Hacettepe Journal of Mathematics and Statistics
Volume 43 (2) (2014), 227 – 243

BASE AND SUBBASE IN INTUITIONISTIC I-FUZZY TOPOLOGICAL SPACES

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Abstract

In this paper, the concepts of the base and subbase in intuitionistic I-fuzzy topological spaces are introduced, and use them to discuss fuzzy continuous mapping and fuzzy open mapping. We also study the base and subbase in the product of intuitionistic I-fuzzy topological spaces, and T_2 separation in product intuitionistic I-fuzzy topological spaces. Finally, the relation between the generated product intuitionistic I-fuzzy topological spaces and the product generated intuitionistic I-fuzzy topological spaces are studied.

Keywords: Intuitionistic I-fuzzy topological space; Base; Subbase; T_2 separation; Generated Intuitionistic I-fuzzy topological spaces.

2000 AMS Classification: 54A40

1. Introduction

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was first introduced by Atanassov [1]. From then on, this theory has been studied and applied in a variety areas ([4, 14, 18], etc). Among of them, the research of the theory of intuitionistic fuzzy topology is similar to the the theory of fuzzy topology. In fact, Çoker [4] introduced the concept of intuitionistic fuzzy topological spaces, this concept is originated from the fuzzy topology in the sense of Chang [3](in this paper we call it intuitionistic I-topological spaces). Based on Çoker's work [4], many topological properties of intuitionistic I-topological spaces has been discussed ([5, 10, 11, 12, 13]). On the other hand, Šostak [17] proposed a new notion of fuzzy topological spaces, and this new fuzzy topological structure has been accepted widely. Influenced by Šostak's work [17], Çoker [7] gave the notion of intuitionistic fuzzy topological spaces in the sense of Šostak. By the standardized terminology introduced in [16], we will call it intuitionistic I-fuzzy

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topological spaces in this paper. In [15], the authors studied the compactness in intuitionistic I-fuzzy topological spaces.

Recently, Yan and Wang [19] generalized Fang and Yue′s work ([8, 21]) from I-fuzzy topological spaces to intuitionistic I-fuzzy topological spaces. In [19], they introduced the concept of intuitionistic I-fuzzy quasi-coincident neighborhood systems of intuitiostic fuzzy points, and construct the notion of generated intuitionistic I-fuzzy topology by using fuzzifying topologies. As an important result, Yan and Wang proved that the category of intuitionistic I-fuzzy topological spaces is isomorphic to the category of intuitionistic I-fuzzy quasi-coincident neighborhood spaces in [19].

It is well known that base and subbase are very important notions in classical topology. They also discussed in I-fuzzy topological spaces by Fang and Yue [9]. As a subsequent work of Yan and Wang [19], the main purpose of this paper is to introduce the concepts of the base and subbase in intuitionistic I-fuzzy topological spaces, and use them to discuss fuzzy continuous mapping and fuzzy open mapping. Then we also study the base and subbase in the product of intuitionistic I-fuzzy topological spaces, and T_2 separation in product intuitionistic I-fuzzy topological spaces. Finally, we obtain that the generated product intuitionistic I-fuzzy topological spaces is equal to the product generated intuitionistic I-fuzzy topological spaces.

Throughout this paper, let $I = [0, 1]$, X a nonempty set, the family of all fuzzy sets and intuitionistic fuzzy sets on X be denoted by I^X and ζ^X, respectively. The notation $\mathrm{pt}(I^X)$ denotes the set of all fuzzy points on X. For all $\lambda \in I$, $\underline{\lambda}$ denotes the fuzzy set on X which takes the constant value λ. For all $A \in \zeta^X$, let $A =< \mu_A, \gamma_A >$. (For the relating to knowledge of intuitionistic fuzzy sets and intuitionistic I-fuzzy topological spaces, we may refer to [1] and [19].)

2. Some preliminaries

2.1. Definition. ([20]) A fuzzifying topology on a set X is a function $\tau : 2^X \to I$, such that

(1) $\tau(\emptyset) = \tau(X) = 1$;

(2) $\forall A, B \subseteq X$, $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$;

(3) $\forall A_t \subseteq X$, $t \in T$, $\tau(\bigvee_{t\in T} A_t) \geq \bigwedge_{t\in T} \tau(A_t)$.

The pair (X, τ) is called a fuzzifying topological space.

2.2. Definition. ([1, 2]) Let a, b be two real numbers in $[0, 1]$ satisfying the inequality $a + b \leq 1$. Then the pair $< a, b >$ is called an intuitionistic fuzzy pair.

Let $< a_1, b_1 >$, $< a_2, b_2 >$ be two intuitionistic fuzzy pairs, then we define

(1) $< a_1, b_1 >\leq< a_2, b_2 >$ if and only if $a_1 \leq a_2$ and $b_1 \geq b_2$;

(2) $< a_1, b_1 >=< a_2, b_2 >$ if and only if $a_1 = a_2$ and $b_1 = b_2$;

(3) if $< a_j, b_j >_{j\in J}$ is a family of intuitionistic fuzzy pairs, then $\bigvee_{j\in J} < a_j, b_j >=< \bigvee_{j\in J} a_j, \bigwedge_{j\in J} b_j >$, and $\bigwedge_{j\in J} < a_j, b_j >=< \bigwedge_{j\in J} a_j, \bigvee_{j\in J} b_j >$;

(4) the complement of an intuitionistic fuzzy pair $< a, b >$ is the intuitionistic fuzzy pair defined by $\overline{< a, b >} =< b, a >$;

In the following, for convenience, we will use the symbols $1^\sim$ and $0^\sim$ denote the intuitionistic fuzzy pairs $< 1, 0 >$ and $< 0, 1 >$. The family of all intuitionistic fuzzy pairs is denoted by $\mathcal{A}$. It is easy to find that the set of all intuitionistic fuzzy pairs with above order forms a complete lattice, and $1^\sim, 0^\sim$ are its top element and bottom element, respectively.

2.3. Definition. ([4]) Let X, Y be two nonempty sets and $f : X \to Y$ a function, if $B = \{< y, \mu_B(y), \gamma_B(y) >: \ y \in Y\} \in \zeta^Y$, then the preimage of B under f, denoted by $f^\leftarrow(B)$, is the intuitionistic fuzzy set defined by

$$f^\leftarrow(B) = \{< x, f^\leftarrow(\mu_B)(x), f^\leftarrow(\gamma_B)(x) >: \ x \in X\}.$$

Here $f^\leftarrow(\mu_B)(x) = \mu_B(f(x)), \ f^\leftarrow(\gamma_B)(x) = \gamma_B(f(x))$. (This notation is from [16]).

