumeration of the set $\mathcal{O}_D$ according to h_D. Then the corresponding sequence (f_{s_n}) is 1-equivalent to the basis (f_{t_n}).*

(iii) *For every $t \in 2^{<\mathbb{N}}$ there exists a sequence (w_n) in $2^{<\mathbb{N}}$ with $t \sqsubset w_n$ for every $n \in \mathbb{N}$ and such that the sequence (f_{w_n}) is weakly null.*

Proof of Claim 6.25. The proof of part (i) is similar to the proof of Claim 5.13. Indeed, consider the sequence (f_{t_n}) and notice that $\overline{\mathrm{span}}\{f_{t_n} : n \in \mathbb{N}\} = C(2^{\mathbb{N}})$. To see that (f_{t_n}) is a monotone basic sequence let $k, m \in \mathbb{N}$ with $k < m$ and $a_0, \dots, a_m \in \mathbb{R}$. There exists $\sigma \in 2^{\mathbb{N}}$ such that

$$\Big\| \sum_{n=0}^{k} a_n f_{t_n} \Big\|_\infty = \Big| \sum_{n=0}^{k} a_n f_{t_n}(\sigma) \Big|.$$

Let $l, j \in \mathbb{N}$ with $t_l \sqsubset t_j$ (by the properties of h this implies that $l < j$). There exists a node $s \in 2^{<\mathbb{N}}$ with $t_l \sqsubset s$, $|s| = |t_j|$ and such that $f_{t_j}(x) = 0$ for every $x \in V_s$. Using this observation we see that there exists $\tau \in 2^{\mathbb{N}}$ such that $f_{t_n}(\tau) = f_{t_n}(\sigma)$ if $0 \leqslant n \leqslant k$ while $f_{t_n}(\tau) = 0$ if $k < n \leqslant m$. Therefore,

$$\Big\| \sum_{n=0}^{k} a_n f_{t_n} \Big\|_\infty = \Big| \sum_{n=0}^{k} a_n f_{t_n}(\sigma) \Big| = \Big| \sum_{n=0}^{m} a_n f_{t_n}(\tau) \Big| \leqslant \Big\| \sum_{n=0}^{m} a_n f_{t_n} \Big\|_\infty.$$

This shows that (f_{t_n}) is a normalized monotone Schauder basis of $C(2^{\mathbb{N}})$. Part (ii) follows using essentially the same argument. Finally, part (iii) is an immediate consequence of the relevant definitions. The claim is proved. □

After this preliminary discussion we are ready to proceed to the proof of part (i). Clearly it is enough to show that if the space X contains an isomorphic copy of $C(2^{\mathbb{N}})$, then so does Y. So, let Z be a subspace of X which is isomorphic to $C(2^{\mathbb{N}})$. We fix an isomorphism $T\colon C(2^{\mathbb{N}}) \to Z$ and we set $K = \|T\| \cdot \|T^{-1}\|$. Also let $Q\colon X \to X/Y$ be the natural quotient map. The basic step for constructing a subspace Y' of Y which is isomorphic to $C(2^{\mathbb{N}})$ is given in the following claim.

Claim 6.26. *Let (z_n) be a normalized weakly null sequence in Z and let $r > 0$ be arbitrary. Then there exist $k \in \mathbb{N}$ and a vector $y \in Y$ such that $\|z_k - y\| < r$.*

Proof of Claim 6.26. Consider the sequence $\big(Q(z_n)\big)$. By our assumptions, it is weakly null. The space X/Y has the Schur property. Hence, $\lim \|Q(z_n)\| = 0$. Let $k \in \mathbb{N}$ with $\|Q(z_k)\| < r$. By definition, there exists a vector $y \in Y$ such that $\|Q(z_k)\| \leqslant \|z_k - y\| < r$. The claim is proved. □

Using part (iii) of Claim 6.25 and Claim 6.26, we may construct, recursively, a dyadic subtree $D = \{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ and a family $\{y_t : t \in 2^{<\mathbb{N}}\}$ in Y such that, setting $z_t = T(f_{s_t})/\|T(f_{s_t})\|$ for every $t \in 2^{<\mathbb{N}}$, we have

$$\sum_{t \in 2^{<\mathbb{N}}} \|z_t - y_t\| < \frac{1}{2K}.$$

By [LT, Proposition 1.a.9] and part (ii) of Claim 6.25, if (t_n) is the enumeration of the set $\mathcal{O}$ according to the bijection h, then the corresponding sequence (y_{t_n}) is equivalent to the sequence (f_{t_n}). By part (i) of Claim 6.25, it follows that the subspace $Y' = \overline{\mathrm{span}}\{y_{t_n} : n \in \mathbb{N}\}$ of Y is isomorphic to $C(2^{\mathbb{N}})$. The proof of part (i) is completed.

(ii) We argue by contradiction. So assume that there exists a subspace Z of X which is isomorphic to E. As in part (i), we denote by $Q\colon X \to X/Y$ the natural quotient map. The fact that the space E does not contain ℓ_1 yields that the operator $Q\colon Z \to X/Y$ is strictly singular. This, in turn, implies that

$$\mathrm{dist}(S_{Z'}, S_Y) = \min\big\{\|z - y\| : z \in Z', y \in Y \text{ and } \|z\| = \|y\| = 1\big\} = 0$$

for every infinite-dimensional subspace Z' of Z. Hence, there exist a subspace Z'' of Z and a subspace Y' of Y which are isomorphic. Since E is minimal, we see that Z'' must contain an isomorphic copy of E. Hence so does Y, a contradiction. The proof of Proposition 6.24 is completed. □

6.6.2 Applications

We start with the following corollary.

Corollary 6.27. **[D3]** *Let A be an analytic subset of* NU*. Then there exists an analytic subset A' of* NU *with the following properties.*

(i) *Every $Y \in A'$ has a Schauder basis.*

(ii) *For every $X \in A$ there exists $Y \in A'$ containing an isometric copy of X.*

Proof. Let $\mathcal{L}_2$ be the analytic subset of SB × SB obtained by Theorem 6.18 applied for $\lambda = 2$. We define A' by

$$Y \in A' \Leftrightarrow \exists X \in \mathrm{SB}\ [X \in A \text{ and } (X,Y) \in \mathcal{L}_2].$$

Clearly A' is analytic. By Theorem 6.1, part (i) of Proposition 6.24 and Theorem B.12, the set A' is as desired. The proof is completed. □

Let X be a Banach space with a Schauder basis. As in Section 2.4, for every normalized Schauder basis (e_n) of X and every separable Banach space Y by $T_{\mathrm{NC}}\big(Y, X, (e_n)\big)$ we shall denote the tree defined in (2.10).

Now let X be a non-universal separable Banach space and $\lambda > 1$. By Theorem 6.1 and part (i) of Proposition 6.24, the space $\mathcal{L}_\lambda[X]$ is non-universal. We have the following quantitative refinement of this fact.

Corollary 6.28. **[D3]** *Let (e_n) be a normalized Schauder basis of $C(2^{\mathbb{N}})$ and let $\lambda > 1$. Then there exists a map $f_\lambda\colon \omega_1 \to \omega_1$ such that for every $\xi < \omega_1$ and every separable Banach space X with $o\big(T_{\mathrm{NC}}(X, C(2^{\mathbb{N}}), (e_n))\big) \leqslant \xi$ we have $o\big(T_{\mathrm{NC}}(\mathcal{L}_\lambda[X], C(2^{\mathbb{N}}), (e_n))\big) \leqslant f_\lambda(\xi)$.*

In particular, there exists a map $f\colon \omega_1 \to \omega_1$ such that for every countable ordinal ξ, every separable Banach space X with $o\big(T_{\mathrm{NC}}(X, C(2^{\mathbb{N}}), (e_n))\big) \leqslant \xi$ embeds isometrically into a Banach space Y with a Schauder basis satisfying $o\big(T_{\mathrm{NC}}(Y, C(2^{\mathbb{N}}), (e_n))\big) \leqslant f(\xi)$.

Proof. By Theorem 2.17, the map

$$\mathrm{NU} \ni X \mapsto o\big(T_{\mathrm{NC}}(X, C(2^{\mathbb{N}}), (e_n))\big)$$

is a $\mathbf{\Pi}^1_1$-rank on NU. Let $\xi < \omega_1$ be arbitrary and set

$$A_\xi = \big\{X \in \mathrm{NU} : o\big(T_{\mathrm{NC}}(X, C(2^{\mathbb{N}}), (e_n))\big) \leqslant \xi\big\}.$$

By part (i) of Theorem A.2, the set A_ξ is analytic (in fact Borel). Let $\mathcal{L}_\lambda$ be the analytic subset of $\mathrm{SB}\times\mathrm{SB}$ obtained by Theorem 6.18 applied for the given λ. We define B_ξ by

$$Y \in B_\xi \Leftrightarrow \exists X \in \mathrm{SB}\ [X \in A_\xi \text{ and } (X,Y) \in \mathcal{L}_\lambda].$$

As in Corollary 6.27, we see that B_ξ is an analytic subset of NU. By part (ii) of Theorem A.2, we have

$$\sup\big\{o\big(T_{\mathrm{NC}}(Y, C(2^{\mathbb{N}}), (e_n))\big) : Y \in B_\xi\big\} = \zeta_\xi < \omega_1.$$

We set $f_\lambda(\xi) = \zeta_\xi$. Clearly the map f_λ is as desired. The proof is completed. □

We close this subsection by presenting the following analogues of Corollaries 6.27 and 6.28 for the class NC_X. They are both derived using identical arguments as above.

Corollary 6.29. **[D3]** *Let X be a minimal Banach space not containing ℓ_1. Also let A be an analytic subset of NC_X. Then there exists an analytic subset A' of NC_X with the following properties.*

(i) *Every $Y \in A'$ has a Schauder basis.*

(ii) *For every $Z \in A$ there exists $Y \in A'$ containing an isometric copy of Z.*

Corollary 6.30. **[D3]** *Let X be a minimal Banach space with a Schauder basis and not containing ℓ_1. Let (e_n) be a normalized Schauder basis of X and let $\lambda > 1$. Then there exists a map $f^X_\lambda \colon \omega_1 \to \omega_1$ such that for every $\xi < \omega_1$ and every separable Banach space Z with $o\big(T_{\mathrm{NC}}(Z, X, (e_n))\big) \leqslant \xi$ we have $o\big(T_{\mathrm{NC}}(\mathcal{L}_\lambda[Z], X, (e_n))\big) \leqslant f^X_\lambda(\xi)$.*

In particular, there exists a map $f_X \colon \omega_1 \to \omega_1$ such that for every countable ordinal ξ, every separable Banach space Z with $o\big(T_{\mathrm{NC}}(Z, X, (e_n))\big) \leqslant \xi$ embeds isometrically into a Banach space Y with a Schauder basis satisfying $o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) \leqslant f_X(\xi)$.

6.7 Comments and Remarks

1. As we have already mentioned, the method of extending operators presented in Section 6.1 was invented by Kisliakov [Ki]. The reader can find in [Ki] and [Pi1] further structural properties. We refer to the monograph of Pisier [Pi2] for more information.

2. The Bourgain–Pisier construction was the outcome of the combination of two major achievements of Banach space theory during the 1980s. The first one is the Bourgain–Delbaen space [BD], the first example of a $\mathcal{L}_\infty$-space not containing an isomorphic copy of c_0. The second one is Pisier's scheme developed in [Pi1] for producing counterexamples to a conjecture of Grothendieck.

Our presentation follows closely the original one. We notice, however, that the proof of Theorem 6.13 differs slightly from the one in [BP].

3. Theorem 6.18 and its consequences are taken from [D3]. We should point out that no explicit bounds for the functions f_λ and f_λ^X, obtained by Corollaries 6.28 and 6.30 respectively, are known.

Chapter 7

Strongly bounded classes of Banach spaces

Let (P) be a property of Banach spaces and suppose that we are given a class of separable Banach spaces such that every space in the class has property (P). The main problem addressed in the chapter is whether we can find a separable Banach space Y which has property (P) and contains an isomorphic copy of every member of the given class. We will consider quite classical properties of Banach spaces such as "being reflexive", "having separable dual" and "being non-universal".

We will characterize those classes of separable Banach spaces admitting such a universal space. The characterization will be a byproduct of a structural property satisfied by the corresponding classes REFL, SD and NU introduced in Chapter 2. In abstract form this structural property is isolated in the following definition.

Definition 7.1. **[AD]** *Let* $\mathcal{C} \subseteq \mathrm{SB}$*. We say that the class* $\mathcal{C}$ *is* strongly bounded *if for every analytic subset* A *of* $\mathcal{C}$ *there exists* $Y \in \mathcal{C}$ *that contains an isomorphic copy of every* $X \in A$*.*

The verification that a certain class is strongly bounded proceeds in two steps. In the first step one treats analytic classes of Banach spaces with a Schauder basis. The universal space in such a case is constructed using the tools presented in Chapters 3 and 4. In the second step one reduces the general case to the case of spaces with a Schauder basis using the embedding results, as well as their parameterized versions, developed in Chapters 5 and 6.

7.1 Analytic classes of separable Banach spaces and Schauder tree bases

This section is devoted to the proof of the following correspondence principle between analytic classes of Banach spaces with a Schauder basis and Schauder tree bases (see Definition 3.1).

Lemma 7.2. [**AD**] *Let A be an analytic subset of* SB *such that every $Y \in A$ has a Schauder basis. Then there exist a separable Banach space X, a pruned B-tree T on $\mathbb{N} \times \mathbb{N}$ and a normalized sequence $(x_t)_{t\in T}$ in X such that the following are satisfied.*

(i) *The family $\mathfrak{X} = (X, \mathbb{N} \times \mathbb{N}, T, (x_t)_{t\in T})$ is a Schauder tree basis.*

(ii) *For every $Y \in A$ there exists $\sigma \in [T]$ with $Y \cong X_\sigma$.*

(iii) *For every $\sigma \in [T]$ there exists $Y \in A$ with $X_\sigma \cong Y$.*

Lemma 7.2 will be our basic tool for constructing universal spaces for certain classes of separable Banach spaces. Its proof is based on a technique in descriptive set theory, introduced by Solovay, known as "unfolding".

Proof of Lemma 7.2. Let U be Pełczyński's universal space for basic sequences described in Theorem 1.9. Recall that the basis (u_n) of U is normalized and bi-monotone, and notice that these properties are inherited by the subsequences of (u_n). As in Section 2.5, for every $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ we set $U_L = \overline{\text{span}}\{u_{l_n} : n \in \mathbb{N}\}$. By identifying the space U with one of its isometric copies in $C(2^\mathbb{N})$, we see that the map $\Phi\colon [\mathbb{N}]^\infty \to \text{SB}$ defined by $\Phi(L) = U_L$ is Borel.

