Aust. J. Phys., 1989, **42**, 129–45

Baryon Structure and QCD

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Abstract

From QCD we derive a three-body Faddeev-type formulation of baryons, as qqq colour-singlet states bound by gluon exchange, which is covariant, has dynamically hidden chiral symmetry and incorporates the colour dynamics. The formulation exploits the dynamical role of colour $\overline{\mathbf{3}}$ diquark substructure in baryons to simplify computations. For non-zero current quark masses the $J^P = \frac{1}{2}^+$ and $\frac{1}{2}^-$ baryon octet mass formulae are shown to satisfy the Gell-Mann–Okubo and the Coleman-Glashow multiplet mass relationships. The $\frac{1}{2}^+$ baryon multiplet mass formulae in conjunction with the mass formulae for the Nambu-Goldstone boson multiplet are used to extract from the corresponding experimental data the chiral-limit $\frac{1}{2}^+$ multiplet mass of $M_{X^+} = 912$ MeV and two parameters characterising the baryon wavefunction for this multiplet. An analysis of the incomplete experimental $\frac{1}{2}^-$ mass spectrum yields a chiral mass of $M_{X^-} \approx 1511$ MeV together with structure information for this multiplet.

1. Introduction

Quantum chromodynamics (QCD) is the model of hadrons in which colour-octet gluon exchange between colour-triplet quarks leads to colour-singlet bound states—the hadrons. The mesons are $\overline{q}q$ bound states and the baryons are qqq bound states with some meson dressing. There are three key dynamical aspects of QCD needed to understand baryons. The first is the colour algebra. In a colour-singlet qqq state any two quarks are in a colour $\overline{3}$ state and for such states colour-octet spin-1 gluon exchange produces an attractive force between the two quarks, and so all three quarks are mutually attracted. Second and of great importance to all hadrons containing low mass quarks are the consequences of the approximate chiral symmetry of the QCD action. In the chiral limit (i.e. massless quarks) the dynamics induces this chiral symmetry to become a hidden symmetry, which in part means that the vacuum is degenerate. Though this degeneracy is lifted by current quark masses the fact that for the low mass u,d and s quarks the vacuum is almost degenerate has important consequences. In the meson sector this hidden chiral symmetry largely determines the pseudoscalar meson properties since they are the Nambu-Goldstone bosons associated with this realisation of the chiral symmetry. For the baryons it is necessary for any approximate treatment of the quark-gluon dynamics leading to baryon structure to be chirally covariant,

0004-9506/89/020129$03.00

that is, to respect this hidden chiral symmetry. It is known, for example, that bag models which have a scalar confining potential do not respect this symmetry, and various ad hoc schemes to restore chiral covariance have been employed in these phenomenologies. Our analysis of baryon structure avoids such problems since it does not assume any such potentials. The requirement of chiral invariance also determines the baryon- Nambu-Goldstone coupling, which in the world of nuclear physics means the nucleon-pion coupling. The third dynamical requirement of hadron structure calculations in QCD is that they be Lorentz covariant. Most of the models of baryons studied up to now were not covariant. This leads to various difficulties such as the need to remove centre-of-mass motion energies from calculated masses and related problems in the calculation of structure functions.

Of course an important requirement of any calculation of hadron properties in the context of QCD is that the treatment must be derivable from QCD. While approximations will always be employed a necessary property of such calculations is that the derivation provides a systematic scheme for extending the calculations in a manner controlled by QCD itself.

We present here an approximation scheme for baryons which satisfies the above requirements; that is, it is derived from QCD in a systematic way, includes the colour algebra, the hidden chiral symmetry and is Lorentz covariant. This means we must treat the baryons as a chirally-covariant relativistic three-body problem, that is, using covariant Faddeev type coupled integral equations, with the quarks bound by gluon exchange and with each quark self-interacting via gluons, this last aspect being intimately related to the hidden chiral symmetry. This produces a bare baryon, and chiral covariance will, to a large extent, then control the meson dressing of this bare baryon by Nambu-Goldstone bosons. Such a three-body problem would lead to excessively difficult computations were it not for an ideal feature of baryons related to the colour algebra, namely that any two of the quarks are necessarily in a $\overline{3}$ colour state and for such states gluon exchange leads to bound states known as diquark states, and for the scalar spin-0 diquarks their masses are such that they play a significant dynamical role in the first baryon multiplet. For higher mass multiplets the pseudoscalar and spin-1 diquark states come into play. This means that in baryons the $q-q$ propagator is dominated by the diquark pole term. This naturally allows the separation-of-variables approximation, which played such a useful role in non-relativistic three-body theory. This allows the reduction of the three-body problem to a non-local covariant two-body problem. We illustrate the general techniques by considering the first baryon octet. After determining the chiral covariance of the formulation we include explicit chiral symmetry breaking by giving the quarks small current masses. The baryon octet mass formulae are then determined and shown to satisfy the Gell-Mann–Okubo and Coleman-Glashow mass relationships. These quark mass effects are treated as perturbations on the chirally covariant solutions and the mass formulae are used to determine from the experimental data the nucleon mass in the chiral limit. This mass will be needed for comparison with calculations to be reported elsewhere.

In Section 2 we derive from QCD the three-body formulation of the baryon structure problem. For simplicity we use the Feynman rules for QCD and we work in the Euclidean metric. The reader is referred to Cahill *et al.* (1987) for details of an analogous treatment of the mesons and the diquarks. In this way it is made clear which class of diagrams is retained as our first-stage approximation. This will allow some insight to be gained into this class by comparison of our results with experiment before further classes of diagrams are included. In this section we also use the separable approximation which is based on the dynamical role of the diquarks. This is chirally invariant, includes the special aspects of the colour algebra needed for baryons, and is Lorentz is covariant. Our general formulation provides a very useful starting point for the systematic study of baryon structure in the context of QCD. In Section 3 the baryon octet mass splittings due to current quark masses are determined. While these give splittings well known from early phenomenologies, they nevertheless formed a simple but important check on the formulation. They are also used here to extract from experimental data various parameters characterising the $\frac{1}{2}^+$ and $\frac{1}{2}^-$ baryon octets in the chiral limit.