If $A = \{< x, \mu_A(x), \gamma_A(x) >: \ x \in X\} \in \zeta^X$, then the image A under f, denoted by $f^\to(A)$ is the intuitionistic fuzzy set defined by

$$f^\to(A) = \{< y, f^\to(\mu_A)(y), (\underline{1} - f^\to(\underline{1} - \gamma_A))(y) >: \ y \in Y\}.$$

Where

$$f^\to(\mu_A)(y) = \begin{cases} \sup_{x \in f^\leftarrow(y)} \mu_A(x), & \text{if } f^\leftarrow(y) \neq \emptyset, \\ 0, & \text{if } f^\leftarrow(y) = \emptyset. \end{cases}$$

$$\underline{1} - f^\to(\underline{1} - \gamma_A)(y) = \begin{cases} \inf_{x \in f^\leftarrow(y)} \gamma_A(x), & \text{if } f^\leftarrow(y) \neq \emptyset, \\ 1, & \text{if } f^\leftarrow(y) = \emptyset. \end{cases}$$

2.4. Definition. ([7]) Let X be a nonempty set, $\delta : \zeta^X \to \mathcal{A}$ satisfy the following:

(1) $\delta(< \underline{0}, \underline{1} >) = \delta(< \underline{1}, \underline{0} >) = 1^\sim$;

(2) $\forall A, \ B \in \zeta^X, \ \delta(A \bigwedge B) \geq \delta(A) \bigwedge \delta(B)$;

(3) $\forall A_t \in \zeta^X, \ t \in T, \ \delta(\bigvee_{t \in T} A_t) \geq \bigwedge_{t \in T} \delta(A_t)$.

Then δ is called an intuitionistic I-fuzzy topology on X, and the pair (X, δ) is called an intuitionistic I-fuzzy topological space. For any $A \in \zeta^X$, we always suppose that $\delta(A) =< \mu_\delta(A), \ \gamma_\delta(A) >$ later, the number $\mu_\delta(A)$ is called the openness degree of A, while $\gamma_\delta(A)$ is called the nonopenness degree of A. A fuzzy continuous mapping between two intuitionistic I-fuzzy topological spaces (ζ^X, δ_1) and (ζ^Y, δ_2) is a mapping $f : X \to Y$ such that $\delta_1(f^\leftarrow(A)) \geq \delta_2(A)$. The category of intuitionistic I-fuzzy topological spaces and fuzzy continuous mappings is denoted by II-**FTOP**.

2.5. Definition. ([6, 11, 12]) Let X be a nonempty set. An intuitionistic fuzzy point, denoted by $x_{(\alpha,\beta)}$, is an intuitionistic fuzzy set $A = \{< y, \mu_A(y), \gamma_A(y) >: y \in X\}$, such that

$$\mu_A(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

and

$$\gamma_A(y) = \begin{cases} \beta, & \text{if } y = x, \\ 1, & \text{if } y \neq x. \end{cases}$$

Where $x \in X$ is a fixed point, the constants $\alpha \in I_0,\ \beta \in I_1$ and $\alpha + \beta \leq 1$. The set of all intuitionistic fuzzy points $x_{(\alpha,\beta)}$ is denoted by $\mathrm{pt}(\zeta^X)$.

2.6. Definition. ([12]) Let $x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)$ and $A,\ B \in \zeta^X$. We say $x_{(\alpha,\beta)}$ quasi-coincides with A, or $x_{(\alpha,\beta)}$ is quasi-coincident with A, denoted $x_{(\alpha,\beta)}\hat{q}A$, if $\mu_A(x) + \alpha > 1$ and $\gamma_A(x) + \beta < 1$. Say A quasi-coincides with B at x, or say A is quasi-coincident with B at x, $A\hat{q}B$ at x, in short, if $\mu_A(x) + \mu_B(x) > 1$ and $\gamma_A(x) + \gamma_B(x) < 1$. Say A quasi-coincides with B, or A is quasi-coincident with B, if A is quasi-coincident with B at some point $x \in X$.
Relation"does not quasi-coincides with" or "is not quasi-coincident with " is denoted by $\neg\hat{q}$.

It is easily to know for $\forall x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)$, $x_{(\alpha,\beta)}\hat{q} < \underline{1}, \underline{0} >$ and $x_{(\alpha,\beta)}\neg\hat{q} < \underline{0}, \underline{1} >$.

2.7. Definition. ([19]) Let (X, δ) be an intuitionistic I-fuzzy topological space. For all $x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X), U \in \zeta^X$, the mapping $Q^\delta_{x_{(\alpha,\beta)}} : \zeta^X \to \mathcal{A}$ is defined as follows

$$Q^\delta_{x_{(\alpha,\beta)}}(U) = \begin{cases} \bigvee\limits_{x_{(\alpha,\beta)}\widehat{q}\ V \leq U} \delta(V), & x_{(\alpha,\beta)}\widehat{q}\ U; \\ 0^\sim, & x_{(\alpha,\beta)}\neg\widehat{q}\ U. \end{cases}$$

The set of $Q^\delta = \{Q^\delta_{x_{(\alpha,\beta)}} : x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)\}$ is called intuitionistic I-fuzzy quasi-coincident neighborhood system of δ on X.

2.8. Theorem. *([19]) Let (X, δ) be an intuitionistic I-fuzzy topological space, $Q^\delta = \{Q^\delta_{x_{(\alpha,\beta)}} :\ x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)\}$ of maps $Q^\delta_{x_{(\alpha,\beta)}} : \zeta^X \to \mathcal{A}$ defined in Definition 2.7 satisfies: $\forall U, V \in \zeta^X$,*

(1) $Q^\delta_{x_{(\alpha,\beta)}}(\langle\underline{1}, \underline{0}\rangle) = 1^\sim, Q^\delta_{x_{(\alpha,\beta)}}(\langle\underline{0}, \underline{1}\rangle) = 0^\sim$;

(2) $Q^\delta_{x_{(\alpha,\beta)}}(U) > 0^\sim \Rightarrow x_{(\alpha,\beta)}\widehat{q}\ U$;

(3) $Q^\delta_{x_{(\alpha,\beta)}}(U \wedge V) = Q^\delta_{x_{(\alpha,\beta)}}(U) \wedge Q^\delta_{x_{(\alpha,\beta)}}(V)$;

(4) $Q^\delta_{x_{(\alpha,\beta)}}(U) = \bigvee\limits_{x_{(\alpha,\beta)}\widehat{q}\ V \leq U} \bigwedge\limits_{y_{(\lambda,\rho)}\widehat{q}\ V} Q^\delta_{y_{(\lambda,\rho)}}(V)$;

(5) $\delta(U) = \bigwedge\limits_{x_{(\alpha,\beta)}\widehat{q}\ U} Q^\delta_{x_{(\alpha,\beta)}}(U)$.

2.9. Lemma. ([21]) *Suppose that (X, τ) is a fuzzifying topological space, for each $A \in I^X$, let $\omega(\tau)(A) = \bigwedge_{r\in I} \tau(\sigma_r(A))$, where $\sigma_r(A) = \{x : A(x) > r\}$. Then $\omega(\tau)$ is an I-fuzzy topology on X, and $\omega(\tau)$ is called induced I-fuzzy topology determined by fuzzifying topology τ.*

2.10. Definition. ([19]) Let (X, τ) be a fuzzifying topological space, $\omega(\tau)$ is an induced I-fuzzy topology determined by fuzzifying topology τ. For each $A \in \zeta^X$, let $\mathrm{I}\omega(\tau)(A) =< \mu^\tau(A), \gamma^\tau(A) >$, where $\mu^\tau(A) = \omega(\tau)(\mu_A) \wedge \omega(\tau)(\underline{1} - \gamma_A)$, $\gamma^\tau(A) = 1 - \mu^\tau(A)$.We say that $(\zeta^X, \mathrm{I}\omega(\tau))$ is a generated intuitionistic I-fuzzy topological space by fuzzifying topological space (X, τ).

