Bandwidth Efficient Coding:
Theoretical Limits and Real Achievements

Interim Report
Error Control Techniques
for Satellite and Space Communications
NASA Grant NAG5-557
September 1993

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Introduction

- In his seminal 1948 paper "*The Mathematical Theory of Communication*", Claude E. Shannon derived the "channel coding theorem" which gave an explicit upper bound, called the channel capacity, on the rate at which "information" could be transmitted reliably on a given communication channel.

- Shannon's result was an existence theorem and did not give specific codes to achieve the bound. Some skeptics have claimed that the dramatic performance improvements predicted by Shannon are not achievable in practice.

- The advances made in the area of coded modulation in the past decade have made communications engineers optimistic about the possibility of achieving or at least coming close to channel capacity. Here we consider this possibility in the light of current research results.

Channel Capacity

- With respect to coding and coded modulation, the most relevant of Shannon's results is the "noisy channel coding theorem for continuous channels with average power limitations."

- This theorem states that for any transmission rate R less than or equal to the channel capacity, C, there exists a coding scheme that achieves an arbitrarily small probability of error!

- Conversely, if R is greater than C, no coding scheme can achieve reliable communication, *regardless of complexity.*

- Shannon then shows that the capacity, C, of a continuous additive white Gaussian noise (AWGN) channel with bandwidth B and assuming Nyquist signaling is given by

$$C = B \log_2 \left(1 + \frac{E_s}{N_0} \right) \; bits/sec, \tag{1}$$

where E_s is the average signal energy in each signaling interval T and $N_0/2$ is the two sided noise power spectral density.

- This bound represents the absolute best performance possible for a communication system on the AWGN channel.

Restatement of the Capacity Bound

- Shannon's capacity bound can be put in a form more useful for the present discussion by introducing the parameter η, called spectral efficiency, to represent the average number of information bits transmitted per signaling interval.

- From Shannon's bound, it follows that

$$0 \leq R \leq C \ \ bits/sec,$$

and hence

$$0 \leq \eta \leq C/B \ \ bits/signal.$$

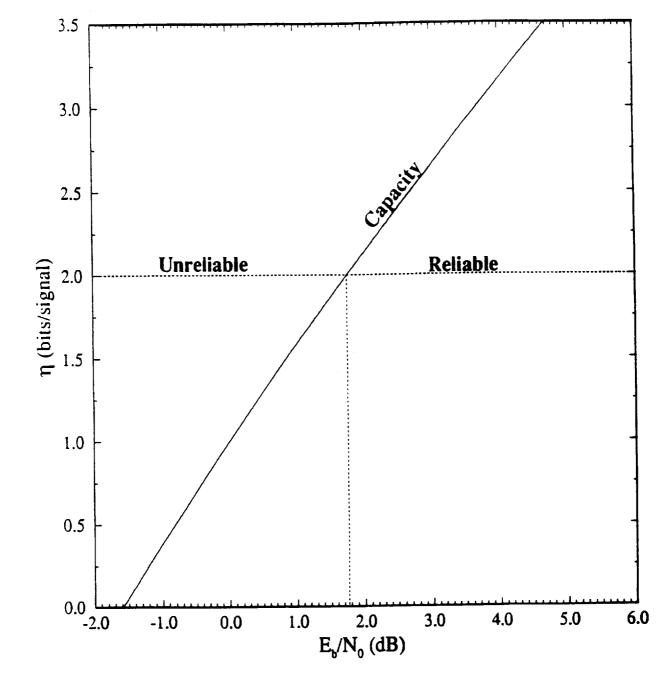- Substituting the relation

$$E_s/N_0 = \eta E_b/N_0,$$

where E_b is the average energy per information bit, into equation (1) and performing some minor manipulations yields

$$E_b/N_0 \geq \frac{2^\eta - 1}{\eta}, \tag{2}$$

which relates the spectral efficiency, η, to the signal-to-noise ratio (SNR), E_b/N_0.