Now let A be as in the statement of the lemma and set

$$A_\cong = \{Z \in \text{SB} : \exists Y \in A \text{ such that } Z \cong Y\}.$$

That is, $A_\cong$ is the isomorphic saturation of A. By property (P7) in Section 2.1.1, the equivalence relation $\cong$ of isomorphism is analytic in $\text{SB} \times \text{SB}$. Therefore, the set $A_\cong$ is analytic. Hence, the set

$$\tilde{A} = \{L \in [\mathbb{N}]^\infty : \exists Y \in A \text{ with } U_L \cong Y\} = \Phi^{-1}(A_\cong)$$

is also analytic. The definition of the set $\tilde{A}$, the universality of the basis (u_n) of the space U and our assumptions on the set A, imply the following.

(a) For every $L \in \tilde{A}$ there exists $Y \in A$ with $U_L \cong Y$.

(b) For every $Y \in A$ there exists $L \in \tilde{A}$ with $Y \cong U_L$.

The space $[\mathbb{N}]^\infty$ is naturally identified as a closed subspace of the Baire space $\mathbb{N}^\mathbb{N}$. Hence, the set $\tilde{A}$ can be seen as an analytic subset of $\mathbb{N}^\mathbb{N}$. By Theorem 1.6, there exists a pruned tree S on $\mathbb{N}\times\mathbb{N}$ such that $\tilde{A} = \mathrm{proj}[S]$. Let $T = S\setminus\{\emptyset\}$; that is, T is the B-tree on $\mathbb{N}\times\mathbb{N}$ corresponding to the tree S (see Section 1.2). Notice that every node t of T is a pair (s,w) of nonempty finite sequences in $\mathbb{N}$ with $|s| = |w|$ and such that s is *strictly increasing*. For every $t = (s,w)\in T$ we set $n_t = \max\{n : n\in s\}$ and we define $x_t = u_{n_t}$. Also we set $X = \overline{\mathrm{span}}\{x_t : t\in T\}$. Using properties (a) and (b) isolated above, it is easy to see that the tree T, the space X and the family $(x_t)_{t\in T}$ are as desired. The proof is completed. □

We will also deal with analytic classes of Banach spaces having shrinking Schauder bases. In these cases we will need the following variant of Lemma 7.2.

Lemma 7.3. **[AD]** *Let A be an analytic subset of* SB *such that every $Y\in A$ has a shrinking Schauder basis. Then there exist a separable Banach space X, a pruned B-tree T on $\mathbb{N}\times\mathbb{N}$ and a normalized sequence $(x_t)_{t\in T}$ in X such that the following are satisfied.*

(i) *The family $\mathfrak{X} = (X,\mathbb{N}\times\mathbb{N},T,(x_t)_{t\in T})$ is a Schauder tree basis.*

(ii) *For every $\sigma\in[T]$ the basic sequence $(x_{\sigma|n})_{n\geqslant 1}$ is shrinking.*

(iii) *For every $Y\in A$ there exists $\sigma\in[T]$ with $Y\cong X_\sigma$.*

(iv) *For every $\sigma\in[T]$ there exists $Y\in A$ with $X_\sigma\cong Y$.*

Proof. The proof follows the lines of the proof of Lemma 7.2. The extra ingredient is the use of the coding of basic sequences developed in Section 2.5. Specifically, let U be Pełczyński's universal space for basic sequences and let (u_n) be the basis of U. As in (2.11), consider the set

$$\mathcal{S} = \big\{L = \{l_0 < l_1 < \cdots\}\in[\mathbb{N}]^\infty : (u_{l_n}) \text{ is shrinking}\big\}.$$

By Theorem 2.20, the set $\mathcal{S}$ is $\mathbf{\Pi}^1_1$ and the map

$$\mathcal{S}\ni L\mapsto \mathrm{Sz}(U_L)$$

is a $\mathbf{\Pi}^1_1$-rank on $\mathcal{S}$.

Now let A be an analytic subset of SD such that every $Y\in A$ has a shrinking Schauder basis. By Theorem 2.11, the Szlenk index is a $\mathbf{\Pi}^1_1$-rank on SD. Hence, by part (ii) of Theorem A.2, there exists a countable ordinal ζ such that

$$\sup\{\mathrm{Sz}(Y) : Y\in A\} = \zeta.$$

Invoking part (i) of Theorem A.2, we see that the set

$$\mathcal{S}_\zeta = \big\{L\in\mathcal{S} : \mathrm{Sz}(U_L)\leqslant\zeta\big\}$$

is a Borel subset of $\mathcal{S}$. By the choice of the ordinal ζ, the definition of the set $\mathcal{S}_\zeta$ and our assumptions on the class A, the following are satisfied.

(a) For every $Y \in A$ there exists $L \in \mathcal{S}_\zeta$ such that $Y \cong U_L$.

(b) For every $L = \{l_0 < l_1 < \cdots\} \in \mathcal{S}_\zeta$ the basic sequence (u_{l_n}) is shrinking.

We define

$$\tilde{A} = \{L \in \mathcal{S}_\zeta : \exists Y \in A \text{ with } U_L \cong Y\}.$$

Using the fact that $\mathcal{S}_\zeta$ is a Borel subset of $[\mathbb{N}]^\infty$ and arguing as in the proof of Lemma 7.2, we see that $\tilde{A}$ is an analytic subset of $[\mathbb{N}]^\infty$. Let S be the pruned tree on $\mathbb{N} \times \mathbb{N}$ such that $\tilde{A} = \mathrm{proj}[S]$ and let $T = S \setminus \{\emptyset\}$ be the pruned B-tree on $\mathbb{N} \times \mathbb{N}$ corresponding to S. We define the family $(x_t)_{t\in T}$ and the space X as in the proof of Lemma 7.2. It is easily verified that the tree T, the space X and the family $(x_t)_{t\in T}$ satisfy all requirements of the lemma. The proof is completed. □

7.2 Reflexive spaces

Reflexive spaces with a Schauder basis

Theorem 7.4. **[AD]** *Let A be an analytic subset of* REFL *and assume that every $Y \in A$ has a Schauder basis. Then there exists a reflexive Banach space Z with a Schauder basis that contains every $Y \in A$ as a complemented subspace.*

Proof. We apply Lemma 7.2 to the analytic class A and we obtain a Schauder tree basis $\mathfrak{X} = (X, \mathbb{N} \times \mathbb{N}, T, (x_t)_{t\in T})$ such that the following are satisfied.

(a) For every $Y \in A$ there exists $\sigma \in [T]$ such that $Y \cong X_\sigma$.

(b) For every $\sigma \in [T]$ there exists $Y \in A$ such that $X_\sigma \cong Y$. In particular, for every $\sigma \in [T]$ the space X_σ is reflexive.

The desired space Z is the 2-amalgamation space $A_2^{\mathfrak{X}}$ of the Schauder tree basis $\mathfrak{X}$ (see Definition 4.2). Indeed observe first that, by the discussion in Section 4.1, the space Z has a Schauder basis. Moreover, by Fact 4.3 and property (a) above, the space Z contains every $Y \in A$ as a complemented subspace. Finally, the fact that Z is reflexive follows by Theorem 4.4 and property (b) above. The proof is completed. □

The class REFL

Theorem 7.5. **[DF]** *Let A be an analytic subset of* REFL. *Then there exists a reflexive Banach space Z with a Schauder basis that contains an isomorphic copy of every $X \in A$.*

In particular, the class REFL *is strongly bounded.*

Proof. We fix an analytic subset A of REFL. We apply Corollary 5.25 and we obtain an analytic subset A' of REFL with the following properties.

(a) Every $Y \in A'$ has a Schauder basis.

(b) For every $X \in A$ there exists $Y \in A'$ containing an isomorphic copy of X.

We apply Theorem 7.4 and we obtain a reflexive Banach space Z with a Schauder basis that contains an isomorphic copy of every $Y \in A'$. By property (b) above, we see that the space Z must also contain an isomorphic copy of every $X \in A$. The proof is completed. □

Consequences

For every $X \in \mathrm{SB}$ let $T_{\mathrm{REFL}}(X)$ be the tree on $\mathbb{N}$ defined in Section 2.2. By Theorem 2.5, the map

$$\mathrm{REFL} \ni X \mapsto o\big(T_{\mathrm{REFL}}(X)\big)$$

is a $\mathbf{\Pi}^1_1$-rank on REFL. This fact and Theorem 7.5 lead to the following characterization of those classes of separable Banach spaces admitting a separable reflexive universal space.

Theorem 7.6. *Let $\mathcal{C} \subseteq \mathrm{SB}$. Then the following are equivalent.*

(i) *There exists a separable reflexive Banach space Z that contains an isomorphic copy of every $X \in \mathcal{C}$.*

(ii) *We have* $\sup\big\{o\big(T_{\mathrm{REFL}}(X)\big) : X \in \mathcal{C}\big\} < \omega_1$.

(iii) *There exists an analytic subset A of* REFL *such that $\mathcal{C} \subseteq A$.*

Proof. (i)⇒(ii) Let Z be a separable reflexive Banach space containing an isomorphic copy of every $X \in A$. There exists a countable ordinal ξ such that $o\big(T_{\mathrm{REFL}}(Z)\big) = \xi$. By Proposition 2.7 and our assumptions, we see that $o\big(T_{\mathrm{REFL}}(X)\big) \leqslant o\big(T_{\mathrm{REFL}}(Z)\big) \leqslant \xi$ for every $X \in \mathcal{C}$. Therefore,

$$\sup\big\{o\big(T_{\mathrm{REFL}}(X)\big) : X \in \mathcal{C}\big\} \leqslant \xi < \omega_1$$

as desired.

(ii)⇒(iii) Let $\xi < \omega_1$ be such that $\sup\big\{o\big(T_{\mathrm{REFL}}(X)\big) : X \in \mathcal{C}\big\} = \xi$ and set

$$A = \big\{Z \in \mathrm{REFL} : o\big(T_{\mathrm{REFL}}(Z)\big) \leqslant \xi\big\}.$$

By part (i) of Theorem A.2, the class A is analytic while, by the choice of the countable ordinal ξ, we obtain that $\mathcal{C} \subseteq A$.

(iii)⇒(i) Let A be an analytic subset of REFL such that $\mathcal{C} \subseteq A$. We apply Theorem 7.5 to the class A and we obtain a reflexive space Z with a Schauder basis that contains an isomorphic copy of every $X \in A$. A fortiori, the space Z contains an isomorphic copy of every $X \in \mathcal{C}$. The proof is completed. □

As in Section 2.2, let UC be the subset of REFL consisting of all separable uniformly convex Banach spaces. It is Borel. Hence, applying Theorem 7.5 to the class UC, we recover the following result due to Odell and Schlumprecht.

Corollary 7.7. **[OS]** *There exists a separable reflexive space R that contains an isomorphic copy of every separable uniformly convex Banach space.*

7.3 Spaces with separable dual

Spaces with a shrinking Schauder basis

Theorem 7.8. **[AD]** *Let A be an analytic subset of* SD *and assume that every $Y \in A$ has a shrinking Schauder basis. Then there exists a Banach space Z with a shrinking Schauder basis that contains every $Y \in A$ as a complemented subspace.*

Proof. Applying Lemma 7.3 to the analytic class A, we obtain a Schauder tree basis $\mathfrak{X} = (X, \mathbb{N} \times \mathbb{N}, T, (x_t)_{t \in T})$ such that the following are satisfied.

(a) For every $Y \in A$ there exists $\sigma \in [T]$ such that $Y \cong X_\sigma$.

(b) For every $\sigma \in [T]$ the basic sequence $(x_{\sigma|n})_{n \geqslant 1}$ is shrinking.

The desired space Z is the ℓ_2 Baire sum of the Schauder tree basis $\mathfrak{X}$ (see Definition 3.2). Notice first that, by the discussion in Section 3.2 and property (a) above, the space Z has a Schauder basis and contains every $Y \in A$ as a complemented subspace. The fact that Z has a shrinking Schauder basis follows by property (b) and Corollary 3.29. The proof is completed. □

The class SD

Theorem 7.9. **[DF]** *Let A be an analytic subset of* SD*. Then there exists a Banach space Z with a shrinking Schauder basis that contains an isomorphic copy of every $X \in A$.*

In particular, the class SD *is strongly bounded.*

Proof. Let A be an analytic subset of SD. By Corollary 5.24, there exists an analytic subset A' of SD with the following properties.

(a) Every $Y \in A'$ has a shrinking Schauder basis.

(b) For every $X \in A$ there exists $Y \in A'$ containing an isomorphic copy of X.

By Theorem 7.4 applied to the analytic class A', we see that there exists a Banach space Z with a shrinking Schauder basis that contains an isomorphic copy of every $Y \in A'$. By (b) above, the space Z is as desired. The proof is completed. □

Consequences

For every $X \in \mathrm{SB}$ let $\mathrm{Sz}(X)$ be the Szlenk index of X. By Theorem 2.11, the map

$$\mathrm{SD} \ni X \mapsto \mathrm{Sz}(X)$$

is a $\mathbf{\Pi}^1_1$-rank on SD. Therefore, by Theorem 7.9 and using identical arguments as in the proof of Theorem 7.6, we obtain the following characterization of those classes of separable Banach spaces admitting a universal space with separable dual.

Theorem 7.10. *Let $\mathcal{C} \subseteq \mathrm{SB}$. Then the following are equivalent.*

(i) *There exists a Banach space Z with separable dual that contains an isomorphic copy of every $X \in \mathcal{C}$.*

(ii) *We have $\sup\big\{\mathrm{Sz}(X) : X \in \mathcal{C}\big\} < \omega_1$.*

(iii) *There exists an analytic subset A of SD such that $\mathcal{C} \subseteq A$.*

Fix a countable ordinal ξ and consider the set

$$A_\xi = \{X \in \mathrm{SB} : \mathrm{Sz}(X) \leqslant \xi\}.$$

By Theorem 2.11 and part (i) of Theorem A.2, we see that A_ξ is a Borel subset of SD. Hence, by Theorem 7.9, we obtain the following corollary.

Corollary 7.11. **[DF]** *There exists a family $\{Y_\xi : \xi < \omega_1\}$ of Banach spaces such that for every countable ordinal ξ the following are satisfied.*

(i) *The space Y_ξ has a shrinking Schauder basis.*

(ii) *If X is a separable Banach space with $\mathrm{Sz}(X) \leqslant \xi$, then Y_ξ contains an isomorphic copy of the space X.*

7.4 Non-universal spaces

Non-universal spaces with a Schauder basis

Theorem 7.12. **[AD]** *Let A be an analytic subset of NU and assume that every $Y \in A$ has a Schauder basis. Then there exists a non-universal Banach space Z with a Schauder basis that contains every $Y \in A$ as a complemented subspace.*

Let X, Y, Z be Banach spaces and let $T\colon X \to Y$ be a bounded linear operator. We say that the operator T *fixes a copy* of Z if there exists a subspace E of X which is isomorphic to Z and is such that the operator $T\colon E \to Y$ is an isomorphic embedding. For the proof of Theorem 7.12 we will need the following classical result due to Rosenthal.