2.11. Lemma. ([19]) *Let (X, τ) be a fuzzifying topological space, then*

(1) $\forall A \subseteq X$, $\mu^\tau(< 1_A, 1_{A^c} >) = \tau(A)$.

(2) $\forall A =< \underline{\alpha}, \underline{\beta} >\in \zeta^X$, $\mathrm{I}\omega(\tau)(A) = 1^\sim$.

2.12. Lemma. ([19]) *Suppose that* (ζ^X, δ) *is an intuitionistic* I*-fuzzy topological space, for each* $A \subseteq X$*, let* $[\delta](A) = \mu_\delta(< 1_A, 1_{A^c} >)$*. Then* $[\delta]$ *is a fuzzifying topology on* X*.*

2.13. Lemma. ([19]) *Let* (X, τ) *be a fuzzifying topological space and* $(X, \mathrm{I}\omega(\tau))$ *a generated intuitionistic* I*-fuzzy topological space. Then* $[\mathrm{I}\omega(\tau)] = \tau$*.*

3. Base and subbase in Intuitionistic I-fuzzy topological spaces

3.1. Definition. Let (X, τ) be an intuitionistic I-fuzzy topological space and $\mathcal{B} : \zeta^X \to \mathcal{A}$. $\mathcal{B}$ is called a base of τ if $\mathcal{B}$ satisfies the following condition

$$\tau(U) = \bigvee_{\bigvee\limits_{\lambda \in K} B_\lambda = U} \bigwedge_{\lambda \in K} \mathcal{B}(B_\lambda), \forall\, U \in \zeta^X.$$

3.2. Definition. Let (X, τ) be an intuitionistic I-fuzzy topological space and $\varphi : \zeta^X \to \mathcal{A}$, φ is called a subbase of τ if $\varphi^{(\sqcap)} : \zeta^X \to \mathcal{A}$ is a base, where $\varphi^{(\sqcap)}(A) = \bigvee\limits_{\sqcap\{B_\lambda : \lambda \in E\} = A} \bigwedge\limits_{\lambda \in E} \varphi(B_\lambda)$, for all $A \in \zeta^X$ with $(\sqcap)$ standing for "finite intersection".

3.3. Theorem. *Suppose that* $\mathcal{B} : \zeta^X \to \mathcal{A}$*. Then* $\mathcal{B}$ *is a base of some intuitionistic* I*-fuzzy topology, if* $\mathcal{B}$ *satisfies the following condition*

(1) $\mathcal{B}(0_\sim) = \mathcal{B}(1_\sim) = 1^\sim$,

(2) $\forall\, U, V \in \zeta^X$, $\mathcal{B}(U \wedge V) \geq \mathcal{B}(U) \wedge \mathcal{B}(V)$.

Proof. For $\forall\, A \in \zeta^X$, let $\tau(A) = \bigvee\limits_{\bigvee\limits_{\lambda \in K} B_\lambda = A} \bigwedge\limits_{\lambda \in K} \mathcal{B}(B_\lambda)$. To show that $\mathcal{B}$ is a base of τ, we only need to prove τ is an intuitionistic I-fuzzy topology on X. For all $U, V \in \zeta^X$,

$$\begin{aligned}
\tau(U) \wedge \tau(V) &= \Big(\bigvee_{\bigvee\limits_{\alpha \in K_1} A_\alpha = U} \bigwedge_{\alpha \in K_1} \mathcal{B}(A_\alpha)\Big) \wedge \Big(\bigvee_{\bigvee\limits_{\beta \in K_2} B_\beta = V} \bigwedge_{\beta \in K_2} \mathcal{B}(B_\beta)\Big) \\
&= \bigvee_{\bigvee\limits_{\alpha \in K_1} A_\alpha = U,\ \bigvee\limits_{\beta \in K_2} B_\beta = V} \Big(\Big(\bigwedge_{\alpha \in K_1} \mathcal{B}(A_\alpha)\Big) \wedge \Big(\bigwedge_{\beta \in K_2} \mathcal{B}(B_\beta)\Big)\Big) \\
&\leq \bigvee_{\bigvee\limits_{\alpha \in K_1, \beta \in K_2} (A_\alpha \wedge B_\beta) = U \wedge V} \Big(\bigwedge_{\alpha \in K_1, \beta \in K_2} \mathcal{B}(A_\alpha \wedge B_\beta)\Big) \\
&\leq \bigvee_{\bigvee\limits_{\lambda \in K} C_\lambda = U \wedge V} \bigwedge_{\lambda \in K} \mathcal{B}(C_\lambda) \\
&= \tau(U \wedge V).
\end{aligned}$$

For all $\{A_\lambda : \lambda \in K\} \subseteq \zeta^X$, Let $\mathcal{B}_\lambda = \{\{B_{\delta_\lambda} : \delta_\lambda \in K_\lambda\} : \bigvee\limits_{\delta_\lambda \in K_\lambda} B_{\delta_\lambda} = A_\lambda\}$, then

$$\tau\Big(\bigvee_{\lambda \in K} A_\lambda\Big) = \bigvee_{\bigvee\limits_{\delta \in K_1} B_\delta = \bigvee\limits_{\lambda \in K} A_\lambda} \bigwedge_{\delta \in K_1} \mathcal{B}(B_\delta).$$

For all $f \in \prod_{\lambda \in K} \mathcal{B}_\lambda$, we have

$$\bigvee_{\lambda \in K} \bigvee_{B_{\delta_\lambda} \in f(\lambda)} B_{\delta_\lambda} = \bigvee_{\lambda \in K} A_\lambda.$$

Therefore,

$$\begin{aligned}
\mu_{\tau(\bigvee_{\lambda \in K} A_\lambda)} &= \bigvee_{\bigvee_{\delta \in K_1} B_\delta = \bigvee_{\lambda \in K} A_\lambda} \bigwedge_{\delta \in K_1} \mu_{\mathcal{B}(B_\delta)} \\
&\geq \bigvee_{f \in \prod_{\lambda \in K} \mathcal{B}_\lambda} \bigwedge_{\lambda \in K} \bigwedge_{B_{\delta_\lambda} \in f(\lambda)} \mu_{\mathcal{B}(B_{\delta_\lambda})} \\
&= \bigwedge_{\lambda \in K} \bigvee_{\{B_{\delta_\lambda} : \delta_\lambda \in K_\lambda\} \in \mathcal{B}_\lambda} \bigwedge_{\delta_\lambda \in K_\lambda} \mu_{\mathcal{B}(B_{\delta_\lambda})} \\
&= \bigwedge_{\lambda \in E} \mu_{\tau(A_\lambda)}.
\end{aligned}$$

