BANACH SPACES WITH THE EXTENSION PROPERTY

BY

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It is the object of this note to complete a characterization of those Banach spaces B with the Hahn-Banach extension property: each bounded linear function F on a subspace of any Banach space C with values in B has a linear extension F' carrying all of C into B such that $\|F'\| = \|F\|$. It is shown here that:

THEOREM. *Each such space B is equivalent to the space C_X of continuous real-valued functions on an extremally disconnected compact Hausdorff space X, C_X having the usual supremum norm.*

Recently, in these Transactions, Nachbin [N] and, independently, Goodner [G] have shown that if B has the extension property and if its unit sphere has an extreme point, then B is equivalent to a function space of this sort; both authors have also proved that such a function space has the extension property. The above theorem simply omits the extreme point hypothesis, and so establishes the equivalence.

My original proof, of which the proof given here is a distillate, depends on an idea of Jerison [J]. Briefly, letting X be the weak* closure of the set of extreme points of the unit sphere of the adjoint B^*, B can be shown equivalent to the space of all weak* continuous real functions f on X such that $f(x) = -f(-x)$, and then properties of X are deduced which imply the theorem. The same idea occurs implicitly in the proof below.

NOTE. Goodner asks [G, p. 107] if every Banach space having the extension property is equivalent to the conjugate of an abstract (L)-space. It is known (this is not my contribution) that the Birkhoff-Ulam example ([B, p. 186] or [HT, p. 490]) answers this question in the negative, the pertinent Banach space being the bounded Borel functions on $[0, 1]$ modulo those functions vanishing except on a set of the first category, with $\|f\| = \inf \{K : |f(x)| \leqq K$ *save on a set of first category*$\}$.

1. **Preliminary definitions and remarks.** A point x is an *extreme point* of a convex subset K of a real linear space if x is not an interior point of any line segment contained in K (i.e., if $x = ty + (1-t)z$, $0 < t < 1$, $y \in K$, and $z \in K$, then $x = y = z$). A set L is a *support* of K if L is a convex, nonvoid subset of K such that each line segment contained in K which has an interior point in L is contained in L. If x is an extreme point of L and L is a sup-

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port of K, then x is an extreme point of K. If F is a linear function carrying a convex set K into a convex set M and L is a support of M, then $F^{-1}(L)\cap K$ is either void or a support of K.

For each Banach space B the adjoint space is denoted by B^* and the weak* topology for B^* is the topology of pointwise convergence of functionals. Each convex, norm-bounded, weak* closed subset K (=convex, weak* compact subset) is, according to the classic theorem of Krein and Milman, the smallest convex weak* closed set which contains all extreme points of K. (See, for example, [K].) If F is a bounded linear function on B to a Banach space C, then F^*, the adjoint function, carries C^* into B^* in a weak* continuous fashion, and in particular, the image of the unit sphere of C^* is weak* compact.

A compact Hausdorff space is *extremally disconnected* if the closure of each open set is open. If X is a compact Hausdorff space, then C_X is the Banach space of all real-valued continuous functions on X, with the usual supremum norm. For each $x\in X$ there is assigned a functional e_x, by setting $e_x(f)=f(x)$ for $f\in C_X$. This functional e_x is the *evaluation at* x. It is known (see [AK]) that the set of extreme points of the unit sphere of C_X^* is precisely $E\cup(-E)$, where E is the set of all evaluations. Moreover, if E has the relativized weak* topology, then the function e carrying x into e_x maps X homeomorphically onto E.

2. Proof of the theorem. Let B be a Banach space with the property: if H is a linear isometry of B into a Banach space C, then there is a linear map G of norm one carrying C onto B such that GH is the identity map of B onto itself. Let X be the weak* closure of the set of all extreme points of the unit sphere of B^*. Then X is weak* compact. In what follows, a subset of X is "open" if it is "open in the relativized weak* topology for X," and the closure U^c of a subset U of X is the weak* closure of U.

Suppose, now, that U and V are open subsets of X such that both $U\cap V$ and $[-(U\cup V)]\cap(U\cup V)$ are void, and $[-(U\cup V)]\cup(U\cup V)$ is dense in X. We construct a space Y, by setting $Y=(\{0\}\times U^c)\cup(\{1\}\times V^c)$, so that Y consists of disjoint copies of U^c and V^c. The set Y is topologized by agreeing that if U_1 is open in U^c and V_1 is open in V^c, then $\{0\}\times U_1$ and $\{1\}\times V_1$ are each open in Y. Let H be the map of B into C_Y defined, for $b\in B$, $u\in U^c$, $v\in V^c$ by: $H(b)((0,u))=u(b)$, $H(b)((1,v))=v(b)$. The basic result about this construction is:

LEMMA. *The map H is a linear isometry of B onto C_Y. Moreover, $U^c\cap V^c$ and $[-(U^c\cup V^c)]\cap(U^c\cup V^c)$ are void, and H^* maps the set of evaluations in C_Y^* weak* homeomorphically onto $U^c\cup V^c$.*

