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Abstract

In this paper, we explore certain Banach spaces of analytic functions. In particular, we study the space A^{-1}, demonstrating some of its basic properties including non-separability. We ask the question: Given a class $\mathcal{C}$ of analytic functions on the unit disk $\mathbb{D}$ and a sequence $\{z_n\}$ of points in the disk, is there an non-zero analytic function $f \in \mathcal{C}$ with $f(z_n) = 0$ for all n? Finally, we explore the M_z invariant subspaces of A^{-1}, demonstrating that they may possess the codimension-2 property.

This paper is part of the requirements for honors in mathematics. The signatures below, by the advisor, a departmental reader, and a representative of the departmental honors committee, demonstrate that Michael T. Nimchek has met all the requirements needed to receive honors in mathematics.

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BANACH SPACES OF ANALYTIC FUNCTIONS

MICHAEL NIMCHEK

ABSTRACT. In this paper, we explore certain Banach spaces of analytic functions. In particular, we study the space A^{-1}, demonstrating some of its basic properties including non-separability. We ask the question: Given a class $\mathcal{C}$ of analytic functions on the unit disk $\mathbb{D}$ and a sequence $\{z_n\}$ of points in the disk, is there an non-zero analytic function $f \in \mathcal{C}$ with $f(z_n) = 0$ for all n? Finally, we explore the M_z invariant subspaces of A^{-1}, demonstrating that they may possess the codimension-2 property.

1. INTRODUCTION

In this paper we will study the space A^{-1} consisting of analytic functions f defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which

$$\sup_{z \in \mathbb{D}} (1 - |z|)|f(z)| < +\infty.$$

In particular, we will demonstrate that

- A^{-1} is a non-separable Banach space.
- The closure of the polynomials in the norm of A^{-1} is the space

$$\{f \in A^{-1} : \lim_{|z| \to 1} (1 - |z|)|f(z)| = 0\}.$$

- The zero sets of A^{-1} are very complicated. In particular, the union of two zero sets is not necessarily a zero set. In fact, it may be a set that can "sample" the norm (see Section 6).
- We will also explore the invariant subspaces for the linear transformation

$$M_z : A^{-1} \to A^{-1} \text{ such that } M_z(f) = zf.$$

  We will focus our attention on the (closed) subspaces $\mathcal{S} \subset A^{-1}$ for which $M_z\mathcal{S} \subset \mathcal{S}$, the invariant subspaces for the linear transformation M_z on A^{-1}. In particular, we will show that these M_z invariant subspaces of A^{-1} can have the codimension-2 property, that is, the quotient space $\mathcal{S}/M_z\mathcal{S}$ is two dimensional. This result had previously been observed by Hakan Hedenmalm in other spaces of analytic functions [5]. It is intriguing because, for many spaces of analytic functions, their M_z invariant subspaces must always have the codimension-1 property [1] [8].

Our paper is organized as follows:

- In Section 2 we discuss basic properties of metric spaces and define the metric spaces we will use throughout the paper.
- Section 3 discusses vector spaces and quotient spaces. This background discussion is necessary in order to understand the results in Section 6.
- In Section 4 we discuss Banach spaces of analytic functions. We demonstrate that A^{-1} is a non-separable Banach space and identify the closure of the polynomials.
- Section 5 discusses the zero sets of various spaces of functions, including A^{-1}.
- Finally, in Section 6 we consider the M_z invariant subspaces of A^{-1} and prove that they may have the codimension-2 property.

2. Metric Spaces

In this section, we define some basic terminology and give examples of metric spaces.

Definition 2.1. A *metric space* is a set X and distance function

$$d : X \to \mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$$

which satisfy the following for all $x, y, z \in X$

$$d(x,y) \geq 0 \tag{2.1}$$

$$d(x,y) = 0 \Leftrightarrow x = y \tag{2.2}$$

$$d(x,y) = d(y,x) \tag{2.3}$$

$$d(x,y) + d(y,z) \geq d(x,z) \tag{2.4}$$

This last item is the familiar "triangle inequality" [4], p. 11.

Example 2.2. The real numbers $\mathbb{R}$ form a metric space under the absolute value of subtraction, that is

$$d(x,y) = |y-x|$$

(1) $|y-x| \geq 0$
(2) $|y-x| = 0 \Leftrightarrow x = y$
(3) $|y-x| = |x-y|$
(4) $|y-x| + |z-y| \geq |z-x|$

All of this we know from the basic properties of numbers.

Example 2.3. Let $n \in \mathbb{N}$ and define $\mathbb{R}^n$ to be

$$\mathbb{R}^n = \{(x_1, \cdots, x_n) : x_i \in \mathbb{R}\}.$$

This is the familiar n-dimensional Euclidean space and forms a metric space under

$$d(x,y) = \sqrt{\sum_{j=1}^{n}(x_j - y_j)^2} = \|x-y\|, x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n).$$

For this metric, (2.1) and (2.3) are obvious.
To see (2.2), if $d(x,y) = 0 \Rightarrow \sum_{j=1}^{n}(x_j - y_j)^2 = 0 \Rightarrow x_j = y_j \forall j \Rightarrow x = y$
Conversely, if $x = y \Rightarrow x_j = y_j \forall j \Rightarrow d(x,y) = 0$
To verify (2.4), first note that

$$\|x+y\|^2 = \sum_{j=1}^{n}(x_j + y_j)^2 = \sum_{j=1}^{n}(x_j^2 + 2x_jy_j + y_j^2) = \|x\|^2 + 2 < x,y > + \|y\|^2$$

where $\sum_{j=1}^{n} x_jy_j$ is the inner-product $< x, y >$.
By the Cauchy-Schwartz inequality
$| < x,y > | \leq \|x\|\|y\| \Rightarrow \|x+y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$

$$\Rightarrow \|x+y\| \leq \|x\| + \|y\|.$$

Thus, for $x, y, z \in \mathbb{R}^n$
$d(x,z) = \|x - y + y - z\| = \|(x-y) + (y-z)\| \leq \|x-y\| + \|y-z\| = d(x,y) + d(y,z).$

Example 2.4. The complex numbers $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$ (as usual $i = \sqrt{-1}$) form a metric space under

$$d(z,w) = |z-w| = \sqrt{(x_z - x_w)^2 + (y_z - y_w)^2}$$

where $z = x_z + iy_z$ and $w = x_w + iy_w$. Since $x + iy$ can be identified with the vector $(x,y) \in \mathbb{R}^2$, this distance function is the 2-dimensional metric previously discussed.

Our next examples of metric spaces consist of spaces of complex valued functions. In particular we will look at classes of functions

$$f : \mathbb{D} \to \mathbb{C},$$

where

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

is the "unit disk" and $\mathbb{C}$ (as above) denotes the complex numbers. Also, we use the notation $|z| = |x + iy| = \sqrt{x^2 + y^2}$ to denote the modulus of a complex number.

Example 2.5. Let $C^\infty(\mathbb{D})$ define the complex valued functions on the disk whose partial derivatives (of all orders) exist and are continuous on $\mathbb{D}$. The partial derivatives are taken with respect to the functions' real and imaginary components.

We wish to define a distance function that will make $C^\infty(\mathbb{D})$ a metric space. The choice that seems immediately obvious is

$$d(f,g) = \sup_{z\in\mathbb{D}} |g(z) - f(z)|.$$

But let $g(z) = \frac{1}{1-z}$ and $f(z) = 0$, both of which are in $C^\infty(\mathbb{D})$. Then $|g(z) - f(z)| \to +\infty$ as $z \to 1$ and so

$$d(f,g) = \sup_{z\in\mathbb{D}} |g(z) - f(z)| = +\infty$$

which cannot be possible since the "distance function" must be finite valued. Clearly, we must try something different.

Fortunately, we can write $\mathbb{D}$ as an infinite union of compact subsets K_n where

$$K_n = \{z \in \mathbb{D} : |z| \leq 1 - \frac{1}{n}\} = \overline{B}(0; 1 - \frac{1}{n})$$

for each $n \in \mathbb{N}$. Clearly,

$$\mathbb{D} = \bigcup_{n=1}^{\infty} K_n \quad \text{with} \quad K_{n+1} \supset K_n.$$

Now for each $f, g \in C^\infty(\mathbb{D})$ define

$$\rho_n(f,g) = \sup_{z\in K_n} |g(z) - f(z)|.$$

Note that $\rho_n(f,g) < \infty$ since K_n is compact and f and g are continuous on K_n. Also notice that

$$\frac{\rho_n(f,g)}{1+\rho_n(f,g)} \leq 1 \;\; \forall\, n \in \mathbb{N}$$

and so we can define

$$\rho(f,g) = \sum_{n=1}^{\infty} (\frac{1}{2})^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)}.$$

To show that $\rho(f,g)$ is a metric for $C^\infty(\mathbb{D})$, we first prove the following two lemmas.

Lemma 2.6. *Given $A \subset \mathbb{D}$, let X_A be the set of bounded functions on A. Then for $f, g \in X_A$*

$$\rho_A(f,g) = \sup_{z\in A} |g(z) - f(z)|$$

serves as a metric for X_A.

Proof. Conditions (2.1), (2.2) and (2.3) are obvious. To prove (2.4), the triangle inequality, consider $f, g, h \in X_A$ and note that

$$\begin{aligned}\rho_A(f,h) &= \sup_{z\in A} |h(z) - g(z) + g(z) - f(z)| \leq \sup_{z\in A}(|h(z) - g(z)| + |g(z) - f(z)|)\\ &\leq \sup_{z\in A} |h(z) - g(z)| + \sup_{z\in A} |g(z) - f(z)| = \rho_A(f,g) + \rho_A(g,h).\end{aligned}$$

This shows that ρ_A is a metric for X_A. $\square$

Lemma 2.7. *Let $\rho(x,y)$ be a metric on a set A and let f be a real-valued function satisfying the following four properties:*

(1) $f(u) \geq 0 \; \forall \, u \geq 0$
(2) $f(0) = 0$
(3) *f is strictly increasing on the interval* $(0, \infty)$
(4) $f(u+v) \leq f(u) + f(v) \; \forall \, u, v \geq 0$

Then $\sigma(x,y) = f(\rho(x,y))$ is a metric on A.

Proof. Let $x, y, z \in A$. Then, since $\rho(x,y) \geq 0$, we have

$$\sigma(x,y) = f(\rho(x,y)) \geq 0$$

so (2.1) is established.

Next, note that if $\sigma(x,y) = f(\rho(x,y)) = 0$ this implies that $\rho(x,y) = 0$ since f is strictly increasing and $f(0) = 0$. But since $\rho(x,y)$ is a metric, then $\rho(x,y) = 0$ implies that $x = y$. Conversely, if $x = y$ then $\rho(x,y) = 0$ which, by (2), implies that $f(\rho(x,y)) = \sigma(x,y) = 0$, thus establishing (2.2).

Condition (2.3) is obvious.

To prove the triangle inequality (2.4) note that

$$\sigma(x,y) + \sigma(y,z) = f(\rho(x,y)) + f(\rho(y,z)) \geq f(\rho(x,y) + \rho(y,z))$$

by property (4) for f. But, since ρ is a metric, this implies $\rho(x,y) + \rho(y,z) \geq \rho(x,z)$. Thus, since f is strictly increasing

$$f(\rho(x,y) + \rho(y,z)) \geq f(\rho(x,z))$$

and therefore

$$\sigma(x,y) + \sigma(y,z) \geq \sigma(x,z)$$

which establishes (2.4). So σ is a metric on A. $\square$

Corollary 2.8. *If $d(x,y)$ is a metric on a set A then*

$$\mu(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

is also a metric on A.

Proof. Let

$$f(u) = \frac{u}{1+u}.$$

Properties (1), (2) and (3) for f in Lemma 2.7 are obvious. To prove property (4), note that

$$f(u+v) = \frac{u+v}{1+u+v} = \frac{u}{1+u+v} + \frac{v}{1+u+v} \leq \frac{u}{1+u} + \frac{v}{1+v}$$

since $u, v \geq 0$.

Thus, by Lemma 2.7, $\mu = f(d)$ is a metric on A. $\square$

Theorem 2.9. *$C^\infty(\mathbb{D})$ is a metric space under*

$$\rho(f,g) = \sum_{n=1}^{\infty}(\frac{1}{2})^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)}$$

where $\rho_n(f,g) = \sup_{z\in K_n} |g(z) - f(z)|$ and $K_n = \overline{B}(0; 1-\frac{1}{n})$

Proof. To prove this result, we first make the following observations:

(1) $\rho_n(f,g) = \rho_{\overline{B}(0;1-\frac{1}{n})}(f,g)$ and thus by Lemma 2.6, $\rho_n(f,g)$ is a metric on the set of bounded functions on K_n.
(2) $\frac{\rho_n(f,g)}{1+\rho_n(f,g)}$ is also a metric on the set of bounded functions on K_n by Corollary 2.8.
(3) Since $0 < \frac{\rho_n(f,g)}{1+\rho_n(f,g)} < 1$ and $\sum_{n=1}^{\infty}(\frac{1}{2})^n = 1$ then $\rho(f,g) = \sum_{n=1}^{\infty}(\frac{1}{2})^n\frac{\rho_n(f,g)}{1+\rho_n(f,g)}$ is finite $\forall\, f,g \in C^\infty(\mathbb{D})$.

It suffices to prove the triangle inequality, since (2.1), (2.2) and (2.3) are obvious. For $f,g,h \in C^\infty(\mathbb{D})$, we know by result (2) just demonstrated that

$$\frac{\rho_n(f,h)}{1+\rho_n(f,h)} \leq \frac{\rho_n(f,g)}{1+\rho_n(f,g)} + \frac{\rho_n(g,h)}{1+\rho_n(g,h)}$$

where this is true $\forall\, n \in \mathbb{N}$.

Thus, multiplying by $(\frac{1}{2})^n$ and summing over n yields

$$\rho(f,h) \leq \rho(f,g) + \rho(g,h)$$

where convergence of the sum is guaranteed by result (3) above. Thus ρ is a metric for $C^\infty(\mathbb{D})$. □

Definition 2.10. We say a function $f \in C^\infty(\mathbb{D})$ is *analytic* on $\mathbb{D}$ if f satisfies the Cauchy-Riemann partial differential equation

$$\overline{\partial} f(x,y) = \frac{1}{2}(\frac{\partial f(x,y)}{\partial x} + i\frac{\partial f(x,y)}{\partial y}) = 0. \tag{2.5}$$

We will denote the space of analytic functions by $H(\mathbb{D})$. (We remark that the symbol "H" is used since these functions are also called "holomorphic".) It is easily verified that the same metric discovered for $C^\infty(\mathbb{D})$ also forms a metric for $H(\mathbb{D})$. The following are examples of analytic functions since each satisfies the Cauchy-Riemann p.d.e.

(1) $f(z) = z^2$
(2) $f(z) = e^z$
(3) $f(z) = \sin z$

The following C^∞ functions are not analytic:

(1) $f(x,y) = y$, since $\overline{\partial} f = i/2 \not\equiv 0$
(2) $f(x,y) = x^2 + y^2$, since $\overline{\partial} f = x + iy = z \not\equiv 0$

Example 2.11. The bounded analytic functions

$$H^\infty(\mathbb{D}) = \{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f(z)| < +\infty \; \forall z \in \mathbb{D}\}$$

By Lemma 2.6

$$d(f,g) = \sup_{z\in\mathbb{D}} |g(z) - f(z)|$$

forms a metric for $H^\infty(\mathbb{D})$ because we have now restricted ourselves to functions that are bounded.

The following space of analytic functions will be the focus of most of this paper.

Definition 2.12. $A^{-1} = \{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}}(1-|z|)|f(z)| < +\infty\}$

Lemma 2.13. *$d(f,g) = \sup_{z\in\mathbb{D}}(1-|z|)|g(z) - f(z)|$ forms a metric for A^{-1}.*

Proof. Clearly conditions (2.1), (2.2) and (2.3) hold. We just need to check (2.4), the triangle inequality. Let $f, g, h \in A^{-1}$. Then

$$d(f,g) + d(g,h) = \sup_{z\in\mathbb{D}}(1-|z|)|g(z) - f(z)| + \sup_{z\in\mathbb{D}}(1-|z|)|h(z) - g(z)|$$

$$\geq \sup_{z\in\mathbb{D}}(1-|z|)(|g(z) - f(z)| + |h(z) - g(z)|)$$

$$\geq \sup_{z\in\mathbb{D}}(1-|z|)|h(z) - f(z)| = d(f,h).$$

This proves d is a metric on A^{-1}. □

We conclude this section by defining some terms that will be used throughout the rest of this paper.

Definition 2.14. Given a set X with metric d, we define the following [4]:

(1) A set $A \subset X$ is *open* if for each $x \in A \; \exists \, \epsilon > 0$ such that

$$\{y \in X : d(x,y) < \epsilon\} = B(x;\epsilon) \subset A$$

(2) A set $B \subset X$ is *closed* if its complement $X \backslash F$ is open.
(3) A sequence $\{x_n\}$ in X *converges* to x, that is, $x_n \to x$ or $x = \lim_{n\to\infty} x_n$, if for every $\epsilon > 0 \; \exists N \in \mathbb{N}$ such that $d(x, x_n) < \epsilon \; \forall n \geq N$.
(4) A sequence $\{x_n\}$ in X is *Cauchy* if for every $\epsilon > 0 \; \exists N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon \; \forall m, n \geq N$.
(5) X is said to be a *complete* metric space if each Cauchy sequence converges in X. [4], p. 12, 18.