Similarly, we have

$$\gamma_{\tau(\bigvee_{\lambda \in K} A_\lambda)} \leq \bigvee_{\lambda \in K} \gamma_{\tau(A_\lambda)}.$$

Hence

$$\tau(\bigvee_{\lambda \in K} A_\lambda) \geq \bigwedge_{\lambda \in K} \tau(A_\lambda).$$

This means that τ is an intuitionistic I-fuzzy topology on X and $\mathcal{B}$ is a base of τ. □

3.4. Theorem. *Let $(X,\tau),(Y,\delta)$ be two intuitionistic I-fuzzy topology spaces and δ generated by its subbase φ. The mapping $f : (X,\tau) \to (Y,\delta)$ satisfies $\varphi(U) \leq \tau(f^{\leftarrow}(U))$, for all $U \in \zeta^Y$. Then f is fuzzy continuous, i.e., $\delta(U) \leq \tau(f^{\leftarrow}(U)), \forall\, U \in \zeta^Y$.*

Proof. $\forall\, U \in \zeta^Y$,

$$\begin{aligned}
\delta(U) &= \bigvee_{\bigvee_{\lambda \in K} A_\lambda = U} \bigwedge_{\lambda \in K} \bigvee_{\sqcap\{B_\mu : \mu \in K_\lambda\} = A_\lambda} \bigwedge_{\mu \in K_\lambda} \varphi(B_\mu) \\
&\leq \bigvee_{\bigvee_{\lambda \in K} A_\lambda = U} \bigwedge_{\lambda \in K} \bigvee_{\sqcap\{B_\mu : \mu \in K_\lambda\} = A_\lambda} \bigwedge_{\mu \in K_\lambda} \tau(f^{\leftarrow}(B_\mu)) \\
&\leq \bigvee_{\bigvee_{\lambda \in K} A_\lambda = U} \bigwedge_{\lambda \in K} \tau(f^{\leftarrow}(A_\lambda)) \\
&\leq \bigvee_{\bigvee_{\lambda \in K} A_\lambda = U} \tau(f^{\leftarrow}(\bigvee_{\lambda \in K} A_\lambda)) \\
&= \tau(f^{\leftarrow}(U)).
\end{aligned}$$

This completes the proof. □

3.5. Theorem. *Suppose that* (X,τ), (Y,δ) *are two intuitionistic* I*-fuzzy topology spaces and* τ *is generated by its base* $\mathcal{B}$*. If the mapping* $f:(X,\tau)\to(Y,\delta)$ *satisfies* $\mathcal{B}(U)\le\delta(f^{\to}(U))$*, for all* $U\in\zeta^X$*. Then* f *is fuzzy open, i.e.,* $\forall\, W\in\zeta^X, \tau(W)\le\delta(f^{\to}(W))$.

Proof. $\forall\, W\in\zeta^X$,

$$\begin{aligned}\tau(W) &= \bigvee_{\bigvee_{\lambda\in K} A_\lambda=W}\ \bigwedge_{\lambda\in K}\mathcal{B}(A_\lambda)\\ &\le \bigvee_{\bigvee_{\lambda\in K} A_\lambda=W}\ \bigwedge_{\lambda\in K}\delta(f^{\to}(A_\lambda))\\ &\le \bigvee_{\bigvee_{\lambda\in K} A_\lambda=W}\ \delta(f^{\to}(\bigvee_{\lambda\in K}A_\lambda))\\ &= \delta(f^{\to}(W)).\end{aligned}$$

Therefore, f is open. □

3.5. Theorem. *Let* $(X,\tau),(Y,\delta)$ *be two intuitionistic* I*-fuzzy topology spaces and* $f:(X,\tau)\to(Y,\delta)$ *intuitionistic* I*-fuzzy continuous,* $Z\subseteq X$*. Then* $f|_Z:(Z,\tau|_Z)\to(Y,\delta)$ *is continuous, where* $(f|_Z)(x)=f(x),(\tau|_Z)(A)=\vee\{\tau(U):U|_Z=A\}$*, for all* $x\in Z, A\in\zeta^Z$.

Proof. $\forall\, W\in\zeta^Z,(f|_Z)^{\leftarrow}(W)=f^{\leftarrow}(W)|_Z$, we have

$$\begin{aligned}(\tau|_Z)((f|_Z)^{\leftarrow}(W)) &= \vee\{\tau(U):U|_Z=(f|_Z)^{\leftarrow}(W)\}\\ &\ge \tau(f^{\leftarrow}(W))\\ &\ge \delta(W).\end{aligned}$$

Then $f|_Z$ is intuitionistic I-fuzzy continuous. □

3.6. Theorem. *Let* (X,τ) *be an intuitionistic* I*-fuzzy topology space and* τ *generated by its base* $\mathcal{B}$*,* $\mathcal{B}|_Y(U)=\vee\{\mathcal{B}(W):W|_Y=U\}$*, for* $Y\subseteq X, U\in\zeta^Y$*. Then* $\mathcal{B}|_Y$ *is a base of* $\tau|_Y$.

Proof. For $\forall\, U\in\zeta^X,(\tau|_Y)(U)=\bigvee_{V|_Y=U}\tau(V)=\bigvee_{V|_Y=U}\ \bigvee_{\bigvee_{\lambda\in K}A_\lambda=V}\ \bigwedge_{\lambda\in K}\mathcal{B}(A_\lambda)$. It remains to show the following equality

$$\bigvee_{V|_Y=U}\ \bigvee_{\bigvee_{\lambda\in K}A_\lambda=V}\ \bigwedge_{\lambda\in K}\mathcal{B}(A_\lambda)=\bigvee_{\bigvee_{\lambda\in K}B_\lambda=U}\ \bigwedge_{\lambda\in K}\ \bigvee_{W|_Y=B_\lambda}\mathcal{B}(W).$$

In one hand, for all $V\in\zeta^X$ with $V|_Y=U$, and $\bigvee_{\lambda\in K}A_\lambda=V$, we have $\bigvee_{\lambda\in K}A_\lambda|_Y=U$. Put $B_\lambda=A_\lambda|_Y$, clearly $\bigvee_{\lambda\in K}B_\lambda=U$. Then

$$\bigvee_{\bigvee_{\lambda\in K}B_\lambda=U}\ \bigwedge_{\lambda\in K}\ \bigvee_{W|_Y=B_\lambda}\mathcal{B}(W)\ge\bigwedge_{\lambda\in K}\mathcal{B}(A_\lambda).$$

Thus,

$$\bigvee_{V|_Y=U}\ \bigvee_{\bigvee\limits_{\lambda\in K}A_\lambda=V}\ \bigwedge_{\lambda\in K}\mathcal{B}(A_\lambda)\leq\bigvee_{\bigvee\limits_{\lambda\in K}B_\lambda=U}\ \bigwedge_{\lambda\in K}\ \bigvee_{W|_Y=B_\lambda}\mathcal{B}(W).$$

On the other hand, $\forall\ a\in(0,1], a<\bigvee_{\bigvee\limits_{\lambda\in K}B_\lambda=U}\ \bigwedge_{\lambda\in K}\ \bigvee_{W|_Y=B_\lambda}\mu_{\mathcal{B}(W)}$, there exists a family of $\{B_\lambda:\lambda\in K\}\subseteq\zeta^Y$, such that

(1) $\bigvee\limits_{\lambda\in K}B_\lambda=U$;

(2) $\forall\ \lambda\in K$, there exists $W_\lambda\in\zeta^X$ with $W_\lambda|_Y=B_\lambda$ such that $a<\mu_{\mathcal{B}(W_\lambda)}$.

