BANACH SPACES WHOSE DUALS ARE L_1 SPACES AND THEIR REPRESENTING MATRICES

BY

A. J. LAZAR and J. LINDENSTRAUSS

University of Washington, Seattle, Wash. and Louisiana State University, Baton Rouge, La., U.S.A. Hebrew University, Jerusalem, Israel, and University of California, Berkeley, Calif., U.S.A.

1. Introduction

It is a matter of general agreement that the $L_p(\mu)$ spaces ($1 \leqslant p \leqslant \infty$ and μ a measure) and the $C(K)$ spaces (K compact Hausdorff) are among the most important Banach spaces. A central part of Banach space theory is devoted to the investigation of the special properties of these spaces and some closely related spaces. This part of Banach space theory is often called the theory of the classical Banach spaces. It is our feeling that in order to get a well rounded theory of the classical Banach spaces, in the framework of the isometric theory, it is worthwhile to take as the main objects of the investigation the class of Banach spaces X for which $X^* = L_p(\mu)$ for some $1 \leqslant p \leqslant \infty$ and some measure μ. Let us examine briefly the relation of this latter class of spaces to those mentioned in the first sentence. Since for $1 < p < \infty$ the $L_p(\mu)$ spaces are reflexive it is clear that $X^* = L_p(\mu)$ if and only if $X = L_q(\mu)$ ($p^{-1} + q^{-1} = 1$). Grothendieck [6] proved the non obvious fact that if $X^* = L_\infty(\mu)$ then $X = L_1(\mu)$. Well-known results of F. Riesz and Kakutani show that if $X = C(K)$ then $X^* = L_1(\mu)$ for a suitable μ. There are, however, Banach spaces X which are not isometric to $C(K)$ spaces while their duals are $L_1(\mu)$ spaces. These are thus the only spaces which should be included in the geometric theory of the classical Banach spaces and which are not "classical" in the strict sense.

The most important geometric properties of the Banach spaces $C(K)$ are shared exactly by the class of all spaces whose duals are $L_1(\mu)$ spaces. Examples of such properties are the extension properties for compact operators which were studied in [14]. The $C(K)$ spaces are singled out from all the spaces whose duals are $L_1(\mu)$ mainly by the fact that they have natural additional structure as algebras or vector lattices. (There is, though, also a pure Banach space theoretic property which singles out the $C(K)$ spaces among the general preduals of L_1, cf. section 4 below.) Another fact which shows that the extension of the notion of classical spaces which we use, is a natural one, is the following (cf. [16] and [12] and their references): Let X be a separable Banach space. Then X^* is an $L_p(\mu)$ space if and only if X can be represented as $X=\overline{\bigcup_{n=1}^{\infty} E_n}$ where $E_1 \subset E_2 \subset E_3 \ldots$ and for every n, E_n is isometric to the n-dimensional L_q space (i.e. $E_n = l_q^n$ in the usual notation) where $p^{-1}+q^{-1}=1$ ($q=1$ resp. ∞ if $p=\infty$ resp. 1). Let us mention in passing that for the isomorphic (rather than isometric) theory of Banach spaces there is a different natural setting for the study of the classical Banach spaces namely that of the $\mathcal{L}_p$ spaces which were introduced in [16]. In the present paper we shall be concerned only with the isometric theory of the classical spaces and more precisely with the study of those spaces which are not strictly "classical" i.e. those whose duals are L_1 spaces.

Several subclasses of the class of spaces whose duals are L_1 spaces (besides the $C(K)$ spaces) have already been studied extensively in the literature and were found to be of importance in other areas of mathematical analysis. This in particular is the case for those spaces which can be ordered in a way compatible with the duality and the natural order of $L_1(\mu)$ i.e. the simplex spaces (see section 4 below for details). Most of the results which were proved for simplex spaces can be extended to the general case of spaces whose duals are L_1 spaces. In section 2 we extend the separation theorem of Edwards [3] and the selection theorem of the first named author [10] to this general setting. Among the corollaries of the selection theorem is the result that every separable Banach space X whose dual is a nonseparable $L_1(\mu)$ space contains a subspace isometric to $C(K)$ with K the Cantor set and hence contains a copy of every separable Banach space. It seems to us that the setting of section 2 is the natural one for the separation and selection theorems in Banach space theory. This setting is probably also the natural one in other contexts. For example the difficulties encountered in the theory of polytopes of Alfsen [1] and Phelps [20] seem to stem from the fact that their starting point was the Choquet simplexes instead of unit balls of $L_1(\mu)$ in a w^* compact topology in which the positive cone is not necessarily closed. It is also very likely that general preduals of $L_1(\mu)$ will find a natural role in areas like potential theory or C^* algebras which will generalize the present role of Choquet simplexes in these areas.

In section 3 the generalized separation theorem of section 2 is used in proving a theorem which characterizes preduals of $L_1(\mu)$ by the structure of their finite-dimensional subspaces. (A weaker version of this result was proved in [12].) For separable infinite-dimensional preduals of $L_1(\mu)$ the characterization theorem implies easily the existence of a representation of the type mentioned already above. Every such space X can be written as $\overline{\bigcup_{n=1}^{\infty} E_n}$ where $E_n \subset E_{n+1}$ and $E_n = l_\infty^n$ (the space of n-tuples $x=(\lambda_1, \ldots, \lambda_n)$ with

$\|x\| = \max |\lambda_i|$) for every n. We thus obtain a different and more transparent proof of the results of Michael and Pełczyński [19].

Section 4, which is to a large extent an introduction to section 5, is devoted to the question of functional representation of the preduals of $L_1(\mu)$. We give there the definitions of the main classes of spaces whose duals are L_1 spaces and recall some results of [17]. We also make there some comments on the relation between simplex spaces and the general preduals of $L_1(\mu)$.

Every representation of a space X as $\bigcup_{n=1}^{\infty} E_n$ with $E_n \subset E_{n+1}$ and $E_n = l_\infty^n$ gives rise in a natural way to a triangular matrix $A = \{a_{i,n}\}_{n=1,2,3,\dots}^{1 \le i \le n}$ with $\Sigma_{i=1} |a_{i,n}| \le 1$ for every n. We can thus associate with every infinite-dimensional separable predual of an L_1 space a class of such traingular matrices called the *representing matrices* of X. Conversely every such matrix A is a representing matrix for a uniquely defined space whose dual is an L_1 space. Section 5 is devoted to the study of some special examples, as well as the proof of some consequences, of this correspondence between preduals of $L_1(\mu)$ and their representing matrices. We show in particular that every metrizable infinite-dimensional Choquet simplex is an inverse limit of a system $\Delta_1 \xleftarrow{\varphi_1} \Delta_2 \xleftarrow{\varphi_2} \Delta_3 \longleftarrow \dots$ where for every n, Δ_n is an n- dimensional simplex, and φ_n is a surjective affine map. We also give a simple proof of the existence of a space constructed first by Gurariĭ [7].

2. Separation and selection theorems

Let us first introduce some definitions. The unit ball of a Banach space X is denoted by $B(X)$. A face F of $B(X^*)$ will be said to be essentially w^*-closed if conv $(F \cup -F)$ is w^*-closed. Obviously a w^*-closed face is essentially w^*-closed and simple examples show that the converse is false. A subset H of $B(X^*)$ is called a facial section if there is a face F of $B(X^*)$ such that $H = \text{conv}\,(F \cup -F)$. A real or vector valued function f defined on a symmetric (with respect to 0) set in a linear space is said to be symmetric if $f(-x) = -f(x)$ for every x in the domain of definition of f. All the Banach spaces we consider in this paper will be over the reals.

LEMMA 2.1. *Let X be a Banach space whose dual is an L_1 space. Let F be an essentially w^*-closed face of $B(X^*)$. Then the linear subspace V of X^* algebraically spanned by F is w^*-closed and $V \cap B(X^*) = \text{conv}\,(F \cup -F)$.*

Proof. By a classical result [cf. 2, p. 43] the first assertion is a consequence of the second. Clearly we have to prove only that $V \cap B(X^*) = \text{conv}\,(F \cup -F)$.

Let G be a maximal proper face of $B(X^*)$ containing F. We order X^* by taking as the positive cone the cone generated by G. With this order X^* is an abstract L space (i.e. order isomorphic and isometric to $L_1(\mu)$ for some μ). Indeed, we assumed that X^* is isometric to $L_1(\nu)$ for some measure ν defined on a measure space Ω. We may assume that ν is well behaved in the sense that $X^{**}=L_\infty(\nu)$. Since G is a maximal face of $B(X^*)$ there is an extreme point ψ of $B(X^{**})$ such that $G=B(X^*)\cap\{x^*; \psi(x^*)=1\}$. Since $\psi\in\operatorname{ext} B(X^{**})$, $|\psi(t)|=1$ a.e. Define the measure μ on Ω by $d\mu=\psi d\nu$. It is clear that the positive cone of $L_1(\mu)$ is the cone generated by G. Let $x^*\in V\cap B(X^*)$, $x^*\neq 0$. Then there are x_1^*, $x_2^*\in F$ and α_1, $\alpha_2\geqslant 0$ such that $x^*=\alpha_1x_1^*-\alpha_2x_2^*$. The case $\alpha_1\cdot\alpha_2=0$ can be easily dealt with so we assume $\alpha_1\cdot\alpha_2>0$. It follows that $x^{*+}=x^*\vee 0\leqslant\alpha_1x_1^*$ i.e. $\alpha_1x_1^*=x^{*+}+\alpha y^*$ with $y^*\in G$ and $\alpha\geqslant 0$. Suppose that $x^{*+}\neq 0$. Then $x^{*+}/\|x^{*+}\|\in G$ and $x_1^*=(\|x^{*+}\|/\alpha_1)(x^{*+}/\|x^{*+}\|)+(\alpha/\alpha_1)y^*$.

Since X^* is an L space, $\alpha_1=\alpha+\|x^{*+}\|$ and hence $x^{*+}/\|x^{*+}\|\in F$ (recall that F is a face of $B(X^*)$). Similarly if $x^{*-}=-(x^*\wedge 0)\neq 0$ then $x^{*-}/\|x^{*-}\|\in F$. Thus if $x^{*+}\neq 0$ and $x^{*-}\neq 0$ we have $x^*=\|x^{*+}\|(x^{*+}/\|x^{*+}\|)+\|x^{*-}\|(x^{*-}/\|x^{*-}\|)\in\operatorname{conv}(F\cup -F)$ (recall that $x^*\in B(X^*)$ i.e. $\|x^{*+}\|+\|x^{*-}\|\leqslant 1$). The inclusion relation obviously holds also if either x^{*+} or x^{*-} are 0 and this proves the lemma.

We shall now prove the generalization of Edwards separation theorem [3] to our present setting.

THEOREM 2.1. *Let X be a Banach space with $X^*=L_1(\mu)$ for some μ. Let g: $B(X^*)\to(-\infty,\infty]$ be a concave w^*-lower semicontinuous function satisfying*

$$g(x^*)+g(-x^*)\geqslant 0,\quad x^*\in B(X^*). \tag{2.1}$$

Let F be an essentially w^*-closed face of $B(X^*)$ and assume that f is a w^*-continuous affine symmetric real-valued function on $H=\operatorname{conv}(F\cup -F)$ such that $f\leqslant g|_H$. Then there exists a w^*-continuous affine symmetric extension of f to a function h on $B(X^*)$ such that $h\leqslant g$.

Remark. Clearly, a function h on $B(X^*)$ is w^*-continuous affine and symmetric if and only if $h(x^*)=x^*(x)$ for some $x\in X$.