Theorem 7.13. **[Ro1]** *Let X be a Banach space and let $T\colon C(2^{\mathbb{N}}) \to X$ be a bounded linear operator. If T fixes a copy of ℓ_1, then T fixes a copy of $C(2^{\mathbb{N}})$.*

We are ready to proceed to the proof of Theorem 7.12.

Proof of Theorem 7.12. We apply Lemma 7.2 to the analytic class A and we obtain a Schauder tree basis $\mathfrak{X} = (X, \mathbb{N}\times\mathbb{N}, T, (x_t)_{t\in T})$ such that the following are satisfied.

(a) For every $Y \in A$ there exists $\sigma \in [T]$ such that $Y \cong X_\sigma$.

(b) For every $\sigma \in [T]$ the space X_σ is non-universal.

The desired space Z is the ℓ_2 Baire sum of the Schauder tree basis $\mathfrak{X}$. Indeed, the space Z has a Schauder basis and contains every $Y \in A$ as a complemented subspace. What remains is to check that the space Z is non-universal.

We will argue by contradiction. So assume, towards a contradiction, that there exists a subspace E of Z which is isomorphic to $C(2^{\mathbb{N}})$. Let E' be a subspace of E which is isomorphic to ℓ_1. By Theorem 3.23, we see that the subspace E' of Z is not X-singular. In other words and according to Definition 3.4, there exist $\sigma \in [T]$ and a further subspace E'' of E' such that the operator $P_\sigma\colon E'' \to \mathcal{X}_\sigma$ is an isomorphic embedding. Clearly, we may additionally assume that E'' is isomorphic to ℓ_1. Now consider the operator $P_\sigma\colon E \to \mathcal{X}_\sigma$. What we have just proved is that the operator $P_\sigma\colon E \to \mathcal{X}_\sigma$ fixes a copy of ℓ_1. By Theorem 7.13, the operator $P_\sigma\colon E \to \mathcal{X}_\sigma$ must also fix a copy of $C(2^{\mathbb{N}})$. Since the spaces $\mathcal{X}_\sigma$ and X_σ are isometric, this implies that the space X_σ is universal. This is a contradiction by property (b) above. Therefore, Z is non-universal. The proof of Theorem 7.12 is completed. □

The class NU

Theorem 7.14. **[D3]** *Let A be an analytic subset of* NU. *Then there exists a non-universal Banach space Z with a Schauder basis that contains an isomorphic copy of every $X \in A$.*

In particular, the class NU *is strongly bounded.*

Proof. Let A be an analytic subset of NU. We apply Corollary 6.27 and we obtain an analytic subset A' of NU with the following properties.

(a) Every $Y \in A'$ has a Schauder basis.

(b) For every $X \in A$ there exists $Y \in A'$ containing an isomorphic copy of X.

Next, we apply Theorem 7.12 to the analytic class A' and we obtain a non-universal Banach space Z with a Schauder basis that contains every $Y \in A'$. By (b) above, the space Z is as desired. The proof is completed. $\square$

Consequences

In what follows let (e_n) denote a fixed normalized Schauder basis of $C(2^{\mathbb{N}})$. As in Section 2.4, for every for every separable Banach space Y by $T_{\mathrm{NC}}\big(Y, C(2^{\mathbb{N}}), (e_n)\big)$ we denote the tree defined in (2.10). By Theorem 2.17, the map

$$\mathrm{NU} \ni Y \mapsto o\big(T_{\mathrm{NC}}(Y, C(2^{\mathbb{N}}), (e_n))\big)$$

is a $\mathbf{\Pi}^1_1$-rank on NU. Arguing as in the proof of Theorem 7.6, we obtain the following characterization of those classes of separable Banach spaces admitting a separable universal Banach space which is not universal for all separable Banach spaces.

Theorem 7.15. **[D3]** *Let $\mathcal{C} \subseteq \mathrm{SB}$. Then the following are equivalent.*

(i) *There exists a non-universal separable Banach space Z that contains an isomorphic copy of every $X \in \mathcal{C}$.*

(ii) *We have $\sup\big\{o\big(T_{\mathrm{NC}}(X, C(2^{\mathbb{N}}), (e_n))\big) : X \in \mathcal{C}\big\} < \omega_1$.*

(iii) *There exists an analytic subset A of NU such that $\mathcal{C} \subseteq A$.*

The following consequence of Theorems 6.1 and 7.14 shows that the class of $\mathcal{L}_\infty$-spaces is "generic".

Corollary 7.16. **[D3]** *For every $\lambda > 1$ there exists a family $\{Y^\lambda_\xi : \xi < \omega_1\}$ of separable Banach spaces with the following properties.*

(i) *For every $\xi < \omega_1$ the space Y^λ_ξ is non-universal and $\mathcal{L}_{\infty,\lambda+}$.*

(ii) *If $\xi < \zeta < \omega_1$, then Y^λ_ξ is contained in Y^λ_ζ.*

(iii) *If X is a separable Banach space with $o\big(T_{\mathrm{NC}}(X, C(2^{\mathbb{N}}), (e_n))\big) \leqslant \xi$, then Y_ξ contains an isomorphic copy of X.*

Proof. Fix $\lambda > 1$. The family $\{Y^\lambda_\xi : \xi < \omega_1\}$ will be constructed by transfinite recursion on countable ordinals. As the first step is identical to the general one we may assume that for some countable ordinal ξ and every $\zeta < \xi$ the space Y^λ_ζ has been constructed. We set

$$C = \big\{X \in \mathrm{NU} : o\big(T_{\mathrm{NC}}(Y, C(2^{\mathbb{N}}), (e_n))\big) \leqslant \xi\big\} \cup \{Y^\lambda_\zeta : \zeta < \xi\}.$$

By Theorem 2.17 and part (i) of Theorem A.2, C is an analytic subset of NU. We apply Theorem 7.14 and we obtain a separable non-universal space Z that contains an isomorphic copy of every $X \in C$. Next, we apply Theorem 6.1 to the space Z, and we define Y_ξ^λ to be the space $\mathcal{L}_\lambda[Z]$. This completes the recursive construction. Using Theorem 6.1 and Proposition 6.24, it is easily seen that the family $\{Y_\xi^\lambda : \xi < \omega_1\}$ is as desired. The proof is completed. □

7.5 Spaces not containing a minimal space X

Spaces with a Schauder basis and not containing a minimal space X

Theorem 7.17. [AD] *Let X be a minimal Banach space. Let A be an analytic subset of NC_X and assume that every $Y \in A$ has a Schauder basis. Then there exists a Banach space $Z \in \mathrm{NC}_X$ with a Schauder basis that contains every $Y \in A$ as a complemented subspace.*

Proof. The space X is minimal. Therefore, there exists $1 < q < +\infty$ such that the space X does not contain an isomorphic copy of ℓ_q. We fix such a q. Applying Lemma 7.2 to the analytic class A, we obtain a Schauder tree basis $\mathfrak{X} = (X, \mathbb{N} \times \mathbb{N}, T, (x_t)_{t \in T})$ with the following properties.

(a) For every $Y \in A$ there exists $\sigma \in [T]$ such that $Y \cong X_\sigma$.

(b) For every $\sigma \in [T]$ we have $X_\sigma \in \mathrm{NC}_X$.

The desired space Z is the q-amalgamation space $A_q^{\mathfrak{X}}$ of the Schauder tree basis $\mathfrak{X}$. Indeed, the space $A_q^{\mathfrak{X}}$ has a Schauder basis and contains every $Y \in A$ as a complemented subspace. We will verify that Z does not contain an isomorphic copy of the minimal Banach space X.

As in the proof of Theorem 7.12, we will argue by contradiction. So assume that there exists a subspace E of Z which is isomorphic to X. There exist a block subspace E' of Z and a subspace X' of X such that $E' \cong X'$. By the choice of q and the minimality of X, we see that E' does not contain an isomorphic copy of ℓ_q. Applying Corollary 4.7 and invoking property (iii) in Lemma 7.2, we obtain $k \in \mathbb{N}$ and $Y_0, \dots, Y_k \in A$ such that E' is isomorphic to a subspace of $\sum_{n=0}^{k} \oplus Y_n$. By Lemma B.6, there exist $n_0 \in \{0, \dots, k\}$ and a subspace E'' of E' such that E'' is isomorphic to a subspace of Y_{n_0}. Since X is minimal, this implies that Y_{n_0} must contain an isomorphic copy of X. This contradicts our assumptions on the class A. Hence, Z does not contain X. The proof is completed. □

The class NC_X with X minimal not containing ℓ_1

Theorem 7.18. [D3] *Let X be a minimal Banach space not containing ℓ_1. Let A be an analytic subset of NC_X. Then there exists a Banach space $Z \in \mathrm{NC}_X$*

with a Schauder basis that contains an isomorphic copy of every $Y \in A$.

In particular, the class NC_X *is strongly bounded.*

Proof. Fix an analytic subset A of NC_X. We apply Corollary 6.29 and we obtain an analytic subset A' of NC_X with the following properties.

(a) Every $E \in A'$ has a Schauder basis.

(b) For every $Y \in A$ there exists $E \in A'$ containing an isomorphic copy of Y.

We apply Theorem 7.17 to the analytic class A' and we obtain a Banach space $Z \in \mathrm{NC}_X$ with a Schauder basis which is universal for the class A'. Invoking property (b) above, we see that the space Z is the desired one. The proof is completed. □

Consequences

Fix a minimal space X with a Schauder basis and not containing ℓ_1. Let (e_n) be a normalized Schauder basis of X. For every separable Banach space Y by $T_{\mathrm{NC}}\big(Y, X, (e_n)\big)$ we denote the tree defined in (2.10). By Theorem 2.17, the map

$$\mathrm{NC}_X \ni Y \mapsto o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big)$$

is a $\mathbf{\Pi}^1_1$-rank on NC_X. Using the same analysis as in Section 7.4, we obtain the following analogues of Theorem 7.15 and Corollary 7.16 respectively.

Theorem 7.19. **[D3]** *Let X be a minimal Banach space with a Schauder basis and not containing ℓ_1 and let (e_n) be a normalized Schauder basis of X. Let $\mathcal{C} \subseteq \mathrm{SB}$. Then the following are equivalent.*

(i) *There exists a Banach space $Z \in \mathrm{NC}_X$ that contains an isomorphic copy of every $Y \in \mathcal{C}$.*

(ii) *We have* $\sup\big\{o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) : Y \in \mathcal{C}\big\} < \omega_1$.

(iii) *There exists an analytic subset A of NC_X such that $\mathcal{C} \subseteq A$.*

Corollary 7.20. **[D3]** *Let X be a minimal Banach space with a Schauder basis and not containing ℓ_1 and let (e_n) be a normalized Schauder basis of X. Then for every $\lambda > 1$ there exists a family $\{Y^\lambda_\xi : \xi < \omega_1\}$ of separable Banach spaces with the following properties.*

(i) *For every $\xi < \omega_1$ the space Y^λ_ξ is $\mathcal{L}_{\infty,\lambda+}$ and does not contain an isomorphic copy of X.*

(ii) *If $\xi < \zeta < \omega_1$, then Y^λ_ξ is contained in Y^λ_ζ.*

(iii) *If Y is a separable Banach space with $o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) \leqslant \xi$, then Y_ξ contains an isomorphic copy of Y.*

7.6 Comments and Remarks

1. The results contained in this chapter are particular instances of a more general phenomenon in descriptive set theory known as *strong boundedness*. The phenomenon was first touched upon by Kechris and Woodin in [KW2], and it was independently rediscovered in the context of Banach space theory in [AD]. Abstractly, one has a $\mathbf{\Pi}^1_1$ set B, a natural notion of embedding between elements of B and a canonical $\mathbf{\Pi}^1_1$-rank ϕ on B which is coherent with the embedding, in the sense that if $x, y \in B$ and x embeds into y, then $\phi(x) \leqslant \phi(y)$. The strong boundedness of B is the fact that for every analytic subset A of B there exists $y \in B$ such that x embeds into y for every $x \in A$. Basic examples of strongly bounded classes are the well-orderings WO and the well-founded trees WF (however, in these cases, strong boundedness is easily seen to be equivalent to boundedness). Beside the examples of strongly bounded classes coming from Banach space theory, another example of a strongly bounded class consisting of topological spaces was discovered in [D2].

Although our description of a strongly bounded class referred to a $\mathbf{\Pi}^1_1$ set, we should point out that the phenomenon has been verified for more complicated sets. For instance, the strongly bounded class discovered in [KW2] was a $\mathbf{\Pi}^1_2$ set. This has also been encountered in Banach space theory. In particular, it was shown in [DL] that the class US consisting of all unconditionally saturated separable Banach spaces is strongly bounded (recall that an infinite-dimensional Banach space X is said to be *unconditionally saturated* if every infinite-dimensional subspace Y of X contains an unconditional basic sequence). The class US is $\mathbf{\Pi}^1_2$.

2. The first step towards the structural results presented in these notes was made in [AD]. The machinery developed in [AD] was able to treat the case of analytic classes of Banach spaces with a Schauder basis. In particular, Theorems 7.4, 7.8, 7.12 and 7.17 are taken from [AD]. In these important special cases one can actually construct a *complementably* universal space. By a classical result due to Johnson and Szankowski [JS], one cannot expect such a strong property for the general case.

3. As we have already noted, Corollary 7.7 is due to Odell and Schlumprecht [OS]. The problem of the existence of a separable reflexive Banach space which is universal for the separable uniformly convex spaces had been asked by Bourgain [Bou1].

4. Corollary 7.11 is taken from [DF] and answers a problem posed by Rosenthal (see [Ro3]).

5. Theorems 7.15 and 7.19 are taken from [D3]. Although the corresponding results, Theorems 7.6 and 7.10 respectively, were not explicitly isolated in [DF] they are implicitly contained in that work.

6. Theorem 7.14 is taken from [D3]. The problem whether the class NU is strongly bounded had been asked by Kechris in the 1980s.

Appendix A

Rank theory

Definitions and basic properties

Definable ranks are fundamental tools in descriptive set theory. In our presentation we will concentrate only on the first level of the projective hierarchy.