Let $V=\bigvee\limits_{\lambda\in E}W_\lambda$, it is clear $V|_Y=U$ and $\bigwedge\limits_{\lambda\in K}\mu_{\mathcal{B}(W_\lambda)}\geq a$. Then

$$\bigvee_{V|_Y=U}\ \bigvee_{\bigvee\limits_{\lambda\in K}A_\lambda=V}\ \bigwedge_{\lambda\in K}\mu_{\mathcal{B}(A_\lambda)}\geq a.$$

By the arbitrariness of a, we have

$$\bigvee_{V|_Y=U}\ \bigvee_{\bigvee\limits_{\lambda\in K}A_\lambda=V}\ \bigwedge_{\lambda\in K}\mu_{\mathcal{B}(A_\lambda)}\geq\bigvee_{\bigvee\limits_{\lambda\in K}B_\lambda=U}\ \bigwedge_{\lambda\in K}\ \bigvee_{W|_Y=B_\lambda}\mu_{\mathcal{B}(W)}.$$

Similarly, we may obtain that

$$\bigwedge_{V|_Y=U}\ \bigwedge_{\bigvee\limits_{\lambda\in K}A_\lambda=V}\ \bigvee_{\lambda\in K}\gamma_{\mathcal{B}(A_\lambda)}\leq\bigwedge_{\bigvee\limits_{\lambda\in K}B_\lambda=U}\ \bigvee_{\lambda\in K}\ \bigwedge_{W|_Y=B_\lambda}\gamma_{\mathcal{B}(W)}.$$

So we have

$$\bigvee_{V|_Y=U}\ \bigvee_{\bigvee\limits_{\lambda\in K}A_\lambda=V}\ \bigwedge_{\lambda\in K}\mathcal{B}(A_\lambda)\geq\bigvee_{\bigvee\limits_{\lambda\in K}B_\lambda=U}\ \bigwedge_{\lambda\in K}\ \bigvee_{W|_Y=B_\lambda}\mathcal{B}(W).$$

Therefore,

$$\bigvee_{V|_Y=U}\ \bigvee_{\bigvee\limits_{\lambda\in K}A_\lambda=V}\ \bigwedge_{\lambda\in K}\mathcal{B}(A_\lambda)=\bigvee_{\bigvee\limits_{\lambda\in K}B_\lambda=U}\ \bigwedge_{\lambda\in K}\ \bigvee_{W|_Y=B_\lambda}\mathcal{B}(W).$$

This means that $\mathcal{B}|_Y$ is a base of $\tau|_Y$. □

3.7. Theorem. *Let $\{(X_\alpha,\tau_\alpha)\}_{\alpha\in J}$ be a family of intuitionistic I-fuzzy topology spaces and $P_\beta:\prod\limits_{\alpha\in J}X_\alpha\to X_\beta$ the projection. For all $W\in\zeta^{\prod\limits_{\alpha\in J}X_\alpha}$, $\varphi(W)=\bigvee\limits_{\alpha\in J}\ \bigvee\limits_{P_\alpha^{\leftarrow}(U)=W}\tau_\alpha(U)$. Then φ is a subbase of some intuitionistic I-fuzzy topology τ, here τ is called the product intuitionistic I-fuzzy topologies of $\{\tau_\alpha:\alpha\in J\}$ and denoted by $\tau=\prod\limits_{\alpha\in J}\tau_\alpha$.*

Proof. We need to prove $\varphi^{(\sqcap)}$ is a subbase of τ.

$$\begin{aligned}\varphi^{(\sqcap)}(1_{\sim}) &= \bigvee_{\sqcap\{B_\lambda:\lambda\in E\}=1_{\sim}} \bigwedge_{\lambda\in E} \varphi(B_\lambda)\\ &= \bigvee_{\sqcap\{B_\lambda:\lambda\in E\}=1_{\sim}} \bigwedge_{\lambda\in E} \bigvee_{\alpha\in J} \bigvee_{P_\alpha^{\leftarrow}(U)=B_\lambda} \tau_\alpha(U)\\ &= 1^{\sim}.\end{aligned}$$

Similarly, $\varphi^{(\sqcap)}(0_{\sim}) = 1^{\sim}$. For all $U, V \in \zeta^{\prod_{\alpha\in J} X_\alpha}$, we have

$$\begin{aligned}\varphi^{(\sqcap)}(U) \wedge \varphi^{(\sqcap)}(V) &= (\bigvee_{\sqcap\{B_\alpha:\alpha\in E_1\}=U} \bigwedge_{\alpha\in E_1} \varphi(B_\alpha)) \wedge (\bigvee_{\sqcap\{C_\beta:\beta\in E_2\}=V} \bigwedge_{\beta\in E_2} \varphi(C_\beta))\\ &= \bigvee_{\sqcap\{B_\alpha:\alpha\in E_1\}=U} \bigvee_{\sqcap\{C_\beta:\beta\in E_2\}=V} ((\bigwedge_{\alpha\in E_1} \varphi(B_\alpha)) \wedge (\bigwedge_{\beta\in E_2} \varphi(C_\beta)))\\ &\leq \bigvee_{\sqcap\{B_\lambda:\lambda\in E\}=U\wedge V} \bigwedge_{\lambda\in E} \varphi(B_\lambda)\\ &= \varphi^{(\sqcap)}(U \wedge V).\end{aligned}$$

Hence, $\varphi^{(\sqcap)}$is a base of τ, i.e., φ is a subbase of τ. And by Theorem 3.3 we have

$$\begin{aligned}\tau(A) &= \bigvee_{\bigvee_{\lambda\in K} B_\lambda=A} \bigwedge_{\lambda\in K} \varphi^{(\sqcap)}(B_\lambda)\\ &= \bigvee_{\bigvee_{\lambda\in K} B_\lambda=A} \bigwedge_{\lambda\in K} \bigvee_{\sqcap\{C_\rho:\rho\in E\}=B_\lambda} \bigwedge_{\rho\in E} \varphi(C_\rho)\\ &= \bigvee_{\bigvee_{\lambda\in K} B_\lambda=A} \bigwedge_{\lambda\in K} \bigvee_{\sqcap\{C_\rho:\rho\in E\}=B_\lambda} \bigwedge_{\rho\in E} \bigvee_{\alpha\in J} \bigvee_{P_\alpha^{\leftarrow}(V)=C_\rho} \tau_\alpha(V).\end{aligned}$$

□

By the above discussions, we easily obtain the following corollary.