Proof. It is easy to see that without loss of generality we may assume that g is finite and bounded from above on $B(X^*)$. We shall prove first the existence of h under the assumption that for some $\varepsilon>0$

$$g(x^*)+g(-x^*)\geqslant 2\varepsilon,\quad x^*\in B(X^*)\quad\text{and}\quad g(x^*)\geqslant f(x^*)+\varepsilon,\ x^*\in H. \tag{2.2}$$

We may also assume that g is continuous. Indeed, let f' be a continuous extension of f to $B(X^*)$ with $g(x^*)>f'(x^*)>-g(-x^*)$ and put $f_1(x^*)=\frac{1}{2}(f'(x^*)-f'(-x^*))$. Then f_1 is continuous symmetric on $B(X^*)$ and $f_1(x^*)<g(x^*)$, $x^*\in B(X^*)$. By a result of Mokobodzki

[5, Lemma 5.2] and a routine compactness argument we may find a continuous concave function g_1 on $B(X^*)$ such that $g_1 \leqslant g$ and $f_1(x^*) < g_1(x^*)$, $x^* \in B(X^*)$. Clearly g_1 satisfies (2.2) for some ε'.

Thus, let g be continuous concave and assume (2.2) holds for it. Let G be a maximal proper face of $B(X^*)$ which contains F and order X^* by taking the cone generated by G as the positive cone. Let V be the subspace of X^* spanned by F. The function f admits a w^*-continuous linear extension to V which we shall also denote by f. (Observe that f can be represented by an element of space $X/V^\perp$ where $V^\perp$ is the subspace of X orthogonal to V). Consider the following sets in $X^* \times R$ (topologized by the product topology with X^* taken in the w^*-topology).

$$A_1 = \{(x^*, g(x^*));\, x^* \in B(X^*)\}$$

$$A_2 = \{(x^*, f(x^*));\, x^* \in V\}.$$

We want to prove that the closed convex hull of A_1 is disjoint from A_2. From Lemma 9.6 in [21] it follows that for any $x^* \in B(X^*)$ one has

$$\inf \{r\colon (x^*, r) \in \overline{\operatorname{conv}}\,(A_1)\} = \inf \{r\colon (x^*, r) \in \operatorname{conv}\,(A_1)\}.$$

It is enough then to show that if $(x^*, r) \in \operatorname{conv}(A_1)$ and $x^* \in H$ then $r \geqslant \varepsilon + f(x^*)$.

Let $x^* \in H$ and $(x^*, r) \in \operatorname{conv}(A_1)$. Then there are $\{x^*\}_{i=1}^n$ in $B(X^*)$ and $\{\mu_i\}_{i=1}^n$ with $\mu_i \geqslant 0$, $\Sigma\mu_i = 1$, $x^* = \Sigma_{i=1}^n \mu_i x_i^*$ and $r = \Sigma_{i=1}^n \mu_i g(x_i^*)$. Since $B(X^*) = \operatorname{conv}(G \cup -G)$ and $H = \operatorname{conv}(F \cup -F)$ there are $y_i^*, z_i^* \in G$, $y^*, z^* \in F$ and $1 \geqslant \lambda_i, \lambda \geqslant 0$ such that $x_i^* = \lambda_i y_i^* + (1-\lambda_i)(-z_i^*)$, $i = 1, 2, \ldots n$, and $x^* = \lambda y^* + (1-\lambda)(-z^*)$. It follows that

$$\sum_{i=1}^n \mu_i \lambda_i y_i^* + (1-\lambda) z^* = \sum_{i=1}^n \mu_i (1-\lambda_i) z_i^* + \lambda y^*, \tag{2.3}$$

and by the fact that g is concave,

$$\sum_{i=1}^n \mu_i (1-\lambda_i) g(-z_i^*) + \sum_{i=1}^n \mu_i \lambda_i g(y_i^*) \leqslant r. \tag{2.4}$$

From (2.3) and the decomposition lemma for vector lattices (cf. e.g. [21, Lemma 9.1]) there are $u_{i,j}^* \in G$ and $\alpha_{i,j} \geqslant 0$, $i, j = 1, 2, \ldots, n+1$ such that

$$\mu_i \lambda_i y_i^* = \sum_{j=1}^{n+1} \alpha_{i,j} u_{i,j}^*, \quad 1 \leqslant i \leqslant n \tag{2.5}$$

$$\mu_j (1-\lambda_j) z_j^* = \sum_{i=1}^{n+1} \alpha_{i,j} u_{i,j}^*, \quad 1 \leqslant j \leqslant n \tag{2.6}$$

$$\lambda y^* = \sum_{i=1}^{n+1} \alpha_{i,n+1} u^*_{i,n+1}, \quad (1-\lambda) z^* = \sum_{j=1}^{n+1} \alpha_{n+1,j} u^*_{n+1,j}. \tag{2.7}$$

From these equations and the additivity of the norm on the non-negative elements in X^* we get

$$\mu_i \lambda_i = \sum_{j=1}^{n+1} \alpha_{i,j}, \quad \mu_j(1-\lambda_j) = \sum_{i=1}^{n+1} \alpha_{i,j}, \tag{2.8}$$

$$\lambda = \sum_{i=1}^{n+1} \alpha_{i,n+1}, \quad 1-\lambda = \sum_{j=1}^{n+1} \alpha_{n+1,j}. \tag{2.9}$$

From (2.5), (2.6) and (2.8) it follows that

$$\mu_i \lambda_i g(y^*_i) \geqslant \sum_{j=1}^{n+1} \alpha_{i,j} g(u^*_{i,j}), \quad 1 \leqslant i \leqslant n,$$

$$\mu_j(1-\lambda_j) g(-z^*_j) \geqslant \sum_{i=1}^{n+1} \alpha_{i,j} g(-u^*_{i,j}), \quad 1 \leqslant j \leqslant n.$$

By (2.2) and (2.4)

$$r \geqslant \sum_{i=1}^{n} \sum_{j=1}^{n+1} \alpha_{i,j} g(u^*_{i,j}) + \sum_{j=1}^{n} \sum_{i=1}^{n+1} \alpha_{i,j} g(-u^*_{i,j}) \geqslant 2\varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{i,j} = \sum_{i=1}^{n} \alpha_{i,n+1} g(u^*_{i,n+1}) + \sum_{j=1}^{n} \alpha_{n+1,j} g(u^*_{n+1,j})$$

$$\geqslant 2\varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{i,j} + \sum_{i=1}^{n+1} \alpha_{i,n+1} f(u^*_{i,n+1}) + \varepsilon \sum_{i+1}^{n} \alpha_{i,n+1} - \sum_{j=1}^{n+1} \alpha_{n+1,j} f(u^*_{n+1,j}) + \varepsilon \sum_{j=1}^{n} \alpha_{n+1,j}.$$

(We used in the last step also the fact that by (2.7) and the fact that F is a face of $B(X^*)$, $u^*_{i,n+1}$ and $u^*_{n+1,j}$ belong to F whenever $\alpha_{i,n+1} \neq 0$ resp. $\alpha_{n+1,j} \neq 0$.) Hence by (2.7), (2.8) and (2.9)

$$r \geqslant \lambda f(y^*) - (1-\lambda) f(z^*) + \varepsilon \left(\sum_{i=1}^{n} \lambda_i \mu_i + \sum_{j=1}^{n} (1-\lambda_j) u_j \right) = f(x^*) + \varepsilon,$$

and this proves our assertion concerning $\overline{\operatorname{conv} A_1}$. The linear manifold A_2 is closed by Lemma 2.1 and $\overline{\operatorname{conv} A_1}$ is compact. Thus there is a closed hyperplane C in $X^* \times R$ such that $C \cap \overline{\operatorname{conv} A_1} = \phi$ and $A_2 \subset C$. We define now h on $B(X^*)$ by the requirement that $(x^*, h(x^*)) \in C$, $x^* \in B(X^*)$. It is obvious that this h has all the required properties.

We pass now to the proof of the theorem in the general case (i.e. if we do not assume (2.2)). The technique for doing this is similar to that used by Edwards in [3]. Let f and g be as in the statement of the theorem.

By the preceding argument there is a w^*-continuous h_1 on X^* such that $h_1 |_{B(X^*)} \leqslant g + \frac{2}{3}$

and $h_1|_H=f$. Assume that we have found $\{h_i\}_{i=1}^n$, all w^*-continuous linear extensions of f so that

$$h_i|_{B(X^*)} \leqslant g+(\tfrac{2}{3})^i \quad i=1,2,\dots,n$$

$$\|h_i-h_{i-1}\| \leqslant (\tfrac{2}{3})^{i+1} \quad i=2,3,\dots,n.$$

Let $g_{n+1}=\min[h_n+(\frac{2}{3})^{n+1},\ g+(\frac{2}{3})^{n+1}]$; then g_{n+1} is concave, w^*-lower semicontinuous, and $g_{n+1}\geqslant f+(\frac{2}{3})^{n+1}$ on H. Furthermore, $g_{n+1}(x^*)+g_{n+1}(-x^*)\geqslant \frac{1}{3}(\frac{2}{3})^n$ for x^* in $B(X^*)$. Indeed, if $h_n(x^*)\leqslant g(x^*)$ and $h_n(-x^*)\leqslant g(-x^*)$, then $g_{n+1}(x^*)+g_{n+1}(-x^*)=(\frac{2}{3})^{n+1}$ while if $h_n(x^*)>g(x^*)$, then $g(-x^*)\geqslant -g(x^*)$ and $|h_n(x^*)-g(x^*)|\leqslant(\frac{2}{3})^n$, hence $g_{n+1}(x^*)+g_{n+1}(-x^*)\geqslant \min[h_n(x^*), g(x^*)]-\max[h_n(x^*), g(x^*)]+2(\frac{2}{3})^{n+1}=2(\frac{2}{3})^{n+1}-|h_n(x^*)-g(x^*)|\geqslant\frac{1}{3}(\frac{2}{3})^n$. A similar argument applies if $h_n(-x^*)>g(-x^*)$. Thus g_{n+1} satisfies (2.2) for some $\varepsilon>0$ and hence there exists a w^*-continuous linear extension h_{n+1} of f such that $h_{n+1}|_{B(X^*)}\leqslant g+(\frac{2}{3})^{n+1}$ and, for x^* in $B(X^*)$, we have $h_n(x^*)+(\frac{2}{3})^{n+1}\geqslant h_{n+1}(x^*)=-h_{n+1}(-x^*)\geqslant h_n(x^*)-(\frac{2}{3})^n$, i.e. $\|h_{n+1}-h_n\|\leqslant(\frac{2}{3})^{n+1}$. The sequence $\{h_i\}_{i=1}^\infty$ converges in norm to an h which has all the desired properties.

Remark. From the first part of the proof it follows that if g satisfies (2.2) for a suitable $\varepsilon>0$ then h can be chosen such that $g(x^*)>h(x^*)$ for every $x^*\in B(X^*)$.

We pass now to the proof of a selection theorem for certain set valued maps defined on $B(X^*)$. This theorem is a generalization of [10, Theorem 3.1] which in turn partially generalized a selection theorem of Michael [18]. The proof given here is a modification of Léger's [13] simpler proof of [10, Theorem 3.1]. First we need some additional notations. If E is a locally convex space we denote by $c(E)$ the set of all convex non empty subsets of E and by $\bar{c}(E)$ the set of all closed sets in $c(E)$. A map φ from a convex subset C of a linear space into $c(E)$ is called convex if

$$\lambda\varphi(x_1)+(1-\lambda)\varphi(x_2)\subset\varphi(\lambda x_1+(1-\lambda)x_2),\quad 0\leqslant\lambda\leqslant 1,\quad x_1,x_2\in C.$$

The map φ is said to be lower semicontinuous if

$$\{x;\ \varphi(x)\cap U\neq\emptyset\}\text{ is open for every open } U \text{ in } E.$$

We say that φ is symmetric if $-\varphi(x)=\varphi(-x)$ whenever $x, -x\in C$. By a selection for φ we mean a map $f\colon C\to E$ such that $f(x)\in\varphi(x)$ for every $x\in C$.

THEOREM 2.2. *Let X be a Banach space such that X^* is an $L_1(\mu)$ space and let E be a Frechét space. Let $\varphi\colon B(X^*)\to\bar{c}(E)$ be a convex symmetric w^*-lower semicontinuous map. Then φ admits a w^*-continuous affine symmetric selection h. Moreover, if F is an essentially closed* *face of* $B(X^*)$, $H = \text{conv}\,(F \cup -F)$ *and* $f: H \to E$ *a* w^**-continuous affine symmetric selection of* $\varphi|_H$ *then the selection* h *can be chosen so that* $h|_H = f$.