(6) The *closure* of a set $A \subset X$ is the set

$$\bigcap\{B : B \text{ is closed and } B \supset A\}.$$

By the completeness axioms for $\mathbb{R}$, the spaces $\mathbb{R}^n$ and $\mathbb{C}$ are complete. But the fact that $C^\infty(\mathbb{D})$ and $H(\mathbb{D})$ are complete is not at all transparent. For a method of demonstrating the completeness of these metric spaces, we refer the reader to Conway [4], p. 151-152.

3. Vector Spaces

In this section we define numerous important terms that will be used throughout the rest of the paper. We begin with the standard definition of a vector space [7], p. 154.

Definition 3.1. A set V is a vector space over the complex numbers if it satisfies the following for all vectors $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$:

(1) $x + y$ is a unique vector in V.
(2) $x + y = y + x$.
(3) $(x + y) + z = x + (y + z)$.
(4) There exists $0 \in V$ such that $x + 0 = x \; \forall \, x \in V$.
(5) For all $x \in V \; \exists \; -x \in V$ such that $x + (-x) = 0$.
(6) αx is a unique vector in V.
(7) $\alpha(x + y) = \alpha x + \alpha y$.
(8) $(\alpha + \beta)x = \alpha x + \beta x$.
(9) $(\alpha\beta)x = \alpha(\beta x)$.
(10) The product of x and unity equals x.

Note that items (1) and (6) imply respectively that a vector space is closed under addition and multiplication by a complex scalar.

Example 3.2. We shall demonstrate that the following vector spaces are closed under addition and multiplication by a scalar. The reader may verify that these sets also satisfy the other properties of a vector space. Let $\alpha \in \mathbb{C}$ for the remainder of this example.

(1) Let $f, g \in C^\infty(\mathbb{D})$. Then the partial derivatives (of all orders) of both f and g exist and are continuous on $\mathbb{D}$. But by the basic properties of derivatives, this implies that the partial derivatives (of all orders) of $f + g$ also exist and are continuous on $\mathbb{D}$. This implies $f + g \in C^\infty(\mathbb{D})$.
Also, since the partial derivatives (of all orders) of f exist and are continuous on $\mathbb{D}$, then clearly the partial derivatives (of all orders) of αf also exist and are continuous on $\mathbb{D}$. So $\alpha f \in C^\infty(\mathbb{D})$. Thus, $C^\infty(\mathbb{D})$ is closed under addition and scalar multiplication.
(2) Let $f, g \in H(\mathbb{D})$. Then both f and g satisfy (2.5), the Cauchy-Riemann equation. But again, by elementary properties of derivatives, this implies that $f + g$ also satisfies

Cauchy-Riemann. This implies that $f+g \in H(\mathbb{D})$. Also, it is obvious that αf also satisfies (2.5), so $\alpha f \in H(\mathbb{D})$. Thus, $H(\mathbb{D})$ is closed under addition and scalar multiplication.

(3) Let $f, g \in H^\infty(\mathbb{D})$. Since $\sup_{z\in\mathbb{D}} |f(z)| < +\infty$ and $\sup_{z\in\mathbb{D}} |g(z)| < +\infty$, then by the triangle inequality

$$\sup_{z\in\mathbb{D}} |(f+g)(z)| \leq \sup_{z\in\mathbb{D}} |f(z)| + \sup_{z\in\mathbb{D}} |g(z)| < +\infty$$

which demonstrates that $f+g \in H^\infty(\mathbb{D})$. Also,

$$\sup_{z\in\mathbb{D}} |(\alpha f)(z)| = |\alpha| \sup_{z\in\mathbb{D}} |f(z)| < +\infty$$

so $\alpha f \in H^\infty(\mathbb{D})$. Thus, $H^\infty(\mathbb{D})$ is closed under addition and scalar multiplication.

(4) Let $f, g \in A^{-1}$. Then by the triangle inequality,

$$\sup_{z\in\mathbb{D}}(1-|z|)|(f+g)(z)| \leq \sup_{z\in\mathbb{D}}(1-|z|)|f(z)| + \sup_{z\in\mathbb{D}}(1-|z|)|g(z)| < +\infty$$

which demonstrates that $f+g \in A^{-1}$. Also,

$$\sup_{z\in\mathbb{D}}(1-|z|)|(\alpha f)(z)| = |\alpha| \sup_{z\in\mathbb{D}}(1-|z|)|f(z)| < +\infty$$

so $\alpha f \in A^{-1}$. Thus, A^{-1} is closed under addition and scalar multiplication.

(5) $\mathbb{C}$ is obviously closed under addition and multiplication.

The following definitions will be used later in the paper.

Definition 3.3. Let V be a vector space over $\mathbb{C}$. Then $W \subset V$ is a *subspace* of V if it is also a vector space over $\mathbb{C}$ with the same operations of addition and scalar multiplication as on V [6] p. 34.

Example 3.4. The reader may verify that the set $K = \{f \in A^{-1} : f(0) = 0\}$ is a subspace of A^{-1}. Specifically, note that if $f, g \in K$ then $(f+g)(0) = f(0) + g(0) = 0$, so K is closed under addition. Also, given $f \in K$ and $c \in \mathbb{C}$ then $(cf)(0) = cf(0) = c0 = 0$, which implies that K is closed under scalar multiplication.

Definition 3.5. Let V be a vector space over $\mathbb{C}$ with $S \subset V$. Then the intersection W of all subspaces of V which contain S is the *span* of S [6] p. 36.

Definition 3.6. Let V be a vector space over $\mathbb{C}$ and $S \subset V$. Then S is *linearly independent* if for all distinct $s_1, s_2, ..., s_n \in S$, $c_1 s_1 + c_2 s_2 + ... + c_n s_n = 0$ implies that $c_1 = c_2 = ... = 0$. Otherwise, S is *linearly dependent* [6] p. 40.

Example 3.7. Fix an $n \in \mathbb{N}$. Consider the set of functions $P = \{1, z, z^2, ..., z^n\}$ and note that $P \subset A^{-1}$. We proceed to show that P is linearly independent. Given $c_0, c_1, ..., c_n \in \mathbb{C}$ then it must be proved that if $g(z) = c_0 + c_1 z + c_2 z^2 + ... + c_n z^n = 0 \ \forall \ z \in \mathbb{D}$ this implies that

$c_0 = c_1 = ... = c_n = 0$. Since $g \equiv 0$, clearly $g(0) = 0$. But $g(0) = c_0 + c_1(0) + c_2(0) + ... + c_n(0) = 0$ further implies that $c_0 = 0$. Also, since $g \equiv 0$, this implies that $g'(0) = 0$, where $g'(z) = c_1 + 2c_2z + ... + nc_nz^{n-1}$. Thus, $g'(0) = c_1 + 2c_2(0) + ... + nc_n(0) = 0$ implies that $c_1 = 0$. Continuing by induction, it is easily seen that since $g \equiv 0$, this implies that $g^{(k)}(0) = 0$ for all $k \leq n$ which further implies that $c_k = 0$ for all $k \leq n$. Therefore, $c_0 = c_1 = ... = c_n = 0$, which proves that P is linearly independent.

Since P is also a subset of the spaces $C^\infty(\mathbb{D})$, $H(\mathbb{D})$, and $H^\infty(\mathbb{D})$, it follows that P is also linearly independent in these spaces.

Definition 3.8. A linearly independent set of vectors which spans a vector space V is a *basis* for V [6] p. 41.

Definition 3.9. The *dimension* of a vector space V is equal to the number of elements in any basis of V.

This definition is well-defined since, given a basis for a vector space, the number of elements in any other basis must be the same.

Example 3.10. (1) It is easy to see that $\mathbb{C}$, treated as a vector space over the complex numbers, is spanned by unity. Note that there are no strict subspaces of $\mathbb{C}$ which contain one, so the "intersection" of all "subspaces" of $\mathbb{C}$ which contain the number one is simply $\mathbb{C}$, which demonstrates that one spans $\mathbb{C}$. Since one is obviously linearly independent, it serves as a basis for $\mathbb{C}$, which implies that $\mathbb{C}$ has a dimension of one.

(2) Consider the set $\mathbb{C} \times \mathbb{C} = (x, y)\ \forall\ x, y \in \mathbb{C}$, the set of all ordered pairs of complex numbers. We leave it to the reader to verify that $\mathbb{C} \times \mathbb{C}$ is indeed a vector space. Since $(c_1, c_2) = c_1(1, 0) + c_2(0, 1)$, this implies that $(1, 0)$ and $(0, 1)$ span $\mathbb{C} \times \mathbb{C}$. Clearly, if $c_1(1, 0) + c_2(0, 1) = (0, 0)$ then $c_1 = c_2 = 0$, and therefore $(1, 0)$ and $(0, 1)$ are a basis for $\mathbb{C} \times \mathbb{C}$. This implies that $\mathbb{C} \times \mathbb{C}$ has a dimension of two.

(3) We proceed to demonstrate that the vector spaces $C^\infty(\mathbb{D})$, $H(\mathbb{D})$, $H^\infty(\mathbb{D})$ and A^{-1} are all of infinite dimension. Recall from the previous example that the set $P = \{1, z, z^2, z^3, ..., z^n\}$ belongs to all four of these spaces and, given any n, is linearly independent. Thus, there can be no finite set of functions which spans these spaces, which implies there is no finite basis, which proves that the spaces are not of finite dimension.

Definition 3.11. Let V and W be vector spaces over $\mathbb{C}$. A *linear transformation* from V into W is a function $T : V \to W$ such that $T(cx + y) = cT(x) + T(y)\ \forall\ x, y \in V, c \in \mathbb{C}$.

Example 3.12. (1) Fix $\alpha \in \mathbb{C}$ and define $T : \mathbb{C} \to \mathbb{C}$ by $T(z) = \alpha z$. Then

$$T(cz_1 + z_2) = \alpha(cz_1 + z_2) = c\alpha z_1 + \alpha z_2 = cT(z_1) + T(z_2)$$

which demonstrates that T is a linear transformation.

(2) Define $T : A^{-1} \to A^{-1}$ by $T(f(z)) = zf(z)$. First, it is not immediately obvious that if $f \in A^{-1}$ then $T(f) \in A^{-1}$. So given $f \in A^{-1}$ then

$$\|T(f)\| = \sup_{z\in\mathbb{D}}(1-|z|)|zf(z)| = \sup_{z\in\mathbb{D}}(1-|z|)|z||f(z)| \leq \sup_{z\in\mathbb{D}}(1-|z|)|f(z)|$$

which proves that $zf \in A^{-1}$.

The following demonstrates that T is indeed a linear transformation.

$$T(cf+g) = z(cf+g) = zcf + zg = c(zf) + zg = cT(f) + T(g).$$

(3) We leave it to the reader to demonstrate similarly that $T(f) = zf$ is a linear transformation from $C^\infty(\mathbb{D}) \to C^\infty(\mathbb{D})$, $H(\mathbb{D}) \to H(\mathbb{D})$, and $H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$.

Definition 3.13. Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. Then the *kernel* of T consists of all vectors $v \in V$ such that $T(v) = 0$ [7] p. 309.

Definition 3.14. Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. Then the *range* of T consists of the $w \in W$ for which there exists a vector $v \in V$ such that $T(v) = w$ [7], p. 311.

The following two lemmas are elementary results of linear algebra. We state them here without proof.

Lemma 3.15. *The kernel K of a linear transformation $T : V \to W$ is a subspace of V.*

Lemma 3.16. *The range R of a linear transformation $T : V \to W$ is a subspace of W.*

Example 3.17. Let $T : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ be defined as $T(c_1, c_2) = (c_1, 0)$. This example will first demonstrate that T is a linear transformation and will then proceed to calculate its kernel and range.

Let $x, y \in \mathbb{C} \times \mathbb{C}$ and let $c \in \mathbb{C}$. Then

$$T(cx+y) = T(c(c_1,c_2) + (c_3,c_4)) = T(cc_1 + c_3, cc_2 + c_4)$$

$$= (cc_1 + c_3, 0) = (cc_1, 0) + (c_3, 0) = T(c(c_1,c_2)) + T(c_3,c_4) = cT(x) + T(y)$$

which demonstrates that T is a linear transformation.

Keeping the notation that $x = (c_1, c_2)$, since $T(x) = (c_1, 0)$ the kernel K of T consists of all points in $\mathbb{C} \times \mathbb{C}$ such that $T(x) = (0,0)$. It is easy to see that

$$K(T) = (0, c_2)\ \forall\, c_2 \in \mathbb{C}$$

since $T(0, c_2) = (0,0)$.

Since $T(c_1, c_2) = (c_1, 0)$, the range of T is simply the set of points $(c_1, 0)$ for all $c_1 \in \mathbb{C}$. To see this, note that the second element of the ordered pair of the range must be zero because there are no points in $\mathbb{C} \times \mathbb{C}$ such that T maps them to any ordered pair the second element of which does not equal zero.

Definition 3.18. Let V be a vector space with subsets $S_1, S_2, ..., S_k$. Then the set of all sums $s_1 + s_2 + ... + s_k$ of vectors $s_i \in S_i$ is the *sum* of the sets $S_1, S_2, ..., S_k$, and is denoted as $S_1 + S_2 + ... + S_k$ [6], p. 37.

Definition 3.19. Let $W_1, W_2, ..., W_k$ be subspaces of a vector space V. These subspaces are *independent* if for all $w_i \in W_i$ then

$$w_1 + w_2 + ... + w_k = 0$$

implies that each $w_i = 0$.

Definition 3.20. Let V be a vector space with subspaces $W_1, W_2, ..., W_k$. The sum of these subspaces is a *direct sum* if $W_1, W_2, ..., W_k$ are independent. This direct sum is denoted $W_1 \oplus W_2 \oplus ... \oplus W_k$ [6], p. 210.

Lemma 3.21. *Two subspaces W_1 and W_2 of a vector space V are independent if and only if $W_1 \cap W_2 = 0$.*

Proof. Suppose W_1 and W_2 are independent and let $w \in W_1 \cap W_2$. Then $w = w_2$ for some vector $w_2 \in W_2$. Thus, $w + (-w_2) = 0$, and since $w \in W_1$, this implies by the definition of independence that $w = w_2 = 0$.

Conversely, let $W_1 \cap W_2 = 0$ and suppose that W_1 and W_2 are not independent. Then there exists $w_1 \in W_1$ and $w_2 \in W_2$ such that if $w_1 + w_2 = 0$ then either w_1 or w_2 does not equal zero. Assuming without loss of generality that $w_1 \neq 0$, then $w_1 = -w_2 \neq 0$. But since $-w_2 \in W_2$, this implies that $W_1 \cap W_2 \neq 0$, which is a contradiction. $\square$

Corollary 3.22. *If W_1 and W_2 are subspaces of a vector space V, then the sum of W_1 and W_2 is a direct sum if and only if $W_1 \cap W_2 = 0$.*

In section six, we will have occasion to use this interpretation of the direct sum of two subspaces.

Definition 3.23. (1) Given vector spaces V and W, a one-to-one linear transformation T from V onto W is called an *isomorphism* of V onto W.

(2) A vector space V is *isomorphic* to a vector space W if there exists an isomorphism of V onto W.

We state the following elementary results from linear algebra without proof.

Lemma 3.24. (1) *If V is isomorphic to W then W is isomorphic to V.*

(2) *If V is isomorphic to W, then both V and W are vector spaces of the same dimension.*

We conclude this section with a discussion of quotient spaces.

Definition 3.25. Let W be a subspace of V. Then the *quotient* of V and W is

$$V/W = \bigcup_{v \in V} (v + W)$$

Lemma 3.26. *Let W be a subspace of V and let $v_1, v_2 \in V$. Then*

$$v_1 + W = v_2 + W \Leftrightarrow v_1 - v_2 \in W.$$

Proof. First assume that $v_1 + W = v_2 + W$. Then for all $w_1 \in W$ there exists a $w_2 \in W$ such that $v_1 + w_1 = v_2 + w_2$. Thus, $v_1 - v_2 = w_2 - w_1 \in W$ since W is a vector space.

Conversely, assume that $w = v_1 - v_2 \in W$. Then $v_1 = w + v_2$. So given $w_1 \in W$, $v_1 + w_1 = v_2 + (w + w_1)$. But $w + w_1 \in W$ since W is a vector space, which suffices to prove that $v_1 + W = v_2 + W$. $\square$

Lemma 3.27. *Let W be a subspace over $\mathbb{C}$ of V and let $v_\alpha, v_\beta \in V$. Also, let $c \in \mathbb{C}$. Then V/W is a vector space if addition and scalar multiplication are defined as follows:*

$$(v_\alpha + W) + (v_\beta + W) = (v_\alpha + v_\beta) + W$$

$$c(v_\alpha + W) = (cv_\alpha) + W$$

Proof. It is not transparent that these operations of addition and scalar multiplication are well defined. If $v_\alpha + W = v_a + W$ and $v_\beta + W = v_b + W$ then it must be shown both that $(v_\alpha + v_\beta) + W = (v_a + v_b) + W$ and $cv_\alpha + W = cv_a + W$.