3.8. Corollary. *Let* $(\prod_{\alpha\in J} X_\alpha, \prod_{\alpha\in J} \tau_\alpha)$ *be the product space of a family of intuitionistic I-fuzzy topology spaces* $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in J}$. *Then* $P_\beta : (\prod_{\alpha\in J} X_\alpha, \prod_{\alpha\in J} \tau_\alpha) \to (X_\beta, \tau_\beta)$ *is continuous, for all* $\beta \in J$.

Proof. $\forall\ U \in \zeta^{X_\beta}$,

$$\begin{aligned}\tau(P_\beta^{\leftarrow}(U)) &= \bigvee_{\bigvee_{\lambda\in K} B_\lambda=P_\beta^{\leftarrow}(U)} \bigwedge_{\lambda\in K} \bigvee_{\sqcap\{C_\rho:\rho\in E\}=B_\lambda} \bigwedge_{\rho\in E} \bigvee_{\alpha\in J} \bigvee_{P_\alpha^{\leftarrow}(V)=C_\rho} \tau_\alpha(V)\\ &\geq \tau_\beta(U)\end{aligned}$$

Therefore, P_β is continuous. □

4. Applications in product Intuitionistic I-fuzzy topological space

4.1. Definition. Let (X,τ) be an intuitionistic I-fuzzy topology space. The degree to which two distinguished intuitionistic fuzzy points $x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X)(x \neq y)$ are T_2 is defined as follows

$$T_2(x_{(\alpha,\beta)}, y_{(\lambda,\rho)}) = \bigvee_{U\wedge V=0_\sim} (Q_{x_{(\alpha,\beta)}}(U) \wedge Q_{y_{(\lambda,\rho)}}(V)).$$

The degree to which (X,τ) is T_2 is defined by

$$T_2(X,\tau) = \bigwedge \big\{T_2(x_{(\alpha,\beta)}, y_{(\lambda,\rho)}) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y\big\}.$$

4.2. Theorem. *Let $(X, \mathrm{I}\omega(\tau))$ be a generated intuitionistic I-fuzzy topological space by fuzzifying topological space (X,τ) and $T_2(X,\mathrm{I}\omega(\tau)) \triangleq \langle \mu_{T_2(X,\mathrm{I}\omega(\tau))}, \gamma_{T_2(X,\mathrm{I}\omega(\tau))}\rangle$. Then $\mu_{T_2(X,\mathrm{I}\omega(\tau))} = T_2(X,\tau)$.*

Proof. For all $x,y \in X, x \neq y$, and each $a < \bigwedge\big\{ \bigvee_{U\wedge V=0_\sim} \big(\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}\big) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y\big\}$, there exists $U, V \in \zeta^X$ with $U \wedge V = 0_\sim$ such that $a < \mu_{Q_{x_{(1,0)}}(U)}, a < \mu_{Q_{y_{(1,0)}}(V)}$. Then there exists $U_1, V_1 \in \zeta^X$, such that

$$x_{(1,0)}\widehat{q}\, U_1 \leq U,\ a < \omega(\tau)(\mu_{U_1}),$$
$$y_{(1,0)}\widehat{q}\, V_1 \leq V,\ a < \omega(\tau)(\mu_{V_1}).$$

Denote $A = \sigma_0(\mu_{U_1}), B = \sigma_0(\mu_{V_1})$, it is clear that $x \in A, y \in B$. From the fact $U \wedge V = 0_\sim$, it implies $\mu_{U_1} \wedge \mu_{V_1} = \underline{0}$. Then we have $\sigma_0(\mu_{U_1}) \wedge \sigma_0(\mu_{V_1}) = \emptyset$, i.e.,$A \wedge B = \emptyset$.

$$a < \omega(\tau)(\mu_{U_1}) = \bigwedge_{r\in I} \tau(\sigma_r(\mu_{U_1})) \leq \tau(\sigma_0(\mu_{U_1})) = \tau(A).$$

Thus

$$a < \bigvee_{x\in U\subseteq A} \tau(U) = N_x(A).$$

Similarly, we have $a < N_y(B)$. Hence

$$a < \bigvee_{A\cap B=\emptyset} (N_x(A) \wedge N_y(B)).$$

Then

$$a \leq \bigwedge\big\{ \bigvee_{A\cap B=\emptyset} (N_x(A) \wedge N_y(B)) : x,y \in X, x \neq y\big\}.$$

Therefore,

$$\bigwedge\big\{ \bigvee_{U\wedge V=0_\sim} \big(\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}\big) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y\big\}$$
$$\leq \bigwedge\big\{ \bigvee_{A\cap B=\emptyset} (N_x(A) \wedge N_y(B)) : x,y \in X, x \neq y\big\}.$$

On the other hand, for all $x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y$, and $a < \bigwedge\big\{ \bigvee_{A\cap B=\emptyset} (N_x(A) \wedge N_y(B)) : x,y \in X, x \neq y\big\}$, there exists $A, B \in 2^X, A \wedge B = \emptyset$, such that $a < N_x(A), a < N_y(B)$. Then there exists $A_1, B_1 \in 2^X$, such that

$$x \in A_1 \subseteq A,\ a < \tau(A_1),$$

$$y \in B_1 \subseteq B,\ a < \tau(B_1).$$

Let $U = \langle 1_{A_1}, 1_{A_1^c}\rangle, V = \langle 1_{B_1}, 1_{B_1^c}\rangle$, where A_1^c is the complement of A_1, then $x_{(\alpha,\beta)}\widehat{q}\ U, y_{(\lambda,\rho)}\widehat{q}\ V$. In fact, $1_{A_1}(x) = 1 > 1-\alpha, 1_{A_1^c}(x) = 0 < 1-\beta$. Thus $x_{(\alpha,\beta)}\widehat{q}\ U$. Similarly, we have $y_{(\lambda,\rho)}\widehat{q}\ V$. By $A \wedge B = \emptyset$, we have $A_1 \wedge B_1 = \emptyset$. Then for all $z \in X$, we obtain

$$(1_{A_1} \wedge 1_{B_1})(z) = 1_{A_1}(z) \wedge 1_{B_1}(z) = 0,$$
$$(1_{A_1^c} \vee 1_{B_1^c})(z) = 1_{A_1^c}(z) \vee 1_{B_1^c}(z) = 1.$$

Hence

$$1_{A_1} \wedge 1_{B_1} = \underline{0},\ 1_{A_1^c} \vee 1_{B_1^c} = \underline{1}.$$

Since $\forall\ r \in I_1, \sigma_r(1_{A_1}) = A_1$, we have

$$\omega(\tau)(1_{A_1}) = \bigwedge_{r \in I_1} \tau(\sigma_r(1_{A_1})) = \tau(A_1).$$

By $\underline{1} - 1_{A_1^c} = 1_{A_1}$, and $a < \tau(A_1)$, we have

$$\begin{aligned} a &< \omega(\tau)(1_{A_1}) \wedge \omega(\tau)(\underline{1} - 1_{A_1^c}) \\ &= \omega(\tau)(\mu_U) \wedge \omega(\tau)(\underline{1} - \gamma_U). \end{aligned}$$