The proof of Theorem 2.2 is based on the following result which ensures the existence of approximate selections.

LEMMA 2.2. *Let* X *be a Banach space with* $X^* = L_1(\mu)$, *let* E *be a locally convex space and let* $\varphi: B(X^*) \to c(E)$ *be convex symmetric and* w^**-lower semicontinuous. Let* F, H *and* f *be as in the statement of Theorem 2.2. Then for every neighborhood* U *of the origin in* E *there is a* w^**-continuous symmetric affine* $h: B(X^*) \to E$ *such that* $h(x^*) \in \varphi(x^*) + U$, $x^* \in B(X^*)$ *and* $h(x^*) - f(x^*) \in U$ *whenever* $x^* \in H$.

Proof. We shall first prove the lemma if $\dim E < \infty$ by induction on the dimension. Assume that $E = R$ i.e. $\dim E = 1$ and let U be a symmetric open interval in R. Define

$$g(x^*) = \sup\,(\varphi(x^*) + U), \quad x^* \in B(X^*).$$

Then g is a w^*-lower semicontinuous concave function which satisfies (2.2) for a suitable $\varepsilon > 0$. The existence of a suitable h follows from Theorem 2.1 and the remark following its proof.

Assume now that the lemma is valid for R^n. Let $E = R^{n+1} = R \times R^n$ and let p, q be the canonic projections of E onto R and R^n respectively. Let U_p and U_q be symmetric neighborhoods of the origin in R and R^n respectively so that $U_p \times U_q$ is contained in the given neighborhood U in E. By Theorem 2.1 and the preceding argument there is a w^*-continuous affine symmetric function $k: B(X^*) \to R$ which extends $p \circ f$ and for which $k(x^*) \in p \circ \varphi(x^*) + \tfrac{1}{2} U_p$ for every $x^* \in B(X^*)$. The map $\tilde{\varphi}: B(X^*) \to c(E)$ defined by $\tilde{\varphi}(x^*) = p^{-1}(k(x^*) + \tfrac{1}{2} U_p)$ is convex symmetric and w^*-lower semicontinuous. Moreover, its graph, that is the set $\{x^*, p^{-1}(k(x^*) + \tfrac{1}{2} U_p));\, x^* \in B(X^*)\}$ is open in $B(X^*) \times R^{n+1}$. It is easy to check (cf. [5, Lemma 8.2]) that the map $\psi: B(X^*) \to c(E)$, defined by

$$\psi(x^*) = p^{-1}(k(x^*) + \tfrac{1}{2} U_p) \cap \varphi(x^*)$$

is convex symmetric and w^*-lower semicontinuous. The same properties are shared by the map $q \circ \psi: B(X^*) \to c(R^n)$, and $q \circ f$ is a selection of $q \circ \psi|_H$. From the induction hypothesis it follows that there exists a w^*-continuous affine symmetric selection τ of $x^* \to q \circ \psi(x^*) + U_q$ such that $\tau(x^*) \in q \circ f(x^*) + U_q$ for $x^* \in H$. It is easy to check that $h: B(X^*) \to E = R \times R^n$ defined by $h(x^*) = (k(x^*), \tau(x^*))$ has all the desired properties.

We pass now to the general case. We assume as we clearly may that U is an absolutely convex neighborhood of 0 in E. For $y \in E$ define

$$G_y = \{x^* \in B(X^*);\, y \in \varphi(x^*) + \tfrac{1}{2}U\}$$

$$G'_y = \{x^* \in H;\, y \in f(x^*) + \tfrac{1}{4}U\}.$$

Since φ is w^*-lower semicontinuous and f is w^*-continuous, $\{G_y\}_{y\in E}$ and $\{G'_y\}_{y\in E}$ are open coverings of $B(X^*)$ and H respectively. Hence there are $\{y_i\}_{i=1}^n$ in E such that $B(X^*) = \bigcup_{i=1}^n G_{y_i}$ and $H = \bigcup_{i=1}^n G'_{y_i}$. Consider the maps ψ_1: $B(X^*) \to c(R^n)$ and ψ_2: $H \to c(R^n)$ defined by

$$\psi_1(x^*) = \{\lambda = (\lambda_i) \in R^n;\, \Sigma_{i=1}^n \lambda_i y_i \in \varphi(x^*) + \tfrac{1}{2}U\}$$

and
$$\psi_2(x^*) = \{\lambda = (\lambda_i) \in R^n;\, \Sigma_{i=1}^n \lambda_i y_i \in f(x^*) + \tfrac{1}{4}U\}.$$

It is easy to check that both maps are convex symmetric and w^*-lower semicontinuous. Let W be an absolutely convex neighbourhood of the origin in R^n such that $(\mu_i) \in W \to \Sigma_{i=1}^n \mu_i y_i \in \frac{1}{4}U$. By the first part of the proof the map $x^* \to \psi_2(x^*) + W$ of H into $c(R^n)$ admits a w^*-continuous affine symmetric selection k (note that by Lemma 2.1 and elementary properties of L_1 spaces the subspace V spanned by H is a dual L_1 space). Clearly $k(x^*) \in \psi_1(x^*)$ for $x^* \in H$. Applying again the finite-dimensional case, which we already established, we get a w^*-continuous symmetric affine $h' = (h'_1, \dots, h'_n)$: $B(X^*) \to R^n$ such that $h'(x^*) \in \psi_1(x^*) + W$, $x^* \in B(X^*)$ and $h'(x^*) - k(x^*) \in W$ for $x^* \in H$. It is easy to check that $h(x^*) = \Sigma_{i=1}^n h'_i(x^*) y_i$ has all the desired properties.

Proof of Theorem 2.2. Clearly it suffices to prove only the second assertion of the theorem. Let $\{p_n\}_{n=1}^\infty$ be an increasing sequence of seminorms on E which determines its topology. For $y \in E$ and $r > 0$ put $B_n(y, r) = \{z \in E: p_n(y, z) < r\}$. By repeated use of Lemma 2.2 we construct a sequence $\{h_n\}_{n=1}^\infty$ of w^*-continuous affine symmetric functions from $B(X^*)$ to E so that

$$h_1(x^*) \in \varphi(x^*) + B_1(0, \tfrac{1}{2}), \quad x^* \in B(X^*);$$

$$h_1(x^*) - f(x^*) \in B_1(0, \tfrac{1}{2}), \quad x^* \in H$$

and for $n > 1$

$$h_n(x^*) \in \varphi(x^*) \cap B_{n-1}(h_{n-1}(x^*), 2^{-n+1}) + B_n(0, 2^{-n}), \; x^* \in B(X^*);$$

$$h_n(x^*) - f(x^*) \in B_n(0, 2^{-n}).$$

The completeness of E guarantees that the sequence $\{h_n\}_{n=1}^\infty$ converges uniformly to a function h: $B(X^*) \to E$ which has all the desired properties.

We state now without proof a few corollaries of Theorem 2.2. Their proofs are similar to those of the corresponding results for simplexes (cf. [10]).

COROLLARY 1. *Let X^* be an L_1 space. Let H be a w^*-metrizable and closed facial section of $B(X^*)$. Then there is a w^*-continuous affine symmetric map of $B(X^*)$ onto H whose restriction to H is the identity.*

Corollary 1 may be phrased also as follows. Let X and H be as above and let V be the subspace of X^* spanned by H. By Lemma 2.1 $V=(X/V^{\perp})^*$ where $V^{\perp}$ is the annihilator of V in X. Denote by $\psi: X\to X/V^{\perp}$ the natural quotient map. With this notation Corollary 1 asserts that there is an isometry into $T: X/V^{\perp}\to X$ so that $\psi\circ T$ is the identity of $X/V^{\perp}$. It follows that $T\circ\psi$ is a projection of norm 1 from X onto $T(X/V^{\perp})$.

COROLLARY 2. *Let X be a Banach space such that $X^*=L_1(\mu)$ for some μ. Let $Y\subset Z$ be Banach spaces so that every $y^*\in Y^*$ has a unique norm preserving extension to an element $\hat{y}^*\in Z^*$. Assume also that the map $y^*\to\hat{y}^*$ is continuous in the respective norm topologies. Then every compact operator from Y to X has a compact norm preserving extension to an operator from Z to X.*

Let us single out the following special case of Corollary 1 above.

THEOREM 2.3. *Let X be a separable Banach space such that X^* is a non-separable $L_1(\mu)$ space. Then X contains a subspace isometric to $C(K)$. K the Cantor set, on which there is a projection of norm 1.*

Proof. Since X is separable and X^* is not separable the set ext $B(X^*)$ is in its w^* topology an uncountable complete metric space (cf. [15]). The map $x^*\to -x^*$ is a homeomorphism of ext $B(X^*)$ onto itself. Hence there is a relatively closed uncountable subset K_1 of ext $B(X^*)$ such that $K_1\cap(-K_1)=\varnothing$. By a classical result [9, p. 408, p. 445] K_1 has a subset K homeomorphic to the Cantor set. By [24], $F=\overline{\text{conv}\,(K)}$ (the closure in the w^*-topology) is a w^*-closed face of $B(X^*)$. Apply now Corollary 1 to the facial section $H=\text{conv}\,(F\cup -F)$ of $B(X^*)$. It is easy to verify that (with the notation of the remark after the statement of the Corollary) $X/V^{\perp}$ is isometric to $C(K)$ and thus X has a subspace isometric to $C(K)$ on which there is a projection of norm 1.

In connection with Theorem 2.3 let us mention two other recent results.

1. Zippin [23]. Every infinite-dimensional Banach space whose dual is an L_1 space has a subspace isometric to c_0 (the space of all sequences tending to 0).

2. Lazar [11]. Every infinite-dimensional Banach space whose dual is an L_1 space and which is not polyhedral (i.e. it has finite dimensional subspaces whose unit balls are not polytopes) has a subspace isometric to c (the space of convergent sequences).

3. The structure theorem and the existence of representing matrices

An important property of Banach spaces whose duals are L_1 spaces is that they are rich with finite dimensional subspaces which are isometric to l_∞^m for some m (recall that l_∞^m is the space of n tuples $\lambda=(\lambda_1, \lambda_2, ..., \lambda_m)$ with $\|\lambda\|=\max_{1\leqslant i\leqslant m}|\lambda_i|$). A strong result in this direction is:

THEOREM 3.1. *Let X be a Banach space such that X^* is an L_1 space. Let F_1 and F_2 be finite-dimensional subspaces of X such that the unit cell of F_1 is a polytope. Then for every $\varepsilon>0$ there exists a subspace E of X such that $E\supset F_1$, $E=l_\infty^m$ for some $m<\infty$ and $d(x, E)\leqslant\varepsilon$ for every $x\in F_2$ with $\|x\|\leqslant 1$.*

By $d(x, E)$ we denote the distance of x from E, i.e. inf $\{\|x-y\|; y\in E\}$. Before we prove the theorem let us make some remarks. Since the unit ball of every subspace of l_∞^n is a polytope the assumption we made on F_1 is clearly necessary. Theorem 3.1 is a simultaneous generalization of [14, Cor. 2 to Theorem 7.9] which is the special case $F_2=\{0\}$ of the theorem and the main result of [12] which is weaker than the case $F_1=\{0\}$ of the theorem. Theorem 3.1 is actually a characterization of spaces whose duals are L_1 spaces. Even if we assume only that X has the property expressed in the theorem with $F_1=\{0\}$ it follows that X^* is an L_1 space (cf. [12] and [14, Theorem 6.1]).

The proof of Theorem 3.1 is based on two lemmas. This first one is a generalization of Lemma 2.1 of [10].