First consider addition. Since by the previous lemma $v_\alpha - v_a \in W$ and $v_\beta - v_b \in W$ then clearly $((v_\alpha - v_a) + (v_\beta - v_b)) \in W$. Or equivalently, $((v_\alpha + v_\beta) - (v_a + v_b)) \in W$. But this implies by the previous lemma that $(v_\alpha + v_\beta) + W = (v_a + v_b) + W$, which shows closure under addition.

Now consider scalar multiplication. Again, $v_\alpha - v_a \in W$ so clearly $c(v_\alpha - v_a) \in W$. Or equivalently, $cv_\alpha - cv_a \in W$. So according to the previous lemma, $cv_\alpha + W = cv_a + W$, which shows closure under scalar multiplication. We leave it to the reader to test that V/W satisfies the ten properties of a vector space with respect to these well defined operations. $\square$

In section six we will make frequent use of the following famous result from basic algebra.

Theorem 3.28 (First Homomorphism Theorem). *Let V and W be vector spaces over $\mathbb{C}$. If there exists a linear transformation $\phi : V \to W$ then the quotient space $V/K(\phi)$ is isomorphic to $R(\phi)$, where $K(\phi)$ denotes the kernel of ϕ and $R(\phi)$ denotes the range of ϕ.*

Proof. Let ϕ be a linear transformation from V to W. Then by the definition of a quotient space,

$$V/K(\phi) = \bigcup_{v \in V} (v + K(\phi)).$$

For any $v \in V$, define a new function $\tilde{\phi} : V/K(\phi) \to R(\phi)$ by

$$\tilde{\phi}(v + K(\phi)) = \phi(v).$$

If we can show that $\tilde{\phi}$ is a well defined bijective linear transformation then this will prove that $V/K(\phi)$ is isomorphic to $R(\phi)$.

(1) First we show that $\tilde{\phi}$ is well defined. Let $v_1, v_2 \in V$ and suppose that $v_1 + K(\phi) = v_2 + K(\phi)$. Then, by Lemma 3.26 this implies that $v_1 - v_2 \in K(\phi)$. Thus, $\phi(v_1 - v_2) = 0$, and since ϕ is a linear transformation, $\phi(v_1) - \phi(v_2) = 0$, or equivalently,

$$\phi(v_1) = \tilde{\phi}(v_1 + K(\phi)) = \phi(v_2) = \tilde{\phi}(v_2 + K(\phi))$$

which demonstrates that $\tilde{\phi}$ is well defined.

(2) Next we show that $\tilde{\phi}$ is a linear transformation. Let $c \in \mathbb{C}$ and $v_1, v_2 \in V$. Then

$$\tilde{\phi}(c(v_1 + K(\phi)) + (v_2 + K(\phi))) = \tilde{\phi}((cv_1 + v_2) + K(\phi))$$
$$= \phi(cv_1 + v_2) = c\phi(v_1) + \phi(v_2) = c\tilde{\phi}(v_1 + K(\phi)) + \tilde{\phi}(v_2 + K(\phi))$$

which demonstrates that $\tilde{\phi}$ is a linear transformation.

(3) Clearly, $R(\tilde{\phi}) = R(\phi)$.

(4) All that remains is to show that $\tilde{\phi}$ is one-to-one. Let $v_1, v_2 \in V$ and suppose that $\tilde{\phi}(v_1 + K(\phi)) = \tilde{\phi}(v_2 + K(\phi))$. Then $\phi(v_1) = \phi(v_2)$, so $\phi(v_1) - \phi(v_2) = 0$. Since ϕ is a linear transformation, $\phi(v_1 - v_2) = 0$. Therefore, $v_1 - v_2 \in K(\phi)$. But according to Lemma 3.26 this implies that $v_1 + K(\phi) = v_2 + K(\phi)$, which demonstrates that $\tilde{\phi}$ is indeed one-to-one.

This completes the proof. $\square$

Example 3.29. Let $\phi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be defined by $\phi(c_1, c_2) = c_1$ for $c_1, c_2 \in \mathbb{C}$. Then, recalling the definition of the kernel K of a linear transformation, it is clear that $K(\phi) = 0 \times \mathbb{C}$. Thus, by the third homomorphism theorem, $\mathbb{C} \times \mathbb{C}/0 \times \mathbb{C}$ is isomorphic to $\mathbb{C}$. Since $\mathbb{C}$ has a dimension of one, then by Lemma 3.24, $\mathbb{C} \times \mathbb{C}/0 \times \mathbb{C}$ also has a dimension of one.

4. Banach Spaces

Definition 4.1. A *norm* of a vector space X is a function $\varrho : X \to \mathbb{R}^+$ satisfying the following for all $x \in X$

$$\varrho(x) \geq 0 \tag{4.1}$$

$$\varrho(x) = 0 \Leftrightarrow x = 0 \tag{4.2}$$

$$\varrho(ax) = |a|\varrho(x) \text{ where } a \in \mathbb{C} \tag{4.3}$$

$$\varrho(x_1) + \varrho(x_2) \geq \varrho(x_1 + x_2) \;\; \forall\, x_1, x_2 \in X \tag{4.4}$$

We remark that (given $x, y \in X$) if $\varrho(x)$ is a norm for X then clearly $\rho(x,y) = \varrho(y-x)$ serves as a metric for X.

X is called a *normed space* if $\rho(x,y) = \varrho(y-x)$ is a metric for X and $\varrho(x)$ is a norm for X.

Definition 4.2. A vector space X is a *Banach space* if it is both normed and complete, where completeness implies that all Cauchy sequences converge in X.

We indicated at the end of the section two that $H(\mathbb{D})$ is a complete metric space under the distance function

$$\rho(f,g) = \sum_{n=1}^{\infty}(\frac{1}{2})^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)}$$

where $\rho_n(f,g) = \sup_{z\in K_n} |g(z) - f(z)|$ and $K_n = \overline{B}(0; 1 - \frac{1}{n})$ as before.

Is this metric of the form $\rho(f,g) = \varrho(g-f)$ with $\varrho(f)$ being a norm? To see that it is not, let

$$h_n(f) = \sup_{z\in K_n} |f(z)|$$

and

$$h(f) = \sum_{n=1}^{\infty}(\frac{1}{2})^n \frac{h_n(f)}{1+h_n(f)}$$

Then

$$\rho(f,g) = h(g-f).$$

Thus, if $H(\mathbb{D})$ is a normed space under the metric $\rho(f,g)$ then $h(f)$ must satisfy all four of the properties for a norm. We shall demonstrate that the third property $h(af) = |a|h(f)$ is not necessarily satisfied.

Let $f = 1$ and $a = 2$. Note that $f \in H(\mathbb{D})$. Then

$$h_n(af) = h_n(2) = \sup_{z\in K_n} |2| = 2$$

and similarly

$$h_n(f) = h_n(1) = 1$$

Thus

$$h(af) = h(2) = \sum_{n=1}^{\infty}(\frac{1}{2})^n \frac{2}{2+1} = \frac{2}{3}\sum_{n=1}^{\infty}(\frac{1}{2})^n = \frac{2}{3}$$

But

$$|a|h(f) = 2h(1) = 2\sum_{n=1}^{\infty}(\frac{1}{2})^n \frac{1}{1+1} = \sum_{n=1}^{\infty}(\frac{1}{2})^n = 1$$

Since $1 \neq \frac{2}{3}$ this implies

$$h(af) \not\equiv |a|h(f) \; \forall \, f \in H(\mathbb{D})$$

Therefore, since $H(\mathbb{D})$ is not normed, it is not a Banach space.

This next result can be found in [4], p. 145.

Proposition 4.3. *A sequence converges in the metric of $H(\mathbb{D}) \Leftrightarrow$ the sequence converges uniformly on all compact subsets of $\mathbb{D}$.*

We use this well-known result to prove the following.

Theorem 4.4. *$H^\infty(\mathbb{D})$ is a Banach space.*

Proof. Let $\varrho(f) = \|f\| = \sup_{z\in\mathbb{D}} |f(z)|$ and note that by Lemma 2.6 this satisfies the properties of a norm for $H^\infty(\mathbb{D})$. It remains to be shown that $H^\infty(\mathbb{D})$ is complete.

Recall that

$$\rho(f,g) = \|g - f\| = \sup_{z\in\mathbb{D}} |g(z) - f(z)|$$

is a metric for $H^\infty(\mathbb{D})$. Let $\{f_n\}$ be a Cauchy sequence in $H^\infty(\mathbb{D})$. We proceed to first prove the following useful results:

$$\sup_n \|f_n\| < +\infty \tag{4.5}$$

$$\text{Given fixed } z_0 \in \mathbb{D},\ \exists f(z_0) : f_n(z_0) \to f(z_0) \text{ as } n \to +\infty. \tag{4.6}$$

$$\|f\| < +\infty \tag{4.7}$$

$$f_n(z) \to f(z)\ \forall\, z \in \mathbb{D} \tag{4.8}$$

$$f \in H(\mathbb{D}) \tag{4.9}$$

Since $\{f_n\}$ is Cauchy then $\exists\ N \in \mathbb{N}$ such that $\forall\, m,n \geq N$, $\|f_n - f_m\| \leq 1$. So $\forall\, n \geq N$

$$\|f_n\| = \|f_n - f_N + f_N\| \leq \|f_n - f_N\| + \|f_N\| \leq 1 + \|f_N\|$$

Let

$$C = \max_{1\leq n\leq N} \{\|f_n\|\}$$

Then

$$\|f_n\| \leq M = \max\{C, 1 + \|f_N\|\}\ \forall\, n$$

Therefore

$$\sup_n \|f_n\| \leq M < +\infty,$$

which proves (4.5).

Fix $z_0 \in \mathbb{D}$. Given $\epsilon > 0, \exists\ N \in \mathbb{N}$ such that $\forall\, m,n \geq N$, $\|f_n - f_m\| \leq \epsilon$.

Now $\forall\, m,n \geq N$

$$|f_n(z_0) - f_m(z_0)| \leq \sup_{z\in\mathbb{D}} |f_n(z) - f_m(z)| = \|f_n - f_m\| \leq \epsilon$$

so $\{f_n(z_0)\}$ is a Cauchy sequence in $\mathbb{C}$. Thus, by the completeness of $\mathbb{C}$

$$\exists\ f(z_0) : f_n(z_0) \to f(z_0),$$

which proves (4.6)

Note that for fixed $z_0 \in \mathbb{D}$

$$|f(z_0)| = \lim_{n\to+\infty} |f_n(z_0)| \le \lim_{n\to+\infty} \sup_{z\in\mathbb{D}} |f_n(z)| \le \sup_n \|f_n\|$$

But we have already shown that

$$\sup_n \|f_n\| \le M < \infty$$

But since z_0 was chosen arbitrarily, this implies

$$|f(z)| < M \; \forall z \in \mathbb{D}$$

and so $\|f\| < +\infty$, which proves (4.7).

Let $\epsilon > 0$ be given. Since $\{f_n\}$ is a Cauchy sequence then $\exists\, N \in \mathbb{N}$ such that $\forall\, m,n \ge N$ $\|f_n - f_m\| < \epsilon$. Thus for fixed $z_0 \in \mathbb{D}$,

$$|f_n(z_0) - f(z_0)| = \lim_{m\to+\infty} |f_m(z_0) - f_n(z_0)| \le \sup_{m,n\ge N} |f_m(z_0) - f_n(z_0)|$$

$$\le \sup_{m,n\ge N} \sup_{z\in\mathbb{D}} |f_m(z) - f_n(z)| = \sup_{m,n\ge N} \|f_m - f_n\| < \epsilon.$$

But since z_0 was chosen arbitrarily we have

$$|f_n(z) - f(z)| < \epsilon \,\forall\, z \in \mathbb{D}.$$

Thus

$$\sup_{z\in\mathbb{D}} |f_n(z) - f(z)| < \epsilon$$

that is,

$$\|f_n - f\| < \epsilon \,\forall\, n \ge N$$

or equivalently

$$f_n \to f$$

in the metric of $H^\infty(\mathbb{D})$, which proves (4.8).

Let $K \subset \mathbb{D}$ be compact. Then

$$\sup_{z\in K} |f_n(z) - f_m(z)| \le \sup_{z\in\mathbb{D}} |f_n(z) - f_m(z)| = \|f_n - f_m\|$$

Since $\|f_n - f_m\| \to 0$ as $m,n \to +\infty$ then $\{f_n\}$ is Cauchy with respect to $H(\mathbb{D})$. But since $H(\mathbb{D})$ is a complete metric space, there exists $g \in H(\mathbb{D})$ such that $f_n \to g$ uniformly on compact subsets of $\mathbb{D}$ in the metric of $H(\mathbb{D})$ by Proposition 4.3. But $f_n(z_0) \to f(z_0)\; \forall\, z_0 \in \mathbb{D}$. Since z_0, as a single point, is a compact subset of $\mathbb{D}$, then $f_n(z_0) \to g(z_0)$. Thus $f(z_0) = g(z_0)$. Since z_0 is arbitrary we have $f(z) = g(z)\; \forall\, z \in \mathbb{D}$. Therefore,

$$f \in H(\mathbb{D})$$

which proves (4.9).

This demonstrates that $H^\infty(\mathbb{D})$ is complete and thus a Banach space. $\square$

Lemma 4.5. *A^{-1} is a Banach space.*

Proof. The proof that A^{-1} is complete is essentially the same as the completeness proof for $H^\infty(\mathbb{D})$. We will prove that

$$\varrho(f) = \sup_{z\in\mathbb{D}}(1-|z|)|f(z)|$$

is a norm for A^{-1}. Recalling the four properties of a norm from the beginning of this section:

(1) Condition (4.1) is obvious

(2) For condition (4.2) note that if $\varrho(f) = \sup_{z\in\mathbb{D}}(1-|z|)|f(z)| = 0 \Rightarrow f \equiv 0$ since $|z|$ is strictly less than 1. Thus, $f \equiv 0$.
Conversely, if $f \equiv 0$ then $|f(z)| = 0\ \forall\ z \in \mathbb{D}$ and so $\sup_{z\in\mathbb{D}}(1-|z|)|f(z)| = \varrho(f) = 0$.

(3) For condition (4.3) notice that

$$\varrho(af) = \sup_{z\in\mathbb{D}}(1-|z|)|af(z)| = \sup_{z\in\mathbb{D}}(1-|z|)|a||f(z)| = |a|\sup_{z\in\mathbb{D}}(1-|z|)|f(z)| = |a|\varrho(f).$$

(4) To prove condition (4.4) notice that

$$\begin{aligned}\varrho(f_1+f_2) &= \sup_{z\in\mathbb{D}}(1-|z|)|f_1(z)+f_2(z)| \le \sup_{z\in\mathbb{D}}(1-|z|)(|f_1(z)|+|f_2(z)|)\\ &\le \sup_{z\in\mathbb{D}}(1-|z|)|f_1(z)| + \sup_{z\in\mathbb{D}}|f_2(z)| = \varrho(f_1)+\varrho(f_2).\end{aligned}$$

Thus, A^{-1} is normed. Since it is also complete, it is a Banach space. $\square$

Definition 4.6. A set Y is *dense* in a complete metric space X if the closure of Y equals X.

Definition 4.7. A complete metric space is *separable* if it contains a countable dense set.

Lemma 4.8. *Let X be complete with metric d. Let $\{x_t\} \subset X$ with $t \in [0,1]$ and*

$$d(x_t, x_s) \ge 1\ \forall\ s \ne t.$$

Then X is not separable.

Proof. Let Y be dense in X. For each $t \in [0,1]$ form an open ball around x_t of radius $\frac{1}{2}$, denoted $B(x_t; \frac{1}{2})$. Since $d(x_t, x_s) \ge 1\ \forall\ s \ne t$ we have

$$B(x_t; \frac{1}{2}) \bigcap B(x_s; \frac{1}{2}) = \emptyset\ \forall\ s \ne t.$$

Since the closure of Y equals X (as the result of Y being dense in X) then given x_t, there exists $\{y_s\} \subset Y$ with $s \in \mathbb{N}$ such that $y_s \to x_t$ as $s \to +\infty$. Thus, there exists $y_t \in Y$ with

$$y_t \in B(x_t; \frac{1}{2}).$$

Thus, Y must contain an uncountable number of elements and therefore X is not separable. $\square$

Using an idea from [2] which shows the non-separability of a different space, we proceed to demonstrate the non-separability of A^{-1}.

Theorem 4.9. *A^{-1} is not separable.*

Proof. Let a be a point on the unit circle, so $|a| = 1$. Define

$$g_a(z) = \frac{a^2}{(1+az)(1-az)} \tag{4.10}$$

By partial fractions we obtain

$$g_a(z) = \frac{a^2}{2(1+az)} + \frac{a^2}{2(1-az)}.$$

We proceed to demonstrate first that $g_a(z) \in A^{-1}$. By the triangle inequality,

$$|1-az| \geq 1 - |az| = 1 - |z|$$

and similarly

$$|1+az| \geq 1 - |z|.$$

Thus

$$\frac{1}{|1-az|} \leq \frac{1}{1-|z|} \quad \text{and} \quad \frac{1}{|1+az|} \leq \frac{1}{1-|z|}$$

This implies that

$$\sup_{z\in\mathbb{D}}(1-|z|)|g_a(z)| = \sup_{z\in\mathbb{D}}(1-|z|)(\frac{1}{2})(\frac{1}{|1+az|} + \frac{1}{|1-az|})$$

$$\leq \sup_{z\in\mathbb{D}}(1-|z|)(\frac{1}{2})(\frac{2}{1-|z|}) = 1 < +\infty$$

which proves that $g_a(z) \in A^{-1}$.