2. Baryons

Consider the three constituent quarks forming a 'bare' baryon and, as in Fig.1*a*, the self-interaction of one of the quarks by way of the emission and absorption of gluons, which are themselves also self-interacting. Summing such diagrams gives a quark propagator which is a matrix in spin, flavour and colour space and which may be written $G(q) = (i\not{q} + \Sigma(q))^{-1}$. The integral equation which determines $\Sigma(q)$ is (in some unspecified gauge), for zero bare masses,

$$\Sigma(p) = \int \frac{d^4 q}{(2\pi)^4} D(p-q)^{ab}_{\mu\nu} \frac{\lambda^a \gamma^\mu}{2} \frac{1}{i\not{q} + \Sigma(q)} \frac{\gamma^\nu \lambda^b}{2} + ..., \tag{1}$$

where ... denote diagrams involving higher order gluon n-point functions D^n and where $D^{ab}_{\mu\nu} = \delta_{ab} D_{\mu\nu}$ is the 2-point function. In those gauges which are not ghost free the ghost 'particles' contribute to the D^n. However in our approach these n-point gluon functions do not involve $\bar{q}q$ loops, which are included through meson dressing at a later stage. Here $\{\lambda^a/2, a = 1, .., 8\}$ are the hermitian generators of the $SU(3)$ colour group of QCD and $[\lambda^a, \lambda^b] = 2if_{abc}\lambda^c$. The $\lambda^a\gamma^\mu$ in (1) of course specify the coupling between the colour-octet spin-1 gluons and the colour-triplet quarks. The algebra of the Euclidean Dirac matrices is $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$, $C = \gamma^2\gamma^4$ is the charge conjugation matrix and $C^{-1}\gamma^\mu C = -\gamma^{\mu T}$.

The solution of (1) has the form (Cahill *et al.* 1985, 1987, 1988; Roberts and Cahill 1987)

$$\Sigma(q) = \{i[A(q^2) - 1]\not{q} + VB(q^2)\}\mathbf{1_c}, \tag{2}$$

where $\mathbf{1_c}$ is the unit matrix in colour space and

$$V = \exp(i\sqrt{2}\pi^a F^a \gamma_5),$$

with $\{F^a\} = \{(1/\sqrt{N_F})\mathbf{1}, \sqrt{2}T^1, ...\}$ where $\{T^c\}$ are the hermitian generators of the flavour group $SU(N_F)$, with $[T^a, T^b] = if_{abc}T^c$. The $\{\pi^a\}$ are arbitrary real constants. The matrix V occurs because in the chiral limit the QCD action has an exact global $G = U_L(N_F) \otimes U_R(N_F)$ chiral symmetry which is now realised as a hidden symmetry. The degenerate vacuum manifold associated with such a hidden symmetry is parametrised by the $\{\pi^a\}$ and in fact is the coset space

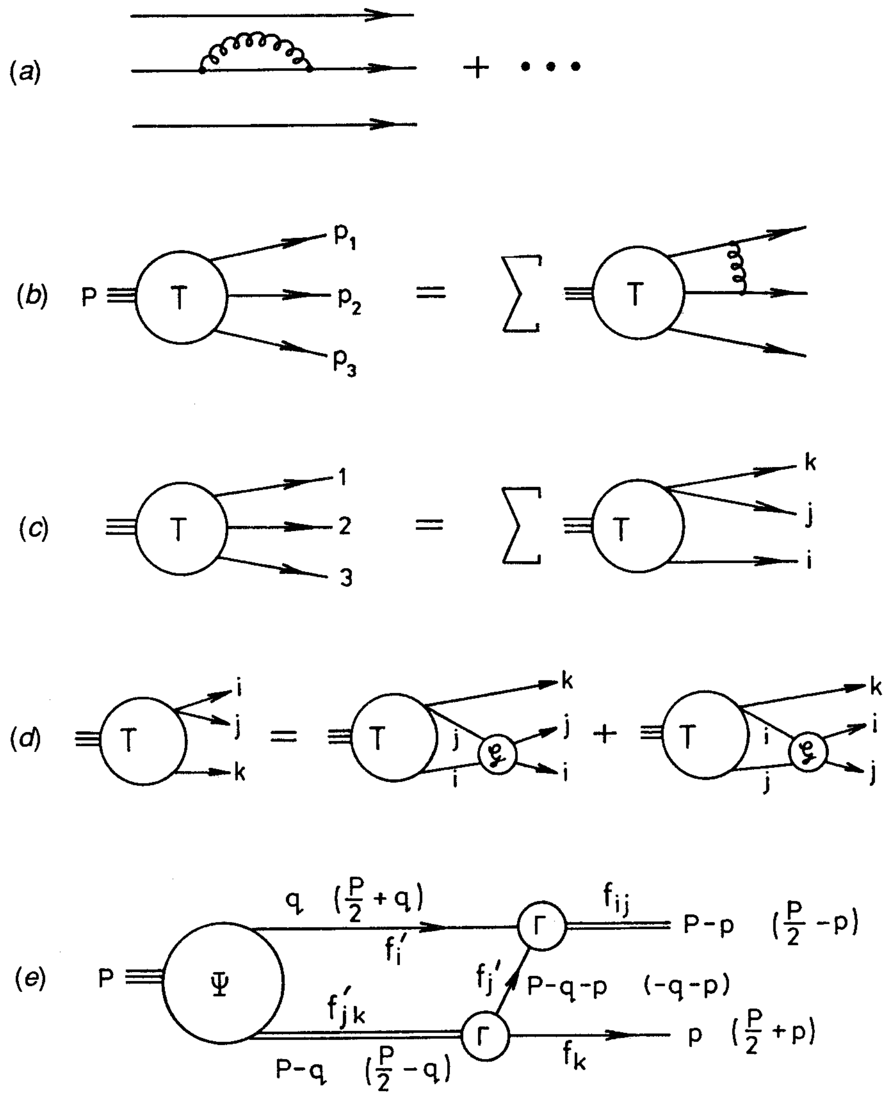

Fig. 1. *(a)* Self-interaction of one of the quarks in a baryon via gluons, leading to the quark propagator in (1). *(b)* Integral equation for the baryon form factor $T(P; p_1 p_2 p_3)$ where the interaction is due to gluon exchange between distinct quarks and where the quark propagators are from *(a)*. *(c)* Decomposition of the baryon form factor into three terms, each corresponding to the sum of all diagrams in *(b)* ending in a gluon exchange between a particular pair of quarks; $\{ijk\}$ is a cyclic permutation of $\{123\}$. *(d)* Three coupled integral equations for the $\{T^{(i)}\}$, representing (3); $\mathcal{G}$ denotes the amputated quark-quark scattering amplitude. *(e)* Definition of two alternate sets of momenta, and illustration of the quark-exchange nature of the quark-diquark formulation of the baryon; the flavour indices will be helpful in understanding equations (11), (19) and (21).