- The bound of equation (2) manifests the fundamental tradeoff between spectral efficiency and SNR. That is, increased spectral efficiency can be reliably achieved only with a corresponding increase in SNR.
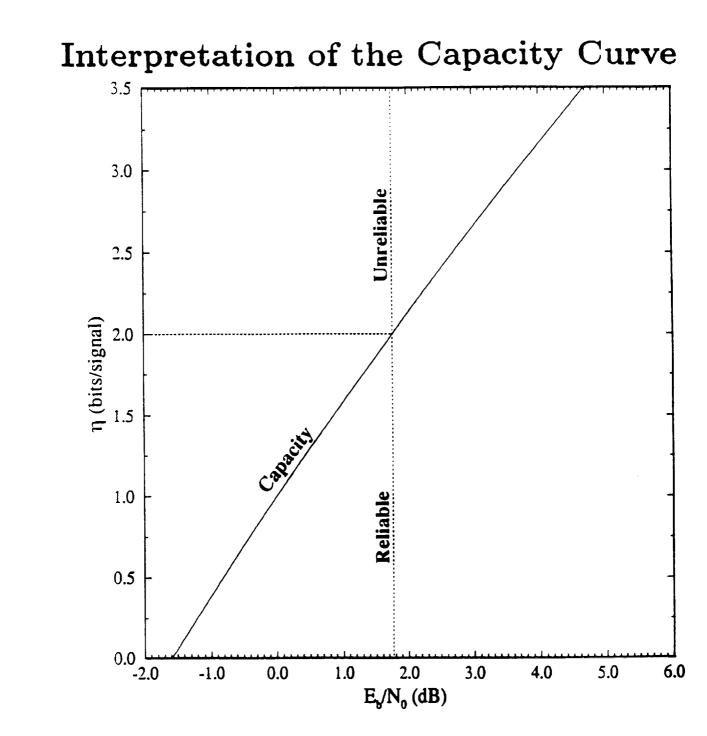
Interpretation of the Capacity Curve

- Shannon's bound gives the minimum signal-to-noise ratio (SNR) required to achieve a specific bandwidth efficiency with an arbitrarily small probability of error.
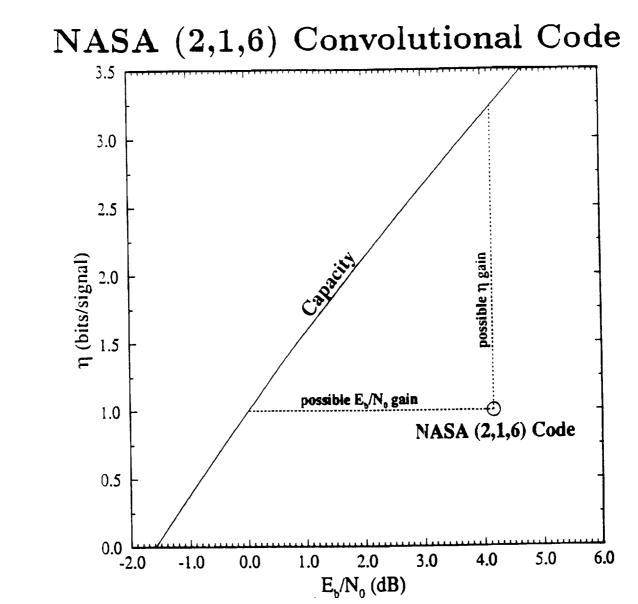
- *Example:* With $\eta = 2$ information bits per channel signal, there exists a coding scheme that operates reliably with an SNR of 1.76dB.