LEMMA 3.1. *Let X be a Banach space whose dual is an L_1 space. Let $\{f_i\}_{i=1}^n$, $\{g_i\}_{i=1}^n$ and $\{u_j\}_{j=1}^n$ be realvalued functions on $B(X^*)$ with $\{f_i\}$ and $\{q_i\}$ affine and w^*-continuous. Let $\{x_k^*\}_{k=1}^p$ be extreme points of $B(X^*)$. Assume that*

$$g_i\leqslant\sum_{j=1}^m \alpha_{i,j}u_j\leqslant f_i,\quad 1\leqslant i\leqslant n \tag{3.1}$$

for some scalars $\alpha_{i,j}$ and that

$$-u_j(x^*)=u_j(-x^*),\quad 1\leqslant j\leqslant m,\quad x^*\in B(X^*). \tag{3.2}$$

Then there are $\{x_j\}_{j=1}^m$ in X such that

$$g_i(x^*)\leqslant\sum_{j=1}^m \alpha_{i,j}x^*(x_j)\leqslant f_i(x^*),\quad 1\leqslant i\leqslant n,\quad x^*\in B(X^*)$$

and $$x_k^*(x_j)=u_j(x_k),\quad 1\leqslant j\leqslant m,\quad 1\leqslant k\leqslant p.$$

Proof. By induction on m. Let $m=1$ so (3.1) reduces to $g_i \leqslant \alpha_{i,1} u_1 \leqslant f_i$, $1 \leqslant i \leqslant n$. There is no loss of generality to assume that $\alpha_{i,1}=1$ for every i. Define g: $B(X^*) \to R$ by $g(x^*) = \min \{f_i(x^*),\ -g_i(-x^*);\ 1 \leqslant i \leqslant n\}$. Then g is w^*-continuous and concave on $B(X^*)$. We have also that $g(x^*) \geqslant u_1(x^*)$ for $x^* \in B(X^*)$ and hence by (3.2) $g(x^*)+g(-x^*) \geqslant 0$, $x^* \in B(X^*)$. By Theorem 2.1 (with $H = \operatorname{conv} \{\pm x_k^*;\ 1 \leqslant k \leqslant p\}$) there is an $x_1 \in X$ such that $x_k^*(x_1) = u_1(x_k^*)$ for every k and $x_1(x^*) \leqslant g(x^*)$, $x^* \in B(X^*)$. This proves the lemma for $m=1$.

Suppose now that the lemma holds for $m-1$. We may assume without loss of generality that $\alpha_{i,m}$ is either 0 or 1 for every i. We get from (3.1) that

$$g_i - \sum_{j=1}^{m-1} \alpha_{i,j} u_j \leqslant u_m \leqslant f_i - \sum_{j=1}^{m-1} \alpha_{i,j} u_j$$

if $\alpha_{i,m} \neq 0$. Hence the functions $\{u_j\}_{j=1}^{m-1}$ satisfy

$$g_r - f_s \leqslant \sum_{j=1}^{m-1} (\alpha_{r,j} - \alpha_{s,j})\, u_j \leqslant f_r - g_s,$$

$$g_r(x^*) + g_s(-x^*) \leqslant \sum_{j=1}^{m-1} (\alpha_{r,j} - \alpha_{s,j})\, u_j(x^*) \leqslant f_r(x^*) + f_s(-x^*), \quad x^* \in B(X^*)$$

if $\alpha_{r,m} = \alpha_{s,m} = 1$, and

$$g_i \leqslant \sum_{j=1}^{m-1} \alpha_{i,j} u_j \leqslant f_i$$

if $\alpha_{i,m} = 0$. By the induction hypothesis we can find $\{x_j\}_{j=1}^{m-1}$ in X such that

$$x_k^*(x_j) = u_j(x_k^*), \quad 1 \leqslant j \leqslant m-1, 1 \leqslant k \leqslant p \tag{3.3}$$

$$g_r(x^*) - \sum_{j=1}^{m-1} \alpha_{r,j}\, x^*(x_j) \leqslant f_s(x^*) - \sum_{j=1}^{m-1} \alpha_{s,j} x^*(x_j), \quad x^* \in B(X^*) \tag{3.4}$$

$$g_r(x^*) + g_s(-x^*) \leqslant \sum_{j=1}^{m-1} (\alpha_{r,j} - \alpha_{s,j})\, x^*(x_j) \leqslant f_r(x^*) + f_s(-x^*), \quad x^* \in B(X^*) \tag{3.5}$$

whenever $\alpha_{r,m} = \alpha_{s,m} = 1$, and

$$g_i(x^*) \leqslant \sum_{j=1}^{m-1} \alpha_{i,j} x^*(x_j) \leqslant f_i(x^*), \quad x^* \in B(X^*) \tag{3.6}$$

if $\alpha_{i,m} = 0$. Put now

$$g(x^*) = \min_{\alpha_{r,m} \neq 0} \left\{ f_r(x^*) - \sum_{j=1}^{m-1} \alpha_{r,j} x^*(x_j),\ -g_r(-x^*) - \sum_{j=1}^{m-1} \alpha_{r,j} x^*(x_j) \right\}.$$

Obviously g is concave and w^*-continuous and by (3.4) and (3.5), $g(x^*)+g(-x^*)\geqslant 0$. By (3.1), (3.2) and (3.3) $g(x_k^*)\geqslant u_m(x_k^*)$, $k=1, 2, ..., p$. Applying once more Theorem 2.1 we find an $x_m \in X$ which together with $\{x_j\}_{j=1}^{m-1}$ satisfies all the requirements of the lemma.

LEMMA 3.2. *Let W be a compact absolutely convex subset of $R^{n+1}=R\times R^n$ such that its canonical projection on R^n is a polytope. Denote by p and q the canonical projections of R^{n+1} onto R and R^n respectively. Then for every $\varepsilon>0$ there are distinct extreme points $\{e_j\}_{j=1}$ of W and realvalued symmetric functions $\{\lambda_j\}_{j=1}^m$ defined on W such that $\lambda_j(e_j)=1$ for every j and, for every $w\in W$*

$$\sum_{j=1}^m |\lambda_j(w)| \leqslant 1, \quad |p(w)-\sum_{=1}^m \lambda_j(w)\,p(e_j)|\leqslant \varepsilon, \quad q(w)=\sum_{j=1}^m \lambda_j(w)\,q(e_j).$$

Proof. Let W' be a symmetric polytope whose vertices are extreme points of W, such that $qW=qW'$ and for every $y\in qW$

$$\{t;\, t\in R,\, (t, y)\in W\}\subset\{t;\, t\in R,\, (t, y)\in W'\}+[-\varepsilon, \varepsilon].$$

Let $\{e_j\}_{j=1}^m$ be vertices of W' so that $W'=\operatorname{conv}\{\pm e_j, 1\leqslant j\leqslant m\}$ and $e_i\pm e_j\neq 0$ for $i\neq j$. For $w\in W'$ define real numbers $\{\lambda_j(w)\}_{i=1}^m$ so that, $\lambda_j(e_j)=1$, $\lambda_j(w)=-\lambda_j(-w)$, $j=1, ..., m$, $\Sigma_{j=1}^m|\lambda_j(w)|=1$ and $w=\Sigma_{j=1}^m \lambda_j(w)e_j$. We extend the functions $\{\lambda_j\}_{j=1}$ to W by defining for $w\in W\sim W'$ $\lambda_j(w)=\lambda_j(w')$, $1\leqslant j\leqslant m$, where $w'\in W'$ is defined by

$$q(w')=q(w) \quad \text{and} \quad |p(w)-p(w')|=\min\{|p(w)-p(w'')|,\, w''\in W',\, q(w'')=q(w)\}.$$

It is easy to check that the λ_j have all the desired properties.

Proof of Theorem 3.1. Let X, F_1, F_2 and $\varepsilon>0$ be given. A simple inductive argument shows that without loss of generality we may assume that $\dim F_2=1$. Let $\{y_i\}_{i=1}^n$ be a basis of F_1 and let z be a unit vector in F_2. Consider the map $T\colon X^*\to R\times R^n$ defined by $Tx^*=(x^*(z), x^*(y_1), x^*(y_2), ..., x^*(y_n))$ and put $W=TB(X^*)$. By Lemma 3.2 there are $\{e_j\}_{j=1}^m\in\operatorname{ext} W$ and symmetric functions $\{\lambda_j\}_{j=1}^m$ defined on W such that $\lambda_j(e_j)=1$ for every j and

$$\sum_{j=1}^m |\lambda_j(Tx^*)|\leqslant 1, \quad x^*\in B(X^*)$$

$$x^*(y_i)=\sum_{j=1}^m \lambda_j(Tx^*)\,e_j^{i+1}, \quad 1\leqslant i\leqslant n,\ x^*\in B(X^*),$$

$$|x^*(z) - \sum_{j=1}^{m} \lambda_j(Tx^*)\, e_j^1| \leqslant \varepsilon, \quad x^* \in B(X^*).$$

Here we used the notation $e_j = (e_j^1, e_j^2, \ldots, e_j^{n+1})$, $1 \leqslant j \leqslant m$. Let u_j: $B(X^*) \to R$ be defined by $u_j(x^*) = \lambda_j(Tx^*)$ and choose $\{x_j^*\}_{j=1}^m \in \operatorname{ext} B(X^*)$ so that $e_j = Tx_j^*$. Then for every j, $u_j(x_j^*) = 1$ and for every $x^* \in B(X^*)$

$$x^*(y_i) = \sum_{j=1}^{m} u_j(x^*)\, e_j^{i+1}, \quad i = 1, \ldots, n$$

$$-\varepsilon \leqslant x^*(z) - \sum_{j=1}^{m} u_j(x^*)\, e_j^1 \leqslant \varepsilon,$$

$$-1 \leqslant \sum_{j=1}^{m} \theta_j u_j(x^*) \leqslant 1, \quad \theta_j = \pm 1, \ldots, m.$$

By Lemma 3.1 there are $\{x_j\}_{j=1}^m$ in X such that

$$y_i = \sum_{j=1}^{m} e_j^{i+1} x_j, \quad i = 1, 2, \ldots, n \tag{3.7}$$

$$\|z - \sum_{j=1}^{m} e_j^1 x_j\| \leqslant \varepsilon, \tag{3.8}$$

$$\|\sum_{j=1}^{m} \theta_j x_j\| \leqslant 1, \quad \theta_j = \pm 1, j = 1, \ldots, m \tag{3.9}$$

and

$$\|x_j\| \geqslant x_j^*(x_j) = 1. \tag{3.10}$$

Let E be the subspace of X spanned by $\{x_j\}_{j=1}^m$. By (3.7) $E \supset F_1$ and by (3.8) $d(x, E) \leqslant \varepsilon$ for every $x \in F_2$ with $\|x\| \leqslant 1$. By (3.9) and (3.10) $E = l_\infty^m$ and this concludes the proof of the theorem.

For separable spaces Theorem 3.1 yields easily a slightly stronger version of the main result of [19].

THEOREM 3.2. *Let X be a separable infinite dimensional Banach space such that X^* is an L_1 space. Let F be a finite-dimensional space whose unit ball is a polytope. Then there exists a sequence $\{E_n\}_{n=1}^\infty$ of finite-dimensional subspaces of X such that $E_1 \supset F$, $E_{n+1} \supset E_n$ and $E_n = l_\infty^{m_n}$ for every n and $X = \overline{\bigcup_{n=1}^\infty E_n}$.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a dense sequence in the unit ball of X. By Theorem 3.1 we can construct inductively a sequence $\{E_n\}_{n=1}$ of finite-dimensional subspaces of X such that

$E_1 \supset F$ and for every n $E_{n+1} \supset E_n$, $E_n = l_\infty^{m_n}$ and $d(x_n, E_n) \leqslant 1/n$. It is clear that $X = \overline{\bigcup_{n=1}^\infty E_n}$ and this concludes the proof.