$G/H = U_A(N_F)$ where $H = U_V(N_F) \subset G$. The implications of this realisation for the meson sector of QCD have been extensively studied in Praschifka *et al.* (1987*a*, 1987*b*) and Roberts *et al.* (1988) and shown to produce the expected effective action for the Nambu-Goldstone (NG) bosons. There the π^a become coordinatised to $\pi^a(x)$ and so the above π^a may properly be considered to describe long-wavelength NG bosons. Approximate forms for the A and B functions have been extensively discussed in Cahill *et al.* (1985, 1987), Praschifka *et al.* (1987*a*, 1987*b*), and Roberts *et al.* (1988). An important aspect to keep in mind is that the quark effective mass function $m(q^2) = B(q^2)A(q^2)^{-1}$ is momentum dependent and $m(q^2) \to 0$ as $q^2 \to \infty$. That is chiral symmetry is restored at large (Euclidean) momenta or short distances, or more formally, G becomes chirally invariant at large q^2. This effect and the momentum scale at which $m(q^2) \approx 0$ play a key role in determining baryon structure. As well the confinement of the quarks is manifested by the absence of a pole in the quark propagator, which is a property of the A and B functions.

The functions $A(q^2)$ and $B(q^2)$ satisfy complicated integral equations whose solution requires a practically unattainable knowledge of all the D^n. While a truncation of (1) is possible an alternative approach is to parametrise QCD in terms of the A and B functions. Since B is the pion form factor (Cahill *et al.* 1987) Γ_π and thus essentially observable and A appears to be slowly varying this approach proves to be meaningful and very useful. It is then possible to define an effective 2-point gluon function D_{eff} so that keeping only the first term in (1), with D replaced by D_{eff}, (1) would then generate the same B. Then from the parametrised A and B we may in practice calculate D_{eff}, and this is then used in the calculation of various meson, diquark and baryon properties. It is possible that this procedure is related to that suggested by Cornwall (1982) which also involves a selective summation to define an effective 2-point function. The systematics of these procedures clearly needs further study, particularly as it provides a practical link between QCD and hadronic observables. It could also be a problem that could be contributed to by the lattice QCD techniques.

A chirally covariant calculation in QCD means that hadronic observables must transform, under a chiral transformation, according to some representation of the chiral group. In particular baryon masses must be chiral invariants. In the present context this means that the baryon masses must be independent of the angle variables π^a, which occur in the quark propagator $G(V)$. One can write $V = P_L U^\dagger + P_R U$, where $P_L = \frac{1}{2}(1-\gamma_5)$, $P_R = \frac{1}{2}(1+\gamma_5)$ and $U = \exp(i\sqrt{2}\pi^a F^a)$. It may then be shown (Roberts and Cahill 1987) that under a chiral transformation $U \to U_L U U_R^\dagger$ which holds also when U is coordinatised. This is the transformation property of $U(x)$ assumed in phenomenological treatments of hadrons. The chiral invariance of the baryon-meson effective action may also be used to place restrictions on phenomenological constructions for baryon-NG meson couplings. See Christos (1987) for a recent phenomenological analysis of these couplings and for references to earlier works.

Consider now gluon exchanges between the three quarks which lead to their binding to form a bare baryon. The baryon form factors and masses are determined by the homogeneous integral equations represented in Fig. 1*b* where the quark propagators are the $G(V)$, where the V dependence arises from (2), and the gluon propagators are D_{eff}. The higher order n-point gluon functions will contribute only in as much that they are implicitly included in D_{eff}. All residual n-point contributions and other graphs are to be treated as perturbations on the formalism presented here. The $\overline{q}q$ dressing of the bare baryons will follow from the non-local meson-baryon effective action which may be determined by our formalism.

Before proceeding we show that the baryon mass spectrum is independent of the choice of the matrix V, that is, of the choice of the vacuum from the vacuum manifold, or more simply, that the baryon masses, in the chiral limit, are chiral invariants. To do this consider the amplitudes $T(P^2)^V_{IJK}$ satisfying the integral equation represented in Fig. 1*b*, where $P^2 = -M_B^2$ with M_B one of the baryon masses for which these homogeneous equations have a solution. The V dependence of these amplitudes arises from that of the quark propagators $G(V)$. We wish to show that the $\{M_B\}$ are independent of V. First note that

$$G(q;V) = (iA(q^2)\not{q} + VB(q^2))^{-1} = \zeta^{\dagger} G(q;\mathbf{1})\zeta^{\dagger}, \tag{3}$$

where $\zeta \equiv \sqrt{V}$. The integral equation of Fig. 1*b* has the form

$$\int K(P;V)_{IJK;LMN} T^V_{LMN} = T^V_{IJK},$$

where only the spin and flavour labels of each quark are explicitly shown. Because $G(V)$ is a unit matrix in colour space the $\{M_B\}$ are manifestly colour invariants. One can easily show that

$$K(P;V)_{IJK;LMN} = \zeta_{II'}\zeta_{JJ'}\zeta_{KK'}K(P;\mathbf{1})_{I'J'K';L'M'N'}\zeta^{\dagger}_{L'L}\zeta^{\dagger}_{M'M}\zeta^{\dagger}_{N'N}, \tag{4}$$

which follows from (3) and because the interactions in K are vector couplings of the gluons to the quarks, and $\gamma^{\mu}\zeta^{\dagger} = \zeta\gamma^{\mu}$. This result is not restricted to those ladder diagrams in Fig. 1*b*, but is easily extended to include all possible diagrams involving any of the gluon n-point functions. It then follows from covariance and the similarity transformation of (4) that the baryon mass spectrum is invariant under the choice of V, and from now on we choose $V = \mathbf{1}$. However we note that a chiral transformation $U_L \otimes U_R$, for which $U \to U_L U U_R^{\dagger}$, and hence which transforms V as

$$V \to (U_L P_R + U_R P_L)V(U_R^{\dagger} P_R + U_L^{\dagger} P_L),$$

thus induces a mapping of, say, the $J^P = \frac{1}{2}^+$ baryon manifold into itself. In contrast Christos (1987) used both the $\frac{1}{2}^+$ and $\frac{1}{2}^-$ manifolds to carry a linear representation of the chiral group in the phenomenological construction of baryon-NG boson couplings. It may well be the case that Christos' representation implicitly assumes that the $\frac{1}{2}^+$ and $\frac{1}{2}^-$ baryons also have the same form factor. The construction of chirally invariant effective actions for the coupling of baryons and mesons thus leaves many problems unresolved.

