Band gaps in the relaxed linear micromorphic continuum

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Abstract

In this note we show that the relaxed linear micromorphic model recently proposed by the authors can be suitably used to describe the presence of band-gaps in metamaterials with microstructures in which strong contrasts of the mechanical properties are present (e.g. phononic crystals and lattice structures). This relaxed micromorphic model only provides 6 constitutive parameters instead of 18 parameters needed in Mindlin- and Eringen-type classical micromorphic models. We show that the onset of band-gaps is related to a unique constitutive parameter, the *Cosserat couple modulus* μ_c which starts to account for band-gaps when reaching a suitable threshold value. The limited number of parameters of our model, as well as the specific effect of some of them on wave propagation can be seen as an important step towards indirect measurement campaigns.

Keywords: relaxed micromorphic continuum, Cosserat couple modulus, wave band-gaps.

1 Introduction

Micromorphic models were originally proposed by Mindlin [16] and Eringen [5] in order to study materials with microstructure while remaining in the simplified framework of macroscopic continuum theories. Nevertheless, the huge number of material parameters (18 in the linear-isotropic case) limited up to now the application of these powerful theories to describe the behavior of real metamaterials (see e.g. [18]). In this paper, we propose to use the newly developed relaxed micromorphic model presented in [6, 19] to study wave propagation in microstructured materials which exhibit frequency band-gaps. The proposed relaxed model only counts 6 constitutive parameters and is fully able to account for the effect of microstructure on the macroscopic mechanical behavior of considered media. It is known that some materials like phononic crystals and lattice structures (see e.g. [24]), granular assemblies with defects (see e.g. [9, 13, 14, 15]) and composites ([4, 23]) can inhibit wave propagation in particular frequency ranges (band-gaps). The aim of this note is to show that the proposed relaxed model allows for describing frequency band-gaps by "switching on" a unique constitutive parameter which is known as *Cosserat Couple modulus* μ_c (see e.g. [7, 8, 17, 20, 21]). The limited number of constitutive parameters makes possible the future conception of direct and indirect measurements on real materials exhibiting frequency band-gaps. On the other hand, the generality of the proposed relaxed model can also be seen as a tool to aid the engineering design of new metamaterials with improved band-gap capabilities. Materials of this type could be used as an alternative to piezoelectric materials which are used today for vibration control and which are for this reason extensively studied in the literature (see e.g. [1, 3, 11, 12, 22, 25]). Because of the possible interest of our findings in a linear modelling framework, we summarize in this communication the main novel results on band gaps related to our relaxed micromorphic model.

2 Equations of motion in strong form

As shown in [19] the equations of motion of the considered linear relaxed isotropic micromorphic continuum read

$$\begin{aligned}\rho\,\mathbf{u}_{tt} =&\operatorname{Div}\left[2\,\mu_e\,\operatorname{sym}(\nabla\mathbf{u}-\mathbf{P})+\lambda_e\operatorname{tr}(\nabla\mathbf{u}-\mathbf{P})\,\mathbb{1}+2\,\mu_c\operatorname{skew}(\nabla\mathbf{u}-\mathbf{P})\right]\\ \eta\,\mathbf{P}_{tt} =& -\alpha_c\operatorname{Curl}(\operatorname{Curl}\mathbf{P})+2\,\mu_e\operatorname{sym}(\nabla\mathbf{u}-\mathbf{P})+\lambda_e\operatorname{tr}(\nabla\mathbf{u}-\mathbf{P})\,\mathbb{1}-2\,\mu_h\operatorname{sym}\mathbf{P}-\lambda_h\operatorname{tr}\mathbf{P}\,\mathbb{1}+2\,\mu_c\operatorname{skew}(\nabla\mathbf{u}-\mathbf{P}),\end{aligned}\tag{1}$$