Definition A.1. *Let X be a Polish space and let B be a $\mathbf{\Pi}^1_1$ subset of X. A map $\phi\colon B \to \omega_1$ is said to be a $\mathbf{\Pi}^1_1$-*rank *on B if there are relations $\leqslant_\Sigma, \leqslant_\Pi \subseteq X \times X$ in $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ respectively, such that for every $y \in B$ we have*

$$\begin{aligned}\phi(x) \leqslant \phi(y) \quad &\Leftrightarrow \quad (x \in B) \text{ and } \phi(x) \leqslant \phi(y) \\ &\Leftrightarrow \quad x \leqslant_\Sigma y \Leftrightarrow x \leqslant_\Pi y.\end{aligned}$$

The basic properties of $\mathbf{\Pi}^1_1$-ranks are summarized below.

Theorem A.2. *Let X be a Polish space, let B be a $\mathbf{\Pi}^1_1$ subset of X and let $\phi\colon B \to \omega_1$ be a $\mathbf{\Pi}^1_1$-rank on B. Then the following hold.*

(i) *For every countable ordinal ξ the set $B_\xi = \{x \in B : \phi(x) \leqslant \xi\}$ is Borel.*

(ii) *For every analytic subset A of B we have $\sup\{\phi(x) : x \in A\} < \omega_1$.*

(iii) *B is Borel if and only if $\sup\{\phi(x) : x \in B\} < \omega_1$.*

Property (i) in Theorem A.2 follows from the fact that $\mathbf{\Delta}^1_1 = B(X)$. For every $\xi < \omega_1$ the set B_ξ is called the ξ-*resolvent* of B. Property (ii) is known as *boundedness*. It is a consequence of a more general result concerning the length of definable well-founded relations due to Kunen and Martin (see [Ke, Theorems 35.23 and 31.1]). Part (iii) follows easily by part (ii). The following fact will be useful in the discussion below.

Fact A.3. *Let X be a Polish space, let B be a $\mathbf{\Pi}^1_1$ subset of X and let $\phi\colon B \to \omega_1$. Then ϕ is a $\mathbf{\Pi}^1_1$-rank on B if and only if there are relations $\leqslant'_\Sigma, <'_\Sigma \subseteq X \times X$ both in $\mathbf{\Sigma}^1_1$ such that for every $y \in B$ we have*

$$(x \in B) \text{ and } \phi(x) \leqslant \phi(y) \Leftrightarrow x \leqslant'_\Sigma y$$

and

$$(x \in B) \text{ and } \phi(x) < \phi(y) \Leftrightarrow x <'_\Sigma y.$$

Proof. First, assume that ϕ is a $\mathbf{\Pi}^1_1$-rank on B and let $\leqslant_\Sigma, \leqslant_\Pi$ be the associated relations described in Definition A.1. Define $<'_\Sigma$ by

$$x <'_\Sigma y \Leftrightarrow (x \leqslant_\Sigma y) \text{ and } \neg(y \leqslant_\Pi x)$$

and set $\leqslant'_\Sigma = \leqslant_\Sigma$. It is easy to see that $\leqslant'_\Sigma$ and $<'_\Sigma$ are as desired.

Conversely, set $\leqslant_\Sigma = \leqslant'_\Sigma$ and define

$$x \leqslant_\Pi y \Leftrightarrow (x \in B) \text{ and } \neg(y <'_\Sigma x).$$

Again it is easily verified that the relations $\leqslant_\Sigma$ and $\leqslant_\Pi$ witness that ϕ is a $\mathbf{\Pi}^1_1$-rank on B. The proof is completed. □

Well-founded trees

The following theorem provides the archetypical example of a $\mathbf{\Pi}^1_1$-rank.

Theorem A.4. *Let Λ be a countable set. Then the set* $\mathrm{WF}(\Lambda)$ *is* $\mathbf{\Pi}^1_1$ *and the map $T \mapsto o(T)$ is a $\mathbf{\Pi}^1_1$-rank on* $\mathrm{WF}(\Lambda)$.

Proof. To see that $\mathrm{WF}(\Lambda)$ is $\mathbf{\Pi}^1_1$ notice that

$$T \in \mathrm{WF}(\Lambda) \Leftrightarrow \forall \sigma \in \Lambda^{\mathbb{N}}\ \exists k \in \mathbb{N} \text{ with } \sigma|k \notin T.$$

We proceed to show that $T \mapsto o(T)$ is a $\mathbf{\Pi}^1_1$-rank on $\mathrm{WF}(\Lambda)$. We will use Fact A.3. Specifically, consider the relations $\leqslant_\Sigma$ and $<_\Sigma$ in $\mathrm{Tr}(\Lambda) \times \mathrm{Tr}(\Lambda)$ defined by

$$S \leqslant_\Sigma T \Leftrightarrow T \notin \mathrm{WF}(\Lambda) \text{ or } [S, T \in \mathrm{WF}(\Lambda) \text{ and } o(S) \leqslant o(T)]$$

and

$$S <_\Sigma T \Leftrightarrow T \notin \mathrm{WF}(\Lambda) \text{ or } [S, T \in \mathrm{WF}(\Lambda) \text{ and } o(S) < o(T)].$$

By Proposition 1.5, we have

$$S \leqslant_\Sigma T \Leftrightarrow \exists f\colon S \to T \text{ monotone},$$

and so, the relation $\leqslant_\Sigma$ is $\mathbf{\Sigma}^1_1$. For every $T \in \mathrm{Tr}(\Lambda)$ and every $\lambda \in \Lambda$ we set $T_\lambda = \{t : \lambda^\frown t \in T\}$. Observe that if $T \in \mathrm{WF}(\Lambda)$, then $o(T) = \sup\{o(T_\lambda) : \lambda \in$

$\Lambda\} + 1$, while if $T \in \mathrm{IF}(\Lambda)$, then there exists $\lambda \in \Lambda$ such that $T_\lambda \in \mathrm{IF}(\Lambda)$. Using these remarks and invoking again Proposition 1.5, we see that

$$S <_\Sigma T \Leftrightarrow \exists \lambda \in \Lambda \text{ and } \exists f \colon S \to T_\lambda \text{ monotone.}$$

The proof is completed. □

Let A be an analytic subset of WF. By part (ii) of Theorem A.2 and Theorem A.4, there exists a well-founded tree S on $\mathbb{N}$ such that $o(T) \leqslant o(S)$ for every $T \in A$. The following parameterized version of this fact is useful in applications.

Theorem A.5. *Let X be a standard Borel space and let $A \subseteq X \times \mathrm{Tr}$ be analytic. Then there exists a Borel map $f \colon X \to \mathrm{Tr}$ such that for every $x \in X$, if the section $A_x = \{T : (x,T) \in A\}$ of A at x is a subset of* WF*, then $f(x) \in \mathrm{WF}$ and $o\big(f(x)\big) \geqslant \sup\{o(T) : T \in A_x\}$, while if $A_x \cap \mathrm{IF} \neq \emptyset$, then $f(x) \in \mathrm{IF}$.*

Proof. All uncountable standard Borel spaces are Borel isomorphic (see [Ke, Theorem 15.6]). Hence, we may assume that $X = \mathbb{N}^{\mathbb{N}}$. In this case we will show that the map f can be chosen to be continuous. So let $A \subseteq \mathbb{N}^{\mathbb{N}} \times \mathrm{Tr}$ be analytic. There exists $F \subseteq \mathbb{N}^{\mathbb{N}} \times \mathrm{Tr} \times \mathbb{N}^{\mathbb{N}}$ closed with $A = \mathrm{proj}_{\mathbb{N}^{\mathbb{N}} \times \mathrm{Tr}} F$. For every $x \in \mathbb{N}^{\mathbb{N}}$ we define $T_x \in \mathrm{Tr}(\mathbb{N} \times \mathbb{N})$ by

$$T_x = \big\{(t,s) \ : \ |t| = |s| = n \text{ and } \exists (y,T,z) \in F \text{ with } x|n = y|n, \ t \in T \text{ and } s = z|n\big\}.$$

The map $h \colon \mathbb{N}^{\mathbb{N}} \to \mathrm{Tr}(\mathbb{N} \times \mathbb{N})$ defined by $h(x) = T_x$ is easily seen to be continuous.

Claim A.6. *For every $x \in \mathbb{N}^{\mathbb{N}}$ we have $T_x \in \mathrm{WF}(\mathbb{N}\times\mathbb{N})$ if and only if $A_x \subseteq \mathrm{WF}$.*

Proof of Claim A.6. Fix $x \in \mathbb{N}^{\mathbb{N}}$. First assume that T_x is well-founded. For every $T \in A_x$ we select $z \in \mathbb{N}^{\mathbb{N}}$ such that $(x,T,z) \in F$. Define $\phi \colon T \to T_x$ by $\phi(t) = (t, z|n)$ where $n = |t|$. Then ϕ is a well-defined monotone map. Since $T_x \in \mathrm{WF}(\mathbb{N} \times \mathbb{N})$, by Proposition 1.5, we see that $T \in \mathrm{WF}$ and $o(T) \leqslant o(T_x)$.

Conversely, assume that $T_x \in \mathrm{IF}(\mathbb{N}\times\mathbb{N})$. Let $\big((t_n, s_n)\big)$ be an infinite branch of T_x. For every $n \in \mathbb{N}$ there exist $y_n \in \mathbb{N}^{\mathbb{N}}$, $T_n \in \mathrm{Tr}$ and $z_n \in \mathbb{N}^{\mathbb{N}}$ such that $(y_n, T_n, z_n) \in F$, $y_n|n = x|n$, $t_n \in T_n$ and $z_n|n = s_n$. It follows that $y_n \to x$ and $z_n \to z$ where $z = \bigcup_n s_n \in \mathbb{N}^{\mathbb{N}}$. Moreover, by passing to subsequences if necessary, we may assume that there exists $T \in \mathrm{Tr}$ such that $T_n \to T$ (the space of trees is compact). The set F is closed. Hence $(x,T,z) \in F$ and so $T \in A_x$. As every $T \in \mathrm{Tr}$ is downwards closed, we see that $t_n \in T_k$ for every $k \geqslant n$. This implies that $t_n \in T$ for every $n \in \mathbb{N}$; that is, the tree T is ill-founded. The claim is proved. □

Notice that, by the proof of the above claim, we have that if $A_x \subseteq \mathrm{WF}$, then $\sup\{o(T) : T \in A_x\} \leqslant o(T_x)$. Now let $g \colon \mathrm{Tr}(\mathbb{N} \times \mathbb{N}) \to \mathrm{Tr}$ be any continuous map satisfying the following properties.

(a) We have $T \in \mathrm{WF}(\mathbb{N}\times\mathbb{N})$ if and only if $g(T) \in \mathrm{WF}$.

(b) For every $T \in \mathrm{Tr}(\mathbb{N}\times\mathbb{N})$ we have $o(T) \leqslant o\big(g(T)\big)$.

We define $f\colon \mathbb{N}^{\mathbb{N}} \to \mathrm{Tr}$ by $f(x) = g(T_x)$. Clearly f is as desired. The proof of Theorem A.5 is completed. □

Reductions

We recall the following notion.

Definition A.7. *Let X and Y be Polish spaces, and let $A \subseteq X$ and $B \subseteq Y$. We say that A is* Wadge (*respectively,* Borel) reducible *to B if there exists a continuous* (*respectively, Borel*) *map $f\colon X \to Y$ such that $f^{-1}(B) = A$.*

The link between the concept of Borel reducibility and $\mathbf{\Pi}^1_1$-ranks is given in the following fact. Its proof is straightforward.

Fact A.8. *Let X and Y be Polish spaces, and let $A \subseteq X$ and $B \subseteq Y$. Assume that A is Borel reducible to B via a Borel map $f\colon X \to Y$. Assume, moreover, that B is $\mathbf{\Pi}^1_1$ and that $\phi\colon B \to \omega_1$ is a $\mathbf{\Pi}^1_1$-rank on B. Then A is $\mathbf{\Pi}^1_1$ and the map $\psi\colon A \to \omega_1$ defined by $\psi(x) = \phi\big(f(x)\big)$ is a $\mathbf{\Pi}^1_1$-rank on A.*

Theorem A.4 combined with Fact A.8 gives us a powerful method for constructing $\mathbf{\Pi}^1_1$-ranks on $\mathbf{\Pi}^1_1$ sets. Simply find a reduction of the set in question to WF and then assign to every point the order of the well-founded tree to which the point is reduced. We will illustrate this method by showing the following fundamental result.

Theorem A.9. *Let X be a Polish space and let $B \subseteq X$ be a $\mathbf{\Pi}^1_1$ set. Then there exists a $\mathbf{\Pi}^1_1$-rank on B.*

Proof. We have already indicated that, by Theorem A.4 and Fact A.8, it is enough to find a Borel reduction of B to WF. Invoking the fact that all uncountable Polish spaces are Borel isomorphic, we may assume that X is the Baire space $\mathbb{N}^{\mathbb{N}}$. In this case we will show that B is Wadge reducible to WF. In particular, by Theorem 1.6 and the fact that B is $\mathbf{\Pi}^1_1$, there exists a pruned tree T on $\mathbb{N}\times\mathbb{N}$ such that $B^{\mathrm{c}} = p[T]$. For every $\sigma \in \mathbb{N}^{\mathbb{N}}$ we set

$$T(\sigma) = \{t \in \mathbb{N}^{<\mathbb{N}} : |t| = k \text{ and } (\sigma|k, t) \in T\} \in \mathrm{Tr}.$$

The tree $T(\sigma)$ is usually called the section tree of T at σ. It is easy to see that the map $f\colon \mathbb{N}^{\mathbb{N}} \to \mathrm{Tr}$ defined by $f(\sigma) = T(\sigma)$ is continuous. Observe that

$$\sigma \notin B \Leftrightarrow \exists \tau \in \mathbb{N}^{\mathbb{N}} \text{ with } (\sigma,\tau) \in [T] \Leftrightarrow T(\sigma) \in \mathrm{IF}$$

and so $f^{-1}(\mathrm{WF}) = B$. The proof is completed. □

Derivatives

The most frequently met construction of an ordinal ranking in analysis involves some kind of derivation procedure. The main result in this subsection asserts that if the derivative is sufficiently definable (in particular, if it is Borel), then the associated rank is actually a $\mathbf{\Pi}^1_1$-rank. A typical example is the Cantor–Bendixson derivative of a compact subset K of a Polish space X and the corresponding Cantor–Bendixson rank on the set of all countable compact subsets of X.

Specifically, let X be a Polish space. A map $D\colon K(X) \to K(X)$ is said to be a *derivative* on $K(X)$ if $D(K) \subseteq K$ for all $K \in K(X)$, and $D(K_1) \subseteq D(K_2)$ if $K_1 \subseteq K_2$. For every $K \in K(X)$ by transfinite recursion we define the *iterated derivatives* $\big(D^\xi(K) : \xi < \omega_1\big)$ of K by

$$D^0(K) = K,\ D^{\xi+1}(K) = D\big(D^\xi(K)\big) \text{ and } D^\lambda(K) = \bigcap_{\xi<\lambda} D^\xi(K) \text{ if } \lambda \text{ is limit.}$$

Clearly $\big(D^\xi(K) : \xi < \omega_1\big)$ is a transfinite decreasing sequence of compact subsets of X, and so, it is eventually constant. The *D-rank* of K, denoted by $|K|_D$, is defined to be the least ordinal ξ such that $D^\xi(K) = D^{\xi+1}(K)$. Also we set $D^\infty(K) = D^{|K|_D}(K)$.