- Conversely, any coding scheme sending $\eta = 2$ information bits per signal with an SNR *less than* 1.76dB will be unreliable, *regardless of complexity.*

Interpretation of the Capacity Curve

- **Alternatively, Shannon's bound gives the maximum achievable spectral efficiency for a specific signal-to-noise ratio (SNR).**

- *Example:* **With an SNR of $E_b/N_0 = 1.76$db, there exists a coding scheme cabable of transmitting reliably with a spectral efficiency of $\eta = 2$ bits per signal.**

- **Conversely, any coding scheme operating with an SNR of $E_b/N_0 = 1.76$dB and attempting to transmit *more than* $\eta = 2$ bits per signal will be unrealiable *regardless of complexity.***
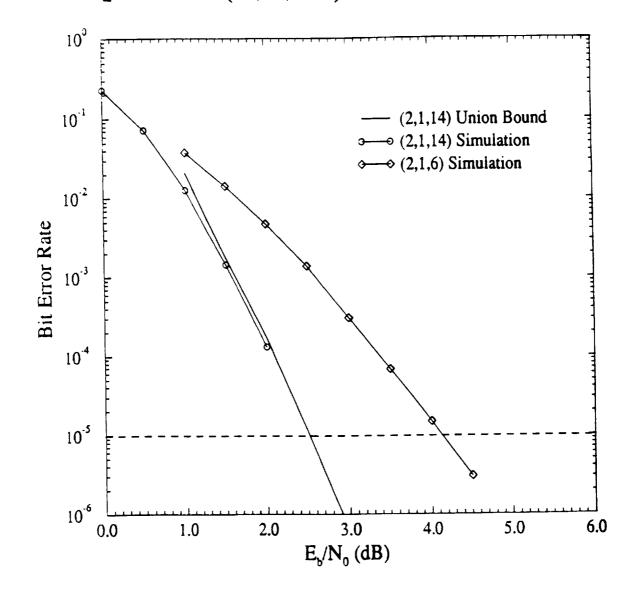
NASA (2,1,6) Convolutional Code

- Historically, most real communications systems operated well below capacity.

- *Example*: The NASA standard rate (2,1,6) convolutional code with QPSK modulation achieves a spectral efficiency of $\eta = 1$ bit/signal and requires a signal-to-noise ratio (SNR) of $E_b/N_0 = 4.15$ dB to achieve error free (10^{-5} bit error rate) communication.

- An ideal system operating with the same $E_b/N_0 = 4.15$dB can achieve error free communication with a spectral efficiency as high as $\eta = C = 3.235$ bits per signal.

OR

An ideal system operating with the same spectral efficiency of $\eta = 1$ bit per signal would require an SNR of only $E_b/N_0 = 0.0$ dB.
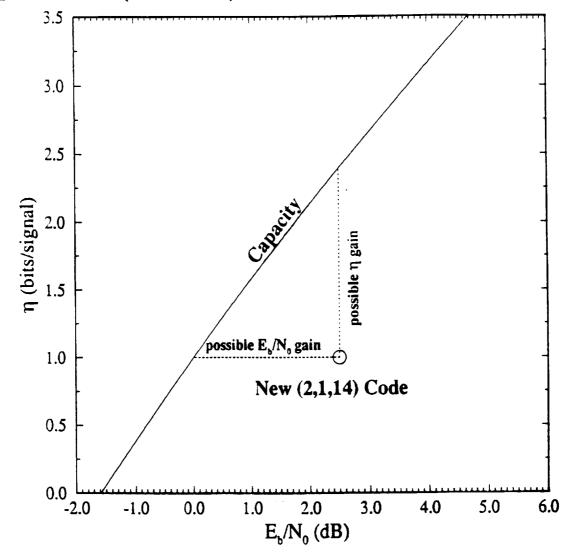
A New Optimal (2,1,14) Convolutional Code

- A computer search has found the optimum distance spectrum (ODS) (2,1,14) convolutional code.

- This code has optimum minimum free Hamming distance, $d_{free} = 18$, and the smallest number of nearest neighbors, $N_{free} = 26$, of any constraint length 15 code.