This result is now used to prove that A^{-1} is not separable. Let b be another point on the unit circle distinct from a.

$$g_a(z) - g_b(z) = \frac{a^2}{(1+az)(1-az)} - \frac{b^2}{(1+bz)(1-bz)} = \frac{a^2-b^2}{(1-a^2z^2)(1-b^2z^2)}$$

Let $0 \leq r < 1$ so that $r\overline{a}$ is a line segment in $\mathbb{D}$ from the origin in the direction of $\overline{a}$ (where $\overline{a}$ denotes the complex conjugate of a). Then

$$\|g_a - g_b\| = \sup_{z\in\mathbb{D}}(1-|z|)|\frac{a^2-b^2}{(1-a^2z^2)(1-b^2z^2)}| \geq \sup_{\{z=r\overline{a}:0\leq r<1\}}(1-|z|)|\frac{a^2-b^2}{(1-a^2z^2)(1-b^2z^2)}|$$

$$= \sup_{0\leq r<1}(1-r)|\frac{a^2-b^2}{(1-a^2r^2\overline{a^2})(1-b^2r^2\overline{a^2})}| = \sup_{0\leq r<1}(1-r)|\frac{a^2-b^2}{(1-r^2)(1-r^2b^2\overline{a^2})}|$$

since $a^2\overline{a^2} = |a|^4 = 1$. But this last result is greater than

$$\sup_{0\leq r<1}|\frac{a^2-b^2}{1-r^2b^2\overline{a^2}}| \tag{4.11}$$

because $0 \le r < 1$ implies that $1 - r^2 > 1 - r$. Now since $|\overline{a}| = |a| = 1$, (4.11) equals

$$\sup_{0\le r<1} \left|\frac{|\overline{a}|^2 a^2 - |\overline{a}|^2 b^2}{1 - r^2 b^2 \overline{a}^2}\right| = \sup_{0\le r<1} \left|\frac{1 - \overline{a}^2 b^2}{1 - r^2 \overline{a}^2 b^2}\right| \ge \lim_{r\to 1} \left|\frac{1 - \overline{a}^2 b^2}{1 - r^2 \overline{a}^2 b^2}\right| = 1.$$

We have now demonstrated that

$$\|g_a - g_b\| \ge 1 \;\forall\, a \ne b.$$

Since the unit circle contains an uncountable number of points that may be indexed according to $[0, 1]$, this implies by Lemma 4.8 that A^{-1} is not separable. □

Having discovered various properties about A^{-1}, we will now investigate an important subspace of A^{-1}. The following subspace is endowed with the same norm as A^{-1}.

Definition 4.10. $A_0^{-1} = \{f \in A^{-1} : \lim_{|z|\to 1}(1 - |z|)|f(z)| = 0\}$

Lemma 4.11. *A_0^{-1} is closed.*

Proof. Let $\{f_n\}$ be Cauchy in A_0^{-1} and note that $A_0^{-1} \subset A^{-1}$. Thus, $f_n \to f \in A^{-1}$. We must show additionally that $f \in A_0^{-1}$.

$$\lim_{|z|\to 1}(1 - |z|)|f(z)| = \lim_{|z|\to 1}(1 - |z|)|f(z) - f_n(z) + f_n(z)|$$

$$\le \lim_{|z|\to 1}(1 - |z|)|f(z) - f_n(z)| + \lim_{|z|\to 1}(1 - |z|)|f_n(z)|.$$

Let $\epsilon > 0$ be given. Since $f_n \to f$ in A^{-1} there exists $N \in \mathbb{N}$ such that for all $n \ge N$ we have

$$\|f(z) - f_n(z)\| = \sup_{z\in\mathbb{D}}(1 - |z|)|f(z) - f_n(z)| < \epsilon.$$

Since

$$\lim_{|z|\to 1}(1 - |z|)|f(z) - f_n(z)| \le \sup_{z\in\mathbb{D}}(1 - |z|)|f(z) - f_n(z)|$$

this implies

$$\lim_{|z|\to 1}(1 - |z|)|f(z)| \le \epsilon + \lim_{|z|\to 1}(1 - |z|)|f_n(z)| = \epsilon + 0 = \epsilon$$

recalling that $f_n \in A_0^{-1}$. Thus, $f \in A_0^{-1}$ which proves closure. □

Lemma 4.12. $A_0^{-1} \supset H^\infty(\mathbb{D})$

Proof. Let $f \in H^\infty(\mathbb{D})$. Then $\sup_{z\in\mathbb{D}} |f(z)| = C < \infty. \Rightarrow |f(z)| \le C \;\forall z \in \mathbb{D}$. Thus,

$$\lim_{|z|\to 1}(1 - |z|)|f(z)| \le C \lim_{|z|\to 1}(1 - |z|) = 0.$$

This implies that $f \in A_0^{-1}$ which completes the proof. □

Example 4.13. This example will demonstrate that A_0^{-1} is not equal to A^{-1} but is in fact a strict subset.

By the technique used at the beginning of Theorem 4.9, it is easily verified that $f(z) = \frac{1}{1-z} \in A^{-1}$. We will show that f does not belong to A_0^{-1}. Let $\{z_n\} = 1 - \frac{1}{n}$. Then $\{z_n\} \to 1$ as $n \to +\infty$ but

$$\lim_{n\to+\infty}(1-|z_n|)|\frac{1}{1-z_n}| = \lim_{n\to+\infty}(\frac{1}{n})(\frac{1}{\frac{1}{n}}) = \lim_{n\to+\infty}\frac{n}{n} = 1 \neq 0.$$

Thus, $f(z) = \frac{1}{1-z}$ is not in A_0^{-1}.

This example, together with the lemma preceding it, allow us to identify the relationships between all of the classes of analytic functions that have been discussed.

$$H(\mathbb{D}) \supset A^{-1} \supset A_0^{-1} \supset H^\infty(\mathbb{D}) \tag{4.12}$$

The following discussion may appear at first to be unrelated to what has been discussed thus far, but will ultimately be utilized to determine whether or not A_0^{-1} is separable. (Of course, we have already demonstrated that A^{-1} is *not* separable).

Lemma 4.14. *Let $0 < r < 1$ and let $\mathbb{D}_r$ denote the open disk of radius $\frac{1}{r}$ about the origin. Also, let $f \in H(\mathbb{D}_r)$ and let*

$$p_n(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k$$

Then $p_n \to f$ in A^{-1}.

Proof. The notation $\overline{\mathbb{D}}$ will be used to denote the closure of $\mathbb{D}$. It is obvious that $\overline{\mathbb{D}}$ is a compact subset of $\mathbb{D}_r$ since $0 < r < 1$. Also, note that p_n is the first n terms of the familiar Taylor series expansion of f, which converges uniformly on compact subsets of $\mathbb{D}_r$ to f, [4] p. 72. Thus, given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|f(z) - p_n(z)| < \epsilon \; \forall \; z \in \overline{\mathbb{D}}.$$

Multiplying by $1 - |z|$, and noting that $1 - |z| \leq 1 \; \forall \; z \in \mathbb{D}$, yields

$$(1-|z|)|f(z) - p_n(z)| < (1-|z|)\epsilon \leq \epsilon \; \forall \; z \in \mathbb{D} \text{ and } n \geq N.$$

Thus, $p_n \to f$ as $n \to +\infty$ in the norm of A^{-1}. □

Lemma 4.15. *Let $0 < r < 1$ with $f \in A^{-1}$ and $z \in \mathbb{D}$. Also, let f_r denote $f(rz)$. Then $f_r \in A_0^{-1}$.*

Proof. The proof is almost trivial. Since $0 < r < 1$, it is clear that $f(rz)$ must be bounded, that is, $f_r \in H^\infty(\mathbb{D})$. But by Lemma 4.12, this implies $f_r \in A_0^{-1}$. □

Theorem 4.16. *Let $0 < r < 1$ and $f \in A_0^{-1}$ with $f_r = f(rz)$ as before. Then $f_r \to f$ in the norm of A^{-1}.*

Proof. Let $\epsilon > 0$ be given. $f \in A_0^{-1} \Rightarrow \lim_{|z|\to 1}(1-|z|)|f(z)| = 0$. Thus, there exists $\delta > 0$ such that for all $1-\delta < |z| < 1$ we have

$$(1-|z|)|f(z)| < \frac{\epsilon}{4} \tag{4.13}$$

Note that

$$\mathbb{D} = \{|z| \le 1 - \frac{\delta}{2}\} \bigcup \{1 - \frac{\delta}{2} < |z| < 1\}$$

which implies the following:

$$\sup_{z\in\mathbb{D}}(1-|z|)|f(rz)-f(z)| \le \sup_{|z|\le 1-\frac{\delta}{2}}(1-|z|)|f(rz)-f(z)| + \sup_{1-\frac{\delta}{2}<|z|<1}(1-|z|)|f(rz)-f(z)| \tag{4.14}$$

The theorem is proved if we can show convergence to zero as $r \to 1$ of the left hand side of this equation. To accomplish this, we will prove that both terms on the right hand side converge to zero as $r \to 1$.

We will now prove convergence for the first term on the right hand side of (4.14). Clearly, $f(z) \in H(\mathbb{D})$ since $A^{-1} \subset H(\mathbb{D})$. Let $K = \{|z| \le 1 - \frac{\delta}{2}\}$ and note that K is a compact subset of $\mathbb{D}$. Thus, f is uniformly continuous on K. So given $\epsilon > 0$ there exists a $\delta_K > 0$ such that, for $z, w \in K$ we have

$$|f(z) - f(w)| < \frac{\epsilon}{2} \ \forall \ |z-w| < \delta_K \tag{4.15}$$

Now consider that

$$|rz - z| = |z||r-1| = (1-r)|z| \tag{4.16}$$

since $0 < r < 1$. Fix r_0 near unity such that $1 - r_0 \le \delta_K$. So for all $z \in K$ (noting that this implies $|z| < 1$) we have, by (4.16), $|r_0 z - z| < 1 - r_0 \le \delta_K$. Therefore, for all $z \in K$ and for all $r > r_0$, by (4.15),

$$|f(z) - f(rz)| < \frac{\epsilon}{2}$$

that is, $f(rz) \to f(z)$ uniformly on K. This implies

$$\sup_{|z|<1-\frac{\delta}{2}}(1-|z|)|f(rz)-f(z)| < \frac{\epsilon}{2} \ \forall \ r > r_0 \tag{4.17}$$

which demonstrates that the first term of the right hand side of (4.14) is bounded above by $\frac{\epsilon}{2}$.

It remains to be shown that the second term is similarly bounded. Let $1 - \frac{\delta}{2} < |z| < 1$ and let

$$r_1 = \frac{1-\delta}{1-\frac{\delta}{2}}.$$

Then for all $1 > r > r_1$

$$|rz| > |r_1||z| > |\frac{1-\delta}{1-\frac{\delta}{2}}||1-\frac{\delta}{2}| = |1-\delta| = 1-\delta \tag{4.18}$$

(since obviously $0 < \delta < 1$). Since $r < 1$, this implies $|rz| < |z|$ which implies $-|rz| > -|z|$. Using this, together with (4.18) and (4.13), we find that

$$(1-|z|)|f(rz)| < (1-|rz|)|f(rz)| \le \frac{\epsilon}{4}.$$

In conjunction with (4.13), this demonstrates that

$$(1-|z|)||f(z)| - |f(rz)|| < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} \ \forall \ 1-\frac{\delta}{2} < |z| < 1.$$

And therefore, utilizing the triangle inequality,

$$\sup_{1-\frac{\delta}{2}<|z|<1} (1-|z|)|f(z) - f(rz)| < \frac{\epsilon}{2} \tag{4.19}$$

which demonstrates convergence for the second term on the right hand side of (4.14). Finally, by (4.17) and (4.19) it is clear that $\forall\, r > \max\{r_0, r_1\}$

$$\sup_{z\in\mathbb{D}}(1-|z|)|f(z) - f(rz)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which proves that $f_r \to f$ in the norm of A^{-1}. □

The following discussion of polynomials is motivated in the hope that it will shed more insight into the space A_0^{-1}. Specifically, we will eventually relate A_0^{-1} to polynomials.

Definition 4.17. (1) A *polynomial* is a function of the form

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

where $a_j \in \mathbb{C}\ \forall\ 0 \le j \le n$. We denote the set of polynomials by $\mathcal{P}$.

(2) Let $\mathbb{Q}$ denote the rational numbers with respect to the complex plane, that is,

$$\mathbb{Q} = \{z \in \mathbb{C} \text{ such that both } Re(z) \text{ and } Im(z) \text{ are rational}\}.$$

Lemma 4.18. *Given a polynomial $p(z) = a_0 + a_1 z + \dots + a_n z^n$, let $\{r_{ij}\}$ be $n+1$ sequences such that $r_{ij} \in \mathbb{Q}$ and $r_{ij} \to a_j$ for each j as $i \to \infty$. Also let $p_i(z) = r_{i0} + r_{i1}z + \dots + r_{in}z^n$. Then $p_i(z) \to p(z)$ in A^{-1}.*

Proof.

$$\begin{aligned}
\lim_{i\to\infty} \|p_i(z) - p(z)\| &= \lim_{i\to\infty} \sup_{z\in\mathbb{D}}(1-|z|)|(r_{i0} + r_{i1}z + \dots + r_{in}z^n) - (a_0 + a_1 z + \dots + a_n z^n)| \\
&= \lim_{i\to\infty} \sup_{z\in\mathbb{D}}(1-|z|)|(r_{i0} - a_0) + (r_{i1} - a_1)z + \dots + (r_{in} - a_n)z^n| \\
&\le \lim_{i\to\infty} \sup_{z\in\mathbb{D}}(1-|z|)(|r_{i0} - a_0| + |r_{i1} - a_1||z| + \dots + |r_{in} - a_n||z^n|) \\
&\le \lim_{i\to\infty}(|r_{i0} - a_0| + |r_{i1} - a_1| + \dots + |r_{in} - a_n|) = 0.
\end{aligned}$$

$$\Rightarrow p_i(z) \;\to\; p(z) \text{ in } A^{-1},$$

which completes the proof. □

Lemma 4.19. *Let $\overline{\mathcal{P}}$ denote the closure of $\mathcal{P}$ in the norm of A^{-1}. Then $\overline{\mathcal{P}} \neq A^{-1}$.*

Proof. Let $\mathcal{P}_{\mathbb{Q}}$ denote the set of polynomials with rational coefficients (in the sense defined by Definition 4.17) and let $\overline{\mathcal{P}_{\mathbb{Q}}}$ denote the closure of these rational polynomials. Lemma 4.18 implies that

$$\overline{\mathcal{P}_{\mathbb{Q}}} = \overline{\mathcal{P}} \tag{4.20}$$

that is, the rational polynomials can approximate any polynomial. Note that $\mathcal{P}_{\mathbb{Q}}$ is a countable set (because is only contains polynomials with rational coefficients). Thus, if $\overline{\mathcal{P}_{\mathbb{Q}}} = A^{-1}$ then $\mathcal{P}_{\mathbb{Q}}$ is dense in A^{-1} which would imply that A^{-1} is separable, which contradicts Theorem 4.9. Thus, $\overline{\mathcal{P}_{\mathbb{Q}}}$ is a strict subset of A^{-1} which by (4.20) implies that $\overline{\mathcal{P}}$ is also a strict subset of A^{-1} and thus does not equal A^{-1}. □

The following theorem relates the polynomials to A_0^{-1}.

Theorem 4.20. $\overline{\mathcal{P}} = A_0^{-1}$

Proof. Let $f \in A_0^{-1}$. Then given $0 < r < 1$, we know that $f_r \in A_0^{-1}$ and $f_r \to f$ as $r \to 1$ by Lemma 4.15 and Theorem 4.16 respectively. Also, $p_n \to f_r$ where $p_n(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k$ according to Lemma 4.14. Consider,

$$\|p_n - f\| = \|p_n - f_r + f_r - f\| \leq \|p_n - f_r\| + \|f_r - f\| \tag{4.21}$$

Since $f_r \to f$ by Theorem 4.16 then given $\epsilon > 0$ there exists $\delta > 0$ such that for all $r > 1 - \delta$ this implies $\|f_r - f\| < \frac{\epsilon}{2}$. Also, since $p_n \to f_r$ then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ this implies $\|p_n - f_r\| < \frac{\epsilon}{2}$. Thus, by (4.21), $\|p_n - f\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ and therefore $p_n \to f$. Since $p_n \in \mathcal{P}$ this demonstrates that

$$A_0^{-1} \subset \overline{\mathcal{P}}. \tag{4.22}$$

Conversely, since $\mathcal{P} \in H^\infty(\mathbb{D})$ and $H^\infty(\mathbb{D}) \subset A_0^{-1}$, this implies that $\mathcal{P} \in A_0^{-1}$. And since, according to Lemma 4.11, A_0^{-1} is closed, this implies that $\overline{\mathcal{P}} \subset \overline{A_0^{-1}} = A_0^{-1}$.