In applications we often need to deal with parameterized derivatives. In particular, let X and Y be Polish spaces. A map $\mathbb{D}\colon Y \times K(X) \to K(X)$ is said to be a *parameterized derivative* if for every $y \in Y$ the map $\mathbb{D}_y\colon K(X) \to K(X)$ defined by $\mathbb{D}_y(K) = \mathbb{D}(y, K)$, is a derivative on $K(X)$. We have the following theorem.

Theorem A.10. **[KW1]** *Let X, Y be Polish spaces, and let $\mathbb{D}\colon Y \times K(X) \to K(X)$ be a parameterized Borel derivative. Then the set*

$$\Omega_{\mathbb{D}} = \big\{(y, K) \in Y \times K(X) : \mathbb{D}^\infty_y(K) = \emptyset\big\}$$

is $\mathbf{\Pi}^1_1$ and the map $(y, K) \mapsto |K|_{\mathbb{D}_y}$ is a $\mathbf{\Pi}^1_1$-rank on $\Omega_{\mathbb{D}}$.

Theorem A.10 will be frequently used in the following form.

Theorem A.11. *Let X be a Polish space and let $D_n\colon K(X) \to K(X)$ $(n \in \mathbb{N})$ be a sequence of Borel derivatives on $K(X)$. Then the set*

$$\Omega = \{K \in K(X) : D^\infty_n(K) = \emptyset\ \forall n \in \mathbb{N}\}$$

is $\mathbf{\Pi}^1_1$ and the map $K \mapsto \sup\{|K|_{D_n} : n \in \mathbb{N}\}$ is a $\mathbf{\Pi}^1_1$-rank on Ω.

Proof. Let $n \in \mathbb{N}$ be arbitrary. Applying Theorem A.10 for $Y = \{n\}$ and $\mathbb{D} = D_n$, we see that the set $\Omega_{D_n} = \{K \in K(X) : D^\infty_n(K) = \emptyset\}$ in $\mathbf{\Pi}^1_1$. Since $\Omega = \bigcap_n \Omega_{D_n}$, we conclude that Ω is $\mathbf{\Pi}^1_1$.

For notational convenience we set $\phi(K) = \sup\{|K|_{D_n} : n \in \mathbb{N}\}$ for every $K \in \Omega$. Let $Y = \mathbb{N}$ be equipped with the discrete topology and consider the map $\mathbb{D}\colon Y \times K(X) \to K(X)$ defined by $\mathbb{D}(n, K) = D_n(K)$. Then $\mathbb{D}$ is a parameterized Borel derivative. Invoking Theorem A.10 again, we see that the map $(n, K) \mapsto |K|_{\mathbb{D}_n} = |K|_{D_n}$ is a $\boldsymbol{\Pi}^1_1$-rank on $\Omega_{\mathbb{D}}$. Let $\leqslant_\Sigma$ and $\leqslant_\Pi$ be the associated relations. Observe that for every $K \in \Omega$ we have

$$\begin{aligned}(H \in \Omega) \text{ and } \phi(H) \leqslant \phi(K) \quad &\Leftrightarrow \quad \forall n \in \mathbb{N}\ \exists m \in \mathbb{N} \text{ with } (n, H) \leqslant_\Sigma (m, K)\\ &\Leftrightarrow \quad \forall n \in \mathbb{N}\ \exists m \in \mathbb{N} \text{ with } (n, H) \leqslant_\Pi (m, K).\end{aligned}$$

Hence ϕ is a $\boldsymbol{\Pi}^1_1$-rank on Ω. The proof is completed. □

For the proof of Theorem A.10 we need some preliminary results. We start with the following lemma.

Lemma A.12. *Let X be a Polish space. Then the map $\bigcap\colon K(X)^{\mathbb{N}} \to K(X)$ defined by $\bigcap\big((K_n)\big) = \bigcap_n K_n$, is Borel.*

Proof. By Proposition 1.4, it is enough to show that the set $A_U = \big\{(K_n) \in K(X)^{\mathbb{N}} : U \cap \big(\bigcap_n K_n\big) \neq \emptyset\big\}$ is Borel for every open subset U of X. So, let U be one. Write U as $\bigcup_m F_m$ where each F_m is closed. Notice that

$$(K_n) \in A_U \Leftrightarrow \exists m \in \mathbb{N}\ \forall n \in \mathbb{N} \text{ we have } F_m \cap K_0 \cap \cdots \cap K_n \neq \emptyset.$$

Therefore, A_U is Borel. The proof is completed. □

Let $\alpha \in 2^{\mathbb{N}\times\mathbb{N}}$, that is, α is the characteristic function of a binary relation on $\mathbb{N}$. The *field* $F(\alpha)$ of α is the set $\{n \in \mathbb{N} : \alpha(n, n) = 1\}$. For every $\alpha \in 2^{\mathbb{N}\times\mathbb{N}}$ we define $\leqslant_\alpha \subseteq \mathbb{N} \times \mathbb{N}$ by

$$n \leqslant_\alpha m \Leftrightarrow n, m \in F(\alpha) \text{ and } \alpha(n, m) = 1.$$

Let LO^* be the subset of $2^{\mathbb{N}\times\mathbb{N}}$ consisting of all $\alpha \in 2^{\mathbb{N}\times\mathbb{N}}$ with $0 \in F(\alpha)$ and such that $\leqslant_\alpha$ is a linear ordering on $F(\alpha)$ with 0 as the least element. Notice that the set LO^* is closed in $2^{\mathbb{N}\times\mathbb{N}}$ as

$$\begin{aligned}\alpha \in \mathrm{LO}^* \quad \Leftrightarrow \quad & 0 \in F(\alpha) \text{ and } (\forall n \in F(\alpha)\ 0 \leqslant_\alpha n) \text{ and}\\ & \forall n, m \in F(\alpha)\ [n \leqslant_\alpha m \text{ or } m \leqslant_\alpha n] \text{ and}\\ & \forall n, m \in F(\alpha)\ [n \leqslant_\alpha m \text{ and } m \leqslant_\alpha n \Rightarrow n = m] \text{ and}\\ & \forall n, m, k \in F(\alpha)\ [n \leqslant_\alpha m \text{ and } m \leqslant_\alpha k \Rightarrow n \leqslant_\alpha k].\end{aligned}$$

Also let WO^* be the subset of LO^* consisting of all $\alpha \in \mathrm{LO}^*$ for which $\leqslant_\alpha$ is a well-ordering on $F(\alpha)$. For every $\alpha \in \mathrm{WO}^*$ by $|\alpha|$ we denote the unique countable ordinal which is isomorphic to $(F(\alpha), <_\alpha)$, where $<_\alpha$ is the strict part of $\leqslant_\alpha$ on $F(\alpha)$, that is,

$$n <_\alpha m \Leftrightarrow n \neq m \text{ and } n \leqslant_\alpha m.$$

Notice that $\{|\alpha| : \alpha \in \mathrm{WO}^*\} = \omega_1 \setminus \{0\}$. It can be shown (but it is of no use in the argument below) that the set WO^* is $\mathbf{\Pi}^1_1$ and that the map $\alpha \mapsto |\alpha|$ is a $\mathbf{\Pi}^1_1$-rank on WO^* (see [Ke, Theorem 34.4]).

We fix a Borel map $h\colon \mathrm{LO}^* \to \mathrm{LO}^*$ such that

(a) $\alpha \in \mathrm{WO}^*$ if and only if $h(\alpha) \in \mathrm{WO}^*$, and

(b) for every $\alpha \in \mathrm{WO}^*$ we have $|h(\alpha)| = |\alpha| + 1$.

We are ready to proceed to the proof of Theorem A.10.

Proof of Theorem A.10. First we notice that the set $\Omega_{\mathbb{D}}$ is $\mathbf{\Pi}^1_1$ as

$$(y, K) \notin \Omega_{\mathbb{D}} \Leftrightarrow \exists H \in K(X)\ [\mathbb{D}(y, H) = H \text{ and } H \subseteq K \text{ and } H \neq \emptyset].$$

It remains to show that the map $(y, K) \mapsto |K|_{\mathbb{D}_y}$ is a $\mathbf{\Pi}^1_1$-rank on $\Omega_{\mathbb{D}}$. To this end it is enough to find $\mathbf{\Sigma}^1_1$ relations R and S in $\mathrm{LO}^* \times Y \times K(X)$ such that the following properties are satisfied.

(P1) If $(y, K) \in \Omega_{\mathbb{D}}$ with $K \neq \emptyset$, then for every $\alpha \in \mathrm{LO}^*$ we have

$$(\alpha, y, K) \in R \Leftrightarrow \alpha \in \mathrm{WO}^* \text{ and } |\alpha| \leqslant |K|_{\mathbb{D}_y}. \tag{A.1}$$

(P2) If $\alpha \in \mathrm{WO}^*$, then for every $(y, K) \in Y \times K(X)$ we have

$$(\alpha, y, K) \in S \Leftrightarrow (y, K) \in \Omega_{\mathbb{D}} \text{ and } |K|_{\mathbb{D}_y} = |\alpha|. \tag{A.2}$$

Indeed, assuming that the relations R and S have been defined, we complete the proof as follows. We define $\leqslant_\Sigma$ and $<_\Sigma$ by

$$\begin{aligned}(z, H) \leqslant_\Sigma (y, K) \quad \Leftrightarrow \quad & (K = \emptyset \Rightarrow H = \emptyset) \text{ and } \big[(H = \emptyset) \text{ or} \\ & (\exists \alpha \in \mathrm{LO}^* \text{ with } [(\alpha, y, K) \in R \text{ and } (\alpha, z, H) \in S])\big]\end{aligned}$$

and

$$\begin{aligned}(z, H) <_\Sigma (y, K) \quad \Leftrightarrow \quad & (K \neq \emptyset) \text{ and } \big[(H = \emptyset) \text{ or} \\ & (\exists \alpha \in \mathrm{LO}^* \text{ with } [(h(\alpha), y, K) \in R \text{ and } (\alpha, z, H) \in S])\big]\end{aligned}$$

Then $\leqslant_\Sigma$ and $<_\Sigma$ are both $\mathbf{\Sigma}^1_1$ since the relations R and S are $\mathbf{\Sigma}^1_1$ and the map h is Borel. Moreover, invoking properties (P1) and (P2) above, we see that for every $(y, K) \in \Omega_{\mathbb{D}}$ we have

$$(z, H) \in \Omega_{\mathbb{D}} \text{ and } |H|_{\mathbb{D}_z} \leqslant |K|_{\mathbb{D}_y} \Leftrightarrow (z, H) \leqslant_\Sigma (y, K)$$

and

$$(z,H)\in\Omega_{\mathbb{D}} \text{ and } |H|_{\mathbb{D}_z}<|K|_{\mathbb{D}_y}\Leftrightarrow (z,H)<_{\Sigma}(y,K).$$

By Fact A.3 we conclude that the map $(y,K)\mapsto |K|_{\mathbb{D}_y}$ is a $\mathbf{\Pi}^1_1$-rank on $\Omega_{\mathbb{D}}$.

We proceed to define the relations R and S. For the first one we set

$$(\alpha,y,K)\in R \quad\Leftrightarrow\quad \exists p\in K(X)^{\mathbb{N}} \text{ with } \Big(p(0)=K \text{ and } \Big[\forall m\in F(\alpha) \\ \big(p(m)\neq\emptyset \text{ and } [m\neq 0\Rightarrow p(m)\subseteq \bigcap_{n<_\alpha m}\mathbb{D}(y,p(n))]\big)\Big]\Big).$$

By Lemma A.12 and our assumptions on the map $\mathbb{D}$, we see that R is $\mathbf{\Sigma}^1_1$. We will check that it satisfies property (P1). So let $(y,K)\in\Omega_{\mathbb{D}}$ with $K\neq\emptyset$ and let $\alpha\in \mathrm{LO}^*$. For notational convenience we set $\zeta=|K|_{\mathbb{D}_y}$. First notice that if $\alpha\in\mathrm{WO}^*$ with $|\alpha|\leqslant\zeta$, then clearly $(\alpha,y,K)\in R$. Conversely, assume that $(\alpha,y,K)\in R$. Let $p\in K(X)^{\mathbb{N}}$ witnessing this fact. We observe that for every $m\in F(\alpha)$ with $m\neq 0$ there exists $\xi<\zeta$ such that $\bigcap_{n<_\alpha m}\mathbb{D}(y,p(n))\nsubseteq \mathbb{D}^{\xi+1}_y(K)$. For if not, there would existed $m\in F(\alpha)$ with $m\neq 0$ such that for every $\xi<\zeta$

$$\emptyset\neq p(m)\subseteq\bigcap_{n<_\alpha m}\mathbb{D}(y,p(n))\subseteq\mathbb{D}^{\xi+1}_y(K).$$

This implies that

$$\mathbb{D}^{\infty}_y(K)=\bigcap_{\xi<\zeta}\mathbb{D}^{\xi+1}_y(K)\supseteq p(m)\neq\emptyset$$

contradicting the fact that $(y,K)\in\Omega_{\mathbb{D}}$. We define $f\colon F(\alpha)\to\{\xi:\xi<\zeta\}$ as follows. We set $f(0)=0$. If $m\in F(\alpha)$ with $m\neq 0$, then let

$$f(m)=\text{least } \xi<\zeta \text{ such that } \bigcap_{n<_\alpha m}\mathbb{D}(y,p(n))\nsubseteq\mathbb{D}^{\xi+1}_y(K).$$

Claim A.13. *For every $m,k\in F(\alpha)$ with $m<_\alpha k$ we have $f(m)<f(k)$.*

Proof of Claim A.13. First we notice that for every $k\in F(\alpha)$ with $k\neq 0$ we have

$$\bigcap_{n<_\alpha k}\mathbb{D}(y,p(n))\subseteq\mathbb{D}(y,p(0))=\mathbb{D}^1_y(K).$$

Hence $f(k)>0=f(0)$. Thus, in what follows we may assume that $m\in F(\alpha)$ and $m\neq 0$. By the definitions of the relation R and the map f, we have

$$p(m)\subseteq\bigcap_{n<_\alpha m}\mathbb{D}(y,p(n))\subseteq\bigcap_{\xi<f(m)}\mathbb{D}^{\xi+1}_y(K)=\mathbb{D}^{f(m)}_y(K).$$

Hence $\mathbb{D}(y,p(m))=\mathbb{D}_y(p(m))\subseteq\mathbb{D}^{f(m)+1}_y(K)$ since $\mathbb{D}_y$ is a derivative on $K(X)$. It follows that for every $k\in F(\alpha)$ with $m<_\alpha k$ we have $\bigcap_{n<_\alpha k}\mathbb{D}(y,p(n))\subseteq \mathbb{D}(y,p(m))\subseteq\mathbb{D}^{f(m)+1}_y(K)$. By the definition of the map f, we conclude that $f(k)\geqslant f(m)+1>f(m)$. The claim is proved. □

By the above claim, the map f is order preserving from $(F(\alpha), <_\alpha)$ to ζ. Therefore, $\alpha \in \mathrm{WO}^*$ and $|\alpha| \leqslant \zeta = |K|_{\mathbb{D}_y}$. This completes the proof that the relation R satisfies property (P1).