We now return to the internal dynamical aspects of the baryons. As we have learnt from non-relativistic three-body theory the equations of Fig. 1*b* must be cast into the form of three coupled amplitudes, with each amplitude representing the sum of all diagrams ending in a gluon exchange between a particular pair of quarks,

$$T = \sum_{\text{cyc.perm.}} T(P; p_i p_j p_k)_{IJK},$$

where we use the notation that the i and j quarks scatter last in the amplitude on the RHS, as shown in Fig. 1c. We then find that these three amplitudes satisfy the three coupled integral equations, represented in Fig. 1d,

$$T(P; p_i p_j p_k)_{IJK} = \sum \int \frac{\mathrm{d}^4 q_i}{(2\pi)^4} \mathcal{G}(p_i p_j; q_i q_j)_{IJ,I''J''} G(q_i)_{I''I'} \\ \times G(q_j)_{J''J'} T(P; q_j p_k q_i)_{J'KI'} + j - \text{term}, \tag{5}$$

where ijk (= cyc. perm. 123) label the quarks, $I = \{s_i f_i c_i\}$ are the Dirac, flavour and colour indices of quark i, and $\mathcal{G}$ are the diquark (amputated) propagators. The summation is over all repeated indices $\{I', ..\}$. With P the baryon momentum, the conservation of 4-momentum requires

$$P = p_i + p_j + p_k = q_i + q_j + p_k, \qquad \text{et. cyc.}$$

The baryon mass M, with $P^2 = -M^2$, is determined by the requirement that the coupled homogeneous equations (5) have a non-trivial solution and this requires analytic continuation in the variable P_4 ($P_4 \to iM$ in the rest frame). A feature of these equations is that only the quark and diquark propagators appear.

We now determine the form of (5) for colour-singlet baryons. Then the diquarks are necessarily in colour $\overline{3}$ states and the diquark propagators have the colour structure

$$\mathcal{G}_{IJ,I'J'} = \mathcal{G}_{IJ,I'J'} \sum_{c_{ij}} \frac{1}{\sqrt{2}} \epsilon_{c_{ij} c_i c_j} \frac{1}{\sqrt{2}} \epsilon_{c_{ij} c_i' c_j'}, \tag{6}$$

where on the RHS $\{I, J...\}$ now represent only Dirac and flavour indices and c_{ij} labels the $\overline{3}$ colour states of the ij quark pair. For a colour-singlet baryon we have

$$T_{IJK} = \tfrac{1}{\sqrt{6}} \epsilon_{c_i c_j c_k} T_{IJK}, \tag{7}$$

where again the $\{I, J, K\}$ on the RHS now label Dirac and flavour indices only. Substituting (6) and (7) in equation (5), and using

$$\sum_{a=1,2,3} \epsilon_{abc} \epsilon_{ade} = \delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd} \tag{8}$$

in the colour summations, we obtain (5) again but with $\{I, J, K..\}$ now representing Dirac and flavour indices only. This simple colour algebra calculation demonstrates the important property that in the colour singlet baryon states of

QCD any two constituent quarks are necessarily in a $\overline{3}$ colour state. Of course the diquark colour state effects the determination of the diquark form factor Γ and effective mass m, as shown in Cahill *et al.* (1987). It is a particular feature of QCD that these $\overline{3}$ diquark states are bound by colour-octet gluon exchange.

We now specify the spin and flavour structure of the diquark propagators, and it is now that we restrict the analysis to $N_F = 3$. For simplicity, for the $\frac{1}{2}^+$ baryons, we shall keep only the spin-0 (scalar) diquark state, and the required propagator is, keeping in mind the Pauli principle,

$$\mathcal{G}(p_i p_j; q_i q_j)_{IJ,I'J'} = \frac{1}{f^2}\sum_{f_{ij}}(C\gamma_5\frac{1}{\sqrt{2}}\epsilon_{f_{ij}})_{IJ}(C\gamma_5\frac{1}{\sqrt{2}}\epsilon_{f_{ij}})_{I'J'}$$
$$\times\Gamma(p_i+p_j;\tfrac{1}{2}[p_i-p_j])d(p_i+p_j)\Gamma(p_i+p_j;\tfrac{1}{2}[q_i-q_j])+..,\tag{9}$$

where ϵ_f is the matrix whose mn element is ϵ_{fmn}, $d(Q)=(Q^2+m^2)^{-1}$, $f_{ij}=1,2,3$ labels the flavour state of the ij pair and $f[\Gamma]$ is the normalisation factor for the diquark form factors. The ϵ_{fmn} occur because, for $N_F = 3$, the scalar diquarks transform as $\overline{3}$ flavour states of the $SU(3)$ flavour group. The dependence of these diquark form factors on the total diquark momentum $p_i + p_j$ while required by Lorentz covariance will, as a minor simplifying approximation, be neglected. As indicated in (9) we will keep only the pole part of the diquark propagator, which then enables a separable expansion of the baryon form factor. For the $\frac{1}{2}^-$ baryon multiplet the pseudoscalar diquark states are relevant and for these the only change is that C rather than $C\gamma_5$ appears in (9) together with the appropriate f,Γ and d. Expressions for these quantities may be determined using the techniques of Cahill *et al.* (1987). Our calculations show that the pseudoscalar diquark mass is larger than that of the scalar diquark. Equations (5) and (9) show that the T have the separable form

$$T(P;p_ip_jp_k)_{IJK} = \sum_{f_{ij}}\Gamma(\frac{p_i-p_j}{2})d(p_i+p_j)(C\gamma_5\frac{1}{\sqrt{2}}\epsilon_{f_{ij}})_{IJ}\Psi(P;p_k)_{f_{ij}s_kf_k},\tag{10}$$

and we find that

$$\Psi(P;p_k)_{f_{ij}s_kf_k} = \frac{1}{2f^2}\sum_{f'_{jk}s'_if'_i}\int\frac{d^4q_i}{(2\pi)^4}\Gamma(\frac{q_i-q_j}{2})\Gamma(\frac{q_j-p_k}{2})d(q_j+p_k)$$
$$\times\{(C\gamma_5\epsilon_{f_{ij}}G(q_j,\mathbf{1})\epsilon_{f'_{jk}}C\gamma_5)^TG(q_i,\mathbf{1})\}_{s_kf_k,s'_if'_i}\Psi(P;q_i)_{f'_{jk}s'_if'_i}+j-\text{term}.\tag{11}$$

Here $\Psi(P;p_k)_{f_{ij}s_kf_k}$ is the spinor part (with Dirac index s_k) of the form factor of a spin-$\frac{1}{2}$ baryon, with momentum P, describing the relative motion of the k-th quark, with flavour f_k and the other two quarks forming a diquark in a flavour state labelled by f_{ij}. The baryon spin is carried by that quark which is not forming the diquark. The basic dynamical structure of (11) describes the rearrangement of the three quarks between the diquark state and the quark state.