Together with (4.22), this demonstrates that $\overline{\mathcal{P}} = A_0^{-1}$. □

We conclude this section with the result to which we have been building which demonstrates the separability of the closure of the polynomials.

Theorem 4.21. *A_0^{-1} is separable.*

Proof. The proof is trivial. Since $\overline{\mathcal{P}_{\mathbb{Q}}} = \overline{\mathcal{P}}$ as demonstrated by (4.20), and since $\overline{\mathcal{P}} = A_0^{-1}$, this implies that $\overline{\mathcal{P}_{\mathbb{Q}}} = A_0^{-1}$. Thus, $\mathcal{P}_{\mathbb{Q}}$, the set of rational polynomials, is dense in A_0^{-1}. Since it is also countable, this demonstrates that A_0^{-1} is separable. □

5. Zero Sets

In this section we will explore the zero sets of spaces of analytic functions. But in order to give the reader a better intuitive understanding of the problem, we begin with an example utilizing a space of functions that is not analytic.

Example 5.1. Recall the space $C^\infty(\mathbb{D})$ from Example 5 of Section 2. This is the space of complex valued functions on the disk whose partial derivatives (of all orders) exist and are continuous on $\mathbb{D}$. It is obvious that $C^\infty(\mathbb{D}) \supset H(\mathbb{D})$. The following is an example of a function that is $C^\infty(\mathbb{D})$ but not analytic:

$$f(z) = Im(z). \tag{5.1}$$

Let $Re(z) = x$ and $Im(z) = y$. Then (5.1) implies that $f(z) = 0$ whenever $y = 0$, regardless of the value of x. We can visualize this geometrically by stating that the function f evaluated at any point on the real line within the unit disk is equal to zero. Yet another way of stating this is to say that the open set $(-1, 1)$ of real numbers is the "zero set" of the function f.

Definition 5.2. A set S is *relatively closed* in $\mathbb{D}$ if there exists a set K closed in $\mathbb{C}$ such that $K \cap \mathbb{D} = S$.

Note that if S is relatively closed then $\mathbb{D} \backslash S$ is open.

Theorem 5.3. *Let S be both a strict subset of $\mathbb{D}$ and be relatively closed in $\mathbb{D}$. Then there exists a $f \in C^\infty(\mathbb{D})$ ($f \not\equiv 0$) such that f evaluated at any point of S equals zero.*

Proof. Fix $z_0 \in \mathbb{D}\backslash S$. Since $\mathbb{D}\backslash S$ is open, there exists $\epsilon > 0$ such that $B(z_0; \epsilon) \subset \mathbb{D}\backslash S$. Then

$$f_\epsilon(z) = \begin{cases} \exp\left(\left|\frac{z-z_0}{\epsilon}\right|^2 - 1\right)^{-1} & \text{if } z \in B(z_0;\epsilon) \\ 0 & \text{if } z \in \mathbb{D}\backslash B(z_0;\epsilon) \end{cases}$$

is such that $f \in C^\infty(\mathbb{D})$, $f_\epsilon(z) = 0 \ \forall \ z \in S$, and $f \not\equiv 0$. In particular, note that there is no discontinuity in any of the partial derivatives of $f_\epsilon(z)$ at any of the points z such that $|z - z_0| = \epsilon$. □

This result motivates us to ask the following: Given a set $S \subset \mathbb{D}$ and a space $\mathcal{C}$ of analytic functions, can we find a function $f \in \mathcal{C}$ (with $f \not\equiv 0$) such that S is the zero set of f?

Definition 5.4. Given f analytic in a neighborhood of a point z_0, then z_0 is a *zero of order* m for f if $f(z_0) = f'(z_0) = ... = f^{(m-1)}(z_0) = 0$ but $f^{(m)}(z_0) \neq 0$.

This definition is motivated by the fact that if f is analytic in a neighborhood of z_0 then we know from elementary complex analysis that f has a power series expansion about z_0. (See, for example, [9] p. 200.)

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + ...$$

where

$$a_j = \frac{f^{(j)}(z_0)}{j!} \quad (j = 0, 1, 2, 3, ...)$$

Thus, if f has a zero of order m then $a_0 = a_1 = ... = a_{m-1} = 0$ but $a_m \neq 0$. Thus, we can write f as

$$f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + a_{m+2}(z - z_0)^{m+2} + ... \tag{5.2}$$

From this we observe that so long as $f \not\equiv 0$ then any zero of f must be finite. For if it were infinite, then all of the coefficients in the power series of f would be zero, which would make f identically equal to zero.

Please also observe the following fact which will be used in the next lemma. By simply factoring (5.2) we obtain

$$f(z) = (z - z_0)^m(a_m + a_{m+1}(z - z_0) + a_{m+2}(z - z_0)^2 + ...) = (z - z_0)^m g(z) \tag{5.3}$$

where g is also analytic in a neighborhood of z_0 but is such that $g(z_0) = a_m \neq 0$.

Lemma 5.5. *Let $f \in H(\mathbb{D})$ with $f \not\equiv 0$. Then the zeros of f are isolated.*

Proof. Let $z_0 \in \mathbb{D}$ be a zero of f of order m. Then by (5.3) we can rewrite f as

$$f(z) = (z - z_0)^m g(z)$$

where $g \in H(\mathbb{D})$ and $g(z_0) \neq 0$. Since g is obviously continuous at z_0, there exists a neighborhood about z_0 throughout which g is non-zero. But this implies that f is non-zero in a punctured neighborhood about z_0. (The neighborhood is punctured of course because $f(z_0) = 0$ by hypothesis.) Because f is non-zero in a punctured neighborhood of z_0, this implies that z_0 is isolated from any other zero. And since z_0 is an arbitrary zero of f, this implies that all the zeros of f are isolated. □

The well known Bolzano-Weierstrass theorem is needed to prove our next lemma. We state it here without proof.

Theorem 5.6 (Bolzano-Weierstrass). *Every bounded sequence of complex numbers has a convergent subsequence.*

Lemma 5.7. *If $A \subset \mathbb{D}$ has no accumulation points in $\mathbb{D}$ then A must be countable.*

Proof. Suppose that A has no accumulation points and yet is uncountable. Define sets A_n to be

$$A_n = B(0; 1 - \frac{1}{n}) \bigcap A \quad \forall\, n \geq 2$$

and note that clearly

$$\bigcup_{n=2}^{\infty} A_n = A.$$

We proceed to show that each A_n must be a finite set. We know that $A_n \subset B(0;1-\frac{1}{n})$, which is bounded. So if A_n were an infinite set then by Theorem 5.6, the Bolzano-Weierstrass Theorem, this would imply that there exists a subsequence of A_n which converges to a point in $\overline{B}(0;1-\frac{1}{n}) \subset \mathbb{D}$. But clearly A_n cannot have any accumulation points inside $\mathbb{D}$ because $A_n \subset A$ and A does not accumulate in $\mathbb{D}$ by hypothesis. Thus we reach a contradiction, which demonstrates that A_n must be finite. Since A is the infinite union of all these finite sets, it clearly must be countable. This contradicts our assumption that it was uncountable, and thus completes the proof. □

Corollary 5.8. *Given $f \in H(\mathbb{D})$ with $f \not\equiv 0$ then the zeros of f are countable.*

Proof. Since the zeros of f are isolated by Lemma 5.5 they cannot accumulate in $\mathbb{D}$, which means by the previous lemma that they must be countable. □

The following proposition simply restates these results in a convenient "geometric" form that the reader can easily conceptualize.

Proposition 5.9. *If A is a zero set for $f \in H(\mathbb{D})$ then A must both be countable and may accumulate only on the boundary of $\mathbb{D}$.*

The obvious question is, if we're simply given a countable set A (i.e. - a sequence $\{a_n\} = A$) that accumulates only on the boundary of $\mathbb{D}$, can we find a function $f \in H(\mathbb{D})$ such that A is the zero set of f? In other words, can we make the previous proposition both necessary and sufficient? The answer is "Yes", but it turns out to be a much more difficult task to prove the "sufficient" direction. The result is the famous Weierstrass Factorization Theorem, which we state here without proof, [4] p. 170.

Theorem 5.10. *Given a sequence $\{a_n\} \subset \mathbb{D}$ which accumulates only on the boundary of $\mathbb{D}$, the following non-zero function f is analytic in the unit disk and has zeros only at the points a_n.*

$$f(z) = \prod_{n=0}^{\infty} E_n\left(\frac{a_n(|a_n|-1)}{z|a_n|-a_n}\right) \tag{5.4}$$

where

$$E_0(z) = 1 - z \text{ and } E_n(z) = (1-z)\exp\left[\sum_{j=1}^{n} \frac{z^j}{j}\right] \ \forall\, n \geq 1. \tag{5.5}$$

Note that not only does Weierstrass give us "sufficiency", but as an added bonus he even derives a closed-form expression for a particular analytic function that possesses A as a zero set! Thus, the zeros of analytic functions are completely classified.

The following example is motivated by the desire to obtain a geometric picture of what these Weierstrass products "look" like.

Example 5.11. Unfortunately, we cannot graph a function of a complex variable from the complex plane to the complex plane because this would require four dimensions (that is, two for each plane). However, if instead of mapping complex numbers to complex numbers, we could somehow map complex numbers to real numbers, then we would only need three dimensions in order to visualize the "complex" function. One method by which this is accomplished is to consider the square of the absolute value of the mapping, which is a real number. In other words, given a complex function f, we can graph $|f|^2$ on the z-axis above the complex plane. Note that graphing $|f|^2$ is a reasonable choice for two reasons: first, the absolute value of a complex number does retain some information about the real and imaginary components of the number, and second, since the absolute value involves an awkward square root, squaring the absolute value serves to "smooth" out the graph.

For the sake of simplicity and purposes of visualization, this example does not correspond exactly to the Weierstrass product defined by (5.4). Instead, we consider merely $E_1(z)$ (as defined by (5.5)). The following are graphs corresponding to the Weierstrass product utilizing $E_1(z)$. Specifically, the function being "graphed" is

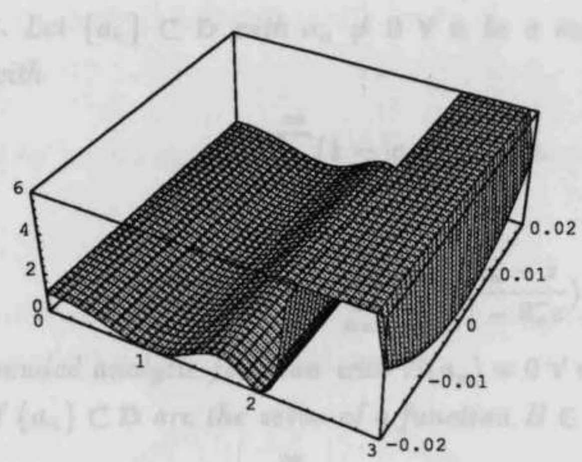
$$f(z) = E_1(z)E_1(\frac{z}{2})E_1(\frac{z}{3}). \tag{5.6}$$

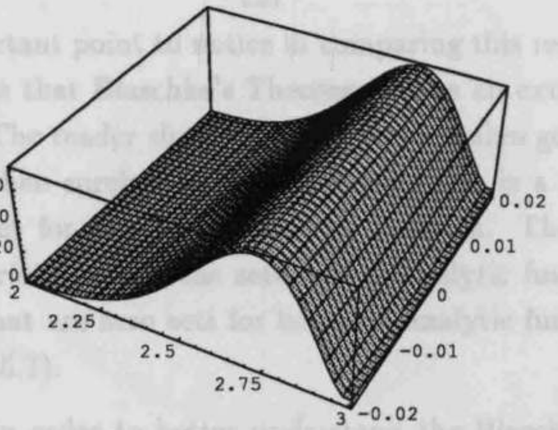
(Of course, we are really graphing the square of the absolute value of this function.) The reader will notice that the zeros of this function are not contained within the unit disk, a result of substituting the simpler functions $E_1(\frac{z}{a_n})$ for

$$E_1(\frac{a_n(|a_n|-1)}{z|a_n|-a_n}).$$

This enables us to conveniently place zeros at $z = 1$, $z = 2$ and $z = 3$.

The first graph shows the zeros of (5.6) that occur at $z = 1$, $z = 2$ and $z = 3$. But because the function becomes so large between $z = 2$ and $z = 3$, it is impossible to both see all three of the zeros and simultaneously to see the maximum of the function between $z = 2$ and $z = 3$. Therefore, we have included a second graph of the function (5.6) which only includes the portion of the graph between $z = 2$ and $z = 3$.





We remark first that the use of this altered Weierstrass product does force the function to equal zero at $z = 1$, $z = 2$ and $z = 3$. But notice how the function "blows up" between $z = 2$ and $z = 3$. This is not particularly surprising - after all, we are working with exponentials - but it indicates the essentially unbounded nature of the Weierstrass product. We have not proved this explicitly, but we use this example as an easy way to show that (5.4) could grow arbitrarily "large" between two elements of the zero sequence $\{a_n\}$ as $|z| \to 1$.

The above example demonstrates why the Weierstrass product sometimes fails to produce bounded analytic functions. The next question is, given a sequence in the unit disk which accumulates on the perimeter, can we find a non-zero bounded analytic function which equals zero when evaluated at the points of the sequence?

The answer to this question resulted in a theorem similar in essence to the Weierstrass Factorization Theorem and was discovered earlier this century by Blaschke. Again, because this is such a well-known classical result, we omit the proof, [4] p. 173.

Theorem 5.12. *Let $\{a_n\} \subset \mathbb{D}$ with $a_n \neq 0 \; \forall \, n$ be a sequence accumulating only on the boundary of $\mathbb{D}$ with*

$$\sum_{n=1}^{\infty}(1 - |a_n|) < +\infty. \tag{5.7}$$

Then

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n}\left(\frac{a_n - z}{1 - \overline{a_n}z}\right) \tag{5.8}$$

is a non-zero bounded analytic function with $B(a_n) = 0 \; \forall \, n$.

Conversely, if $\{a_n\} \subset \mathbb{D}$ are the zeros of a function $B \in H^\infty(\mathbb{D})$ then

$$\sum_{n=1}^{\infty}(1 - |a_n|) < +\infty.$$

The most important point to notice in comparing this result with the Weierstrass Factorization Theorem is that Blaschke's Theorem places an extra convergence restriction, (5.7), on the zero sets. The reader should note that this makes good intuitive sense because, since $H^\infty(\mathbb{D}) \subset H(\mathbb{D})$, then surely not every sequence that is a zero set for an analytic function could be a zero set for a bounded analytic function. Therefore, the idea is to put some kind of extra restriction upon the zero sets of analytic functions in order to pick out only those sequences that are zero sets for bounded analytic functions. This is precisely what is accomplished by (5.7).

Example 5.13. In order to better understand the Blaschke restriction, consider that the sequence

$$\{a_n\} = 1 - \frac{1}{n}$$

does not satisfy (5.7) since

$$\sum_{n=1}^{\infty}(1 - |1 - \frac{1}{n}|) = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

However, the sequence

$$\{a_n\} = 1 - \frac{1}{n^2}$$

does satisfy (5.7) because

$$\sum_{n=1}^{\infty}(1 - |1 - \frac{1}{n^2}|) = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is a convergent series.

Example 5.14. It is also possible to construct more interesting sequences which accumulate at all points on the perimeter of the unit disk. Let $\{z_m\}$ be the following finite sequence containing 2^m elements,

$$\{z_m\} = \{(1 - \frac{1}{m})\exp(\frac{\pi k i}{2^{m-1}}) : 1 \leq k \leq 2^m\}. \tag{5.9}$$

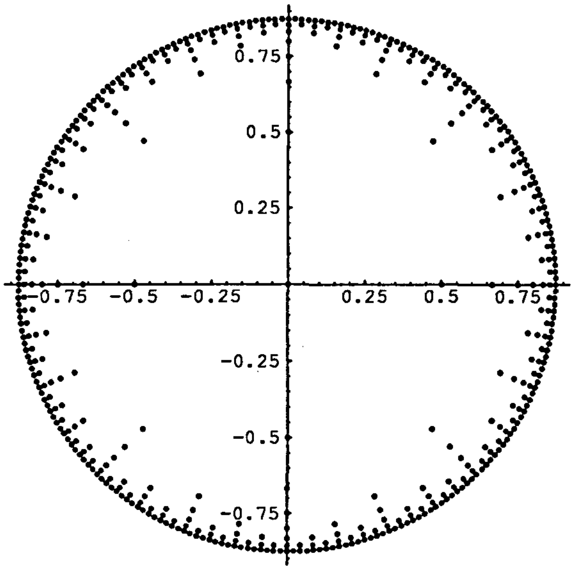
As an example, it is easily verified that

$$\{z_2\} = \{\frac{i}{2}, \frac{-1}{2}, \frac{-i}{2}, \frac{1}{2}\}.$$

Now define $\{a_n\}$ to be the union over all m of the sequences z_m, that is,

$$\{a_n\} = \bigcup_{m=1}^{\infty} \{z_m\} \tag{5.10}$$

where $\{a_n\}$ is indexed such that $\{a_{2^m-1}, a_{2^m}, ..., a_{2^{m+1}-3}, a_{2^{m+1}-2}\} = \{z_m\}$. The following is a graph of the first 510 points of this sequence.