It remains to define the relation S. We set

$$(\alpha, y, K) \in S \quad \Leftrightarrow \quad \exists p \in K(X)^{\mathbb{N}} \text{ with } \Big(p(0) = K \text{ and } \Big[\forall m \in F(\alpha) \\ \big(p(m) \neq \emptyset \text{ and } [m \neq 0 \Rightarrow p(m) = \bigcap_{n <_\alpha m} \mathbb{D}(y, p(n))]\big)\Big] \\ \text{and } \bigcap_{m \in F(\alpha)} \mathbb{D}(y, p(m)) = \emptyset\Big).$$

Invoking Lemma A.12 and the Borelness of the map $\mathbb{D}$, we see that S is $\mathbf{\Sigma}^1_1$. Moreover, it is easily verified that S satisfies property (P2). The proof of Theorem A.10 is completed. □

We close this appendix by mentioning the following result which concerns sets in product spaces with compact sections. Although it is not related to the notion of a $\mathbf{\Pi}^1_1$-rank, it is a very useful tool for checking that various derivatives are Borel. Its proof can be found in [Ke, Theorem 28.8].

Theorem A.14. *Let X and Y be Polish spaces and let $A \subseteq Y \times X$ such that for every $y \in Y$ the section $A_y = \{x \in X : (y, x) \in A\}$ of A at y is compact. Consider the map $\Phi_A \colon Y \to K(X)$ defined by $\Phi_A(y) = A_y$. Then the set A is Borel if and only if Φ_A is a Borel map.*

Appendix B

Banach space theory

B.1 Schauder bases

Definition B.1. *A sequence* (x_n) *in a Banach space* X *is said to be a* Schauder basis *of* X *if for every* $x \in X$ *there exists a unique sequence* (a_n) *of scalars such that* $x = \sum_{n\in\mathbb{N}} a_n x_n$. *A sequence* (x_n) *which is a Schauder basis of its closed linear span is called a* basic sequence.

Let (x_n) be a Schauder basis of a Banach space X. By (x_n^*) we shall denote the sequence of *bi-orthogonal functionals* associated to (x_n). For every subset F of $\mathbb{N}$ by P_F we shall denote the natural projection onto $\overline{\mathrm{span}}\{x_n : n \in F\}$. The *basis constant* of (x_n) is defined to be the number $\sup\{\|P_{\{0,\dots,n\}}\| : n \in \mathbb{N}\}$. If $x = \sum_{n\in\mathbb{N}} a_n x_n$ is a vector in X, then the *support of* x, $\mathrm{supp}(x)$, is defined to be the set $\{n \in \mathbb{N} : a_n \neq 0\}$.

Definition B.2. *Let* (x_n) *be a Schauder basis of a Banach space* X *and* $C \geqslant 1$.

(1) *The basis* (x_n) *is said to be* monotone *if its basis constant is* 1. *It is said to be* bi-monotone *if* $\|P_I\| = 1$ *for every interval* I *of* $\mathbb{N}$.

(2) *The basis* (x_n) *is said to be* C-unconditional *if* $\|P_F\| \leqslant C$ *for every subset* F *of* $\mathbb{N}$. *The basis* (x_n) *is said to be* unconditional *if it is* K*-unconditional for some* $K \geqslant 1$.

(3) *The basis* (x_n) *is said to be* shrinking *if the sequence* (x_n^*) *of bi-orthogonal functionals associated to* (x_n) *is a Schauder basis of* X^*.

(4) *The basis* (x_n) *is said to be* boundedly complete *if for every sequence* (a_n) *of scalars such that* $\sup\big\{\|\sum_{n=0}^{k} a_n x_n\| : k \in \mathbb{N}\big\} < +\infty$ *we have that the series* $\sum_{n\in\mathbb{N}} a_n x_n$ *converges.*

(5) *A sequence* (v_k) *in* X *is said to be* block *if* $\max\{n : n \in \mathrm{supp}(v_k)\} < \min\{n : n \in \mathrm{supp}(v_{k+1})\}$ *for every* $k \in \mathbb{N}$.

Two sequences (x_n) and (y_n), in two Banach spaces X and Y respectively, are said to be C-*equivalent*, where $C \geqslant 1$, if for every $k \in \mathbb{N}$ and every $a_0, \dots, a_k \in \mathbb{R}$ we have

$$\frac{1}{C} \cdot \Big\| \sum_{n=0}^{k} a_n y_n \Big\|_Y \leqslant \Big\| \sum_{n=0}^{k} a_n x_n \Big\|_X \leqslant C \cdot \Big\| \sum_{n=0}^{k} a_n y_n \Big\|_Y .$$

The following stability result is classical and asserts that basic sequences are invariant under small perturbations.

Proposition B.3. *Let* X *be a Banach space and let* (x_n) *be a normalized basic sequence in* X *with basis constant* $K \geqslant 1$. *If* $(y_n)_{n=0}^{l}$ *is a finite sequence in* X *such that*

$$\|x_n - y_n\| \leqslant \frac{1}{2K} \cdot \frac{1}{2^{n+2}}$$

for every $n \in \{0, \dots, l\}$, *then* $(y_n)_{n=0}^{l}$ *is* 2*-equivalent to* $(x_n)_{n=0}^{l}$.

B.2 Operators on Banach spaces

Let X, Y be Banach spaces, and let $\mathcal{L}(X,Y)$ be the Banach space of all bounded linear operators from X to Y. For every $T \in \mathcal{L}(X,Y)$ by $T^* \in \mathcal{L}(Y^*, X^*)$ we shall denote the *dual* operator of T defined by

$$T^*(y^*)(x) = y^*\big(T(x)\big) \text{ for every } x \in X.$$

We recall the following classes of operators.

Definition B.4. *Let* X *and* Y *be Banach spaces and let* $T \in \mathcal{L}(X,Y)$.

(1) T *is said to be a* finite rank operator *if* $\dim\big(T(X)\big) < \infty$.

(2) T *is said to be a* compact operator *if* $T(B_X)$ *is a relatively compact subset of* Y.

(3) T *is said to be a* weakly compact operator *if* $T(B_X)$ *is a relatively weakly compact subset of* Y.

(4) T *is said to be a* strictly singular operator *if for every infinite-dimensional subspace* Z *of* X *the operator* $T\colon Z \to Y$ *is not an isomorphic embedding.*

Clearly every finite rank operator is compact and every compact operator is weakly compact. Strictly singular operators possess some of the strong stability properties of compact operators (they form, for instance, an operator ideal) though they are not well-behaved under duality, as the dual operator of a strictly singular one is not necessarily strictly singular. The following result, due to Kato, shows that strictly singular operators are very "near" to the compact ones (see [LT, Proposition 2.c.4]).

Proposition B.5. *Let X and Y be Banach spaces. Let $T \in \mathcal{L}(X,Y)$ be an operator such that the restriction of T to any finite co-dimensional subspace Z of X is not an isomorphic embedding. Then for every $\varepsilon > 0$ there exists an infinite dimensional subspace Z of X such that $T|_Z$ is compact and $\|T|_Z\| < \varepsilon$. Moreover, if X has a Schauder basis (x_n), then Z can be chosen to be a block subspace.*

The following useful fact is related to Proposition B.5, and it is essentially a Ramsey-type statement.

Lemma B.6. *Let X be a Banach space and let Y be a closed subspace of X. Then for every subspace Z of X there exists a further subspace Z' of Z such that Z' is isomorphic either to a subspace of Y or to a subspace of X/Y.*

In particular, if $Y, X_0, \dots, X_k$ are Banach spaces and $T\colon Y \to \sum_{n=0}^{k} \oplus X_n$ is a continuous linear operator which is not strictly singular, then there exist a subspace Y' of Y and $i \in \{0,\dots,k\}$ such that the operator $P_i \circ T\colon Y' \to X_i$ is an isomorphic embedding, where $P_i\colon \sum_{n=0}^{k} \oplus X_n \to X_i$ stands for the natural projection.

B.3 Interpolation method

Definitions and basic properties

Let $(X, \|\cdot\|)$ be a Banach space and let W be a closed, convex, bounded and symmetric subset of X. For every $n \in \mathbb{N}$ with $n \geqslant 1$ let $\|\cdot\|_n$ be the Minkowski gauge of the set $2^n W + 2^{-n} B_X$; that is,

$$\|x\|_n = \inf\big\{\lambda > 0 : \frac{x}{\lambda} \in 2^n W + 2^{-n} B_X\big\}.$$

Clearly $\|\cdot\|_n$ is an equivalent norm on X. Let $1 < p < +\infty$. For every $x \in X$ we define

$$|x|_p = \Big(\sum_{n=1}^{\infty} \|x\|_n^p\Big)^{1/p}. \tag{B.1}$$

We notice that the map $|\cdot|_p$ is not necessarily a norm on X (and, in fact, it is not for most interesting cases). It is, however, a norm on the vector subspace of X consisting of all $x \in X$ for which $|x|_p < +\infty$. This is essentially the content of the following definition due to Davis, Fiegel, Johnson and Pełczyński.

Definition B.7. **[DFJP]** *Let X, W and $1 < p < +\infty$ be as above. The* p-interpolation space of the pair (X, W)*, denoted by $\Delta_p(X, W)$, is defined to be the vector space*

$$\{x \in X : |x|_p < +\infty\}$$

equipped with the $|\cdot|_p$ norm.

By $J\colon (\Delta_p(X, W), |\cdot|_p) \to (X, \|\cdot\|)$ we denote the inclusion map. Respectively, for every $n \in \mathbb{N}$ by $J_n\colon (\Delta_p(X, W), |\cdot|_p) \to (X, \|\cdot\|_n)$ we denote the inclusion map.

For every pair (X, W) as above and every $1 < p < +\infty$ consider the space

$$Z = \Big(\sum_{n=1}^{\infty} \oplus (X, \|\cdot\|_n)\Big)_{\ell_p}.$$

The space $(\Delta_p(X, W), |\cdot|_p)$ is naturally identified with the "diagonal" subspace $\Delta = \{(x, x, \dots) \in Z : x \in X\}$ of Z via the isometry

$$(\Delta_p(X, W), |\cdot|_p) \ni x \mapsto (x, x, \dots) \in \Delta.$$

It follows that the p-interpolation space of the pair (X, W) is a Banach space. This fact is isolated in the following proposition, in which we also gather some basic properties of the interpolation space.

Proposition B.8. *Let $(X, \|\cdot\|)$ be a Banach space, let W be a closed, convex, bounded and symmetric subset of X, and let $1 < p < +\infty$. We set $Y = (\Delta_p(X, W), |\cdot|_p)$. Then the following hold.*

(i) $W \subseteq B_Y$.

(ii) *The space Y is a Banach space and J is continuous.*

(iii) *The operator $J^{**}\colon Y^{**} \to X^{**}$ is one-to-one and $(J^{**})^{-1}(X) = Y$.*

(iv) *The space Y is reflexive if and only if W is weakly compact.*

(v) *Let τ_X and τ_Y be the relative topologies on B_Y of (X, w) and (Y, w) respectively. Then $\tau_X = \tau_Y$.*

(vi) *The operator $J^*\colon X^* \to Y^*$ has norm dense range.*

Parts (i)–(v) in the above proposition are essentially the content of [DFJP, Lemma 1]. Part (vi) is also well-known. It is a consequence of the fact that the operator J^{**} is one-to-one.

Spaces with a Schauder basis

In what follows let X denote a Banach space with a Schauder basis (x_n). For every $n \in \mathbb{N}$ let P_n be the natural projection onto $\mathrm{span}\{x_k : k \leqslant n\}$. Also let W be a closed, convex, bounded and symmetric subset of X. The following proposition provides sufficient conditions on W so that the p-interpolation space $\Delta_p(X, W)$ will have a basis.

Proposition B.9. **[DFJP]** *Let X and W be as above and $1 < p < +\infty$. Assume that $P_n(W) \subseteq W$ and $\lambda_n x_n \in \mathrm{span}\{W\}$ for some $\lambda_n \in \mathbb{R}$ and every $n \in \mathbb{N}$. We set $z_n = J^{-1}(x_n)$ for every $n \in \mathbb{N}$. Then the sequence (z_n) is a monotone Schauder basis (not normalized) of $\Delta_p(X, W)$. Moreover, if (x_n) is shrinking, then so is (z_n).*

Now assume that X is a Banach space with a *shrinking* Schauder basis (x_n). Let W be a closed, convex, bounded and symmetric subset of X. If W is weakly compact, then, by part (iv) of Proposition B.8, for every $1 < p < +\infty$ the space $\Delta_p(X, W)$ is reflexive. However, the space $\Delta_p(X, W)$ does not necessarily have a basis unless W satisfies $P_n(W) \subseteq W$ for every $n \in \mathbb{N}$. The following lemma shows that we can assume that W has this property without harming the basic topological assumption on W.

Lemma B.10. **[DFJP]** *Let X be a Banach space with a shrinking Schauder basis (x_n) and let W be a weakly compact subset of X. Then the set*

$$W' = W \cup \bigcup_{n \in \mathbb{N}} P_n(W)$$

is also weakly compact.

B.4 Local theory of infinite-dimensional Banach spaces

Let X and Y be two isomorphic Banach spaces (not necessarily infinite dimensional). The *Banach–Mazur distance* between X and Y is defined by

$$d(X, Y) = \inf\big\{\|T\| \cdot \|T^{-1}\| : T\colon X \to Y \text{ is an isomorphism}\big\}. \tag{B.2}$$

Now let $\lambda \geqslant 1$. An infinite-dimensional Banach space X is said to be a *$\mathcal{L}_{\infty,\lambda}$-space* if for every finite-dimensional subspace F of X there exists a finite-dimensional subspace G of X with $F \subseteq G$ and such that $d(G, \ell_\infty^n) \leqslant \lambda$, where $n = \dim(G)$. The space X is said to be a *$\mathcal{L}_{\infty,\lambda+}$-space* if X is a $\mathcal{L}_{\infty,\theta}$-space for any $\theta > \lambda$. Finally, the space X is said to be a *$\mathcal{L}_\infty$-space* if X is a $\mathcal{L}_{\infty,\lambda}$-space for some $\lambda \geqslant 1$. The class of $\mathcal{L}_\infty$-spaces was introduced by Lindenstrauss and Pełczyński [LP1].