- The performance of this code is 1.65 dB better than the (2,1,6) code, but is still 2.5dB away from capacity.

Optimal (2,1,14) Convolutional Code

- The new optimal (2,1,14) convolutional code requires a signal-to-noise ratio (SNR) of $E_b/N_0 = 2.5$ dB to achieve error free (10^{-5} bit error rate) communication.

- An ideal system operating with the same $E_b/N_0 = 2.5$dB can achieve error free communication with a spectral efficiency as high as $\eta = C = 2.4$ bits per signal.

OR

An ideal system operating with the same spectral efficiency of $\eta = 1$ bit per signal would require an SNR of only $E_b/N_0 = 0.0$ dB.

Practical Bounds

- In real communication systems, there are many practical considerations that take precedence over Shannon's bound in design decisions.

- For example, satellite communication systems that use nonlinear travelling wave tube amplifiers (TWTA's) require constant envelope signaling such as M-ary phase shift keying (MPSK).

- Thus, even *if* Shannon's results firmly stated that capacity at a spectral efficiency of $\eta = 3$ bits per signal can be achieved with a (4,3,8) convolutional code using 16 QAM, it would not be feasible to do so on the TWTA satellite link.

- It therefore seems reasonable to ask what is the minimum SNR required to achieve reliable communication, *given* a particular modulation scheme and a spectral efficiency, η.
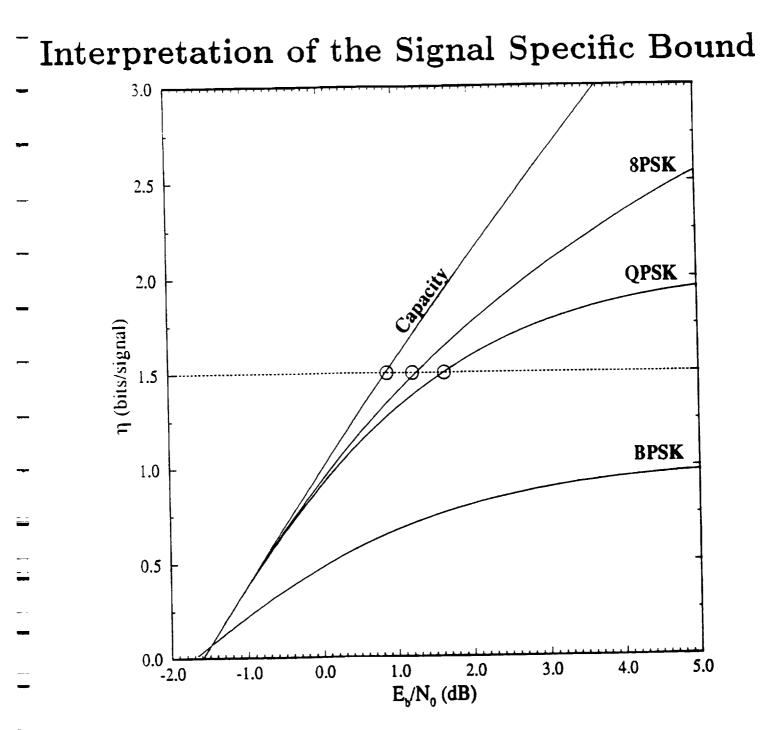
A Signal Specific Bound

- For the discrete input, continuous output, memoryless AWGN channel with M-ary one dimensional amplitude modulation (AM) or two dimensional (PSK, QAM) modulation and equiprobable signaling, the capacity bound becomes