For the baryon octet we decompose Ψ into the generators of $SU(N_F = 3)$ given in the Appendix and which are chosen to facilitate the analysis of the mass splitting of this baryon multiplet in Section 3:

$$\Psi(P; p_k)_{f_{ij}f_k} = \sum_a \Psi(P; p_k)^a \mathcal{T}^a_{f_k f_{ij}}$$

with the s_k index suppressed, and (11) becomes, using $\mathrm{tr}(\mathcal{T}^a \mathcal{T}^{bT}) = \delta^{ab}$,

$$\Psi(P; p_k)^a = -\frac{1}{2f^2} \sum \int \frac{\mathrm{d}^4 q_i}{(2\pi)^4} \Gamma(\frac{q_j - p_k}{2}) \Gamma(\frac{q_i - q_j}{2}) d(q_j + p_k) \mathcal{T}^a_{f_k f_{ij}} \mathcal{T}^b_{f'_i f'_{jk}}$$
$$\times \epsilon_{f_{ij} f'_i f'_j} \epsilon_{f'_{jk} f'_j f_k}\, G(q_j) G(q_i) \Psi(P; q_i)^b + j - \text{term}, \qquad (12)$$

where the minus sign comes from

$$(C\gamma_5 G(q) C\gamma_5)^T = -G(q).$$

For the $\frac{1}{2}^-$ baryons the corresponding statement is

$$(CG(q)C)^T = -G(-q).$$

Using (8) for the flavour summations and $\mathrm{tr}\mathcal{T}^a = 0$ equation (12) becomes

$$\Psi(P; p_k)^a = +\frac{1}{2f^2} \int \frac{\mathrm{d}^4 q_i}{(2\pi)^4} \Gamma(\frac{q_j - p_k}{2}) \Gamma(\frac{q_i - q_j}{2}) d(q_j + p_k)$$
$$\times G(q_j) G(q_i) \Psi(P; q_i)^a + j - \text{term}, \qquad (13)$$

which has solutions for which $\Psi(P; p_k)^a = \Psi(P; p_k)$, for all a, as expected for a baryon flavour octet, and then using the relationships between the various momenta we finally obtain

$$\Psi(P; p) = \frac{1}{f^2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \Gamma(\frac{P - p}{2} - q) \Gamma(\frac{P - q}{2} - p) d(P - q)$$
$$\times G(P - p - q) G(q) \Psi(P; q). \qquad (14)$$

Fig. 1*e* shows another choice of momentum variables, and as discussed in Burden *et al.* (1989, present issue p. 147), the choice of momentum variables is important in the numerical solution of (14). We note that for a flavour-singlet baryon we obtain, in an analogous calculation, (14) again but with the coefficient on the RHS multiplied by −2. The change of sign presumably means that this state is unbound.

Equation (14) is a linear integral equation for the spinor Ψ where the kernel describes the 'exchange' of a quark from a diquark to a quark, to form a new diquark, as shown in Fig. 1*e*. The extended nature of the diquark state is described by the form factor Γ. It may offer some physical insight to consider the once iterated form of (14),

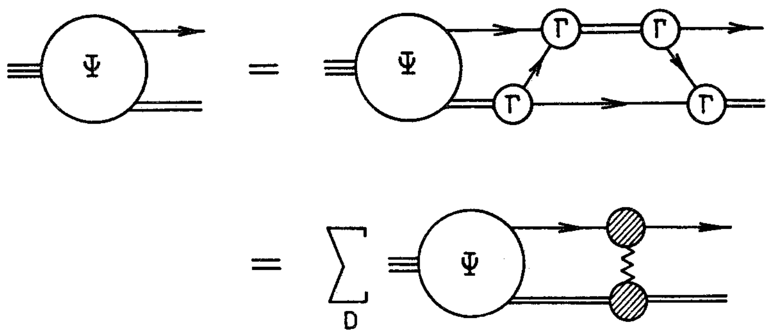

Fig. 2. Iteration of (14) produces an effective non-local exchange interaction between the quark and the diquark, (15). This interaction may be decomposed into scalar, vector, ... parts, denoted by the D summation, by means of a Dirac matrix decomposition.

$$\Psi(P;p) = \int \frac{d^4q}{(2\pi)^4} \mathcal{K}(P,p,q) d(P-q) G(q) \Psi(P;q), \tag{15}$$

where the detailed form of the effective kernel is easily obtained from (14) and which is illustrated in Fig. 2. This kernel describes an effective interaction between a quark and a diquark, arising from a double quark 'exchange'. This $\mathcal{K}$, which is a matrix acting on spinors, may be decomposed into scalar, vector,... parts by using the completeness of the set $\{\mathbf{1}, \gamma^\mu, ...\}$ giving

$$\mathcal{K}(P,p,q) = \mathcal{K}_0(P,p,q)\mathbf{1} + \mathcal{K}_\mu(P,p,q)\gamma^\mu + ...$$

This decomposition is illustrated in the lower part of Fig. 2. Hence our analysis shows that gluon exchange between the three quarks forming a baryon is ultimately manifested as an effective non-local scalar + vector +... exchange interaction between the quark and diquark. It is important to note however that this effective interaction is not related to meson exchange of scalar, vector,... mesons which in the past have often been used in phenomenological models for baryons. If we were to construct a local approximation to $\mathcal{K}_0$,... and to consider the dependence, after Fourier transforming, on the distance between the quark and diquark we would find that this dependence is controlled by the form of the quark effective mass function $m(q^2)$. It is possible that this x dependence is responsible for the success of some phenomenological models for baryons. At this stage it is also useful to compare our results with some particular phenomenological models for baryon structure. Lichtenberg *et al.* (1982) have considered the quark-diquark picture of the baryon. However they used a gluon exchange between the quark and diquark, with an effective colour charge for the diquark. We believe that this approach is inconsistent with a proper Faddeev approach to three-body problems and the subsequent reduction to an effective two-body problem. Our analysis is also to be contrasted with the work of Weber (1987) who used the light-front formalism to achieve a reduction in the complexity of the calculations, but with a phenomenological confining scalar potential and at short distances a vector hyperfine interaction.