So,

$$a < \bigvee_{x_{(\alpha,\beta)}\widehat{q}\ W \subseteq U} (\omega(\tau)(\mu_W) \wedge \omega(\tau)(\underline{1} - \gamma_W)) = \mu_{Q_{x_{(\alpha,\beta)}}(U)}.$$

Similarly, we have $a < \mu_{Q_{y_{(\lambda,\rho)}}(V)}$. This deduces that

$$a < \bigvee_{U \wedge V = 0_\sim} \big(\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}\big).$$

Furthermore, we may obtain

$$a \leq \bigwedge\big\{ \bigvee_{U \wedge V = 0_\sim} \big(\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}\big) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y\big\}.$$

Hence

$$\begin{aligned} &\bigwedge\big\{ \bigvee_{U \wedge V = 0_\sim} \big(\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}\big) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y\big\} \\ &\geq \bigwedge\big\{ \bigvee_{A \cap B = \emptyset} (N_x(A) \wedge N_y(B)) : x, y \in X, x \neq y\big\}. \end{aligned}$$

This means that $\bigwedge\big\{ \bigvee_{U \wedge V = 0_\sim} \big(\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}\big) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y\big\} = \bigwedge\big\{ \bigvee_{A \cap B = \emptyset} (N_x(A) \wedge N_y(B)) : x, y \in X, x \neq y\big\}$. Therefore we have

$$\mu_{T_2(X, \mathrm{I}\omega(\tau))} = T_2(X, \tau).$$

□

4.3. Lemma. *Let* $(\prod_{j \in J} X_j, \prod_{j \in J} \tau_j)$ *be the product space of a family of intuitionistic I-fuzzy topology spaces* $\{(X_j, \tau_j)\}_{j \in J}$*. Then* $\tau_j(A_j) \leq (\prod_{j \in J} \tau_j)(P_j^{\leftarrow}(A_j))$*, for all* $j \in J, A_j \in \zeta^{X_j}$.

Proof. Let $\prod_{j\in J}\tau_j=\delta$, $x_{(\alpha,\beta)}\widehat{q}\, f^{\leftarrow}(U)\Leftrightarrow f^{\rightarrow}(x_{(\alpha,\beta)})\widehat{q}\, U$. Then for all $j\in J, A_j\in \zeta^{X_j}$, we have

$$\begin{aligned}\delta(P_j^{\leftarrow}(A_j)) &= \bigwedge_{x_{(\alpha,\beta)}\widehat{q}\ P_j^{\leftarrow}(A_j)} Q^{\delta}_{x_{(\alpha,\beta)}}(P_j^{\leftarrow}(A_j))\\ &\geq \bigwedge_{x_{(\alpha,\beta)}\widehat{q}\ P_j^{\leftarrow}(A_j)} Q^{\tau_j}_{P_j^{\rightarrow}(x_{(\alpha,\beta)})}(A_j)\\ &= \bigwedge_{P_j^{\rightarrow}(x_{(\alpha,\beta)})\widehat{q}\ A_j} Q^{\tau_j}_{P_j^{\rightarrow}(x_{(\alpha,\beta)})}(A_j)\\ &\geq \bigwedge_{x^j_{(\alpha,\beta)}\widehat{q}\ A_j} Q^{\tau_j}_{x^j_{(\alpha,\beta)}}(A_j)\\ &= \tau_j(A_j).\end{aligned}$$

This completes the proof. □

4.4. Theorem. *Let $(\prod_{j\in J}X_j,\prod_{j\in J}\tau_j)$ be the product space of a family of intuitionistic I-fuzzy topology spaces $\{(X_j,\tau_j)\}_{j\in J}$. Then $\bigwedge_{j\in J}T_2(X_j,\tau_j)\leq T_2(\prod_{j\in J}X_j,\prod_{j\in J}\tau_j)$.*

Proof. For all $g_{(\alpha,\beta)},h_{(\lambda,\rho)}\in \mathrm{pt}(\zeta^{\prod_{j\in J}X_j})$ and $g\neq h$. Then there exists $j_0\in J$ such that $g(j_0)\neq h(j_0)$, where $g(j_0),h(j_0)\in X_{j_0}$.

For all $U_{j_0},V_{j_0}\in\zeta^{X_{j_0}}$ with $U_{j_0}\wedge V_{j_0}=0_{\sim}^{X_{j_0}}$, we have

$$P_{j_0}^{\leftarrow}(U_{j_0})\wedge P_{j_0}^{\leftarrow}(V_{j_0})=P_{j_0}^{\leftarrow}(U_{j_0}\wedge V_{j_0})=0_{\sim}^{\prod_{j\in J}X_j}.$$

Then $Q_{g(j_0)_{(\alpha,\beta)}}(U_{j_0})\leq Q_{g_{(\alpha,\beta)}}(P_{j_0}^{\leftarrow}(U_{j_0}))$. In fact, if $g(j_0)_{(\alpha,\beta)}\widehat{q}\,U_{j_0}$, then $g_{(\alpha,\beta)}\widehat{q}\,P_{j_0}^{\leftarrow}(U_{j_0})$. For all $V\leq U_{j_0}$, we have $P_{j_0}^{\leftarrow}(V)\leq P_{j_0}^{\leftarrow}(U_{j_0})$. On account of Lemma 4.3, we have

$$\begin{aligned}\bigvee_{g(j_0)_{(\alpha,\beta)}\widehat{q}\ V\leq U_{j_0}}\tau_{j_0}(V) &\leq \bigvee_{g_{(\alpha,\beta)}\widehat{q}\ P_{j_0}^{\leftarrow}(V)\leq P_{j_0}^{\leftarrow}(U_{j_0})}(\prod_{j\in J}\tau_j)(P_{j_0}^{\leftarrow}(V))\\ &\leq \bigvee_{g_{(\alpha,\beta)}\widehat{q}\ G\leq P_{j_0}^{\leftarrow}(U_{j_0})}(\prod_{j\in J}\tau_j)(G),\end{aligned}$$

i.e., $Q_{g(j_0)_{(\alpha,\beta)}}(U_{j_0})\leq Q_{g_{(\alpha,\beta)}}(P_{j_0}^{\leftarrow}(U_{j_0}))$. Thus,

$$\begin{aligned}&\bigvee_{U\wedge V=0_{\sim}^{X_{j_0}}}(Q_{g(j_0)_{(\alpha,\beta)}}(U)\wedge Q_{h(j_0)_{(\lambda,\rho)}}(V))\\ \leq&\bigvee_{P_{j_0}^{\leftarrow}(U)\wedge P_{j_0}^{\leftarrow}(V)=0_{\sim}^{\prod_{j\in J}X_j}}(Q_{g_{(\alpha,\beta)}}(P_{j_0}^{\leftarrow}(U))\wedge Q_{h_{(\lambda,\rho)}}(P_{j_0}^{\leftarrow}(V)))\\ \leq&\bigvee_{G\wedge H=0_{\sim}^{\prod_{j\in J}X_j}}(Q_{g_{(\alpha,\beta)}}(G)\wedge Q_{h_{(\lambda,\rho)}}(H)).\end{aligned}$$