$$\eta^* = \log_2 (M) - \frac{1}{M} \sum_{i=0}^{M-1} E \left\{ \log_2 \sum_{j=0}^{M-1} \exp \left[\frac{|a^i + n - a^j|^2 - |n|^2}{N_0} \right] \right\}, \quad (3)$$
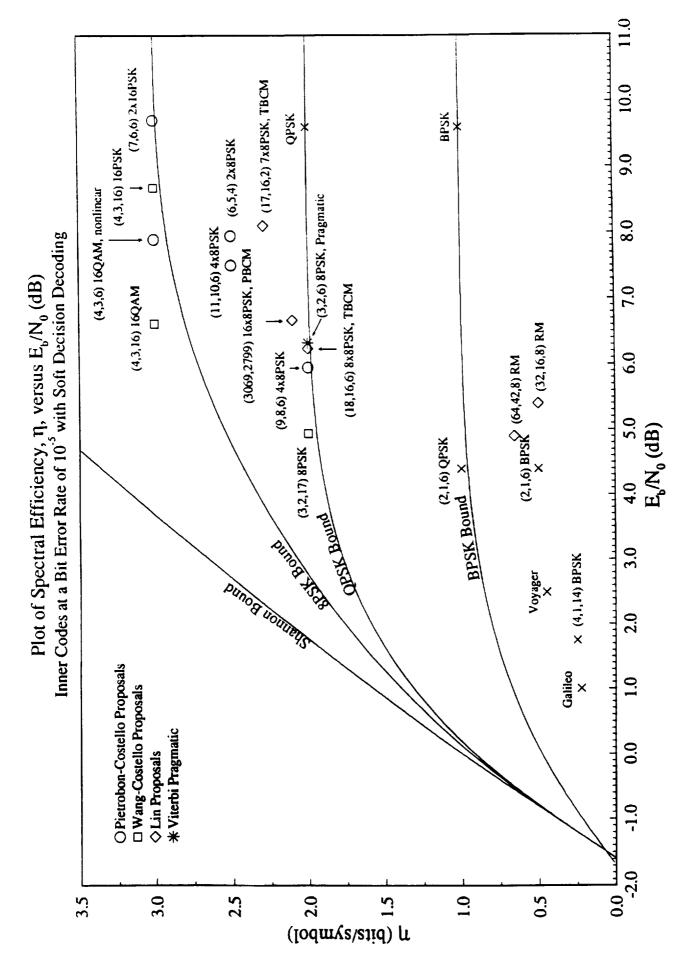
- Here

$$\left\{ a^j, j = 0, 1, \ldots M - 1 \right\} \qquad (4)$$

  is an M-ary modulations set, a^j is a channel signal, n is a Gaussian distributed noise random variable with mean 0 and variance $N_0/2$, and E is the expectation operator.

- For a specified signaling method and spectral efficiency, this bound can be used to compute the minimum SNR required to achieve reliable communication.

Interpretation of the Signal Specific Bound

- To send $\eta = 1.5$ information bits per signaling interval, an ideal system *using QPSK modulation* requires a minimum SNR of $E_b/N_0 \approx$ 1.64dB. This is **0.76 dB more than an ideal system without any modulation constraints.**

- To send $\eta = 1.5$ information bits per signaling interval, an ideal system *using 8PSK modulation* requires a minimum SNR of $E_b/N_0 \approx$ 1.22dB. This is **0.34 dB more than an ideal system without any modulation constraints.**

Plot of Spectral Efficiency, η, versus E_b/N_0 (dB)
Inner Codes at a Bit Error Rate of 10^{-5} with Soft Decision Decoding

Notes About the Graph

All results are with soft decision decoders.

1 NASA Codes: denoted by ×

Galileo: This is the concatenated code used on the Galileo probe.

(4,1,14) BPSK The is the (4,1,14) convolutional code developed by JPL.

Voyager: This is the concatenated code used on the Voyager probe.

(2,1,6) BPSK: This is the NASA standard (2,1,6) convolutional code with BPSK modulation.

(2,1,6) QPSK: This is the NASA standard (2,1,6) convolutional code with QPSK modulation.