In our approach all the effects which Weber models using potentials are generated, in our analysis, by gluon exchanges so long as we include quark self-interactions via gluon exchange.

Equation (14), for the $J^P = \frac{1}{2}^+$ first baryon octet, is the major result of this paper. Analogous equations for the other baryon multiplets are easily derived, but are not presented here, except for the multiplet of parity partners to the above multiplet.

At this point we also comment on the effective mass of the three quarks constituting the bare baryon. It has been known for a long time (Close 1979) that baryons could be understood as three quark bound states, using simple non-relativistic models, provided these quarks were given a 'constituent' mass of some 300 MeV. These masses were also needed to understand nucleon magnetic moments and also, it seems, deep inelastic scattering data. However recently it was discovered that the formulation of QCD used here explains the origin of the quark constituent mass effect. In Praschifka *et al.* (1988) calculations of the scalar diquark form factor are reported where it was observed that the form factor $\Gamma(s)$ exhibits a dramatic peaking for (Euclidean) $s \approx 0.2$ GeV^2 which, in conjunction with the running mass $m(s)$ emerging from (1), was shown to signal the generation of a quark constituent mass of ≈ 270 MeV. Thus the constituent quark mass turns out to be a uniquely quantum field theory effect, being due to the self-interaction of a quark by way of gluon emission and absorption. Initial studies (Burden *et al.* 1989) of the numerical solution of (14) show that, for example, $\Psi(q)^\dagger\Psi(q)$ exhibits peaking for q^2 away from the origin in a similiar manner to the diquark, and so the constituent mass effect in baryon states is seen to emerge naturally from QCD. These constituent mass effects and their relationship to baryon form factors will be important to an understanding of deep inelastic lepton-nucleon scattering data when we finally move on from the simplistic parton type analysis.

3. Baryon Octet Splitting

Here we determine the mass splitting of the baryon octet due to the current masses of the quarks. It will be shown that the baryon masses satisfy the Gell-Mann–Okubo and Coleman-Glashow mass formulae. The magnitude of these mass splittings will provide a test of the quark-diquark picture of the baryons presented herein. We return to (12) which for non-zero quark masses $\{m_f\}$ becomes

$$\Psi(P;p_k)^a = -\frac{1}{2f^2}\sum\int \frac{\mathrm{d}^4 q_i}{(2\pi)^4}\Gamma(\frac{q_j-p_k}{2})\Gamma(\frac{q_i-q_j}{2})d(q_j+p_k)_{f'_{jk}}\mathcal{T}^a_{f_k f_{ij}}\mathcal{T}^b_{f'_i f'_{jk}}$$
$$\times\,\epsilon_{f_{ij}f'_i f'_j}\epsilon_{f'_{jk}f'_j f_k}G(q_j)_{f'_j}G(q_i)_{f'_i}\Psi(P;q_i)^b + j-\text{term}, \qquad (16)$$

where the change is that now the quark and diquark propagators carry flavour labels. To first order in the current masses we can neglect changes in the diquark form factors. As discussed in Cahill *et al.* (1987) when $m_i \neq 0$ the quark propagators become

$$G(q)_f \approx \frac{1}{iA(q)\not{q} + m_f + B(q)}$$
$$= G(q) - m_f G(q) G(q) + ...,\tag{17}$$

to first order in m_f where $G(q) = (iA(q)\not{q} + B(q))^{-1}$. The vacuum is no longer degenerate and (1) (with the current mass term included) determines that $V = \mathbf{1}$. The mass dependence of these quark propagators in turn affects the diquark masses and from Cahill *et al.* (1987) we obtain for the mass of the diquark in flavour state f_{ij} composed of quarks in flavour states f_i and f_j, to first order in the quark masses,

$$m_{f_{ij}}^2 = m^2 + g \sum \frac{m_{f_i} + m_{f_j}}{2} \mid \epsilon_{f_{ij} f_i f_j} \mid,$$

where m is the diquark mass in the chiral limit and where

$$g = \frac{24}{f^2} \int \frac{d^4q}{(2\pi)^4} \frac{\Gamma(q)^2 B(q)}{(A(q)^2 q^2 + B(q)^2)^2}.$$

Then for the diquark propagator

$$d(q)_{f_{ij}} = \frac{1}{q^2 + m_{f_{ij}}^2}$$
$$= d(q) - g \sum \frac{m_{f_i} + m_{f_j}}{2} \mid \epsilon_{f_{ij} f_i f_j} \mid d(q) d(q) + ...\tag{18}$$

Equation (16) then becomes

$$\Psi(P;p)^a = \frac{1}{f^2} \sum \int \frac{d^4q}{(2\pi)^4} \Gamma(\frac{P-p}{2} - q) \Gamma(\frac{P-q}{2} - p) d(P-q) \times G(P-p-q) G(q) \Psi(P;q)^a$$
$$+ \frac{1}{2f^2} \sum \int \frac{d^4q}{(2\pi)^4} \Gamma(\frac{P-p}{2} - q) \Gamma(\frac{P-q}{2} - p) \mathcal{T}^a_{f_k f_{ij}} \mathcal{T}^b_{f_i' f_{jk}'} \epsilon_{f_{ij} f_i' f_j'} \epsilon_{f_{jk}' f_j' f_k}$$
$$\times \left[C(m_{f_j'} + m_{f_i'}) + D(m_{f_i'} + m_{f_j'}) + E(m_{f_j'} + 2m_{f_k} + m_{f_i'}) \right] \Psi(P;q)^b,\tag{19}$$

where

$$C = -d(p-q) G(P-p-q)^2 G(q),$$
$$D = -d(p-q) G(P-p-q) G(q)^2,$$
$$E = -g d(p-q)^2 G(P-q-p) G(q).\tag{20}$$