It follows readily by the above definition that if X is a separable $\mathcal{L}_{\infty,\lambda}$-space, then there exists an increasing (with respect to inclusion) sequence (G_n) of finite-dimensional subspaces of X with $\bigcup_n G_n$ dense in X and such that $d(G_n, \ell_\infty^{m_n}) \leqslant \lambda$, where $m_n = \dim(G_n)$ for every $n \in \mathbb{N}$. It is relatively easy to see that this property actually characterizes separable $\mathcal{L}_\infty$-spaces. Precisely, we have the following fact.

Fact B.11. *Let X be a separable Banach space and $\lambda \geqslant 1$. Assume that there exists an increasing sequence (F_n) of finite-dimensional subspaces of X with $\bigcup_n F_n$ dense in X and such that $d(F_n, \ell_\infty^{m_n}) \leqslant \lambda$, where $m_n = \dim(F_n)$. Then X is a $\mathcal{L}_{\infty,\lambda+}$-space.*

Recall that if F is a finite-dimensional subspace of a Banach space X such that $d(F, \ell_\infty^n) \leqslant \lambda$, where $n = \dim(F)$, then there exists a projection $P\colon X \to F$ with $\|P\| \leqslant \lambda^2$. Therefore, every separable $\mathcal{L}_\infty$-space has a finite dimensional decomposition. In fact, the following stronger structural property is valid due to Johnson, Rosenthal and Zippin.

Theorem B.12. **[JRZ]** *Every separable $\mathcal{L}_\infty$-space has a Schauder basis.*

The book of Bourgain [Bou3] contains a presentation of the theory of $\mathcal{L}_\infty$-spaces and a discussion of many remarkable examples. Further structural properties of $\mathcal{L}_\infty$-spaces, and in particular refinements of Theorem B.12, can be found in [Ro4].

B.5 Theorem 6.13: the Radon–Nikodym property

Our aim in this section is to complete the proof of Theorem 6.13. In particular, we will show the following.

Let $0 < \eta < 1$. Let (F_n, j_n) be a system of isometric embeddings where the sequence (F_n) consists of finite-dimensional Banach spaces and for every $n \in \mathbb{N}$ the isometric embedding $j_n\colon F_n \to F_{n+1}$ is η-admissible. Then the inductive limit of the system (F_n, j_n) has the Radon–Nikodym property.

The Radon–Nikodym property can be defined in many equivalent ways, either geometric or probabilistic. We refer to the monograph of Diestel and Uhl [DU] for more details. We will use, below, the following probabilistic characterization: *a Banach space X has the Radon–Nikodym property if and only if for every probability space (Ω, Σ, μ) and every martingale (M_k) in $L_1(\mu, X)$ satisfying $\sup_k \int \|M_k\|\, d\mu < +\infty$, the martingale (M_k) converges in X μ-almost everywhere.*

The following definition, due to Bourgain and Pisier, will be our basic conceptual tool.

Definition B.13. **[BP]** *Let X be a Banach space, let Y be a subspace of X and denote by $q\colon X \to X/Y$ the natural quotient map. Also let $\delta > 0$. We say that Y is* δ-well-placed *inside X if for every probability space (Ω, Σ, μ) and every $f \in L_1(\mu, X)$ with $\int f\, d\mu \in Y$ we have*

$$\int \|f\|\, d\mu \geqslant \Big\| \int f\, d\mu \Big\| + \delta \int \|q \circ f\|\, d\mu. \tag{B.3}$$

We will isolate some basic properties of δ-well-placed subspaces. To this end, we recall the following standard notation. Given a probability space (Ω, Σ, μ), a sub-σ-algebra Σ' of Σ, a Banach space Z and a function $g \in L_1(\mu, Z)$, by $\mathbb{E}(g \,|\, \Sigma')$ we shall denote the conditional expectation of g relative to Σ'.

Lemma B.14. **[BP]** *Let X be a Banach space, let Y be a subspace of X and denote by $q\colon X \to X/Y$ the natural quotient map. Let $\delta > 0$ and assume that Y is δ-well-placed inside X. Let (Ω, Σ, μ) be a probability space. Then the following are satisfied.*

(i) *For every $g \in L_1(\mu, X)$ we have*

$$\int \|g\|\, d\mu \geqslant \Big\| \int g\, d\mu \Big\| + \delta \int \|q \circ g\|\, d\mu - (2+\delta) \cdot \Big\| q\Big(\int g\, d\mu \Big) \Big\|.$$

(ii) *If Σ' is a sub-σ-algebra of Σ, then for every $g \in L_1(\mu, X)$ we have*

$$\mathbb{E}(\|g\| \,|\, \Sigma') \geqslant \|\mathbb{E}(g \,|\, \Sigma')\| + \delta \cdot \mathbb{E}(\|q \circ g\| \,|\, \Sigma') - (2+\delta) \cdot \|q \circ \mathbb{E}(g \,|\, \Sigma')\|$$

μ-almost everywhere.

(iii) *If Σ' is a sub-σ-algebra of Σ, then for every $g \in L_1(\mu, X)$ we have*

$$\int \|g\|\, d\mu \geqslant \int \|\mathbb{E}(g \,|\, \Sigma')\|\, d\mu + \delta \int \|q \circ g\|\, d\mu - (2+\delta) \int \|q \circ \mathbb{E}(g \,|\, \Sigma')\|\, d\mu.$$

Proof. (i) Let $g \in L_1(\mu, X)$ and set $x = \int g\, d\mu$. Also let $\varepsilon > 0$ be arbitrary. We select $y \in Y$ such that

$$\|x - y\| \leqslant \|q(x)\| + \varepsilon. \tag{B.4}$$

Define $f \in L_1(\mu, X)$ by $f = g - (x-y)\mathbf{1}_\Omega$ and notice that $\int f\, d\mu = y \in Y$. Applying inequality (B.3) to f, we obtain that

$$\int \|f\|\, d\mu \geqslant \|y\| + \delta \int \|q \circ g - q(x)\|\, d\mu. \tag{B.5}$$

Since $g = f - (y-x)\mathbf{1}_\Omega$ and $y = x - (x-y)$, we see that

$$\int \|g\|\, d\mu \geqslant \int \|f\|\, d\mu - \|x-y\|, \tag{B.6}$$

$$\|y\| \geqslant \|x\| - \|x-y\|. \tag{B.7}$$

Finally, notice that

$$\int \|q\circ g - q(x)\|\, d\mu \geqslant \int \|q\circ g\|\, d\mu - \|q(x)\|. \tag{B.8}$$

Combining, successively, inequalities (B.6), (B.5), (B.7), (B.8) and (B.4), we conclude that

$$\int \|g\|\, d\mu \geqslant \Big\| \int g\, d\mu \Big\| + \delta \int \|q\circ g\|\, d\mu - (2+\delta)\cdot \Big\| q\Big(\int g\, d\mu\Big)\Big\| - 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the proof of part (i) is completed.

(ii) Notice that if g is a simple function, then the desired estimate follows by the inequality obtained in part (i). The general case follows using this observation and a standard approximation argument.

(iii) Follows immediately by integrating the estimate in part (ii). The proof is completed. □

The link between the notion of a δ-well-placed subspace and the notion of an η-admissible embedding (see Definition 6.7) is given in the following lemma.

Lemma B.15. **[BP]** *Let $0 < \eta \leqslant 1$ and let X, X' be Banach spaces. Also let $J\colon X \to X'$ be an isometric embedding. Assume that J is η-admissible. Then $J(X)$ is $(1-\eta)$-well-placed inside X'.*

It is easy to see that Lemma B.15 implies Lemma 6.11. The proof given below shows that Lemma B.15 is actually equivalent to Lemma 6.11.

Proof of Lemma B.15. Let $q\colon X' \to X'/J(X)$ denote the natural quotient map. Also let (Ω, Σ, μ) be a probability space, and let $f \in L_1(\mu, X')$ be a simple function with $\int f\, d\mu \in J(X)$. Then inequality (6.2) can be reformulated as

$$\int \|f\|\, d\mu \geqslant \Big\| \int f\, d\mu \Big\| + (1-\eta) \int \|q\circ f\|\, d\mu.$$

In other words, inequality (6.2) implies inequality (B.3) for simple functions. The general case follows by a standard approximation argument. Indeed, let $f \in L_1(\mu, X')$ such that $\int f\, d\mu \in J(X)$. We may select a sequence (f_n) in $L_1(\mu, X')$ consisting of simple functions, such that $f_n \to f$ μ-almost everywhere and $\int \|f_n - f\|\, d\mu \to 0$. By passing to a small perturbation of each f_n, we may assume that $\int f_n\, d\mu \in J(X)$. Hence,

$$\int \|f_n\|\, d\mu \geqslant \Big\| \int f_n\, d\mu \Big\| + (1-\eta) \int \|q \circ f_n\|\, d\mu$$

for every $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ and using the dominated convergence theorem, inequality (B.3) follows. The proof is completed. □

We are ready to proceed to the proof of Theorem 6.13.

Proof of Theorem 6.13*: the Radon–Nikodym property.* Fix $0 < \eta < 1$ and a system (F_n, j_n) of isometric embeddings such that each F_n is finite-dimensional and for every $n \in \mathbb{N}$ the isometric embedding $j_n \colon F_n \to F_{n+1}$ is η-admissible. Let X be the inductive limit of the system (F_n, j_n).

As in Section 6.3, we start by making some simple observations. In particular, we view the sequence (F_n) as being an increasing (with respect to inclusion) sequence of finite-dimensional subspaces of X such that $\bigcup_n F_n$ is dense in X. For every $n \in \mathbb{N}$ by $q_n \colon X \to X/F_n$ we shall denote the natural quotient map, while for every pair $n, m \in \mathbb{N}$ with $n < m$ by $I(n,m) \colon F_n \to F_m$ we shall denote the inclusion operator. As the isometric embedding $I(n, n+1) \colon F_n \to F_{n+1}$ is η-admissible for every $n \in \mathbb{N}$, by Lemma 6.9, we see that the isometric embedding $I(n,m)$ is also η-admissible for every pair $n, m \in \mathbb{N}$ with $n < m$. Applying Lemma B.15, we see that F_n is $(1-\eta)$-well-placed inside F_m. Using a standard approximation argument, we obtain the following basic fact.

Fact B.16. *For every $n \in \mathbb{N}$ the space F_n is $(1-\eta)$-well-placed inside X.*

We are now in the position to argue that the space X has the Radon–Nikodym property. So, let (Ω, Σ, μ) be a probability space and let (Σ_k) be an increasing sequence of sub-σ-algebras of Σ. Let (M_k) be a martingale in $L_1(\mu, X)$ adapted to (Σ_k) and assume that

$$\sup_k \int \|M_k\|\, d\mu = C < \infty. \tag{B.9}$$

We have to show that (M_k) converges in X μ-almost everywhere. The main claim is the following.

Claim B.17. **[BP]** *We have*

$$\lim_{n\to\infty} \lim_{k\to\infty} \int \|q_n \circ M_k\|\, d\mu = 0 \tag{B.10}$$

Proof of Claim B.17*.* Fix $l \in \mathbb{N}$. Notice that $\lim_n q_n \circ M_l = 0$ μ-almost everywhere. Therefore,

$$\lim_{n\to\infty} \int \|q_n \circ M_l\|\, d\mu = 0. \tag{B.11}$$

Let $n \in \mathbb{N}$ be arbitrary. Also let $k > l$ be arbitrary. Then $\mathbb{E}(M_k \,|\, \Sigma_l) = M_l$. By Fact B.16, the subspace F_n is $(1-\eta)$-well-placed inside X. Hence, applying part (iii) of Lemma B.14 to $Y = F_n$, $g = M_k$ and $\Sigma' = \Sigma_l$, we obtain that

$$\int \|M_k\|\, d\mu \geqslant \int \|M_l\|\, d\mu + (1-\eta) \int \|q_n \circ M_k\|\, d\mu - (3-\eta) \int \|q_n \circ M_l\|\, d\mu.$$

Taking the limit superior above first in k, then in n and finally in l and using (B.11), we conclude that

$$0 \geqslant (1-\eta) \limsup_{n\to\infty} \limsup_{k\to\infty} \int \|q_n \circ M_k\|\, d\mu \geqslant 0.$$

The claim is proved. □

Let Y be a subspace of X and denote by $Q\colon X \to X/Y$ the natural quotient map. Notice that the sequence $(Q \circ M_k)$ is a martingale in $L_1(\mu, X/Y)$ adapted to (Σ_k). For every $k \in \mathbb{N}$ we set $g_k = \|Q \circ M_k\| \in L_1(\mu, \mathbb{R})$. The norm $\|\cdot\|$ of X/Y is a convex function, and so, the sequence (g_k) is a sub-martingale. That is, for every $k \in \mathbb{N}$ the inequality $\mathbb{E}(g_{k+1} \,|\, \Sigma_k) \geqslant g_k$ holds μ-almost everywhere. Moreover, condition (B.9) reduces to the fact that $\sup_k \int |g_k|\, d\mu \leqslant C < +\infty$. Hence, by [Bi, Theorem 35.5], we obtain the following fact.

Fact B.18. *The following hold.*

(i) *The sequence $(\|M_k\|)$ is convergent μ-almost everywhere.*

(ii) *For every $n \in \mathbb{N}$ the sequence $(\|q_n \circ M_k\|)$ is convergent μ-almost everywhere.*

For every $n \in \mathbb{N}$ let $h_n = \sup_k \|q_n \circ M_k\|$. Notice that for μ-almost all $\omega \in \Omega$ the sequence $\big(h_n(\omega)\big)$ is decreasing and, consequently, its limit exists.

Claim B.19. *We have $\lim h_n = 0$ μ-almost everywhere.*

Proof of Claim B.19. Assume, towards a contradiction, that the claim is false. Hence we may find $A \in \Sigma$ and $\varepsilon, \delta > 0$ such that

(a) $\mu(A) = \varepsilon$ and

(b) $h_n(\omega) > \delta$ for every $n \in \mathbb{N}$ and every $\omega \in A$.

Moreover, by part (ii) of Fact B.18, we may assume that

(c) the sequence $\big(\|q_n \circ M_k(\omega)\|\big)$ is convergent for every $n \in \mathbb{N}$ and every $\omega \in A$.