The rest of this section is devoted to some simple facts concerning isometric embeddings of an l_∞^n space into an l_∞^m space with $n < m$. Let $\{e_i\}_{i=1}$ be the usual unit vector basis in l_∞^n, i.e. $e_i = (0, 0, ..., 0, 1, 0, ...)$ with 1 only in the i'th place. By an admissible basis in l_∞^n we mean a basis of the form $\{\psi e_i\}_{i=1}^n$ where ψ is an isometry of l_∞^n, i.e. a basis of the form $\{\theta_i e_{\pi(i)}\}_{i=1}^n$ where $\theta_i = \pm 1$ and π a permutation of $\{1, 2, ..., n\}$. It is easy to see (cf. [19]) that if $\{u_i\}_{i=1}^n$ is an admissible basis in l_∞^n and $T: l_\infty^n \to l_\infty^m$ is a linear isometry then there exists an admissible basis $\{v_i\}_{i=1}^m$ in l_∞^m such that $Tu_i = v_i + \sum_{j=n+1}^m a_{i,j} v_j$ with $\sum_{i=1}^n |a_{i,j}| \leqslant 1$ for every $n+1 \leqslant j \leqslant m$. Conversely, for every such $\{v_i\}_{i=1}^m$ and $\{a_{i,j}\}_{1 \leqslant i \leqslant n}^{n+1 \leqslant j \leqslant m}$ the equation above defines an isometric embedding of l_∞^n into l_∞^m. It follows in particular that if $F_j = \operatorname{Span}\{\{Tu_i\}_{i=1}^n, v_{n+1}, v_{n+2}, ..., v_j\} \subset l_\infty^m$ then F_j is isometric to $l_\infty^j (n+1 \leqslant j \leqslant m)$. Thus (as observed in [19]) whenever $F \subset E$ with $F = l_\infty^n$ and $E = l_\infty^m$, there exist $\{F_j\}_{j=n+1}^{m-1}$ such that $F \subset F_{n+1} \subset ... \subset F_{m-1} \subset E$ with $F_j = l_\infty^j$. Hence if in the statement of Theorem 3.2 dim $F = 0$ or 1 we can take the E_n given by the theorem so that dim $E_n = n$ (i.e. $E_n = l_\infty^n$).

Let now $X = \overline{\bigcup_{n=1}^\infty E_n}$ with $E_n \subset E_{n+1}$ and $E_n = l_\infty^n$ for every n. Choose a unit vector $e_{1,1}$ in E_1 (this is just a choice of direction). By the description made above of the general form of an isometry from l_∞^n into l_∞^{n+1} we can choose inductively for every $n \geqslant 2$ admissible bases $\{e_{i,n}\}_{i=1}$ in E_n so that

$$e_{i,n} = e_{i,n+1} + a_{i,n} e_{n+1,n+1}, \quad 1 \leqslant i \leqslant n, \; n = 1, 2, 3, \dots \tag{3.11}$$

and

$$\sum_{i=1}^n |a_{i,n}| \leqslant 1 \quad n = 1, 2, 3, \dots \tag{3.12}$$

The triangular matrix $A = \{a_{i,n}\}_{n=1,2,\dots}^{1 \leqslant i \leqslant n}$ which we associate with X in this manner is called a *representing matrix of X*.

The matrix A is not determined uniquely by the representation $X = \overline{\bigcup_{n=1}^\infty E_n}$. Indeed, in every step of the inductive construction of the bases $\{e_{i,n}\}_{i=1}^n$ we have a choice of sign in the definition of $e_{n,n}$. If for one fixed n we replace $e_{n,n}$ by $-e_{n,n}$ then in order to preserve the validity of (3.11) we have to replace $e_{n,m}$ by $-e_{n,m}$ for every $m > n$. The effect this operation has on the matrix A is to replace $a_{i,n-1}$ by $-a_{i,n-1}$ for $1 \leqslant i \leqslant n-1$ and $a_{n,m}$ by $-a_{n,m}$ for $m \geqslant n$. It is easy to see that up to this operation of changing suitable signs the matrix A is determined uniquely by the representation $X = \overline{\bigcup_{n=1}^\infty E_n}$. In particular to every such representation there corresponds one and only one matrix A which satisfies for every n

$$\sum_{i=1}^{n} a_{i,n} \geqslant 0 \quad \text{and if} \quad \sum_{i=1}^{n} a_{i,n} = 0 \quad \text{then} \quad a_{i_0,n} > 0 \tag{3.13}$$

provided that $a_{i,n} = 0$ for $i \leqslant i_0$ and $|a_{i_0,n}| > 0$. A separable infinite-dimensional Banach space X with $X^* = L_1$ does not have a unique representation as $X = \overline{\bigcup_{n=1}^{\infty} E_n}$ with $E_n \subset E_{n+1}$ and $E_n = l_\infty^n$. Different representations of the same space give rise, in general, to entirely different matrices A. This leads to some difficult problems which will be mentioned in Section 5.

Every triangular matrix A satisfying (3.12) is a representing matrix of a uniquely defined Banach space whose dual is L_1. Indeed, if $\{e_{i,n}\}_{i=1}^{n}$ denotes the unit vector basis of l_∞^n then (3.11) defines for every n an isometric embedding of l_∞^{n-1} in l_∞^n and with identification of l_∞^{n-1} as a subspace of l_∞^n, $n = 2, 3, \ldots$ the Banach space $X = \overline{\bigcup_{n=1}^{\infty} l_\infty}$ has an L_1 dual and A is a matrix representing this X. It follows that there is a one to one correspondence between all representations of all separable infinite dimensional X with L_1 duals as $X = \overline{\bigcup_{n=1}^{\infty} E_n}$ with $E_n \subset E_{n+1}$ and $E_n = l_\infty^n$, and all triangular matrices A satisfying (3.12) and (3.13).

4. Functional representations

In this section we introduce some special classes of spaces whose duals are L_1 spaces and give some orientation on the relation between these classes. The section is essentially a summary of [17] with some additions. Facts brought in this section without a reference or a comment concerning their proof may be either found explicitely in [17] or follow easily from the results of [17]. First the definitions. In cases where there can be no confusion we shall use the same notations for a class of Banach spaces and a general representative of this class. In all the function spaces we take the supremum norm.

$C(K)$ spaces: The spaces of continuous functions on compact Hausdorff spaces K.

$C_0(K)$ spaces: The spaces of continuous functions on compact Hausdorff spaces K which vanish at a fixed point of K. Or, equivalently, the spaces of continuous functions on locally compact Hausdorff spaces which vanish at infinity.

$C_\sigma(K)$ spaces: The spaces of all continuous functions f on compact Hausdorff spaces K which satisfy $f(\sigma k) = -f(k)$ for all $k \in K$, where $\sigma: K \to K$ is a homeomorphism of period 2 (i.e. $\sigma^2 = \text{identity}$).

$C_\Sigma(K)$ spaces: Those $C_\sigma(K)$ spaces in which the homeomorphism σ has no fixed points.

M spaces: Sublattices of $C(K)$ spaces or, equivalently (by [8]) spaces X which can be represented as follows: there is a compact Hausdorff space K and a set A of triples $\{k_\alpha^1, k_\alpha^2, \lambda_\alpha\}_{\alpha \in A}$ with $k_\alpha^1, k_\alpha^2 \in K$ and $\lambda_\alpha \geqslant 0$ such that X is the set of all $f \in C(K)$ which satisfy $f(k_\alpha^1) = \lambda_\alpha f(k_\alpha^2)$ for all $\alpha \in A$.

G spaces: Spaces defined like the explicit definition of M spaces only that now the λ_α are allowed to be arbitrary real numbers (i.e. may also be negative).

$A(S)$ spaces: Spaces of affine continuous functions on compact Choquet simplexes S. (For information on simplexes cf. e.g. [21].)

$A_0(S)$ spaces: Spaces of affine continuous functions on compact Choquet simplexes S which vanish at one fixed extreme point of S (these are the simplex spaces in the terminology of [4]).

For all these classes of Banach spaces the duals are L_1 spaces. The relations between those classes are clarified by the following diagram

$$\begin{tikzcd}
 & A(S) \arrow[r] & A_0(S) \arrow[drr] & & \\
C(K) \arrow[ur] \arrow[r] \arrow[dr] & C_0(K) \arrow[r] \arrow[dr] & M \arrow[u] \arrow[dr] & & \{X\colon X^* = L_1(\mu)\} \\
 & C_\Sigma(K) \arrow[r] & C_\sigma(K) \arrow[r] & G \arrow[ur] &
\end{tikzcd}$$

Here $A \to B$ means that every space of class A is also of class B. From the diagram it is possible also to read the intersection of two classes. It is the common "source" of these classes in the diagram. For example

$$A_0(S) \cap G = M, \quad C_\sigma(K) \cap A(S) = C(K), \quad \text{etc.}$$

Here are properties which characterize some of the classes above among all Banach spaces whose duals are L_1 spaces. Let X be a Banach space such that $X^* = L_1(\mu)$ for some μ. Then

(i) X is an $A(S)$ space if and only if $\operatorname{ext} B(X) \neq \emptyset$.

(ii) X is a $C_\Sigma(K)$ space if and only if $\operatorname{ext} B(X^*)$ is w^*-closed.

(iii) X is a $C(K)$ space if and only if $\operatorname{ext} B(X) \neq \emptyset$ and $\operatorname{ext} B(X^*)$ is w^*-closed.

(iv) X is an $A_0(S)$ space if and only if X can be ordered so that X^* is an $L_1(\mu)$ as an ordered Banach space. Stated otherwise: $X = A_0(S)$ if and only if X^* is isometric to $L_1(\mu)$ in such a way that the positive cone of $L_1(\mu)$ is the image of a w^*-closed set. (This assertion is an immediate consequence of the definition.)

The class of spaces whose duals are L_1 spaces is closed under some natural operations: direct sums with the supremum norm, tensor products with the smallest cross norm and projections of norm 1 which are of particular interest. Let $\mathcal{B}$ be a class of Banach spaces. We denote by $\pi(\mathcal{B})$ the class of all Banach spaces Y for which there is an $X \supset Y$ with $X \in \mathcal{B}$ and a projection of norm 1 from X onto Y. Then

$$\pi(C(K)) = \pi(C_0(K)) = \pi(C_\Sigma(K)) = \pi(C_\sigma(K)) = C_\sigma(K)$$

$$\pi(M) = \pi(G) = G.$$

It is likely that $\pi(A(S)) = \pi(A_0(S)) = \{X; X^* = L_1(\mu)\}$. The next proposition shows that $\pi(A(S)) = \pi(A_0(S))$ and in the next section we show that $\pi(A(S))$ contains all the separable spaces whose dual is an L_1 space.

PROPOSITION 4.1. *Let X be a simplex space. Then there is a simplex S so that X is isometric to a subspace of $Y = A(S)$ on which there is a projection of norm 1.*

Proof. By our assumption $X = A_0(\tilde{S})$ for some simplex $\tilde{S}$. We may clearly assume that the extreme point of $\tilde{S}$ on which all the functions of X vanish is the origin of the linear space V which contains $\tilde{S}$. Let V_1 be a linear space isomorphic to V by a (topological) isomorphism ψ. Let S be the convex hull of $\{(v, 0), v \in \tilde{S}\} \cup \{(0, \psi v), v \in \tilde{S}\}$ in the direct sum $V \oplus V_1$. It is easy to verify that S is a simplex, and that for every $f \in A_0(S)$ there is one and only one function $F = Tf$ in $A(S)$ which satisfies $F(v, 0) = f(v)$ and $F(0, \psi v) = -f(v)$ for every $v \in S$. This map $T: X \to Y = A(S)$ is an isometry. Let P be the operator from Y to X defined by $PF(v) = (F(v, 0) - F(0, \psi v))/2$, $v \in \tilde{S}$. Then TP is a projection of norm 1 from Y onto TX.