Equation (19) involves the matrices

$$Y^{ab} = \sum \mathcal{T}^a_{f_k f_{ij}} \mathcal{T}^b_{f_i' f_{jk}'} \epsilon_{f_{ij} f_i' f_j'} \epsilon_{f_{jk}' f_j' f_k} (m_{f_i'} + m_{f_j'}),$$
$$Z^{ab} = \sum \mathcal{T}^a_{f_k f_{ij}} \mathcal{T}^b_{f_i' f_{jk}'} \epsilon_{f_{ij} f_i' f_j'} \epsilon_{f_{jk}' f_j' f_k} m_{f_k}.\tag{21}$$

The matrix set $\{\mathcal{T}^a\}$ in the Appendix diagonalises Y^{ab} and Z^{ab} independently of the values of $\{m_f\} = \{m_u, m_d, m_s\}$ except for the 3 and 8 (Σ^0 and Λ) states for which, however, this mixing is small. Hence using this matrix set the eight coupled integral equations (19) may be uncoupled. The form of the perturbation due to the current masses is easily seen to give the following mass expressions for the the $J^P = \frac{1}{2}^+$ baryon octet,

$$\begin{aligned}
M_{\Xi^-} &= M_\chi + 2(y+z)m_s + 2ym_d + ...,\\
M_{\Xi^0} &= M_\chi + 2(y+z)m_s + 2ym_u + ...,\\
M_{\Sigma^-} &= M_\chi + 2ym_s + 2(y+z)m_d + ...,\\
M_{\Sigma^0} &= M_\chi + 2ym_s + (y+z)(m_u + m_d) + ...,\\
M_{\Sigma^+} &= M_\chi + 2ym_s + 2(y+z)m_u + ...,\\
M_\Lambda &= M_\chi + (\tfrac{2}{3}y + \tfrac{4}{3}z)m_s + (\tfrac{5}{3}y + \tfrac{1}{3}z)(m_u + m_d) + ...,\\
M_n &= M_\chi + 2(y+z)m_d + 2ym_u + ...,\\
M_p &= M_\chi + 2(y+z)m_u + 2ym_d + ...,
\end{aligned} \tag{22}$$

where as usual M_χ is the chirally invariant mass for this particular baryon multiplet, determined by solving (14), and only terms up to first order in the quark masses are shown. Values for y and z are easily determined by solving (20) and clearly depend on the quark running mass, the diquark mass and form factor, and most importantly on the form of the spinor part of the baryon wave function Ψ . All of these are ultimately determined by the form of the gluon propagator. It is easily checked that these mass formulae satisfy, for any value of y and z, the linear relations

$$\tfrac{1}{2}\left[\frac{M_n + M_p}{2} + \frac{M_{\Xi^-} + M_{\Xi^0}}{2}\right] = \tfrac{1}{4}\left[3M_\Lambda + \frac{M_{\Sigma^+} + M_{\Sigma^0} + M_{\Sigma^-}}{3}\right],$$

which is the Gell-Mann–Okubo mass formula for the baryon octet in which the mass splittings of the isospin multiplets are naturally included (but not electromagnetic contributions to the splittings), and

$$M_n - M_p + M_{\Sigma^+} - M_{\Sigma^-} + M_{\Xi^-} - M_{\Xi^0} = 0,$$

which is the Coleman-Glashow formula. Curiously the Coleman-Glashow formula was originally attributed to electromagnetic interactions between the quarks, but here we see that the same formula also arises from current quark masses.

As M_χ, y and z are properties of the baryon octet in the chiral limit, it will be useful to extract these parameters from the experimental baryon octet mass spectrum. We have performed a least squares fit of the mass formulae in (22) to the known masses. This determines the m_s/m_u versus m_d/m_u plot shown in Fig. 3 and also fixes M_X at 912 MeV and the y/z ratio at 1.96, which is related to the usual F/D ratio by,

$$\frac{F}{D} = \frac{y+z}{z-y} = -3.08 .$$

The quality of the fit may be judged from the results in Table 1, which are independent of the m_d/m_u ratio. The value of this ratio is not determined by the above fit but may be determined when experimental masses for some other multiplet are simultaneously fitted to the appropriate mass formulae. For this the Nambu-Goldstone octet comprising the mesons π^+, π^-, π^0, K^+, K^-, K^0, $\overline{K}^0$, η are particularly useful. Extending the results in Cahill *et al.*

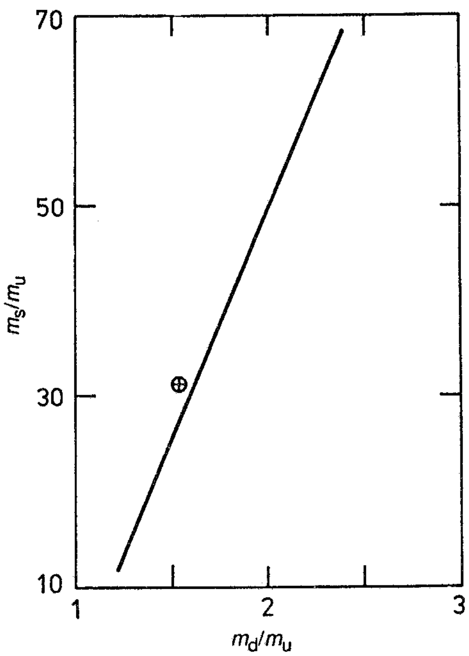

Fig. 3. The m_s/m_u versus m_d/m_u relation from a least squares fit of the baryon octet mass formulae, in (22), to the corresponding $\frac{1}{2}^+$ experimental masses. Also shown is the one point from the Nambu-Goldstone octet (shown by ⊕). Together the baryon and meson plots fix $m_d/m_u \approx 1.6$.