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where $\mathbf{u} \in \mathbb{R}^3$ is the displacement field, $\mathbf{P} \in \mathbb{R}^{3\times 3}$ is the microdeformation tensor (basic kinematical fields), ρ and η are the macro and micro mass densities respectively and all other quantities are the constitutive parameters of the model. As for the operators appearing in (1), we use standard notation. It can be checked that when considering a completely one-dimensional case, the term $\mathrm{Curl}\,(\mathrm{Curl}\,\mathbf{P})$ vanishes and no characteristic length related to microstructure can be accounted for by our model. We need at least the case of plane waves (all the components of $\mathbf{u}$ and $\mathbf{P}$ are non-vanishing, but all depend on one space variable X which is also the direction of propagation of considered wave) to disclose all the characteristic features of the proposed relaxed model. On the other hand, Mindlin-Eringen models allow to account for characteristic lengths even when considering complete one-dimensional cases (all components of the kinematical fields in the plane orthogonal to the direction of propagation are vanishing). This is shown e.g. in [2, 10] in which these fully 1D equations are derived by the standard internal variable theory. We also remark that, in general, the relaxed term $\mathrm{Curl}\,(\mathrm{Curl}\,\mathbf{P})$ in the second of Eqs. (1) is much weaker than the full term $\Delta\mathbf{P}$ appearing in Mindlin and Eringen models. Despite this weaker formulation, we claim that the proposed relaxed model is fully able to account for the presence of microstructure on the overall mechanical behavior of considered continua. In particular, our relaxed model is able to account for the description of frequency band-gaps, while the classical Mindlin- and Eringen- type models are not.

In [19] it is also proved that positive definiteness of the strain energy density associated to Eqs.(1) implies

$$\mu_e > 0, \qquad \mu_c > 0, \qquad 3\lambda_e + 2\mu_e > 0, \qquad \mu_h > 0, \qquad 3\lambda_h + 2\mu_h > 0, \qquad \alpha_c > 0. \tag{2}$$

We limit ourselves to the case of plane waves travelling in an infinite domain, i.e. we suppose that the space dependence of all the introduced kinematical fields is limited only to the space component X which we also suppose to be the direction of propagation of the considered wave. We introduce the new variables

$$P^S := \frac{1}{3}\left(P_{11} + P_{22} + P_{33}\right), \quad P^D := \left(\mathrm{dev}\,\mathrm{sym}\,\mathbf{P}\right)_{11}, \quad P_{(1\xi)} = \left(\mathrm{sym}\,\mathbf{P}\right)_{1\xi}, \quad P_{[1\xi]} = \left(\mathrm{skew}\,\mathbf{P}\right)_{1\xi}, \quad \xi = 1, 2. \tag{3}$$

It is immediate that, according to the Cartan-Lie-algebra decomposition (see e.g. [19]), the component P_{11} of the tensor $\mathbf{P}$ can be rewritten as $P_{11} = P^D + P^S$. We also define the additional variables

$$P^V = P_{22} - P_{33}, \qquad P_{(1\xi)} = \left(\mathrm{sym}\,\mathbf{P}\right)_{1\xi}, \qquad P_{[1\xi]} = \left(\mathrm{skew}\,\mathbf{P}\right)_{1\xi}, \quad \xi = 1, 2. \tag{4}$$

We rewrite the equations of motion (1) in terms of the new variables (3), (4) and, of course, of the displacement variables. Before doing so, we introduce the quantities[1]

$$c_m = \sqrt{\frac{\alpha_c}{\eta}}, \qquad c_s = \sqrt{\frac{\mu_e + \mu_c}{\rho}}, \qquad c_p = \sqrt{\frac{\lambda_e + 2\mu_e}{\rho}},$$
$$\omega_s = \sqrt{\frac{2\left(\mu_e + \mu_h\right)}{\eta}}, \quad \omega = \sqrt{\frac{\left(3\lambda_e + 2\mu_e\right) + \left(3\lambda_h + 2\mu_h\right)}{\eta}}, \quad \omega_r = \sqrt{\frac{2\mu_c}{\eta}}, \quad \omega_l = \sqrt{\frac{\lambda_h + 2\mu_h}{\eta}}, \quad \omega_t = \sqrt{\frac{\mu_h}{\eta}}. \tag{5}$$

With the proposed new choice of variables and recalling that we are considering the case of planar waves, we are able to rewrite the governing equations (1) as different uncoupled sets of equations, namely:

- A set of three equations only involving longitudinal quantities (left) and two sets of three equations only involving transverse quantities in the k-th direction, with $\xi = 2, 3$ (right):

$$\begin{cases} \ddot{u}_1 = c_p^2 u_{1,11} - \frac{2\mu_e}{\rho} P^D_{,1} - \frac{3\lambda_e + 2\mu_e}{\rho} P^S_{,1}, \\ \\ \ddot{P}^D = \frac{4}{3}\frac{\mu_e}{\eta} u_{1,1} + \frac{1}{3} c_m^2 P^D_{,11} - \frac{2}{3} c_m^2 P^S_{,11} - \omega_s^2 P^D, \\ \\ \ddot{P}^S = \frac{3\lambda_e + 2\mu_e}{3\eta} u_{1,1} - \frac{1}{3} c_m^2 P^D_{,11} + \frac{2}{3} c_m^2 P^S_{,11} - \omega_p^2 P^S, \end{cases} \qquad \begin{cases} \ddot{u}_\xi = c_s^2 u_{\xi,11} - \frac{2\mu_e}{\rho} P_{(1\xi),1} + \frac{\eta}{\rho}\omega_r^2 P_{[1\xi],1}, \\ \\ \ddot{P}_{(1\xi)} = \frac{\mu_e}{\eta} u_{\xi,1} + \frac{1}{2} c_m^2 P_{(1\xi),11} + \frac{1}{2} c_m^2 P_{[1\xi],11} - \omega_s^2 P_{(1\xi)}, \\ \\ \ddot{P}_{[1\xi]} = -\frac{1}{2}\omega_r^2 u_{\xi,1} + \frac{1}{2} c_m^2 P_{(1\xi),11} + \frac{1}{2} c_m^2 P_{[1\xi],11} - \omega_r^2 P_{[1\xi]}, \end{cases} \tag{6}$$

- Three uncoupled equations only involving the variables $P_{(23)}$, $P_{[23]}$ and P^V respectively

$$\ddot{P}_{(23)} = -\omega_s^2 P_{(23)} + c_m^2 P_{(23),11}, \quad \ddot{P}_{[23]} = -\omega_r^2 P_{[23]} + c_m^2 P_{[23],11}, \quad \ddot{P}^V = -\omega_s^2 P^V + c_m^2 P^V_{,11}. \tag{7}$$

These 12 scalar differential equations will be used to study wave propagation in our relaxed micromorphic media.

[1] Due to the chosen values of the parameters, which are supposed to satisfy (2), all the introduced characteristic velocities and frequencies are real. Indeed, the condition $(3\lambda_e + 2\mu_e) > 0$, together with the condition $\mu_e > 0$, imply the condition $(\lambda_e + 2\mu_e) > 0$.

3 Plane wave propagation

We now look for a wave form solution of the previously derived equations of motion. We start from the uncoupled equations (7) and assume that the involved unknown variables take the harmonic form

$$P_{(23)} = Re\left\{\beta_{(23)}e^{i(kX-\omega t)}\right\}, \quad P_{[23]} = Re\left\{\beta_{[23]}e^{i(kX-\omega t)}\right\}, \quad P^V = Re\left\{\beta^V e^{i(kX-\omega t)}\right\}, \tag{8}$$

where $\beta_{(23)}$, $\beta_{[23]}$ and β^V are the amplitudes of the three introduced waves. Replacing this wave form in Eqs. (7) and simplifying one obtains the following dispersion relations respectively:

$$\omega(k) = \sqrt{\omega_s^2 + k^2 c_m^2}, \qquad \omega(k) = \sqrt{\omega_r^2 + k^2 c_m^2}, \qquad \omega(k) = \sqrt{\omega_s^2 + k^2 c_m^2}. \tag{9}$$

We notice that for a vanishing wave number ($k = 0$) the dispersion relations for the three considered waves give non-vanishing frequencies so that these waves are so-called *optic waves* with cutoff frequencies ω_s, ω_r and ω_s respectively.

We now introduce the unknown vectors $\mathbf{v}_1 = \left(u_1, P^D, P^S\right)$ and $\mathbf{v}_\xi = \left(u_\xi, P_{(1\xi)}, P_{[1\xi]}\right)$, $\xi = 2, 3$ and look for wave form solutions of equations (6) in the form

$$\mathbf{v}_1 = Re\left\{\boldsymbol{\beta} e^{i(kX-\omega t)}\right\}, \qquad \mathbf{v}_\xi = Re\left\{\boldsymbol{\gamma}^\xi e^{i(kX-\omega t)}\right\}, \ \xi = 2, 3, \tag{10}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)^T$ and $\boldsymbol{\gamma}^\xi = (\gamma_1^\xi, \gamma_2^\xi, \gamma_3^\xi)^T$ are the unknown amplitudes of the considered waves. Replacing this expressions in equations (6) one gets respectively