2 Pietrobon-Costello Codes: denoted by ◯

(6,5,4) 2x8PSK: This is a rate 5/6, 16 state, trellis code using 2x8PSK modulation with Viterbi decoding. It has a spectral efficiency of $\eta = 2.5$ bits/signal and is 45° rotationally invariant. A Viterbi decoder for this code has been built by Steven Pietrobon and tested at New Mexico State.

(9,8,6) 4x8PSK: This is a rate 8/9, 64 state, trellis code using 4x8PSK modulation with Viterbi decoding. It has a spectral efficiency of $\eta = 2.0$ bits/signal and is 45° rotationally invariant.

(11,10,6) 4x8PSK: This is a rate 10/11, 64 state, trellis code using 4x8PSK modulation with Viterbi decoding. It has a spectral efficiency of $\eta = 2.5$ bits/signal and is 45° rotationally invariant.

(7,6,6) 2x16PSK: This is a rate 6/7, 64 state, trellis code using 2x16PSK modulation with Viterbi decoding. It has a spectral efficiency of $\eta = 3.0$ bits/signal and is 45° rotationally invariant.

(4,3,6) 16QAM, nonlinear: This is a nonlinear rate 3/4, 16 state, trellis code using 16QAM modulation with Viterbi decoding. It has a spectral efficiency of $\eta = 3.0$ bits/signal and is 90° rotationally invariant.

3 Wang-Costello Codes: denoted by □

(3,2,17)8PSK: This is a rate 2/3, memory 17, trellis code using 8PSK modulation and sequential decoding with a modified Fano algorithm. It has a spectral efficiency of $\eta = 2.0$ bits/signal and is 180° rotationally invariant.

(4,3,16) 16PSK: This a rate 3/4, memory 16, trellis code using 16PSK modulation and sequential decoding with a modified Fano algorithm. It has a spectral efficiency of $\eta = 3.0$ bits/signal and is 180° rotationally invariant.

(4,3,16) 16QAM: This is a rate 3/4, memory 16, trellis code using 16QAM modulation and sequential decoding with a modified Fano algorithm. It has a spectral efficiency of $\eta = 3.0$ bits/signal and is 180° rotationally invariant.
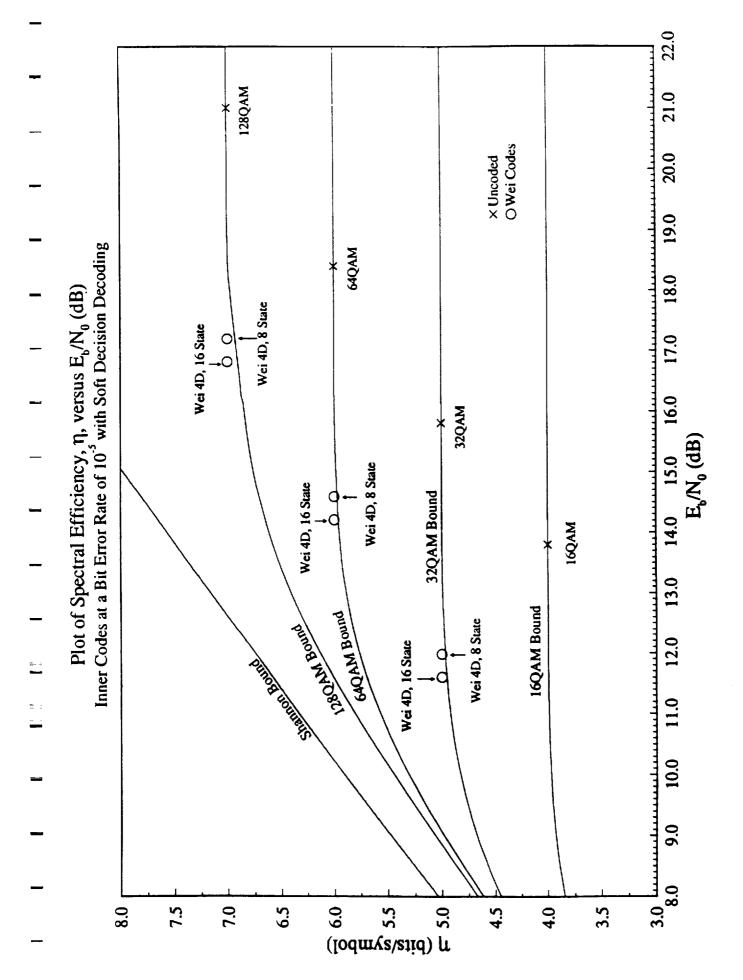
4 Lin Codes: denoted by ◇