Proof. We first verify that H is a linear isometry. The unit sphere S of B^* is weak* compact and, for each $b\in B$, the linear functional b', whose value at $z\in B^*$ is $z(b)$, is weak* continuous, and maps S onto the closed interval

$[-\|b\|, \|b\|]$. The set of points at which the functional b' assumes the value $\|b\|$ is a support of S and hence contains an extreme point x, which is a member of X. Either x or $-x$ belongs to $U^c \cup V^c$, and consequently $\|H(b)\| \geqq |x(b)| = \|b\|$. On the other hand, since $U^c \cup V^c$ is a subset of the unit sphere of B^*, $\|H(b)\| \leqq \|b\|$, so that H is an isometry.

Next, a small calculation. Suppose $e_{(0,u)} \in C_Y^*$ is the evaluation at $(0, u)$, and that $b \in B$. Then $H^*(e_{(0,u)})(b)$ is, by definition of H^*, $e_{(0,u)}(H(b))$, which from the definition of $e_{(0,u)}$ is $H(b)((0, u))$, and using the definition of H this is $u(b)$. Consequently, the valuation at $(0, u)$ maps under H^* onto u, and similarly the evaluation at $(1, v)$ maps onto v.

If $u \in U$ and u is an extreme point of the unit sphere S of B^*, then $H^{*-1}(u)$ intersects the unit sphere T of C_Y^* in a set which is a support of S. This support, being weak* compact, consists of a single point or else contains at least two extreme points (the Krein-Milman theorem). Each extreme point of the support is also an extreme point of T. But the extreme points of T are $\pm$ evaluations, and since $u \notin V^c$, the only extreme point which can map onto u under H^* is $e_{(0,u)}$, in view of the preceding paragraph. Consequently, $H^{*-1}(u) \cap T$ consists of the single point $e_{(0,u)}$ and similarly, if $v \in V$ and v is an extreme point of S, then $H^{*-1}(v) \cap T = \{e_{(1,v)}\}$.

Now let G be a linear function of norm one carrying C_Y onto B so that GH is the identity on B. Then G^* carries the unit sphere S of B^* into the unit sphere T of C_Y^* and $(GH)^* = H^*G^*$ is the identity on B^*. If $u \in U$ and u is an extreme point of S, then necessarily $G^*(u) = e_{(0,u)}$, in view of the preceding paragraph, and if $v \in V$ and v is an extreme point of S, then $G^*(v) = e_{(1,v)}$. Because such points are dense in U and in V the function G^* carries a dense subset of X onto a weak* dense subset of $E \cup (-E)$, where E is the set of evaluations. Because X and $E \cup (-E)$ are weak* compact G^* carries X onto $E \cup (-E)$. Now H^*G^* is the identity on B^*, and if $u \in U$ and u is an extreme point of S, then $G^*H^*(e_{(0,u)}) = G^*(u) = e_{(0,u)}$, and similarly for $v \in V$ and v extreme, so that G^*H^* is the identity on a dense subset of $E \cup (-E)$. Consequently G^* is, on X, a homeomorphism, and H^* is, on $E \cup (-E)$, the inverse of this homeomorphism. From the structure of $E \cup (-E)$ it follows (see preliminary remarks) that $U^c \cap V^c$ and $[-(U^c \cup V^c)] \cap (U^c \cup V^c)$ are void, and it is also clear that H^* maps E homeomorphically onto $U^c \cup V^c$.

It remains to show that H maps B onto C_Y. The image $G^*(S)$ of the unit sphere S of B^* is convex and weak* compact, and each extreme point of the unit sphere T of C_Y^*, as was shown in the preceding paragraph, belongs to $G^*(S)$. From the Krein-Milman theorem it follows that $T \subset G^*(S)$, and since G^* has norm one, $T = G^*(S)$. Since H^*G^* is the identity on B^* and since G^* maps B^* onto C_Y^*, it follows that H^* is 1-1. Because H^* is 1-1 it is true that H maps B onto C_Y, for otherwise there is a nonzero linear functional on C_Y which vanishes on the range of H (a closed subspace) and H^* applied to this functional gives the zero of B^*. The proof of the lemma is then complete.

The theorem is now established as follows. Choose, using Zorn's Lemma, an open subset W of X maximal with respect to the property that $(-W) \cap W$ be void. Then $(-W)\cup W$ is dense in X. Applying the lemma to $U=W$, $V=$void set, it follows that $(-W^c)\cap(W^c)$ is void, and that W^c is open as well as closed in X. Moreover H is an isometry of B onto C_Y, where Y is homeomorphic to W^c. Proceeding, let U be any open subset of W^c and let $V=W^c\backslash U^c$. Applying the lemma again, we see that $U^c\cap V^c$ is void so that U^c is open and it is proven that W^c is extremally disconnected, which establishes the theorem.

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