$$\mathbf{A}_1 \cdot \boldsymbol{\beta} = 0, \qquad \mathbf{A}_\xi \cdot \boldsymbol{\gamma}^\xi = 0, \qquad \xi = 2, 3, \tag{11}$$

where

$$\mathbf{A}_1 = \begin{pmatrix} -\omega^2 + c_p^2 k^2 & i\,k\,2\mu_e/\rho & i\,k\,\left(3\lambda_e + 2\mu_e\right)/\rho \\ -i\,k\,\frac{4}{3}\,\mu_e/\eta & -\omega^2 + \frac{1}{3}k^2 c_m^2 + \omega_s^2 & -\frac{2}{3}\,k^2 c_m^2 \\ -\frac{1}{3}\,i\,k\,\left(3\lambda_e + 2\mu_e\right)/\eta & -\frac{1}{3}\,k^2\,c_m^2 & -\omega^2 + \frac{2}{3}\,k^2\,c_m^2 + \omega_p^2 \end{pmatrix},$$

$$\mathbf{A}_2 = \mathbf{A}_3 = \begin{pmatrix} -\omega^2 + k^2 c_s^2 & i\,k\,2\mu_e/\rho & -i\,k\,\frac{\eta}{\rho}\omega_r^2, \\ -\,i\,k\,2\mu_e/\eta, & -2\omega^2 + k^2 c_m^2 + 2\omega_s^2 & k^2 c_m^2 \\ i\,k\,\omega_r^2 & k^2 c_m^2 & -2\omega^2 + k^2 c_m^2 + 2\omega_r^2 \end{pmatrix}.$$

In order to have non-trivial solutions of the algebraic systems (11), one must impose that

$$\det \mathbf{A}_1 = 0, \qquad \det \mathbf{A}_2 = 0, \qquad \det \mathbf{A}_3 = 0, \tag{12}$$

which are the so-called dispersion relations $\omega = \omega(k)$ for the longitudinal and transverse waves in the relaxed micromorphic continuum.

4 Numerical results

In this section, following Mindlin [16, 5], we will show the dispersion relations $\omega = \omega(k)$ associated to our relaxed micromorphic model.

Parameter	Value	Unit
μ_e	200	MPa
$\lambda_e = 2\mu_e$	400	MPa
$\mu_c = 2.2\mu_e$	440	MPa
μ_h	100	MPa
λ_h	100	MPa
$L_c = L_g$	3	mm

Parameter	Value	Unit
$\alpha_c = \mu_e L_c^2$	1.8×10^{-3}	$MPa\, m^2$
$\alpha_g = \mu_e L_g^2$	1.8×10^{-3}	$MPa\, m^2$
ρ	2000	Kg/m^3
ρ'	2500	Kg/m^3
d	2	mm
$\eta = d^2\,\rho'$	10^{-2}	Kg/m

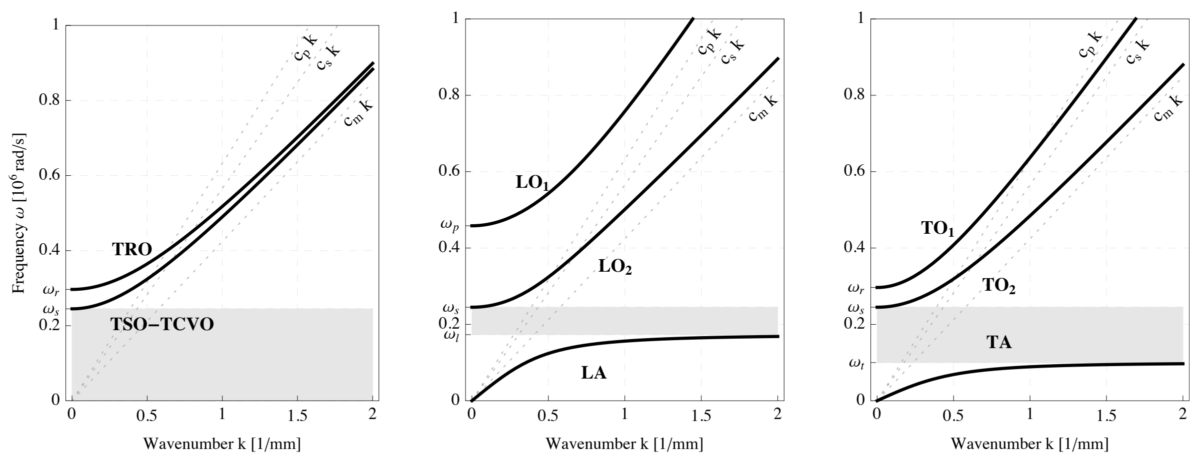
Parameter	Value	Unit
λ	82.5	MPa
μ	66.7	MPa
E	170	MPa
ν	0.28	$-$