We proceed to convince the reader that $\{a_n\}$ accumulates everywhere on the perimeter of the unit disk using a geometric argument. Consider again the finite sequences $\{z_m\}$ defined by (5.9). $\{z_m\}$ contains 2^m points in the unit disk all separated in polar coordinates by a radial angle of $(2\pi)(2^{-m}) = 2^{-m+1}\pi$ at a distance of $\frac{1}{m}$ from the perimeter of the disk. So as m becomes large, the distance from the elements of $\{z_m\}$ to the perimeter becomes small and simultaneously the points are located closer together because the radial angle separating them also becomes small. Thus, as $m \to \infty$ the sequences $\{z_m\}$ start to approach every point on the perimeter of the unit disk since the distance from the points to the perimeter is becoming infinitesimal and the angle between each point is approaching zero. The sequence $\{a_n\}$, which is simply the infinite union of the $\{z_m\}$ as defined by (5.10), must therefore approach every point on the perimeter of $\mathbb{D}$. But since it is a countable sequence with no accumulation points inside the disk, then by the Weierstrass Factorization Theorem we can construct an analytic function with zeros at all of the points of $\{a_n\}$.

Example 5.15. The sequence $\{a_n\}$ from the previous example obviously does not satisfy the Blaschke restriction (5.7) because it is seen to contain the subsequence $\{1-\frac{1}{n}\}$ which has already been shown in Example 5.13 to violate (5.7). So the obvious next question is whether it is possible to construct a sequence analogous to $\{a_n\}$ that accumulates everywhere on the perimeter of $\mathbb{D}$ but also satisfies the Blaschke restriction.

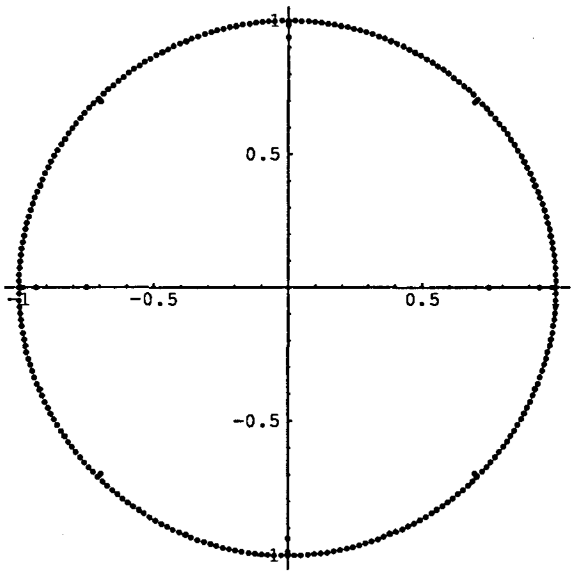
Let $\{y_m\}$ be finite sequences containing 2^m elements and defined by

$$\{y_m\} = \{(1 - \frac{1}{4^m})\exp(\frac{\pi k i}{2^{m-1}}) : 1 \leq k \leq 2^m\}$$

and let $\{b_n\}$ be the infinite union of these sequences

$$\{b_n\} = \bigcup_{m=1}^{\infty} \{y_m\}$$

indexed such that $\{b_{2^m-1}, \ldots, b_{2^{m+1}-2}\} = \{y_m\}$. The following is a graph of the first 510 points of this sequence. Note that this sequence converges to the perimeter more quickly than the sequence in the previous example.



Clearly, by the geometric arguments used in the previous example, $\{b_n\}$ accumulates at every point on the boundary of $\mathbb{D}$. We proceed to demonstrate that $\{b_n\}$ satisfies the Blaschke restriction (5.7). Fix an m and note that the distance from an element of $\{y_m\}$ to the perimeter of $\mathbb{D}$ is 4^{-m}. Since there are 2^m elements in $\{y_m\}$, this implies that the sum of the distances of the elements of $\{y_m\}$ to the perimeter is $2^m 4^{-m} = (\frac{1}{2})^m$. Now since $\{b_n\}$ is the union of all the $\{y_m\}$ this further implies that the sum of the distances of all the elements of $\{b_n\}$ to the perimeter is

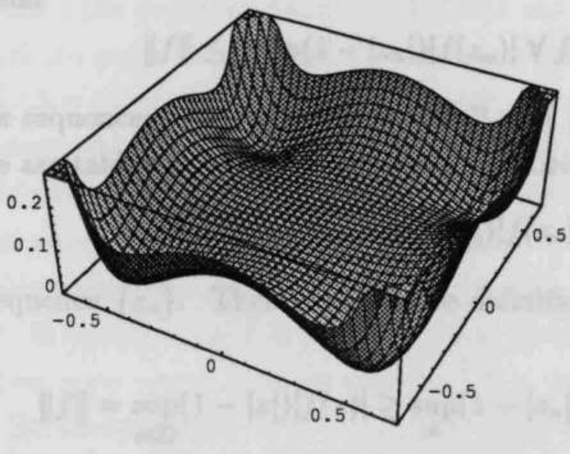
$$\sum_{m=1}^{\infty} (\frac{1}{2})^m = 1$$

thus demonstrating that the sequence $\{b_n\}$ does indeed satisfy the Blaschke restriction. This interesting result implies that one can construct a sequence which accumulates everywhere on the boundary of the disk and still be able to find a bounded analytic function which equals zero when evaluated at each point of the sequence.

Example 5.16. Assume that $a_1 = .5 + .5i$, $a_2 = .5 - .5i$, $a_3 = -.5 + .5i$ and $a_4 = -.5 - .5i$ are the first four elements of a sequence $\{a_n\}$ which satisfies the Blaschke restriction (5.7). To graphically explore the nature of the Blaschke product (5.8), we construct a function based upon the Blaschke product but using only these first four elements of $\{a_n\}$.

$$b(z) = \prod_{n=1}^{4} \frac{|a_n|}{a_n}\left(\frac{a_n - z}{1 - \overline{a_n} z}\right)$$

This function should equal zero at a_1, a_2, a_3 and a_4. Following the pattern of Example 5.11, we plot $|b(z)|^2$ on the z-axis above the complex plane.



The reader can observe how b equals zero at each of the desired points. We also remark that the corners of the graph are on the perimeter of the unit disk (so that the region above which the function is graphed is the square circumscribed within the closed disk). Notice how $b(z)$ does not become arbitrarily large as $|z| \to 1$. This example helps demonstrate visually why $B(z)$ from (5.8) is a bounded analytic function.

Having completely classified the zero sets for $H(\mathbb{D})$ and for $H^\infty(\mathbb{D})$, we now move to A^{-1}, the space that forms the main body of our research. The remainder of this section will be concerned with discussing the zeros of A^{-1}.

The following definitions are valid with respect to any class $\mathcal{C}$ of analytic functions on the unit disk.

Definition 5.17. (1) A sequence $\{z_n\} \subset \mathbb{D}$ is a *vanishing sequence* for $\mathcal{C}$ if there exists $f \in \mathcal{C}$ with $f \not\equiv 0$ such that $f(z_n) = 0 \; \forall \, n$.

(2) A sequence $\{z_n\} \subset \mathbb{D}$ is a *zero sequence* for $\mathcal{C}$ if there exists $f \in \mathcal{C}$ with $f \not\equiv 0$ such that $f^{-1}(\{0\}) = \{z_n\}$.

Though at a superficial first glance these definitions may seem to be describing the same thing in two different ways, closer inspection actually reveals that a zero sequence is a stricter classification than a vanishing sequence. In other words, all zero sequences are vanishing sequences but not all vanishing sequences are zero sequences. To see why, note the requirement for a zero sequence that $f^{-1}(\{0\}) = \{z_n\}$ implies that the points in the sequence $\{z_n\}$ are the only zeros of f, whereas the requirement for a vanishing sequence that $f(z_n) = 0 \ \forall \ n$ leaves open the question of whether or not there are points other than those in the sequence $\{z_n\}$ which may be zeros of f. To give a concrete example, note that any subset of a zero sequence is a vanishing sequence.

The following definition is given with respect to the space A^{-1}.

Definition 5.18. A sequence $\{z_n\} \subset \mathbb{D}$ is a *sampling sequence* if there exists $c > 0$ independent of f such that

$$\|f\| \leq c \sup_n (1 - |z_n|)|f(z_n)| \ \forall \ f \in A^{-1}.$$

Why are these sequences called sampling? Recall that $\|f\| = \sup_{z \in \mathbb{D}}(1 - |z|)|f(z)|$. Note that, because we are taking the supremum over the entire disk we know that

$$\|f\| \geq \sup_n (1 - |z_n|)|f(z_n)|$$

for any given sequence $\{z_n\}$. Therefore, by the definition just given, a sequence $\{z_n\}$ is sampling if

$$\|f\| = \sup_{z \in \mathbb{D}}(1 - |z|)|f(z)| \geq \sup_n (1 - |z_n|)|f(z_n)| \geq \frac{\|f\|}{c}$$

for some $c > 0$ independent of all $f \in A^{-1}$. This can be thought of intuitively as saying that as $\|f\|$ becomes small or large when evaluated at the points of $\{z_n\}$, then $c\|f\|$ also becomes small or large respectively at these points. The new norm $c\|f\|$ then serves as an "equivalent" norm to $\|f\|$. $\{z_n\}$ is called a sampling sequence because this implies that one need only consider those points in $\{z_n\}$ evaluated with respect to the new norm $c\|f\|$ in order to understand the behavior of the original norm $\|f\|$. We don't have to look at the entire disk, we can merely take a "sampling" of points in the disk.

Lemma 5.19. *A sampling sequence is not a vanishing sequence.*

Proof. If $\{z_n\}$ is a sampling sequence then there exists a $c > 0$ such that for all $f \in A^{-1}$, $\|f\| \leq c \sup_n (1 - |z_n|)|f(z_n)|$. Now if $\{z_n\}$ were also a vanishing sequence then there would exist a $g \in A^{-1}$ not identically equal to zero such that $\sup_n (1 - |z_n|)|g(z_n)| = 0 \ \forall \ n$. But this implies that $\|g\| \leq c \sup_n (1 - |z_n|)|g(z_n)| = c(0) = 0$, which means that g is identically equal to zero, a contradiction. $\square$

The following definition is the last one we need in our discussion of sequences.

Definition 5.20. A sequence $\{z_n\} \in \mathbb{D}$ is an *interpolating sequence* for A^{-1} if given any sequence $\{a_n\} \in \mathbb{C}$ with

$$\sup_n (1 - |z_n|)|a_n| < +\infty$$

then there exists a $f \in A^{-1}$ such that $f(z_n) = a_n$.

These sequences are called "interpolating" because for each sequence there exists a function f which can map it essentially anywhere in the plane. Since the sequence can be thus be "inserted" into any arbitrary sequence under a particular mapping, the original sequence is designated as interpolating.

Lemma 5.21. *If $\{z_n\}$ is an interpolating sequence for A^{-1} then it is also a vanishing sequence for A^{-1}.*

Proof. First consider the sequence $\{a_n\} = 0 \ \forall\ n$ and note that clearly $\sup_n(1 - |z_n|)|a_n|$ is bounded. (Indeed, it equals zero.) Since $\{z_n\}$ is interpolating by hypothesis, there exists an $f \in A^{-1}$ with $f(z_n) = a_n = 0$.

The only problem with this is that we have no guarantee that f is not identically equal to zero, which would violate the requirements for $\{z_n\}$ being a vanishing sequence. To overcome this, consider the sequence $\{a_1\} = 1$ and $\{a_n\} = 0 \ \forall\ n \neq 1$. Note again that this sequence satisfies the requirement that $\sup_n(1 - |z_n|)|a_n| < +\infty$. Since $\{z_n\}$ is interpolating, this implies that that there exists an $f \in A^{-1}$ such that $f(z_1) = 1$ and $f(z_n) = 0$ for all $n \neq 1$. Unfortunately, now it no longer appears that $\{z_n\}$ is a vanishing sequence.

But consider the function $g(z) = (z - z_1)f(z)$. Note that this function does equal zero when evaluated at $z = z_1$. Also, g evaluated at any z_n such that $n \neq 1$ must equal zero because f evaluated at these points equals zero. Thus, $g(z_n) = 0 \ \forall\ n$. Also, recall that $f(z_1) = 1$, which implies that $f \not\equiv 0$. Since $z - z_1$ is also not identically zero, this implies that $g \not\equiv 0$. And since g is obviously in A^{-1}, this demonstrates that $\{z_n\}$ is indeed a vanishing sequence. □

Corollary 5.22. *An sampling sequence is not an interpolating sequence.*

Proof. This immediately follows from Lemma 5.19. □

Kristian Seip [10] has not only completely characterized the sampling and interpolating sequences for A^{-1}, but has also constructed interesting examples of them using the Caley Transform

$$\phi(z) = \frac{z - i}{z + i} \tag{5.11}$$

Lemma 5.23. *ϕ defined by (5.11) maps the upper half-plane into the unit disk.*

Proof. Let z be in the upper half-plane. Since i is also in the upper half-plane, they are both above the real axis of the complex plane. Since the real axis perpendicularly bisects the line segment from i to $-i$ on the imaginary axis, this implies that $|z - i| < |z - (-i)| = |z + i|$, which further demonstrates that $|\phi| < 1$, thus completing the proof. □

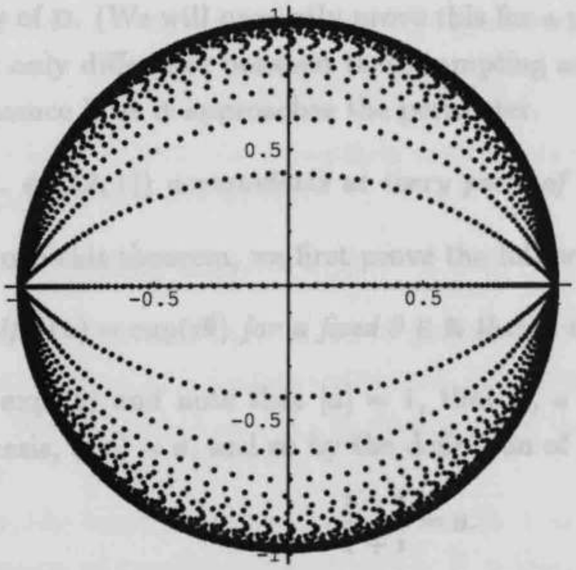
The sequences are generated according to $\phi(\Gamma(a,b))$ where

$$\Gamma(a,b) = a^m(bn+i) \text{ such that } m,n \in \mathbb{Z}, a > 1, \text{ and } b > 0. \tag{5.12}$$

Corollary 5.24. *$\phi(\Gamma(a,b))$ lies in the unit disk.*

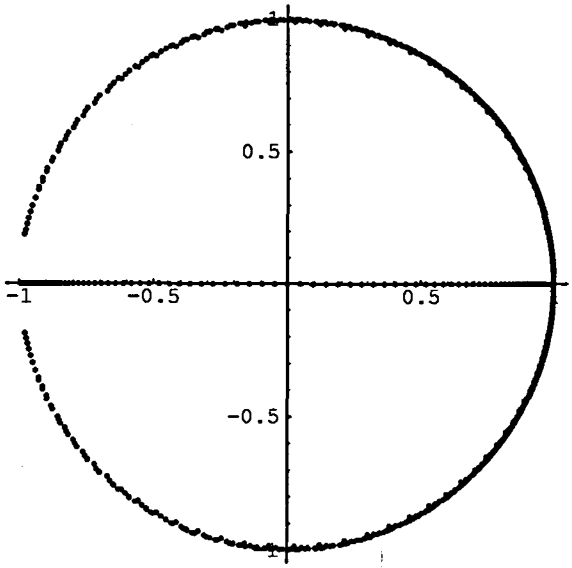
Proof. This immediately follows from Lemma 5.23 since $\Gamma(a,b)$ clearly lies in the upper half-plane. □

Example 5.25. Seip has demonstrated that if $b\log(a) < 2\pi$ then $\phi(\Gamma(a,b))$ is a sampling sequence, whereas if $b\log(a) > 2\pi$ then $\phi(\Gamma(a,b))$ is an interpolating sequence. The following is a graph of 19881 points of a sampling sequence formed by letting $a = 1.1$ and $b = 1$ so that $b\log(a) < 2\pi$.



Notice how "thick" the sequence is as it approaches the perimeter of the disk. This is not surprising, for the sequence must be dense near the perimeter in order for it to contain enough points to effectively sample the norm.

This next graph is 19881 points of an interpolating sequence formed by letting $a = 1.1$ and $b = 75$ so that $b\log(a) > 2\pi$.



Obviously, if we graphed more points we would eventually be able to see accumulation on the perimeter near -1, but notice how "thin" the sequence is as it approaches the perimeter compared with the thick density of the sampling sequence. This is also not surprising, for we expect that an interpolating sequence, as a vanishing sequence, would be too dense near the perimeter.