Table 1. Least squares fit of the baryon mass formulae in (22) to the experimental $J^P = \frac{1}{2}^+$ masses (in MeV)

Baryon	Exp. mass	Theor. mass	% error
Ξ^-	1321.3	1323.4	+0.15
Ξ^0	1314.9	1318.1	+0.25
Σ^-	1197.3	1196.3	−0.09
Σ^0	1192.5	1192.3	−0.01
Σ^+	1189.4	1188.4	−0.08
Λ	1115.6	1109.9	−0.51
n	939.6	941.6	+0.22
p	938.3	938.9	+0.07

(1987) to the case $m_u \neq m_d$ we obtain the mass formulae

$$
\begin{aligned}
m_\pi^2 &= \tfrac{1}{2}\mu(m_u + m_d), \\
m_{K^+}^2 &= \tfrac{1}{2}\mu(m_u + m_s), \\
m_{K^0}^2 &= \tfrac{1}{2}\mu(m_d + m_s), \\
m_\eta^2 &= \tfrac{1}{6}\mu(m_u + m_d + 4m_s),
\end{aligned} \tag{23}
$$

where

$$\mu = \frac{<\overline{q}q>}{f_\pi^2}.$$

Explicit expressions for $<\overline{q}q>$ and f_π in terms of $A(s)$ and $B(s)$ are also given in Cahill *et al.* (1987). We emphasise that while the meson masses of (23) are recognised as well known results from current algebra phenomenology, they have been determined in a manner completely analogous to that which we use to determine the baryon masses of (22). Hence it is completely consistent to use both (22) and (23) together in the analysis of experimental data. Then from (23) and the experimental masses we obtain

$$
\begin{aligned}
\frac{m_d}{m_u} &= \frac{m_{K^0}^2 + m_\pi^2 - m_{K^+}^2}{m_{K^+}^2 + m_\pi^2 - m_{K^0}^2} = 1.53, \\
\frac{m_s}{m_u} &= \frac{m_{K^0}^2 - m_\pi^2 + m_{K^+}^2}{m_{K^+}^2 + m_\pi^2 - m_{K^0}^2} = 31.41
\end{aligned} \tag{24}
$$

in which we have used the average pion mass $m_\pi = 138$ MeV. The above quark mass ratios are also plotted, as one point, in Fig. 3, and we see that the meson and baryon data are remarkably consistent, requiring $m_d/m_u \approx 1.6$. If we use a value of $m_u = 3.5$ MeV, which is consistent with values determined by QCD sum rules (Reinders *et al.* 1985) then we find from the baryon fit that $m_s/m_u = 30.3, m_d = 5.6$ MeV, $m_s = 106$ MeV and, as properties of the $\frac{1}{2}^+$ multiplet, $y = 1.24$ and $z = 0.63$. For the $J^P = \frac{1}{2}^-$ baryon octet the same mass formulae are obtained and using the experimental masses $N(1535)$, $\Lambda(1670)$ and $\Sigma(1750)$ we obtain, using the above quark masses, $y = 1.06$, $z = 0.47$ and $M_{X^-} = 1511$ MeV.

4. Conclusions

We have shown herein that it is possible to express the problem of calculating the structure and masses of bare-baryon states in QCD as a practical three-body Faddeev-type formulation. The most important feature of our approach is that it is systemically derived from QCD and is the beginning of a systematic study of baryon structure. Until now this has not been possible because of the absence of a direct connection between QCD and a viable analytical reformulation of QCD appropriate to baryon structure. The only alternative formulation is the lattice technique which is, however, at its weakest in the chiral limit. In contrast, our analytic approach is particularly suited to the chiral limit and to current quark mass induced perturbations about that limit. The formulation also reveals the three-bound-quark nature of bare baryons, which become 'physical' baryons after meson dressing, but even at the bare baryon level the constituent quark mass effect is emerging. The details of the meson 'dressing' calculations will be presented elsewhere together with a re-evaluation of the parameters M_X, y and z which these dressings necessitate.

Here we have only used the baryon integral equation formulation to determine multiplet splitting and we have shown that the Gell-Mann–Okubo and Coleman-Glashow mass relationships emerge. The derivation of the Gell-Mann–Okubo relation settles a long standing debate concerning the early phenomenological mass relations as to whether these should be linear or quadratic in the masses. Our results here for baryons, and the analogous derivation for mesons, show that one obtains, in QCD, linear mass relations for baryons and quadratic mass relations for mesons. The baryon splitting formulae allow a determination of the baryon chiral mass and the y/z ratio (related to the usual F/D ratio). In conjunction with the Nambu-Goldstone boson splitting formulae and using $m_u = 3.5$ MeV, we determine the values of y and z as well. These various parameters associated with the baryon octets will allow a testing of detailed computations. While we have only considered the $J^P = \frac{1}{2}^+$ and $\frac{1}{2}^-$ baryon multiplets the analysis is easily extended to other baryon multiplets, particularly the $\frac{3}{2}^+$.

Because our formulation is a systematic development from QCD we avoid all of the ad hoc phenomenological assumptions which to date have characterised theoretical investigations of baryons and mesons. It is worthwhile to re-emphasise those features of QCD which have been naturally retained in our analysis; namely the dynamical consequences of the colour algebra, of the chiral invariance and of Lorentz covariance. It is of course the colour algebra which allows three quarks to be bound into colour singlet states by exchange of colour octet gluons. This is because in such states any two quarks are necessarily in a colour $\overline{3}$ state and they are bound into diquark states by the gluon exchange, and hence all three quarks are mutually attracted. Surprisingly many phenomenological models for baryons actually make no use of this significant property of the colour dynamics. Indeed, in the topical Skyrmion model for baryons, in which the baryon is considered to be a mesonic topological soliton, no use is made of those aspects of the colour algebra which are essential to the binding of colour-singlet three-quark states. Indeed only those trivial aspects of the colour algebra which arise in the meson sector are used. Along these lines we have discussed in some detail in Cahill *et al.* (1988) why baryons, as QCD states, cannot be modelled as Skyrmions, but we also note that the Skyrmion model does not lead to the successful baryon multiplet mass splitting formulae of Section 3. Chiral invariance is particularly important as illustrated, for example, by the result that the nucleon mass of 940 MeV would only be reduced to 912 MeV if the quarks were massless. Hence to understand baryon structure it is necessary for any theoretical analysis to be well suited to the chiral limit. As well, the dynamical breaking of chiral symmetry and the consequent emergence of the NG bosons and their coupling to baryons are also most easily studied using the formalism herein. The importance of a Lorentz covariant treatment of baryons needs little emphasis. We note that there is also a natural description of quark confinement in our formulation and that recently the dynamical origin of the quark constituent mass effect has been determined (Praschifka *et al.* 1988).

Acknowledgment

We wish to thank Dr Conrad Burden for checking the analysis.

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Appendix

The following set of matrices $\{\mathcal{T}^a; a = 1,...8\}$ serve as generators for $SU(3)$ and for which $\mathrm{tr}(\mathcal{T}^a \mathcal{T}^{bT}) = \delta_{ab}$:

$$\mathcal{T}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{T}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{T}^4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{T}^5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{T}^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{T}^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\mathcal{T}^3 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{T}^8 = \tfrac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Manuscript received 25 May, accepted 25 November 1988