As mentioned above, a Banach space X with $X^* = L_1$ is an $A(S)$ space if and only if the unit ball of X has at least one extreme point. It thus looks as if the relation between general preduals of $L_1(\mu)$ and $A(S)$ spaces is similar to the relation between general Banach algebras and Banach algebras with identity. I.e. that by suitably adjoining an extreme point to a predual of an L_1 space we get an $A(S)$ space. This is obviously the case if X is a simplex space $A_0(S)$ – we have just to add to X the constant functions. However, in general the situation is not as simple, and it seems that the natural way to reduce (if possible) questions on general preduals of L_1 to $A(S)$ spaces is by using projections of norm 1 (in the sense of Theorem 5.5 below). As an example of the fact that a simple "adjoining of an extreme point" is not possible we take the subspace X of $C(0, 1)$ consisting of all the functions which satisfy $2f(0) = -f(1/3)$ and $2f(1) = -f(2/3)$. Clearly X is a G space of codimension 2 in $C(0, 1)$. There does not exist a Choquet simplex S such that X is isometric to a subspace of codimension one in $A(S)$. Our proof of this fact is not short and instead of presenting here the (quite boring) detailed proof we just state (also without proof) the main proposition on which it is based.

PROPOSITION 4.2. *Let S be a Choquet simplex and let X be a subspace of codimension one in $A(S)$ so that* ext $B(X) = \emptyset$. *Then there is a subset K of* ext $B(X^*)$ *such that*

(1) $K \cup (-K) = \text{ext } B(X^*)$.

(2) *S is affinely homeomorphic either to the w^* closure of* conv K *or to the w^* closure of* conv $(K \cup \{x^*\})$ *for some* $x^* \in B(X^*)$.

(3) *If S is homeomorphic to* $\overline{\text{conv } K}$ *then* $K = \text{ext } \overline{\text{conv } K}$. *If the other possibility of* 2) *holds then* $K \cup \{x^*\} = \text{ext } \overline{\text{conv } (K \cup \{x^*\})}$.

We are convinced (though we did not check it in detail) that there are G spaces which are not isometric even to subspaces of finite codimension in $A(S)$ spaces.

5. Representing matrices—examples and applications

As mentioned at the end of Section 3 every separable infinite dimensional predual X of $L_1(\mu)$ has many representations $X = \overline{\bigcup_{n=1}^{\infty} E_n}$ and thus many representing matrices. (In this section whenever we mention such a representation we shall assume that $E_n \subset E_{n+1}$ and $E_n = l_\infty^n$ for every integer n.) In particular it is clear that X is not affected if we change a finite number of the isometric embeddings $E_n \to E_{n+1}$. Thus X depends only on the assymptotic behaviour of $A = \{a_{i,n}\}$ as $n \to \infty$. We shall give now two simple examples to illustrate this point.

EXAMPLE 5.1. *Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of numbers in the interval* $[0, 1]$. *Let A be the matrix defined by $a_{n,n} = t_n$ and $a_{i,n} = 0$ for $i < n$. Then*

(i) *If the infinite product $\Pi_{n=1}^{\infty} t_n$ converges, A represents the space c of convergent sequences.*

(ii) *If the infinite product $\Pi_{n=1}^{\infty} t_n$ diverges, A represents the space c_0 of sequences converging to* 0.

Proof. Case (i). Let $\{x_n\}_{n=1}^{\infty}$ be the sequence in c defined by

$$x_1 = (1, t_1, t_1 t_2, t_1 t_2 t_3, \ldots)$$
$$x_2 = (0, 1, t_2, t_2 t_3, \ldots)$$
$$\cdots\cdots\cdots$$
$$x_n = (0, 0, \ldots, 0, 1, t_n, t_n t_{n+1}, \ldots)$$
$$\cdots\cdots\cdots$$

It is clear that for every n, $E_n = \text{span } \{x_i\}_{i=1}^{n}$ is isometric to l_∞^n with $\{e_i\}_{i=1}^{n-1} \cup \{x_n\}$ as an admissible basis (e_i is the sequence in c_0 whose i'th coordinate is 1 while all the rest are 0). Since $x_n = e_n + t_n x_{n+1}$, $n = 1, 2, \ldots$, A is a representing matrix of $X = \overline{\bigcup_{n=1}^{\infty} E_n}$. Since

$e_i \in X$ for every i it follows that $c_0 \subset X$. The fact that $\Pi_{n=i}^{\infty} t_n \neq 0$ for sufficiently large i implies that $X \not\subset c_0$ and hence $X = c$.

The proof of case (ii) is similar. In this case $\{x_n\}_{n=1}^{\infty} \subset c_0$ and thus $X = c_0$.

EXAMPLE 5.2. *Let* $\{t_n\}_{n=1}^{\infty}$ *be a sequence of numbers in* $[0, 1]$. *Let* A *be the matrix defined by* $a_{1,n} = t_n$ *and* $a_{i,n} = 0$, $i > 1$. *Then the isometric type of the space which* A *represents determines and is determined by the set of limiting points of the sequence* $\{t_n\}_{n=1}^{\infty}$.

Proof. It is easy to check that A represents the subspace X of the space m of bounded sequences, spanned by the unit vectors $\{e_i\}_{i=2}^{\infty}$ and the vector $u = (1, t_1, t_2, t_3, \ldots)$. The extreme points of $B(X^*)$ are the functionals $\{\pm x_i^*\}_{i=1}^{\infty}$ defined by $x_i^*(\lambda_1, \lambda_2, \ldots) = \lambda_i$. It follows that λ is a limiting point of $\{t_n\}_{n=1}^{\infty}$ if and only if there is a sequence in ext $B(X^*)$ which converges in the w^* topology to a functional of norm λ. Thus the set of limiting points of $\{t_n\}_{n=1}^{\infty}$ is determined by the isometric type of X. Conversely, if $\{t_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ have the same set of limiting points then by a well-known elementary fact there is a permutation π of the set of positive integers such that $t_n - s_{\pi(n)} \to 0$ as $n \to \infty$. This permutation π induces in a natural way an isometry of m which maps the span of $(1, s_1, s_2, \ldots) \cup \{e_i\}_{i=2}^{\infty}$ onto the span of $\{(1, t_1, t_2, \ldots)\} \cup \{e_i\}_{i=2}^{\infty}$.

The reader should note the essential difference between Examples 5.1 and 5.2. While in 5.2 the rate in which $t_n \to 1$ (if at all $t_n \to 1$) is of crucial importance we have that in 5.2 only the set of limiting points itself matters.

It seems to be a very difficult problem to determine the set of all representing matrices of a given separable infinite-dimensional predual of $L_1(\mu)$. We know the answer to this question only for one such space, namely the space of Gurariĭ [7] and even here the situation is not entirely clear. We shall come back to this space at the end of the paper.

The situation is somewhat simpler if we ask the following question. Given a class $\mathcal{B}$ of separable Banach spaces whose duals are L_1 spaces, find a class of matrices $\mathcal{A}$ so that every matrix of $\mathcal{A}$ represents a space in $\mathcal{B}$ and that every space in $\mathcal{B}$ has a representing matrix belonging to $\mathcal{A}$. Our next three theorems give answers to some particular cases of this question.

The set of all representing matrices, i.e. all triangular matrices satisfying (3.12) form a convex set. Its extreme points are easily determined. These are the matrices $\{a_{i,n}\}$ such that for every n there is an $i(n)$ such that $|a_{i(n),n}| = 1$ and $a_{i,n} = 0$ for $i \neq i(n)$. The spaces which admit extreme representing matrices are also easily determined. By the discussion at the end of Section 3 we may assume that (3.13) holds, i.e. that $a_{i(n),n} = 1$ for every n.

THEOREM 5.1. *A Banach space X has a representation by an extreme representing matrix if and only if $X = C(K)$ for some compact metric totally disconnected K.*

Proof. Let K be a totally disconnected compact metric space. There exists a sequence $\{\Pi_n\}_{n=1}^\infty$ of partitions of K into disjoint closed sets so that for every n, Π_n has n elements, Π_{n+1} is a refinement of Π_n and

$$\varrho_n = \max_{A \in \Pi_n} d(A) \to 0 \quad \text{as } n \to \infty$$

(where $d(A)$ denotes the diameter of A). Let E_n be the linear span of the characteristic functions of the sets belonging to Π_n. Then $E_n = l_\infty^n$, $E_n \subset E_{n+1}$ and $C(K) = \overline{\bigcup_{n=1}^\infty E_n}$. It is clear that the matrix A corresponding to this representation of $C(K)$ is extreme.

Conversely let A be an extreme matrix i.e. $a_{i(n),n} = 1$ and $a_{i,n} = 0$ for $i \neq i(n)$. Let K be the set of all sequences of integers $(k_1, k_2, k_3, \ldots)$ with $k_1 = 1$,

$$k_{n+1} = k_n, \quad \text{if} \quad k_n \neq i(n), \quad n = 1, 2, \ldots$$

and
$$k_{n+1} = \text{either } k_n \text{ or } n+1 \text{ if } k_n = i(n),\ n = 1, 2, \ldots$$

Clearly $K \subset \Pi_{n=1}^\infty Z_n$ where $Z_n = \{1, 2, \ldots, n\}$. We take on each Z_n the discrete topology and on K the topology induced by the product topology. With this topology K is a compact metric totally disconnected space. Let $\Pi_n = (A_n^1, A_n^2, \ldots, A_n^n)$ be the partition of K defined by $(k_1, k_2, k_3, \ldots) \in A_n^i \rightleftarrows k_n = i$. The $\{\Pi_n\}_{n=1}^\infty$ have all properties required of the partitions in the first part of the proof. It is clear that the matrix representing $C(K)$ which was constructed above out of the sequence of partitions $\{\Pi_n\}_{n=1}^\infty$ is the matrix A with which we started.

THEOREM 5.2. *A separable infinite-dimensional Banach space X has a representing matrix $A = \{a_{i,n}\}$ with*

$$\sum_i a_{i,n} = 1, \quad n = 1, 2, 3, \ldots \tag{5.1}$$

if and only if $X = A(S)$ for some simplex S.

Proof. Assume that $X = \overline{\bigcup_n E_n}$ is a representation of X corresponding to a matrix A satisfying (5.1). Let $e = e_{1,1}$ be (the positively oriented) unit vector of E_1. It follows from (3.11) by using induction on n that

$$e = \sum_{i=1}^n e_{i,n} \tag{5.2}$$

It follows that $e \in \operatorname{ext} B(E_n)$ for every n and thus for every $x \in B(E_n)$

$$\max(\|e+x\|, \|e-x\|) = \|e\| + \|x\|. \tag{5.3}$$

Since $\bigcup_n E_n$ is dense in X, (5.3) holds for every $x \in B(X)$ and hence $e \in \text{ext } B(X)$. By the characterization of $A(S)$ which was mentioned in Section 4 we deduce that $X = A(S)$ for a suitable simplex S.

Conversely, assume that $X = A(S)$ and let e be an extreme point in the unit ball of X (e.g. the function identically equal to 1). By Theorem 3.2 there exists a representation $X = \overline{\bigcup_{n=1}^{\infty} E_n}$ with $E_1 = \{\lambda e; \lambda \in R\}$. Since $e \in \text{ext } B(X)$ it follows that $e \in \text{ ext } B(E_n)$ for every n. We show now by induction on n that with a suitable choice of signs (of $e_{n,n}$, $n = 1, 2, \ldots$) (5.2) and (5.1) hold. Indeed assume that (5.2) holds for some n. Then by (3.11)

$$e = \sum_{i=1}^{n} e_{i,n} = \sum_{i=1}^{n} e_{i,n+1} + \left(\sum_{i=1}^{n} a_{i,n}\right) e_{n+1,n+1}.$$

Since $e \in \text{ext } B(E_{n+1})$ we get that $\left|\sum_{i=1}^{n} a_{i,n}\right| = 1$ and hence after a change of sign of $e_{n+1,n+1}$ if necessary $\sum_{i=1}^{n} a_{i,n} = 1$. Thus (5.1) holds for n and (5.2) holds for $n+1$. This concludes the proof.