The reason why these sequences are so interesting is because they accumulate everywhere on the boundary of $\mathbb{D}$. (We will presently prove this for a particular choice of a and b.) This implies that the only difference between these sampling and interpolating sequences is how "dense" the sequence is as it approaches the perimeter.

Theorem 5.26. $\phi(\Gamma(2,1))$ *accumulates at every point of the boundary of* $\mathbb{D}$.

In order to prove this theorem, we first prove the following two lemmas.

Lemma 5.27. *If* $\phi(z) = \exp(i\theta)$ *for a fixed* $\theta \in \mathbb{R}$ *then* $z \in \mathbb{R}$.

Proof. Let $a = \exp(i\theta)$ and note that $|a| = 1$, that is, a lies on the perimeter of the unit disk. By hypothesis, $\phi(z) = a$, and so by the definition of ϕ,

$$\frac{z-i}{z+i} = a.$$

One may perform simple algebra (which we leave to the reader to verify) upon this equation to discover that

$$z = \frac{i(1+a)}{1-a}$$

which implies that

$$\overline{z} = \frac{-i(1+\overline{a})}{1-\overline{a}}.$$

We proceed to demonstrate that $z = \overline{z}$, which clearly suffices to prove the lemma. Obtaining a common denominator implies that

$$z - \overline{z} = \frac{i(1+a)(1-\overline{a}) + i(1+\overline{a})(1-a)}{(1-a)(1-\overline{a})}$$

$$= \frac{2i(1-a\overline{a})}{(1-a)(1-\overline{a})} = \frac{2i(1-|a|^2)}{(1-a)(1-\overline{a})} = 0$$

since $|a| = 1$. Thus, $z = \overline{z}$, which proves that z is real. □

Corollary 5.28. *ϕ maps the real line onto every point of the perimeter of $\mathbb{D}$.*

Proof. This follows from the fact that we could choose $a \in bd(\mathbb{D})$ arbitrarily and find a $z \in \mathbb{R}$ such that $\phi(z) = a$. □

Let us pause for a moment to interpret this lemma in the context of what we are ultimately trying to prove. The goal is to show that $\phi(\Gamma(2,1))$ accumulates everywhere on the perimeter of $\mathbb{D}$. The previous lemma, together with its corollary, implies that it suffices to prove that the sequence $\Gamma(2,1)$ accumulates everywhere on the real line.

Lemma 5.29. $\overline{\Gamma(2,1)} \supset \{\frac{k}{2^j}\} \ \forall \ k \in \mathbb{Z}, j \in \mathbb{N}$.

Proof. Recalling (5.12), $\Gamma(2,1) = \{2^m n + 2^m i\} \ \forall \ m, n \in \mathbb{Z}$. Fix $k \in \mathbb{Z}$, $m < 0$ and $j \in \mathbb{N}$ such that $j \leq -m$. Let $n = k2^{-m-j}$. Note that $n \in \mathbb{Z}$ since $2^{-m-j} \in \mathbb{Z}$ due to the fact that $j \leq -m$. Then $2^m n$, the real component of $\Gamma(2,1)$ satisfies

$$2^m n = 2^m(k2^{-m-j}) = k2^{-j}.$$

Also, in the limit as $m \to -\infty$, the imaginary component of $\Gamma(2,1)$, that is, $2^m i$, approaches zero. This suffices to prove the lemma. □

We now proceed to prove Theorem 5.26

Proof. Lemma 5.27 implies that it suffices to prove that $\Gamma(2,1)$ accumulates everywhere on the real line, and Lemma 5.29 demonstrates that $\Gamma(2,1)$ does accumulate at "many" points on the real line. We proceed to use this to demonstrate that $\Gamma(2,1)$ does indeed accumulate everywhere on $\mathbb{R}$.

We know from the basic properties of numbers that any natural number can be written as the sum of powers of two (including $2^0 = 1$). It is thus easy to see that for fixed $j, k \in \mathbb{N}$ such that $k < 2^j$ there exists a sequence $\{a_n\}$ of zeros and ones such that

$$x_j = \sum_{n=1}^{j} \frac{a_n}{2^n} = \frac{k}{2^j}$$

because

$$\frac{a_1}{2} + \frac{a_2}{2^2} + \ldots + \frac{a_j}{2^j} = \frac{2^{j-1}a_1 + 2^{j-2}a_2 + \ldots + 2^0 a_j}{2^j} = \frac{k}{2^j}.$$

Now note that any real number $x \in [0,1]$ can be written as a binary expansion

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

for an appropriate sequence $\{a_n\}$ of zeros and ones. Since $x_j \to x$ as $j \to +\infty$, this demonstrates that $\Gamma(2,1)$ accumulates everywhere on $[0,1]$. And since any real number can be written as the sum of an integer and an element of $[0,1]$, this implies that $\Gamma(2,1)$ accumulates everywhere on $\mathbb{R}$. $\square$

We conclude this section with a theorem which demonstrates that there are vanishing sequences for A^{-1} which are not vanishing for $H^\infty(\mathbb{D})$.

Theorem 5.30. *$\phi(\Gamma(a,b))$ does not satisfy the Blaschke restriction (5.7).*

Proof. It suffices to prove that a subsequence of $\phi(\Gamma(a,b))$ does not satisfy (5.7). Let $n = 0$ and $m < 0$. Then $\Gamma(a,b) \supset \{a^m i\}$. We proceed to demonstrate that $\phi(a^m i)$ does not satisfy (5.7).

$$\phi(a^m i) = \frac{a^m i - i}{a^m i + i} = \frac{a^m - 1}{a^m + 1}$$

implies that

$$1 - |\phi(a^m i)| = \frac{a^m + 1}{a^m + 1} - \frac{a^m - 1}{a^m + 1} = \frac{2}{a^m + 1}.$$

Consider

$$\sum_{m=-1}^{-\infty} \frac{2}{a^m + 1} = \sum_{m'=1}^{+\infty} \frac{2}{a^{-m'} + 1}$$

where we have made a "change of variables" from m to m' to fit the form of (5.7).

But this sum is easily seen to diverge because, since $a > 1$ and $m' \geq 1$, then $a^{-m'} + 1 < 2$, which implies that the elements we are summing over are all greater than one. Since

$$\sum_{m,n} 1 - |\phi(\Gamma(a,b))| > \sum_{m} 1 - |\phi(a^m i)| = \sum_{m'=1}^{\infty} \frac{2}{a^{-m'} + 1} = +\infty$$

this completes the theorem. $\square$

Corollary 5.31. *There exist vanishing sequences for A^{-1} that are not vanishing for $H^\infty(\mathbb{D})$.*

Proof. Since Seip has demonstrated that $\Gamma(a,b)$ is an interpolating sequence if $b \log a > 2\pi$, and since by Lemma 5.21 all interpolating sequences in A^{-1} are vanishing sequences, then this result immediately follows from Theorem 5.12 and the theorem just proved. $\square$

6. Invariant Subspaces

In this section we will explore the invariant subspaces of the linear transformation

$$M_z : A^{-1} \to A^{-1} \text{ such that } M_z(f) = zf.$$

Definition 6.1. A subspace $\mathcal{S}$ of a space of analytic functions is *z-invariant* if given any function $f \in \mathcal{S}$ then $M_z(f) \in \mathcal{S}$ where $M_z(f) = zf$.

For purposes of convenient notation, we write the set of all functions zf such that $f \in \mathcal{F}$ as $z\mathcal{F}$. We encourage the reader to refer back to Example 3.12 which demonstrated that A^{-1} is z-invariant. This example can be used similarly to show that $C^\infty(\mathbb{D})$, $H(\mathbb{D})$, and $H^\infty(\mathbb{D})$ are all z-invariant.

The "shift operator" M_z is an important operator which plays a fundamental role in the theories of functions and operators. It was examined successfully (in a much different setting) by Arne Beurling in 1949 [3]. Since then it has been studied by many others. A general discussion of the shift operator can be found in [11]. For many spaces of analytic functions the M_z invariant subspaces have been completely classified. However, this is not the case for A^{-1}, and in this section we explore the difficulties that arise in the characterization of the M_z invariant subspaces of A^{-1}. In particular, we will give examples of how the M_z invariant subspaces of A^{-1} can be very complicated.

Our ideas are based upon observations made by Hedenmalm of the complexity of the M_z invariant subspaces of a slightly different space [5]. To accomplish this, we will use (as did Hedenmalm) certain ideas of Seip [10].

Lemma 6.2. $zA^{-1} = \{f \in A^{-1} : f(0) = 0\}$

Proof. Let $f \in zA^{-1}$ and $K = \{f \in A^{-1} : f(0) = 0\}$. Then $f = zg$ for some $g \in A^{-1}$. Thus, $f(0) = (zg)(0) = 0g(0) = 0$, which implies $f \in K$.

Conversely, let $f \in K$ so $f(0) = 0$. Since $f \in H(\mathbb{D})$ then f can be written in a Taylor series expansion as [4] p.72

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ where } a_n = \frac{f^{(n)}}{n!}.$$

Since $f(0) = 0$, this implies that $a_0 = 0$, and therefore

$$f(z) = a_1 z + a_2 z^2 + \ldots = z(a_1 + a_2 z + \ldots) = zg(z)$$

where g is clearly in A^{-1}. This demonstrates that $f \in zA^{-1}$, thus completing the proof. $\square$

We leave it to the reader to similarly show that

(1) $zH(\mathbb{D}) = \{f \in H(\mathbb{D}) : f(0) = 0\}$
(2) $zH^\infty(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) : f(0) = 0\}$

Lemma 6.3. $\dim(A^{-1}/zA^{-1}) = 1$

Proof. Since $\mathbb{C}$ has a dimension of one, we will use Theorem 3.28, the first homomorphism theorem, and show that A^{-1}/zA^{-1} is isomorphic to $\mathbb{C}$. Define $\phi : A^{-1} \to \mathbb{C}$ by $\phi(f) = f(0)$. ϕ is clearly a linear transformation since given $c \in \mathbb{C}$ we have

$$\phi(cf + g) = (cf + g)(0) = cf(0) + g(0) = c\phi(f) + \phi(g).$$

By definition, the kernel K of ϕ is

$$K(\phi) = \{f \in A^{-1} : f(0) = 0\}.$$

But by the previous lemma, this implies that $K(\phi) = zA^{-1}$. Thus, according to the first homomorphism theorem,

$$A^{-1}/K(\phi) = A^{-1}/zA^{-1} \approx \mathbb{C}$$

which completes the proof. □

Again, we leave it to the reader to similarly show that

$$\dim(H(\mathbb{D})/zH(\mathbb{D})) = \dim(H^{\infty}(\mathbb{D})/zH^{\infty}(\mathbb{D})) = 1.$$

Definition 6.4. Let V be a vector space over $\mathbb{C}$ and W be a z-invariant subspace of V. Then the *codimension* of W is the dimension of the quotient space W/zW.

The codimension of W/zW can be thought of as measuring the reduction in the dimension of W that results from its multiplication by z. By "dividing" W by its subspace zW, we obtain a space with the dimension of which is equal to difference between the dimensions of W and zW. This is precisely the codimension of W. Recent results of Aleman, Richter, and Ross have demonstrated that there are many subspaces of the holomorphic functions with a codimension equal to unity.

Theorem 6.5 ([1] [8]). *Let $p \geq 1$ and define*

$$D_p = \{f \in H(\mathbb{D}) : \int_{\mathbb{D}}\int |f'(z)|^p dxdy < +\infty\}.$$

Then given any closed z-invariant subspace $\mathcal{S}$ of D_p ($\mathcal{S} \neq 0$), the codimension of $\mathcal{S}/z\mathcal{S}$ is equal to one.

Our ultimate goal in this section is to construct a subspace of A^{-1} which has a codimension not equal to one, for which we will employ an idea of Hedenmalm [5]. The above theorem indicates that this most likely will not be a simple task. For example, at first glance the following lemma does not appear to help us.

Lemma 6.6. *Let $A = \{a_n\}$, a countable sequence such that $a_n \neq 0 \; \forall \, n$. Define*

$$I(A) = \{f \in A^{-1} : f(a_n) = 0 \; \forall \, n\}. \tag{6.1}$$

Then $I(A)$ is a Banach space and $\dim I(A)/zI(A) = 1$.

Proof. We leave it to the reader to verify that $I(A)$ is indeed a subspace of A^{-1}. However, it is not immediately obvious that $I(A)$ is closed. Let f_n be a Cauchy sequence in $I(A)$. Since $I(A) \subset A^{-1}$ and, by Lemma 4.5, A^{-1} is closed, we know that $f_n \to f \in A^{-1}$. We must show additionally that $f \in I(A)$.

Let $a \in A \subset \mathbb{D}$. Then

$$(1-|a|)|f(a)| \leq \sup_{z\in\mathbb{D}}(1-|z|)|f(z)| = \|f\|.$$

Thus

$$|f(a)| \leq \frac{\|f\|}{1-|a|}. \tag{6.2}$$

Now

$$|f(a)| = |f(a) - f_n(a) + f_n(a) \leq |f(a) - f_n(a)| + |f_n(a)| = |f(a) - f_n(a)| = |(f-f_n)(a)|$$

since $f_n(a) = 0$ as a result of being part of a Cauchy sequence in $I(A)$. But by (6.2),

$$|(f-f_n)(a)| \leq \frac{\|f-f_n\|}{1-|a|}$$

which implies that

$$|f(a)| = \lim_{n\to\infty} |f(a)| \leq \lim_{n\to\infty} \frac{\|f-f_n\|}{1-|a|} = 0$$

since $f_n \to f \in A^{-1}$. Therefore $f(a) = 0$, demonstrating that $f \in I(A)$ (since $a \in A$ was chosen arbitrarily).

To show that $\dim I(A)/zI(A) = 1$, we first convince the reader that $I(A)$ is indeed z-invariant. This is easy to see since given $f \in I(A)$ then $f(a_n) = 0 \ \forall\ n$. So therefore $(zf)(a_n) = a_n f(a_n) = a_n 0 = 0 \ \forall\ n$, which demonstrates that $zI(A) \subset I(A)$.

We proceed to use the first homomorphism theorem to demonstrate that $I(A)/zI(A) \approx \mathbb{C}$, which suffices to complete the proof. Let $f \in I(A)$ and define $\phi(f) = f(0)$. Then the kernel K of ϕ is

$$K(\phi) = \{f \in I(A) : f(0) = 0\}$$

so $\dim I(A)/K(\phi) = 1$. It must be shown that $K(\phi) = zI(A)$.

Clearly, $zI(A) \subset K(\phi)$ both because $zI(A) \subset I(A)$ and, given $f \in zI(A)$ then $f = zg$ for some $g \in I(A)$ whereby $f(0) = (zg)(0) = 0g(0) = 0$.

Conversely, let $f \in K(\phi)$. Then $f(a_n) = 0$ and $f(0) = 0$. But we may "divide out the zero" (see Lemma 6.2) to construct a function g such that $g = \frac{f}{z}$, or equivalently, $f = zg$. All that remains is to show that $g \in I(A)$. Since by hypothesis none of the points $\{a_n\}$ equal zero, this implies that $(\frac{f}{z})(a_n) = g(a_n) = 0 \ \forall\ n$, which shows that $g \in I(A)$. Thus, $K(\phi) \subset zI(A)$, thus completing the proof. □

The effort to construct a subspace of A^{-1} with a codimension not equal to one is further complicated by the fact that the sum of two closed z-invariant strict subspaces of A^{-1} is not necessarily also a closed z-invariant strict subspace of A^{-1}, as demonstrated by the following example.

Example 6.7. Let $a, b \in \mathbb{D}$ with $a, b \neq 0$ and $a \neq b$. Then, keeping the same notation as in the previous lemma, let

$$I(a) = \{f \in A^{-1} : f(a) = 0\}$$

$$I(b) = \{g \in A^{-1} : g(b) = 0\}$$

so that

$$I(a) + I(b) = \{f + g \in A^{-1} : f(a) = 0, g(b) = 0\}.$$

We proceed to demonstrate that $I(a) + I(b) = A^{-1}$, and is therefore not a strict subspace of A^{-1}. First note that $f(z) = z - a \in I(a)$ and $g(z) = z - b \in I(b)$. Let h be an arbitrary function in A^{-1}. Then h can be written as the following linear combination of elements of $I(a)$ and $I(b)$. The reader can verify algebraically that the right hand side does indeed reduce to h.

$$h \equiv \frac{h}{b-a}(z-a) + \frac{h}{a-b}(z-b) = \frac{h}{b-a}f + \frac{h}{a-b}g.$$

Note that since $f \in I(a)$ then $(hf)(a) = h(a)f(a) = 0$, which implies that $hf \in I(a)$. Similarly, $hg \in I(b)$. This proves that h can be written as a linear combination of elements of $I(a)$ and $I(b)$, thus demonstrating that $I(a) + I(b) = A^{-1}$.