(32,16,8) RM: This is a rate 16/32=0.5, 64 state Reed-Muller code using BPSK modulation and Viterbi decoding.

(64,42,8) RM: This is a rate 42/64=0.656, 1024 state Reed-Muller code using BPSK modulation and Viterbi decoding.

(17,16,2) 7x8PSK, TBCM: This is rate 16/17, 4 state, block coded modulation scheme using 7x8PSK modulation with Viterbi decoding. It has a spectral efficiency of $\eta = 2.286$ bits/signal and is 180° rotationally invariant.

(18,16,6) 8x8PSK, TBCM: This is a 2-level trellis code using 8x8PSK modulation. The first level has 64 states and is decoded with a Viterbi decoder. The second level has 8 states and is deocded with a Viterbi decoder. It has a spectral efficiency of $\eta = 2.0$ bits/symbol and is 180° rotationally invariant.

(3069,2799) 16x8PSK, PBCM: This is a 3x3 product block coded modulation scheme. The horizontal codes are BCH codes and the vertical code is a 3-level block code. It is decoded using suboptimal multi-stage decoding. It has a spectral efficiency of $\eta = 2.1$ bits/signal and is 45° rotationally invariant.

5 Viterbi Pragmatic Code: denoted by *

(3,2,6) 8PSK, Pragmatic: This is a rate 2/3, 64 state, trellis code using 8PSK modulation and Viterbi decoding. It uses the NASA standard (2,1,6) convolutional code as its basis and is suboptimal. It can be decoded using essentially the same Viterbi decoding chip that is used to decode the NASA standard convolutional code.

Plot of Spectral Efficiency, η, versus E_b/N_0 (dB)
Inner Codes at a Bit Error Rate of 10^{-5} with Soft Decision Decoding

Notes About the Graph

All results are with soft decision decoders.

1   Uncoded Systems: denoted by ×

16QAM: Performance of uncoded 16QAM with a spectral efficiency of $\eta = 4$ bits/symbol from simulation results.

32QAM: Performance of uncoded 32QAM with a spectral efficiency of $\eta = 5$ bits/symbol from simulation results.

64QAM: Performance of uncoded 64QAM with a spectral efficiency of $\eta = 6$ bits/symbol from simulation results.

128QAM: Performance of uncoded 128QAM with a spectral efficiency of $\eta = 7$ bits/symbol from simulation results.

2   Wei Codes: denoted by ○

Wei 4D, 8 state: This is Wei's 8 state code using a 4-dimensional constellation (of any size). Performance taken from "Coset Codes - Part 1: Introduction and Geometrical Classification," by G. David Forney. The performance was estimated taking into account the minimum squared Euclidean distance and the number of nearest neighbors. Thus, this point shows effective coding gain at a bit-error-rate (BER) of 10^{-5}.

Wei 4D, 16 state: This is Wei's 16 state code using a 4-dimensional constellation (of any size). Performance taken from "Coset Codes - Part 1: Introduction and Geometrical Classification," by G. David Forney. The performance was estimated taking into account the minimum squared Euclidean distance and the number of nearest neighbors. Thus, this point shows effective coding gain at a bit-error-rate (BER) of 10^{-5}.
This code with a 2x192QAM constellation is being considered by CCITT for the V.FAST ultimate modem standard.