Table 1: Values of the parameters of the relaxed model used in the numerical simulations (left and center) and corresponding values of the Lamé parameters and of the Young modulus and Poisson ratio (right).

We start by showing in Tab.1 (left and center) the values of the parameters of the relaxed model used in the performed numerical simulations. In order to make the obtained results better exploitable, we also recall that in [19, 18] the following homogenized formulas were obtained which relate the parameters of the relaxed model to the macroscopic Lamé parameters λ and μ which are usually measured by means of standard mechanical tests

$$\mu_e = \frac{\mu_h\,\mu}{\mu_h - \mu}, \qquad 2\mu_e + 3\lambda_e = \frac{(2\mu_h + 3\lambda_h)\,(2\mu + 3\lambda)}{(2\mu_h + 3\lambda_h) - (2\mu + 3\lambda)}. \tag{13}$$

These relationships imply that the following inequalities are satisfied: $\mu_h > \mu$, $3\lambda_h + 2\mu_h > 3\lambda + 2\mu$. It is clear that, once the values of the parameters of the relaxed models are known, the standard Lamé parameters can be calculated by means of formulas (13), which is what was done in Tab.1 (right). For completeness, we also show in the same table the corresponding Young modulus and Poisson ratio, calculated by means of the standard formulas. Figure 1 shows the dispersion relations for the considered relaxed micromorphic continuum. It can be easily noticed



(a) (b) (c)

Figure 1: Dispersion relations $\omega = \omega(k)$ for the relaxed model: uncoupled waves (a), longitudinal waves (b) and transverse waves (c). TRO: transverse rotational optic, TSO: transverse shear optic, TCVO: transverse constant-volume optic, LA: longitudinal acoustic, LO_1-LO_2: first and second longitudinal optic, TA: transverse acoustic, TO_1-TO_2: first and second transverse optic.

from Fig.1 that there exist a frequency range in which no propagative wave can be found. This means that the wavenumber becomes imaginary and only standing waves exist. It is evident (see Eqs. (5)) that, in general, the relative positions of the horizontal asymptotes ω_l and ω_t as well as of the cutoff frequencies ω_s, ω_r and ω_p can vary depending on the values of the constitutive parameters (see (5)). It can also be checked that, in the case in which $\lambda_e > 0$ and $\lambda_h > 0$ one always has $\omega_p > \omega_s > \omega_t$ and $\omega_l > \omega_t$. The relative position of ω_l and of ω_s can vary depending on the values of the parameters λ_h and μ_h. It can be checked that, in order to have a global band-gap, the following conditions must be simultaneously satisfied: $\omega_s > \omega_l$ and $\omega_r > \omega_l$. In terms of the constitutive parameters of the relaxed model, we can say that global band-gaps can exist, in the case in which one considers positive values for the parameters λ_e and λ_h , if and only if we have simultaneously

$$0 < \mu_e < +\infty, \qquad 0 < \lambda_h < 2\mu_e, \qquad \mu_c > \frac{\lambda_h + 2\mu_h}{2}. \tag{14}$$

As far as negative values for λ_e and λ_h are allowed, the conditions for band gaps are not so straightforward as (14), but we do not consider this possibility in this note.

We conclude by saying that the relaxed micromorphic model proposed in [6, 19] is able to describe the presence of frequency band-gaps in which no wave propagation can occur. The presence of band-gaps is intrinsically related to a critical value of the Cosserat couple modulus μ_c (see [7, 8, 17, 20, 21] for its interpretation) which must be greater than a threshold value in order to let band-gaps appear. This parameter can hence be seen as a discreteness quantifier which starts accounting for lattice discreteness as soon as it reaches the threshold value specified in Eq.(14). This fact is a novel feature of the introduced relaxed model: we claim that neither the classical micromorphic continuum model nor the Cosserat and the second gradient ones are able to predict such band-gap phenomena.

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