Example 6.8. Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ with $a_1, a_2, b_1, b_2 \neq 0$, $A \cap B = \emptyset$, $a_1, a_2, b_1, b_2 \in \mathbb{D}$. Then if $h \in A^{-1}$, a calculation with Mathematica shows that

$$h \equiv (c_1 + c_2 z)(z - a_1)(z - a_2)h + (d_1 + d_2 z)(z - b_1)(z - b_2)h$$

where

$$c_1 = \frac{a_1 a_2 - a_1 b_1 - a_2 b_1 + b_1^2 - a_1 b_2 - a_2 b_2 + b_1 b_2 + b_2^2}{(b_1 - a_1)(b_1 - a_2)(b_2 - a_1)(b_2 - a_2)}$$

$$c_2 = \frac{b_1 + b_2 - a_1 - a_2}{(a_1 - b_1)(a_2 - b_1)(a_1 - b_2)(b_2 - a_2)}$$

$$d_1 = \frac{a_1^2 + a_1 a_2 + a_2^2 - a_1 b_1 - a_2 b_1 - a_1 b_2 - a_2 b_2 + b_1 b_2}{(b_1 - a_1)(b_1 - a_2)(b_2 - a_1)(b_2 - a_2)}$$

$$d_2 = \frac{a_1 + a_2 - b_1 - b_2}{(a_2 - b_1)(b_1 - a_1)(b_2 - a_1)(b_2 - a_2)}.$$

Note that h has been written as the sum of two functions; one from $I(A)$ and one from $I(B)$. This implies that $A^{-1} = I(A) + I(B)$.

Similarly, if we let $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_m\}$ for $n, m < +\infty$, $A \cap B = \emptyset$, $a_i, b_j \in \mathbb{D}\ \forall\ 1 \leq i \leq n$ and $1 \leq j \leq m$ and $a_i, b_j \neq 0\ \forall\ 1 \leq i \leq n$ and $1 \leq j \leq m$ then it can be shown similarly that $I(A) + I(B) = A^{-1}$ by letting h be an arbitrary function in A^{-1} and showing that h can be written as a linear combination of functions from $I(A)$ and $I(B)$. Needless to say, the algebra becomes tedious for all but the simplest of examples.

The reason why one can write any function in A^{-1} as a linear combination of functions from $I(A)$ and $I(B)$ (where A and B are finite sequences) is because the sum $I(A) + I(B)$ is not a direct sum, that is, $I(A)$ and $I(B)$ are not independent. To see this using the previous example, note that the function $f(z) = (z - a)(z - b)$ is a non-zero function that is in both $I(a)$ and $I(b)$. Thus, $I(a) \cap I(b) \neq 0$, which by Corollary 3.22 implies that $I(a) + I(b)$ is not a direct sum.

It is now clear that we will have to use infinite sequences $A = \{a_n\}$, $B = \{b_n\}$ to have any hope that $I(A) + I(B)$ will be a strict closed z-invariant subspace of A^{-1}. The following is a result of Kristian Seip which demonstrates that there are vanishing sequences whose union is a sampling sequence.

Theorem 6.9. *(Seip) There exists two sequences $A, B \in \mathbb{D}$ such that*

(1) *no elements of A or B equal zero.*
(2) *$A \cap B = \emptyset$.*
(3) *A and B are both interpolating in A^{-1}.*
(4) *$A \cup B$ is sampling in A^{-1}* [10].

The following lemma demonstrates how this amazing result might be used to overcome the problems we encountered with finite sequences.

Lemma 6.10. *Let A, B be the two sequences guaranteed by Theorem 6.9. Then the sum of $I(A)$ and $I(B)$ is a direct sum, $I(A) \oplus I(B)$.*

Proof. Let $f \in I(A) \cap I(B)$. Then f equals zero when evaluated both at all points of A and at all points of B. Thus, f evaluated at the points of $A \cup B$ equals zero. But since $A \cup B$ is a sampling sequence, then $f \equiv 0$ since $A \cup B$, as a sampling sequence, cannot also be a vanishing sequence, as was demonstrated by Lemma 5.19. Thus, $I(A) \cap I(B) = 0$, which completes the proof. $\square$

For the remainder of this section, A and B will denote the two interpolating sequences guaranteed by Theorem 6.9. Our ultimate goal is to demonstrate that $I(A) \oplus I(B)$ is a Banach space that does not have a codimension with respect to M_z of one. However, first we demonstrate that $I(A) \oplus I(B)$ is indeed z-invariant.

Lemma 6.11. *$I(A) \oplus I(B)$ is z-invariant. Moreover, $z(I(A) \oplus I(B)) = zI(A) \oplus zI(B)$.*

Proof. Let $f \in I(A) \oplus I(B)$. Then $f = f_a + f_b$ where $f_a \in I(A)$ and $f_b \in I(B)$. Thus,

$$zf = z(f_a + f_b) = zf_a + zf_b \tag{6.3}$$

where $zf_a \in I(A)$ and $zf_b \in I(B)$. Thus, $zf \in I(A) \oplus I(B)$, which proves the first part of the lemma.

Next, note that by Lemma 6.10 $I(A) \cap I(B) = 0$. Since $zI(A) \subset I(A)$ and $zI(B) \subset I(B)$, this implies that $zI(A) \cap zI(B) = 0$, so the sum of $zI(A)$ and $zI(B)$ is indeed a direct sum. Furthermore, (6.3) implies that $z(I(A) \oplus I(B)) = zI(A) \oplus zI(B)$. □

Theorem 6.12. *$I(A) \oplus I(B)$ has the codimension-2 property.*

Proof. We will use Theorem 3.28, the first homomorphism theorem, to prove the theorem by demonstrating that there exists a well defined linear transformation $\phi : I(A) \oplus I(B) \to \mathbb{C} \times \mathbb{C}$ such that the kernel K of ϕ is such that $K(\phi) = z(I(A) \oplus I(B))$. So for $f_a \in I(A)$ and $f_b \in I(B)$ let

$$\phi(f_a + f_b) = (f_a(0), f_b(0)).$$

First we show that ϕ is well defined. It is not obvious that for $f_a, f_\alpha \in I(A)$ and $f_b, f_\beta \in I(B)$ where $f_a + f_b = f_\alpha + f_\beta$ then $\phi(f_a + f_b) = \phi(f_\alpha + f_\beta)$. So let $g = f_a - f_\alpha$ and $h = f_b - f_\beta$. Then

$$g + h = (f_a - f_\alpha) + (f_b - f_\beta) = (f_a + f_b) - (f_\alpha + f_\beta) = 0 \tag{6.4}$$

since by hypothesis $f_a + f_b = f_\alpha + f_\beta$. Moreover, note that $g = f_a - f_\alpha \in I(A)$ and $h = f_b - f_\beta \in I(B)$. We proceed to demonstrate that $f_a = f_\alpha$ and $f_b = f_\beta$. Suppose $g = f_a - f_\alpha \neq 0$. Then by (6.4), $-h = f_\beta - f_b = f_a - f_\alpha = g \neq 0$. But this is a contradiction, for it implies that $I(A) \cap I(B) \neq 0$, which by Lemma 6.10 is false. Thus, $f_a - f_\alpha = 0$, or equivalently, $f_a = f_\alpha$. And by (6.4) it then follows that $f_b = f_\beta$. This in turn shows that ϕ is well defined, for it implies that

$$\phi(f_a + f_b) = (f_a(0), f_b(0)) = (f_\alpha(0), f_\beta(0)) = \phi(f_\alpha + f_\beta).$$

Next it must be shown that ϕ is a linear transformation. Let $g, h \in I(A) \oplus I(B)$ and $c \in \mathbb{C}$. Then there exists $f_a, f_\alpha \in I(A)$ and $f_b, f_\beta \in I(B)$ such that $g = f_a + f_b$ and $h = f_\alpha + f_\beta$. Thus,

$$\phi(cg+h) = \phi(c(f_a+f_b)+(f_\alpha+f_\beta)) = \phi((cf_a+f_\alpha)+(cf_b+f_\beta)) = (cf_a(0)+f_\alpha(0), cf_b(0)+f_\beta(0))$$

$$= c(f_a(0), f_b(0)) + (f_\alpha(0), f_\beta(0)) = c\phi(f_a + f_b) + \phi(f_\alpha + f_\beta) = c\phi(g) + \phi(h)$$

which proves that ϕ is a linear transformation.

In order to use the first homomorphism theorem, it is necessary to demonstrate that the range of ϕ is all of $\mathbb{C} \times \mathbb{C}$ and not a strict subset of $\mathbb{C} \times \mathbb{C}$. So given $(c_1, c_2) \in \mathbb{C} \times \mathbb{C}$, we must show that there exists an $g \in I(A)$ and $h \in I(B)$ such that $\phi(g+h) = (g(0), h(0)) = (c_1, c_2)$.

Since by Theorem 6.9 no points of the sequences A or B equal zero, then by simply referring to the definitions of $I(A)$ and $I(B)$ from (6.1) it is clear that there exists $f_a \in I(A)$ and $f_b \in I(B)$ such that $f_a(0) \neq 0$ and $f_b(0) \neq 0$. So suppose $f_a(0) = \lambda_1 \neq 0$ and $f_b(0) = \lambda_2 \neq 0$. Then let $g = \frac{c_1}{\lambda_1} f_a \in I(A)$ and $h = \frac{c_2}{\lambda_2} f_b \in I(B)$. This implies that $g(0) = \frac{c_1}{\lambda_1} f_a(0) = c_1$ and $h(0) = \frac{c_2}{\lambda_2} f_b(0) = c_2$. Thus, $\phi(g+h) = (c_1, c_2)$, which proves that the range of ϕ is all of $\mathbb{C} \times \mathbb{C}$.

At this point we know that $I(A) \oplus I(B)/K(\phi) \approx \mathbb{C} \times \mathbb{C}$. It remains to be shown that the kernel of ϕ is equal to $z(I(A) \oplus I(B))$. So let $f \in K(\phi)$. Then there exists $f_a \in I(A)$ and $f_b \in I(B)$ such that $f = f_a + f_b$ and $\phi(f) = (f_a(0), f_b(0)) = (0,0)$, which implies that $f_a(0) = 0$ and $f_b(0) = 0$. But by Lemma 6.6 we know that $zI(A) = \{g \in I(A) : g(0) = 0\}$ and similarly $zI(B) = \{h \in I(B) : h(0) = 0\}$. Thus, $f_a \in zI(A)$ and $f_b \in zI(B)$. This implies that $f \in zI(A) \oplus zI(B)$, and therefore by Lemma 6.11, $f \in z(I(A) \oplus I(B))$, which demonstrates that $K(\phi) \subset z(I(A) \oplus I(B))$.

Conversely, suppose $f \in z(I(A) \oplus I(B))$. Again, by Lemma 6.11, this means that

$$f \in zI(A) \oplus zI(B) = \{g \in I(A) : g(0) = 0\} \oplus \{h \in I(B) : h(0) = 0\}.$$

Thus, there exists $g \in zI(A)$ and $h \in zI(B)$ such that $f = g + h$ whereby $g(0) = h(0) = 0$. Thus, $\phi(f) = \phi(g+h) = (g(0), h(0)) = (0,0)$ which shows that $f \in K(\phi)$. Therefore, $K(\phi) = z(I(A) \oplus I(B))$.

Since by Example 3.29 we know that $\mathbb{C} \times \mathbb{C}$ has a dimension of two, this suffices to prove that $I(A) \oplus I(B)$ has the codimension-2 property. $\square$

Corollary 6.13. $I(A) \oplus I(B) \neq A^{-1}$.

Proof. Since by Lemma 6.3, A^{-1} has a codimension with respect to z of one, then it clearly cannot be identical to $I(A) \oplus I(B)$. Thus, $I(A) \oplus I(B)$ must be a strict subspace of A^{-1}. $\square$

Having found a strict subspace of A^{-1} with a codimension of two, we have accomplished our goal. As an added bonus, it is not too difficult to show that $I(A) \oplus I(B)$ is a Banach space.

Theorem 6.14. *$I(A) \oplus I(B)$ is closed.*

Proof. Let $g \in I(A)$ and $h \in I(B)$. Then there exists a $c > 0$ independent of g such that

$$\|g\| = \sup_{z \in \mathbb{D}} (1 - |z|)|g(z)| \leq c \sup_{z \in A \cup B} (1 - |z|)|g(z)|$$

since $A \cup B$ is a sampling sequence. But since $g \in I(A)$, this implies that g vanishes on all points in the interpolating sequence A. Thus,

$$c \sup_{z \in A \cup B} (1 - |z|)|g(z)| = c \sup_{z \in B} (1 - |z|)|g(z)|.$$

Moreover, since $h \in I(B)$, this implies that h vanishes on all points in the interpolating sequence B, and therefore

$$c\sup_{z\in B}(1-|z|)|g(z)| = c\sup_{z\in B}(1-|z|)|g(z)+h(z)|$$

since adding h in this context is adding zero. But clearly

$$c\sup_{z\in B}(1-|z|)|g(z)+h(z)| \leq c\sup_{z\in \mathbb{D}}(1-|z|)|g(z)+h(z)|$$

since $\mathbb{D}$ is a larger set than B. Thus,

$$\|g\| \leq c\|g+h\|. \tag{6.5}$$

Similarly for h,

$$\|h\| \leq c \sup_{z\in A\cup B}(1-|z|)|h(z)|$$

where this c is the same as that used above since $A \cup B$ is the same sampling sequence. By the same argument as that just offered,

$$c\sup_{z\in A\cup B}(1-|z|)|h(z)| = c\sup_{z\in A}(1-|z|)|h(z)|$$
$$= c\sup_{z\in A\cup B}(1-|z|)|g(z)+h(z)| \leq c\sup_{z\in \mathbb{D}}(1-|z|)|g(z)+h(z)|$$

which demonstrates that

$$\|h\| \leq c\|g+h\|.$$

Together with (6.5), this implies

$$\|g\|+\|h\| \leq 2c\|g+h\| \Rightarrow \|g+h\| \geq \frac{1}{2c}(\|g\|+\|h\|). \tag{6.6}$$

Let $\{f_n\}$ be a Cauchy sequence in $I(A)\oplus I(B)$. Clearly, since A^{-1} is a Banach space, then $f_n \to f \in A^{-1}$. It must be shown that $f \in I(A)\oplus I(B)$. Now there exists $\{g_n\} \in I(A)$ and $\{h_n\} \in I(B)$ such that $\{f_n\} = \{g_n + h_n\}$. Thus, we must demonstrate that

$$g_n + h_n \to g + h = f \in I(A)\oplus I(B).$$

Given $\epsilon > 0$ there exists $N > 0$ such that for all $m, n \geq N$,

$$\|f_n - f_m\| = \|(g_n+h_n)-(g_m+h_m)\| = \|(g_n-g_m)+(h_n-h_m)\| < \frac{\epsilon}{2c} \tag{6.7}$$

where we may choose this c to be the same as that used above. But by (6.6),

$$\|(g_n-g_m)+(h_n-h_m)\| \geq \frac{1}{2c}(\|g_n-g_m\|+\|h_n-h_m\|)$$

or equivalently,

$$\|g_n-g_m\|+\|h_n-h_m\| \leq 2c\|(g_n-g_m)+(h_n-h_m)\| < 2c\frac{\epsilon}{2c} = \epsilon$$

by (6.7). This clearly implies that $\|g_n - g_m\| < \epsilon$ and $\|h_n - h_m\| < \epsilon$, thus demonstrating that $\{g_n\}$ and $\{h_n\}$ are both individually Cauchy. But since, by Lemma 6.6, both $I(A)$ and

$I(B)$ are Banach spaces, this implies that $\{g_n\}$ is Cauchy in $I(A)$ and $\{h_n\}$ is Cauchy in $I(B)$, or equivalently, $g_n \to g \in I(A)$ and $h_n \to h \in I(B)$. And therefore,

$$f_n = g_n + h_n \to g + h = f \in I(A) \oplus I(B)$$

which completes the proof. □

References

1. A. Aleman, S. Richter, and W. T. Ross, 'Pseudocontinuations and the backwards shift', preprint.
2. J.M. Anderson, J. Clunie, and C. Pommerenke, 'On Bloch functions and normal functions', J. Reine Angew. Math. **270** (1974), 12-37.
3. A. Beurling, 'On two problems concerning linear transformations in Hilbert space', Acta Math. **81** (1949), 239-255.
4. J.B. Conway, *Functions of one Complex Variable*, Springer-Verlag, New York, 1978.
5. H. Hedenmalm, 'An invariant subspace of the Bergman space having the codimension two property', J. Reine Angew. Math. **443** (1993), 1-9.
6. K. Hoffman and R. Kunze, *Linear Algebra*, Prentice-Hall, Englewood Cliffs, NJ, 1971.
7. H.G. Moore and A. Yaqub, *A First Course In Linear Algebra*, Harper-Collins, New York, 1992.
8. S. Richter and A. Shields, 'Bounded analytic functions in the Dirichlet space', Math. Z. **198** (1988), 151-159.
9. E.B. Saff and A.D. Snyder, *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering*, Prentice Hall, Englewood Cliffs, NJ, 1993.
10. K. Seip, 'Beurling type density theorems in the unit disk', Invent. Math. **113** (1993), 21-39.
11. A. Shields, 'Weighted shift operators and analytic function theory', in *Topics In Operator Theory*, C. Pearcy, ed., Mathematical Surveys, Volume 13, AMS, Providence, RI, 1974.

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