COROLLARY. *Let S be a compact metrizable infinite-dimensional Choquet simplex. Then there exists a sequence of affine surjective maps $\varphi_n: \Delta_{n+1} \to \Delta_n$ where Δ_n is an n-dimensional simplex such that S is the inverse limit of the system*

$$\Delta_1 \xleftarrow{\varphi_1} \Delta_2 \xleftarrow{\varphi_2} \Delta_3 \leftarrow \cdots$$

Proof. This is just a restatement of Theorem 5.2 in terms of the dual space. If $T_n: l_\infty^n \to l_\infty^{n+1}$ is given by $T_n e_{i,n} = e_{i,n+1} + a_{i,n} e_{n+1,n+1}$ with $a_{i,n} \geq 0$ and $\sum_{i=1}^{n} a_{i,n} = 1$ then T_n^* maps the positive face of $B(l_\infty^{n+1})$ affinely onto the positive face of $B(l_\infty^n)$.

Remark. Theorem 5.2 can be given also a probabilistic interpretation. Consider a random walk on the integers $1, 2, 3, \ldots$ in which it is impossible to advance. Denote the probability to go from a state n to a state i with $i \leq n$ by $a_{i,n}$. The matrix $A = \{a_{i,n}\}$ gives rise to a unique Choquet simplex S. Conversely every metrizable Choquet simplex gives rise to a family of such random walks. This correspondence gives probabilistic meaning to some quantities which arise naturally in the study of $A(S)$ as a Banach space. We did not, however, find any substantial application of the existence of this relation between Choquet simplexes and random walks.

THEOREM 5.3. *A separable infinite-dimensional Banach space X has a non-negative representing matrix if and only if X can be ordered so that $X^* = L_1(\mu)$ as an ordered Banach space (i.e. X is a simplex space).*

Proof. Assume that the matrix $A=\{a_{i,n}\}$ corresponding to the representation $X=\overline{\bigcup_{n=1}^{\infty} E_n}$ is non negative. In each E_n we introduce an order by taking as the positive cone the set $C_n=\{\sum_{i=1}^{n}\lambda_i e_{i,n};\ \lambda_i\geqslant 0,\ 1\leqslant i\leqslant n\}$. Since A is non negative $C_n\subset C_{n+1}$ for every n. We order X by taking as the positive cone the set $\overline{\bigcup_{n=1}^{\infty} C_n}$. It can be proved directly that this defines a suitable order on X. We find it however simpler to present an indirect proof of this. Let $B=\{b_{i,n}\}$ be the triangular matrix defined by

$$b_{1,1}=1,\quad b_{1,n}=1-\sum_{i=1}^{n-1} a_{i,n-1},\quad n=2,3,\dots$$

$$b_{i,n}=a_{i-1,n-1},\quad 2\leqslant i\leqslant n,\quad n=2,3,\dots$$

By Theorem 5.2 the space $Y=\overline{\bigcup_{n=1} F_n}$ represented by B is an $A(S)$ space for some simplex S. We assume that $A(S)$ is ordered so that the vector e used in the proof of Theorem 5.2 corresponds to the function identically equal to 1 on S (i.e. we take

$$S=\{y^*\in Y^*;\ \|y^*\|=y^*(e)=1\}).$$

Define the functional y_0^* on $\bigcup_n F_n$ by $y_0^*(y)=\lambda_1$ if $y=\sum_{i=1}^{n}\lambda_i f_{i,n}\in F_n$ where $\{f_{i,n}\}_{i=1}^{n}$ is the canonical basis of F_n which is associated to the matrix B. By (3.11) y_0^* is well defined on $\bigcup_n F_n$ and thus defines a unique element of S. Moreover, since $y_0^*\,|\,F_n$ is an extreme point of $B(F_n^*)$ for every n it follows that $y_0^*\in\operatorname{ext} S$ (this fact was exploited in [23]). By the definition of B the subspace X of $A(S)$ consisting of all the vectors which annihilate y_0^* has the given matrix A as a representing matrix. Hence A represents a simplex space $A_0(S)$. Observe also that the order induced by $A(S)$ on X coincides with the order we defined in the beginning of the proof.

Conversely, assume that X is a simplex space. The fact that X has a non-negative representing matrix can be proved by following the proofs of Theorems 3.1 and 3.2, taking care at each step that the embedding constructed in those proofs are in addition non-negative (in l_∞^n we take always the natural order). In the proof of Theorem 3.1 we have only to replace Lemma 3.1 by Lemma 2.1 of [10] and replace Lemma 3.2 by its following variant: Let W be a compact convex subset of $R^{n+1}=R\times R^n$. Let p and q denote the canonical projections of R^{n+1} onto R and R^n respectively, and assume that qW is a polytope. Then for every $\varepsilon>0$ there exist distinct extreme points $\{e_j\}_{j=1}^{m}$ of W and non-negative functions $\{\lambda_j\}_{j=1}^{m}$ defined on W such that $\sum_{j=1}^{m}\lambda_j(w)=1$, $|p(w)-\sum_{j=1}^{m}\lambda_j(w)p(e_j)|<\varepsilon$, $q(w)=\sum_{j=1}^{m}\lambda_j(w)q(e_j)$ for all $w\in W$ and $\lambda_j(e_j)=1$ for every j.

Let A be a representing matrix which satisfies (5.1). By Theorem 5.2 A represents an $A(S)$ space X. There is a simple necessary (though not sufficient) condition which A has to satisfy in order that $X=C(K)$ for some compact metric K.

Theorem 5.4. *Let $A=\{a_{i,n}\}$ be a non-negative triangular matrix such that $\Sigma_{i=1}^{n} a_{i,n}=1$ for every n. Assume that A represents a $C(K)$ space. Then*

$$\lim_{n\to\infty}\left(\sum_{i=1}^{k} a_{i,n}-\max_{1\leqslant i\leqslant k} a_{i,n}\right)=0 \tag{5.4}$$

for every integer k.

Proof. As mentioned in Section 4 $A(S)\cap M=C(K)$. More precisely: Let X be an $A(S)$ space. Then X is a $C(K)$ space if and only if whenever X is ordered so that X^* is order isometric to an L_1 space, then X is a lattice in this order. Let $X=\overline{\bigcup_{n=1}^{\infty} E_n}$ be the representation induced by A. If we take in each E_n its natural order then, as pointed out in the proof of Theorem 5.3, these orders are mutually compatible and induce an order on X such that X^* is order isometric to $L_1(\mu)$ for some μ. Let $x, y\in E_n$ for some n and let $(x\vee y)_m$, $m>n$ be the supremum of x and y in the (lattice) ordering in E_m. Clearly $(x\vee y)_{m+1}\leqslant(x\vee y)_m$ for every $m>n$ and since $\bigcup_m E_m$ is dense in X it follows that $x\vee y$ (maximum in X) exists if and only if $lim_{m\to\infty}(x\vee y)_m$ exists. Consequently, X is a $C(K)$ space if and only if for every $x, y\in\overline{\bigcup_{n=1}^{\infty} E_n}$, $lim_{m\to\infty}(x\vee y)_m$ exists.

Assume now that (5.4) does not hold. It follows that there exist an $\varepsilon>0$, integers $i<j$ and a sequence of integers $\{m_n\}_{n=1}^{\infty}$ such that $a_{i,m_n}\geqslant\varepsilon$ and $a_{j,m_n}\geqslant\varepsilon$ for every n. Take any such m_n and let

$$e_{i,j}=\sum_{k=1}^{m_n}\lambda_k e_{k,m_n},\quad e_{j,j}=\sum_{k=1}^{m_n}\mu_k e_{k,m_n},$$

where $\{e_{i,n}\}_{i=1}^{n}$ denote, as usual, the canonical basis of E_n which corresponds to A. By (3.11) and the fact that A is non-negative we get that

$$\lambda_i=1;\quad \lambda_k=0,\ 1\leqslant k\leqslant j,\ k\neq i;\quad \lambda_k\geqslant 0,\ j+1\leqslant k\leqslant m_n,$$

$$\mu_j=1;\quad \mu_k=0,\ 1\leqslant k\leqslant j-1;\quad \mu_k\geqslant 0,\ j+1\leqslant k\leqslant m_n.$$

Hence
$$(e_{i,j}\vee e_{j,j})_{m_n}=e_{i,m_n}+e_{j,m_n}+\sum_{k=j+1}^{m_n}\max(\lambda_k,\mu_k)\,e_{k,m_n}.$$

By (3.11),

$$e_{i,j}=e_{i,m_n+1}+\sum_{k=j+1}^{m_n}\lambda_k e_{k,m_n+1}+\Big(a_{i,m_n}+\sum_{k=j+1}^{m_n}\lambda_k a_{k,m_n}\Big)e_{m_n+1,m_n+1} \tag{5.5}$$

$$e_{j,j}=e_{j,m_n+1}+\sum_{k=j+1}^{m_n}\mu_k e_{k,m_n+1}+\Big(a_{j,m_n}+\sum_{k=j+1}^{m_n}\lambda_k a_{k,m_n}\Big)e_{m_n+1,m_n+1} \tag{5.6}$$

$$(e_{i,j} \vee e_{j,j})_{m_n} = e_{i,m_n+1} + e_{j,m_n+1} + \sum_{k=j+1}^{m_n} \max(\lambda_k, \mu_k)\, e_{k,m_n+1}$$
$$+ \left(a_{i,m_n} + a_{j,m_n} + \sum_{k=j+1}^{m_n} \max(\lambda_k, \mu_k)\, a_{k,m_n} \right) e_{m_n+1,\, m_n+1}. \tag{5.7}$$

Hence by (5.5) and (5.6)

$$(e_{i,j} \vee e_{j,j})_{m_n+1} = e_{i,m_n+1} + e_{j,m_n+1} + \sum_{k=j+1}^{m_n} \max(\lambda_k, \mu_k)\, e_{k,m_n+1}$$
$$+ \max\left(a_{i,m_n} + \sum_{k=j+1}^{m_n} \lambda_k\, a_{k,m_n},\; a_{j,m_n} + \sum_{k=j+1}^{m_n} \mu_k\, a_{k,m_n} \right) e_{m_n+1,m_n+1}.$$

Comparing the coefficients in (5.7) and (5.8) and recalling that all the numbers involved are non-negative we get that

$$\| (e_{i,j} \vee e_{j,j})_{m_n} - (e_{i,j} \vee e_{j,j})_{m_n+1} \| \geq \min(a_{i,m_n}, a_{j,m_n}) \geq \varepsilon$$

and this contradicts the assumption that A represents a $C(K)$ space.

Remarks. 1. (5.4) is not a sufficient condition for A to represent a $C(K)$ space. By using similar ideas to those used in the proof above it is possible to give a (complicated) necessary and sufficient condition for A (assuming that (5.1) holds) to represent a $C(K)$ space.

2. (5.4) is also a necessary condition for a non-negative matrix A to represent an M space.

We shall now use Theorem 5.3 to settle a question, mentioned in Section 4, in the separable case.

THEOREM 5.5. *Let X be a separable infinite-dimensional Banach space whose dual is an L_1 space. Then there is a metrizable Choquet simplex S so that X is isometric to a subspace of $A(S)$ on which there is a projection of norm 1.*

Proof. Let $X = \overline{\bigcup_{n=1}^{\infty} E_n}$ and let $A = \{a_{i,n}\}$ be a matrix corresponding to this representation. For every integer n let $F_n = l_\infty^{2n}$ and let $\{f_{i,n}\}_{i=1}^{2n}$ be the unit vector basis of F_n. We embed F_n isometrically into F_{n+1} by the operator T_n defined as follows

$$T_n f_{2i-1,n} = \begin{cases} f_{2i-1,n+1} + a_{i,n} f_{2n+1,n+1} & \text{if } a_{i,n} \geq 0 \\ f_{2i-1,n+1} - a_{i,n} f_{2n+2,n+1} & \text{if } a_{i,n} < 0 \end{cases} \qquad i = 1, \ldots, n$$

$$T_n f_{2i,n} = \begin{cases} f_{2i,n+1} + a_{i,n} f_{2n+2,n+1} & \text{if } a_{i,n} \geq 0 \\ f_{2i,n+1} - a_{i,n} f_{2n+1,n+1} & \text{if } a_{i,n} < 0 \end{cases} \qquad i = 1, 2, \ldots, n$$

Since each T_n is positive (in the natural orders in F_n and F_{n+1}) a slight modification of the first part of the proof of Theorem 5.3 shows that $Y=\overline{\bigcup_{n=1}^{\infty} F_n}$ is a simplex space.