Fix $\omega \in A$. For every $n \in \mathbb{N}$ we select $k_n \in \mathbb{N}$ such that $\|q_n \circ M_{k_n}(\omega)\| > \delta$. Notice, first, that there exists an infinite subset L of $\mathbb{N}$ such that $k_n \neq k_m$ for every $n, m \in L$ with $n \neq m$. For if not, by Ramsey's theorem, there would existed an infinite subset M of $\mathbb{N}$ and $k \in \mathbb{N}$ such that $k_n = k$ for every $n \in M$. This clearly contradicts the fact that $\lim_n \|q_n \circ M_k(\omega)\| = 0$.

Now let $n \in \mathbb{N}$ and $m \in L$ with $n < m$. The sequence (F_n) is increasing with respect to inclusion, and so,

$$\|q_n \circ M_{k_m}(\omega)\| \geqslant \|q_m \circ M_{k_m}(\omega)\| > \delta$$

by the choice of k_m. Therefore, for every $\omega \in A$ and every $n \in \mathbb{N}$ the set

$$\big\{k \in \mathbb{N} : \|q_n \circ M_k(\omega)\| > \delta\big\}$$

is infinite. Invoking property (c) above, we obtain that

(d) for every $\omega \in A$ and every $n \in \mathbb{N}$ there exists $l_n \in \mathbb{N}$ (depending on ω) such that $\|q_n \circ M_k(\omega)\| \geqslant \delta$ for every $k \in \mathbb{N}$ with $k \geqslant l_n$.

Combining properties (a) and (d) isolated above, we see that

$$\lim_{n\to\infty} \lim_{k\to\infty} \int \|q_n \circ M_k\| \, d\mu \geqslant \varepsilon \cdot \delta > 0.$$

This contradicts Claim B.17. The claim is proved. □

Claim B.20. *For μ-almost all $\omega \in \Omega$ the set $\{M_k(\omega) : k \in \mathbb{N}\}$ is a relatively norm compact subset of X.*

Proof of Claim B.20. Let $\omega \in \Omega$ be such that

(a) $\lim h_n(\omega) = 0$ and

(b) $\sup\{\|M_k(\omega)\| : k \in \mathbb{N}\} < +\infty$.

We will show that the set $\{M_k(\omega) : k \in \mathbb{N}\}$ is relatively norm compact. By part (i) of Fact B.18 and Claim B.19, this will finish the proof.

To this end we will argue by contradiction. So assume that there exist an infinite subset L of $\mathbb{N}$ and $\varepsilon > 0$ such that

$$\|M_k(\omega) - M_l(\omega)\| > \varepsilon \tag{B.12}$$

for every $k, l \in L$ with $k \neq l$. Since $\lim h_n(\omega) = 0$, there exists $n_0 \in \mathbb{N}$ with $h_{n_0}(\omega) < \varepsilon/4$. It follows that for every $k \in L$ we may find $x_k \in F_{n_0}$ such that

$$\|x_k - M_k(\omega)\| < \varepsilon/4. \tag{B.13}$$

The sequence $(x_k)_{k\in L}$ is bounded and the space F_{n_0} is finite-dimensional. Therefore, there exists an infinite subset M of L such that

$$\|x_k - x_l\| < \varepsilon/4 \tag{B.14}$$

for every $k, l \in M$ with $k \neq l$. Combining (B.13) and (B.14), we see that $\|M_k(\omega) - M_l(\omega)\| < 3\varepsilon/4$ for every $k, l \in M$ with $k \neq l$. This clearly contradicts (B.12). The claim is proved. □

Now let (x_i^*) be a sequence in B_{X^*} which separates the points in X. Invoking (B.9), we see that for every $i \in \mathbb{N}$ the sequence $(x_i^* \circ M_k)$ is a bounded martingale in $L_1(\mu, \mathbb{R})$. Hence, by [Bi, Theorem 35.5], we obtain that

(P) for every $i \in \mathbb{N}$ the sequence $(x_i^* \circ M_k)$ is convergent μ-almost everywhere.

Combining Claim B.20 and property (P) isolated above, we conclude that the martingale (M_k) must be convergent in X μ-almost everywhere. This shows that the space X has the Radon–Nikodym property. The proof of Theorem 6.13 is completed. □

Appendix C

The Kuratowski–Tarski algorithm

We will frequently need to compute the complexity of a given set. To this end we will follow a method, employed by logicians, which is known as the *Kuratowski–Tarski algorithm* (see [Mo] and [Ke]).

We will comment on the method which relies on the use of logical notations in defining sets and functions. For instance, let X be a Polish space, and let $P(x)$ and $Q(x)$ be expressions defining A and B respectively, i.e., $A = \{x \in X : P(x)\}$ and $B = \{x \in X : Q(x)\}$. Then the expression "$P(x)$ and $Q(x)$" defines the set $A \cap B$, the expression "$P(x)$ or $Q(x)$" defines the set $A \cup B$ while "$\neg P(x)$" defines the set A^{c}. In other words, conjunction corresponds to intersection, disjunction to union and negation to complementation.

Now let X and Y be Polish spaces and let $P(x, y)$ be an expression, where x varies over X and y varies over Y, defining the set $A = \{(x, y) : P(x, y)\}$. In this case, the expression "$\exists y \in Y$ with $P(x, y)$" defines the set $B = \mathrm{proj}_X A$. That is, existential quantification corresponds to projection. On the other hand, universal quantification corresponds to the operation of co-projection since the expression "$\forall y \in Y$ we have $P(x, y)$" defines the set $B = (\mathrm{proj}_X A^{\mathrm{c}})^{\mathrm{c}}$.

We will illustrate by an example the above remarks. So, let X, Y and Z be Polish spaces and let $P(x, y, z)$ and $Q(x, y)$ be expressions defining two Borel subsets of $X \times Y \times Z$ and $X \times Y$ respectively. Consider the subset A of X defined by

$$x \in A \Leftrightarrow \exists z \in Z \text{ with } [\forall y \in Y \text{ we have } P(x, y, z) \Leftrightarrow Q(x, y)].$$

The expression "$P(x, y, z) \Leftrightarrow Q(x, y)$" is equivalent to

$$\text{“}[P(x, y, z) \text{ and } Q(x, y)] \text{ or } [\neg P(x, y, z) \text{ and } \neg Q(x, y)]\text{”}$$

and so, it defines a Borel subset A_1 of $X \times Y \times Z$. The formula

$$\text{“}\forall y \in Y \text{ we have } P(x,y,z) \Leftrightarrow Q(x,y)\text{”}$$

defines the co-projection A_2 of A_1. Hence A_2 is $\mathbf{\Pi}^1_1$. As the final quantifier is existential, we conclude that the set A is the projection of A_2, and so, A is $\mathbf{\Sigma}^1_2$.

We point out that it is the reasoning behind the above mentioned method which justifies the use of the notation $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi, \mathbf{\Delta}^0_\xi$ $(1 \leqslant \xi < \omega)$ and $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n$ $(n \geqslant 1)$ for the Borel and projective classes respectively. We refer to [Mo] and the references therein for a detailed explanation.

Appendix D

Open problems

1. Let X be a Banach space with property (S) (see Definition 2.28) and with a Schauder basis. Let (e_n) be a normalized Schauder basis of X. By Theorem 2.29, there exists a map $\phi_X : \omega_1 \times \omega_1 \to \omega_1$ such that for every $\xi, \zeta < \omega_1$ and every $Y, Z \in \mathrm{NC}_X$ with $o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) = \xi$ and $o\big(T_{\mathrm{NC}}(Z, X, (e_n))\big) = \zeta$ we have

$$o\big(T_{\mathrm{NC}}(Y \oplus Z, X, (e_n))\big) \leqslant \phi_X(\xi, \zeta).$$

Problem 1. *Let X be a Banach space with property* (S) *and with a Schauder basis. Find an explicit upper bound for the map ϕ_X.*

Of particular importance are the cases "$X = \ell_2$" and "$X = C(2^{\mathbb{N}})$".

2. Consider the class

$$\mathrm{SSD} = \{X \in \mathrm{SB} : X^{**} \text{ is separable}\}.$$

For every $X \in \mathrm{SSD}$ let

$$\varphi_{\mathrm{SSD}}(X) = \max\big\{\mathrm{Sz}(X), \mathrm{Sz}(X^*)\big\}.$$

The map $\mathrm{SSD} \ni X \mapsto \varphi_{\mathrm{SSD}}(X)$ behaves like a $\mathbf{\Pi}^1_1$-rank for most practical purposes (see [D1]).

It turned out that the rank ϕ_{SSD} is also well-behaved when restricted on the class REFL of separable reflexive Banach spaces. For instance, Odell, Schlumprecht and Zsák [OSZ] have shown that for every countable ordinal ξ the class $\{X \in \mathrm{REFL} : \varphi_{\mathrm{SSD}}(X) \leqslant \xi\}$ is analytic.

Problem 2. *Is the map* $\mathrm{REFL} \ni X \mapsto \varphi_{\mathrm{SSD}}(X)$ *a $\mathbf{\Pi}^1_1$-rank on* REFL*?*

3. Let (e_n) be a normalized Schauder basis of $C(2^{\mathbb{N}})$. By Corollary 6.28, there exists a map $f : \omega_1 \to \omega_1$ such that for every countable ordinal ξ, every separable Banach space X with $o\big(T_{\mathrm{NC}}(X, C(2^{\mathbb{N}}), (e_n))\big) \leqslant \xi$ embeds into a Banach space Y with a Schauder basis satisfying $o\big(T_{\mathrm{NC}}(Y, C(2^{\mathbb{N}}), (e_n))\big) \leqslant f(\xi)$.

Problem 3. *Find an explicit upper bound for the map f.*

4. Let $\mathcal{C}$ be a $\mathbf{\Pi}^1_1$ strongly bounded class of separable Banach spaces. Also let $\phi_{\mathcal{C}}\colon \mathcal{C} \to \omega_1$ be a canonical $\mathbf{\Pi}^1_1$-rank on $\mathcal{C}$. For every $\xi < \omega_1$ we set

$$\mathcal{C}_\xi = \{Z \in \mathcal{C} : \phi_{\mathcal{C}}(Z) \leqslant \xi\}$$

and

$$u_{\mathcal{C}}(\xi) = \min\big\{\phi_{\mathcal{C}}(Y) : Y \in \mathcal{C} \text{ and is universal for the class } \mathcal{C}_\xi\big\}.$$

Notice that $u_{\mathcal{C}}(\xi)$ is well-defined.

Problem 4. *Find explicit upper bounds for the maps u_{REFL}, u_{SD}, u_{NU} and u_{NC_X} where X is a minimal Banach space not containing ℓ_1.*

No bounds are known for u_{NU} and u_{NC_X}. The problem of estimating the values of these maps is related to Problems 1 and 3.

For the classes REFL and SD there are two results which provide almost optimal upper bounds for the corresponding maps u_{REFL} and u_{SD}. The first one is due to Odell, Schlumprecht and Zsák and deals with separable reflexive spaces.

Theorem D.1. [OSZ] *Let $\xi < \omega_1$. Then there exists a separable reflexive space Y satisfying $\max\big\{\mathrm{Sz}(Y), \mathrm{Sz}(Y^*)\big\} \leqslant \omega^{\xi\cdot\omega+1}$ and containing an isomorphic copy of every separable reflexive space Z satisfying $\max\big\{\mathrm{Sz}(Z), \mathrm{Sz}(Z^*)\big\} \leqslant \omega^{\xi\cdot\omega}$.*

The second result is due to Freeman, Odell, Schlumprecht and Zsák and deals with Banach spaces with separable dual.

Theorem D.2. [FOSZ] *Let $\xi < \omega_1$. Then there exists a separable Banach space Y satisfying $\mathrm{Sz}(Y) \leqslant \omega^{\xi\cdot\omega+1}$ and containing an isomorphic copy of every separable space Z satisfying $\mathrm{Sz}(Z) \leqslant \omega^{\xi\cdot\omega}$.*

5. By Theorem 7.18, the class NC_X is strongly bounded for every minimal Banach space X not containing ℓ_1. The following problems are related to the natural question whether the class NC_{ℓ_1} is also strongly bounded.

Problem 5. *Is it true that every separable Banach space X not containing a copy of ℓ_1 embeds into a space Y with a Schauder basis and not containing a copy of ℓ_1?*

Problem 6. *Let (e_n) be the standard unit vector basis of ℓ_1. Does there exist a map $g\colon \omega_1 \to \omega_1$ such that for every countable ordinal ξ and every separable Banach space X with $o\big(T_{\mathrm{NC}}(X, \ell_1, (e_n))\big) \leqslant \xi$ the space X embeds into a Banach space Y with a Schauder basis satisfying $o\big(T_{\mathrm{NC}}(Y, \ell_1, (e_n))\big) \leqslant g(\xi)$?*

Problem 7. *Is the class NC_{ℓ_1} strongly bounded?*

We notice that an affirmative answer to Problem 6 can be used to provide an affirmative answer to Problem 7. (To see this combine Theorem 2.17, Lemma 2.25 and Theorem 7.17.)

It seems reasonable to conjecture that Problems 5, 6 and 7 have an affirmative answer. Our optimism is based on the following facts. Firstly, by Theorem 7.17, Problem 7 is true within the category of Banach spaces with a Schauder basis. Secondly, the "dual" versions of Problems 6 and 7 also have an affirmative answer. Specifically, denoting by (e_n) the standard unit vector basis of ℓ_1, we have the following theorem.

Theorem D.3. [**D4**] *There exists a map* $f\colon \omega_1 \to \omega_1$ *such that for every countable ordinal* ξ *and every separable Banach space* X *with* $o\big(T_{\mathrm{NC}}(X,\ell_1,(e_n))\big) \leqslant \xi$, *the space* X *is a quotient of a Banach space* Y *with a Schauder basis satisfying* $o\big(T_{\mathrm{NC}}(Y,\ell_1,(e_n))\big) \leqslant f(\xi)$.

Theorem D.4. [**D4**] *Let* $\mathcal{C} \subseteq \mathrm{SB}$. *Then the following are equivalent.*

(i) *There exists a separable Banach space* Y *not containing a copy of* ℓ_1 *and such that every space* $X \in \mathcal{C}$ *is a quotient of* Y.

(ii) *We have* $\sup\big\{o\big(T_{\mathrm{NC}}(X,\ell_1,(e_n))\big) : X \in \mathcal{C}\big\} < \omega_1$.

(iii) *There exists an analytic subset* A *of* NC_{ℓ_1} *such that* $\mathcal{C} \subseteq A$.

6. Let

$$\mathrm{S} = \{X \in \mathrm{SB} : X \text{ has a Schauder basis}\}.$$

By Lemma 2.25, the class S is analytic.

Problem 8. *Is the class* S *analytic non-Borel?*

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