For every integer n define the isometry R_n: $E_n \to F_n$ by $R_n e_{i,n} = f_{2i-1,n} - f_{2i,n}$, $1 \leqslant i \leqslant n$, where $\{e_{i,n}\}_{i=1}^n$ is the basis of E_n which corresponds to the matrix A. It is easy to check that the $\{R_n\}_{n=1}^\infty$ are consistent with the embeddings of E_n in E_{n+1} and F_n in F_{n+1} (i.e. that $R_{n+1}E_n = T_n R_n$) for all n. Thus the $\{R_n\}_{n=1}^\infty$ define a unique isometry R: $X \to Y$. For every n, let P_n be the projection of norm 1 in F_n defined by $P_n f_{2i-1,n} = -P_n f_{2i,n} = (f_{2i-1,n} - f_{2i,n})/2$ for $1 \leqslant i \leqslant n$. Clearly $P_n F_n = R_n E_n$ and $T_{n+1} P_n = P_{n+1} T_n$ for every integer n. Hence the $\{P_n\}_{n=1}^\infty$ define a unique projection of norm 1 from Y onto RX. The proof of Theorem 5.5 is completed now by applying Proposition 4.1.

We conclude the paper by a discussion of the space(s) constructed by Gurariĭ [7]. Gurariĭ proved that there is a separable Banach space X which has the following property:

(*) For every finite dimensional Banach spaces $F \supset E$, every isometry T: $E \to X$ and every $\varepsilon > 0$ there is an operator $\tilde{T}$: $F \to X$ such that $\tilde{T}|_E = T$ and $(1-\varepsilon)\|x\| \leqslant \|\tilde{T}x\| \leqslant (1+\varepsilon)\|x\|$ for every $x \in F$.

It is clear (cf. e.g. [14]) that the dual of every space which satisfies (*) is an L_1 space. Gurariĭ observed that there is no separable X which satisfies (*) with $\varepsilon = 0$ (For a stronger result cf. [14, Theorem 7.8].) He observed also that a space satisfying (*) is essentially unique in the following sense. Let X and Y be two separable spaces satisfying (*). Then for every $\varepsilon > 0$ there is an operator $U = U(\varepsilon)$ from X onto Y such that

$$(1-\varepsilon)\|x\| \leqslant \|Ux\| \leqslant (1+\varepsilon)\|x\|$$

for every $x \in X$. We do not know whether the Gurariĭ space is strictly unique (i.e. whether any two separable spaces satisfying (*) are isometric). A functional representation of the Gurariĭ space(s) is not known. It turns out, however, that it is possible to characterize all the representing matrices of this space(s). We shall restrict ourselves here in giving a simple sufficient (but not necessary) condition for a matrix to represent a Gurariĭ space. This will give in particular a much simpler proof of the existence of a Gurariĭ space. (Observe that for this proof of the existence of a Gurariĭ space we do not use the results of Section 3 on the existence of a representing matrix for every separable predual of L_1.)

THEOREM 5.6. *Let $A = \{a_{i,n}\}$ be a triangular matrix with $\sum_{i=1}^n |a_{i,n}| \leqslant 1$ for every n. Assume that the vectors*

$$a^n = (a_{1,n}, a_{2,n}, a_{3,n} \ldots, a_{n,n}, 0, 0, \ldots),\ n = 1, 2, \ldots$$

are dense in the unit ball of l_1. Then A represents a Gurariĭ space.

Proof. Let $X=\overline{\bigcup_{n=1}^{\infty} E_n}$ have A as a representing matrix. By a routine approximation argument it is clear that in order to prove that X has property (*) it is enough to consider only spaces F such that $B(F)$ (and therefore also $B(E)$) is a polytope and isometries T such that $TE\subset E_n$ for some n. Let Y be the quotient space of $F\oplus E_n$ (normed by $\|(f,x)\|=\|f\|+\|x\|$) modulo the subspace $\{(e,-Te);\, e\in E\}$. Denote by ψ the quotient map $F\oplus E_n\to Y$. Clearly ψ is an isometry on $\hat{E}_n=\{(0,x);\, x\in E_n\}$ and on $\hat{E}=\{(e,0);\, e\in E\}$ and $\psi\hat{E}\subset\psi\hat{E}_n$. Define $\hat{T}$: $\psi E_n\to E_n$ by $\hat{T}\psi(0,x)=x$. Then $\hat{T}$ is an isometry and $\hat{T}\psi(e,0)=Te$ for every $e\in E$. Thus in order to show that (*) holds it is enough to prove that given $\varepsilon>0$ there is a $\tilde{T}$: $Y\to X$ such that $\tilde{T}_{\psi(\hat{E}_n)}=\tilde{T}$ and $(1-\varepsilon)\|y\|\leqslant\|\tilde{T}y\|\leqslant(1+\varepsilon)\|y\|$ for $y\in Y$. In other words, it is no loss of generality to assume that $E=E_n$ and T is the identity. Finally, since $B(Y)$ is a polytope $Y\subset l_\infty^m$ for some m and so we may assume that $F=l_\infty^{n+r}$ for some positive integer r. Denote the embedding of $E=E_n$ into F by U. As remarked at the end of Section 3 there is an admissible basis $\{f_i\}_{i=1}^{n+r}$ in F so that

$$Ue_{i,n}=f_i+\sum_{j=1}^{r}\alpha_{i,j}f_{n+j},\quad i=1,2,\dots,n$$

with

$$\sum_{j=1}^{n}|\alpha_{i,j}|\leqslant 1,\quad 1\leqslant j\leqslant r.$$

Let $\varepsilon>0$. Choose $\delta>0$ so that if $|\beta_{i,j}-\alpha_{i,j}|\leqslant\delta$, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant r$ then there is an operator V from F into itself for which

$$V\Big(f_i+\sum_{j=1}^{r}\alpha_{i,j}f_{n+j}\Big)=f_i+\sum_{j=1}^{r}\beta_{i,j}f_{n+j},\ 1\leqslant i\leqslant n$$

and $(1-\varepsilon)\|f\|\leqslant\|Vf\|\leqslant(1+\varepsilon)\|f\|$ for every $f\in F$. By our assumption on the matrix A there are $k_r>k_{r-1}>\dots>k_1>n$ so that for every $1\leqslant j\leqslant r$

$$|\alpha_{i,j}-a_{i,k_j}|\leqslant\delta/2,\quad 1\leqslant i\leqslant n,\ \sum_{i>n}|a_{i,k_j}|\leqslant\delta/2.$$

By (3.11) it follows easily that

$$e_{i,n}=e_{i,k_r+1}+\sum_{m=n+1}^{k_r+1}\gamma_{i,m}e_{m,k_r+1},\quad 1\leqslant i\leqslant n$$

with $|\gamma_{i,k_j+1}-\alpha_{i,j}|\leqslant\delta$, $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant r$. Choose an operator V in F as above where we take $\beta_{i,j}=\gamma_{i,k_j+1}$. Let T_0 be the isometry from F into $E_{k_r+1}\subset X$ defined by

$$T_0f_i=e_{i,n}-\sum_{j=1}^{r}\gamma_{i,k_j+1}e_{k_j,k_r+1},\quad 1\leqslant i\leqslant n,$$

$$T_0 f_{n+j} = e_{k_j+1, k_r+1}, \quad 1 \leqslant j \leqslant r.$$

The operator $\tilde{T} = T_0 V$: $F \to X$ satisfies $\tilde{T}|_E = T$ (i.e. $\tilde{T}U = T$) and

$$(1-\varepsilon)\|f\| \leqslant \|\tilde{T}f\| \leqslant (1+\varepsilon)\|f\|$$

for all $f \in F$, as desired.

Remark. As A. Pełczyński pointed out to one of the authors it is not hard to see that if A is as in the statement of Theorem 5.6 then the space X which is represented by A satisfies that ext $B(X^*)$ is w^* dense in $B(X^*)$. Similarly, if $A = \{a_{i,n}\}$ is non-negative, $\Sigma_{i=1}^n a_{i,n} = 1$ for every n and $a^n = (a_{1,n}, ..., a_{n,n}, 0, \ldots)$ are dense in the positive face of the unit ball of l_1 then A represents a space $A(S)$ with S a Choquet simplex for which ext S is dense in S (i.e. a simplex of the type constructed by Poulsen [22]).

References

[1]. Alfsen, E. M., On the geometry of Choquet simplexes. *Math. Scand.*, 15 (1964), 97–110.
[2]. Day, M. M., *Normed linear spaces*. Springer, Berlin 1958.
[3]. Edwards, D. A., Minimum stable wedges of semicontinuous functions. *Math. Scand.*, 19 (1966), 15–26.
[4]. Effros, E. G., Structure in simplexes. *Acta Math.*, 117 (1967), 103–121.
[5]. Goullet de Rugy, A., *Géométrie des simplexes*. Centre de Documentation Universitaire et S.E.D.E.S., Paris 1969.
[6]. Grothendieck, A., Une caractérisation vectorielle métrique des espaces L_1. *Canad. J. Math.*, 7 (1955), 552–561.
[7]. Gurariĭ, V. I., Space of universal disposition, isotopic spaces and the Mazur problem on rotations of Banach spaces. *Sibirskii Mat. Zh.*, 7 (1966), 1002–1013.
[8]. Kakutani, S., Concrete representation of abstract M spaces. *Ann. of Math.*, 42 (1941), 994–1024.
[9]. Kuratowski, K., *Topology, Vol. 1.* Academic Press New York, 1966.
[10]. Lazar, A. J., Spaces of affine continuous functions on simplexes. *Trans. Amer. Math. Soc.*, 134 (1968), 503–525.
[11]. —— Polyhedral Banach spaces and extensions of compact operators. *Israel J. Math.*, 7 (1969), 357–364.
[12]. Lazar, A. J. & Lindenstrauss, J., On Banach spaces whose duals are L_1 spaces. *Israel J. Math.*, 4 (1966), 205–207.
[13]. Léger, Ch., Une démonstration du théorème de A. J. Lazar. *C. R. Acad. Sci. Paris*, 265 (1967), 830–831.
[14]. Lindenstrauss, J., Extension of compact operators. *Memoirs Amer. Math. Soc.*, No. 48, 1964.
[15]. Lindenstrauss, J. & Phelps, R. R., Extreme point properties of convex bodies in reflexive Banach spaces. *Israel J. Math.*, 6 (1968), 39–48.
[16]. Lindenstrauss, J. & Pełczynski, A., Absolutely summing operators in L_p spaces and their applications. *Studia. Math.*, 29 (1968), 275–326.
[17]. Lindenstrauss, J. & Wulbert, D. E., On the classification of the Banach spaces whose duals are L_1 spaces, *J. Funct. Anal.*, 4 (1969), 332–349.

[18]. MICHAEL, E. A., Continuous selections I. *Ann. of Math.*, 63 (1956), 361–382.
[19]. MICHAEL, E. A. & PEŁCZYNSKI, A., Separable Banach spaces which admit l_n^∞ approximations. *Israel J. Math.*, 4 (1966), 189–198.
[20]. PHELPS, R. R., Infinite-dimensional compact convex polytopes. *Math. Scand.*, 24, (1969), 5–26.
[21]. —— *Lectures on Choquet's theorem.* Van Nostrand, Princeton 1966.
[22]. POULSEN, E. T., A simplex with dense extreme points. *Ann. Inst. Fourier* (*Grenoble*), 11 (1961), 83–87.
[23]. ZIPPIN, M., On some subspaces of Banach spaces whose duals are L_1 spaces. *Proc. Amer Math. Soc.*, 23 (1969), 378–385.
[24]. LAZAR, A. J., The unit ball in conjugate L_1 spaces. (To appear.)

Received February 23, 1970