So we have

$$T_2(g(j_0)_{(\alpha,\beta)}, h(j_0)_{(\lambda,\rho)}) \leq T_2(g_{(\alpha,\beta)}, h_{(\lambda,\rho)}).$$

Thus

$$T_2(X_{j_0}, \tau_{j_0}) \leq T_2(\prod_{j\in J} X_j, \prod_{j\in J} \tau_j).$$

Therefore,

$$\bigwedge_{j\in J} T_2(X_j, \tau_j) \leq T_2(\prod_{j\in J} X_j, \prod_{j\in J} \tau_j).$$

□

4.5. Lemma. *Let $(X, \mathrm{I}\omega(\tau))$ be a generated intuitionistic I-fuzzy topological space by fuzzifying topological space (X, τ). Then*

(1) $\mathrm{I}\omega(\tau)(A) = 1^{\sim}$, *for all* $A = \langle \underline{\alpha}, \underline{\beta} \rangle \in \zeta^X$;

(2) $\forall\ B \subseteq X, \tau(B) = \mu_{\mathrm{I}\omega(\tau)}(\langle 1_B, 1_{B^c}\rangle)$.

Proof. By Lemma 2.11, 2.12 and 2.13, it is easy to prove it. □

4.6. Lemma. *Let (X, δ) be a stratified intuitionistic I-fuzzy topological space (i.e., for all $<\alpha, \beta>\in \mathcal{A}, \delta(<\underline{\alpha}, \underline{\beta}>) = 1^{\sim}$). Then for all $A \in \zeta^X$*

$$\bigwedge_{r\in I} \mu_\delta(\langle 1_{\sigma_r(\mu_A)}, 1_{(\sigma_r(\mu_A))^c}\rangle) \leq \mu_\delta(A).$$

Proof. For all $A \in \zeta^X$, and for any $a < \bigwedge_{r\in I} \mu_\delta(\langle 1_{\sigma_r(\mu_A)}, 1_{(\sigma_r(\mu_A))^c}\rangle), y_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)$ with $y_{(\alpha,\beta)}\widehat{q}\ A$, clearly $\mu_A(y) > 1-\alpha$. Then there exists $\delta > 0$ such that $\mu_A(y) > 1-\alpha+\delta$. Thus $y \in \sigma_{1-\alpha+\delta}(\mu_A)$. So we have

$$y_{(\alpha,\beta)}\widehat{q}\ \langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c}\rangle.$$

Then

$$\begin{aligned} a &< \mu_\delta(\langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c}\rangle) \\ &= \bigwedge_{z_{(\alpha,\beta)}\widehat{q}\ \langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c}\rangle} \mu(Q_{z_{(\alpha,\beta)}}(\langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c}\rangle)). \end{aligned}$$

Therefore,

$$a < \mu(Q_{y_{(\alpha,\beta)}}(\langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c}\rangle)).$$

Since (X, δ) is a stratified intuitionistic I-fuzzy topological space, we have $Q_{y_{(\alpha,\beta)}}(\underline{1-\alpha+\delta}, \underline{\alpha-\delta}\rangle) = 1^{\sim}$. Moreover, it is well known that the following relations hold

$$\underline{1-\alpha+\delta} \wedge 1_{\sigma_{1-\alpha+\delta}(\mu_A)} \leq \mu_A,$$

$$\underline{\alpha-\delta} \vee 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \geq 1-\mu_A \geq \gamma_A.$$

So we have

$$a < \mu(Q_{y_{(\alpha,\beta)}}(\langle \underline{1-\alpha+\delta} \wedge 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, \underline{\alpha-\delta} \vee 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c}\rangle)) \leq \mu(Q_{y_{(\alpha,\beta)}}(A)).$$

Then $a \leq \mu_\delta(A)$. Therefore,

$$\bigwedge_{r\in I} \mu_\delta(\langle 1_{\sigma_r(\mu_A)}, 1_{(\sigma_r(\mu_A))^c}\rangle) \leq \mu_\delta(A).$$

□

4.7. Theorem. *Let* $(\prod_{\alpha\in J} X_\alpha, \prod_{\alpha\in J} \tau_\alpha)$ *be the product space of a family of fuzzifying topological space* $\{(X_\alpha,\tau_\alpha)\}_{\alpha\in J}$. *Then* $(\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A) = \mathrm{I}\omega(\prod_{\alpha\in J} \tau_\alpha)(A)$.

Proof. Let $(\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A) = \langle \mu_{\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))}(A), \gamma_{\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))}(A)\rangle$. For all $a < \mu_{\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))}(A)$, there exists $\{U_j^a\}_{j\in K}$ such that $\bigvee_{j\in K} U_j^a = A$, for each U_j^a, there exists $\{A_{\lambda,j}^a\}_{\lambda\in E}$ such that $\bigwedge_{\lambda\in E} A_{\lambda,j}^a = U_j^a$, where E is an finite index set. In addition, for every $\lambda\in E$, there exists $\alpha \triangleq \alpha(\lambda)\in J$ and $W_\alpha \in \zeta^{X_\alpha}$ with $P_\alpha^{\leftarrow}(W_\alpha) = A_{\lambda,j}^a$ such that $a < \mu(\mathrm{I}\omega(\tau_\alpha)(W_\alpha))$. Then we have

$$a < \omega(\tau_\alpha)(\mu_{W_\alpha}),$$

$$a < \omega(\tau_\alpha)(\underline{1} - \gamma_{W_\alpha}).$$

Thus for all $r\in I$, we have

$$\begin{aligned} a &< \tau_\alpha(\sigma_r(\mu_{W_\alpha})) \\ &\leq (\prod_{\alpha\in J} \tau_\alpha)(P_\alpha^{\leftarrow}(\sigma_r(\mu_{W_\alpha}))) \\ &= (\prod_{\alpha\in J} \tau_\alpha)(\sigma_r(P_\alpha^{\leftarrow}(\mu_{W_\alpha}))) \\ &= (\prod_{\alpha\in J} \tau_\alpha)(\sigma_r(\mu_{A_{\lambda,j}^a})). \end{aligned}$$

Hence

$$\begin{aligned} a &\leq (\prod_{\alpha\in J} \tau_\alpha)(\bigwedge_{\lambda\in E} \sigma_r(\mu_{A_{\lambda,j}^a})) \\ &= (\prod_{\alpha\in J} \tau_\alpha)(\sigma_r(\bigwedge_{\lambda\in E} \mu_{A_{\lambda,j}^a})) \\ &= (\prod_{\alpha\in J} \tau_\alpha)(\sigma_r(\mu_{U_j^a})). \end{aligned}$$

Furthermore

$$\begin{aligned} a &\leq (\prod_{\alpha\in J} \tau_\alpha)(\bigvee_{j\in K} \sigma_r(\mu_{U_j^a})) \\ &= (\prod_{\alpha\in J} \tau_\alpha)(\sigma_r(\bigvee_{j\in K} \mu_{U_j^a})) \\ &= (\prod_{\alpha\in J} \tau_\alpha)(\sigma_r(\mu_A)). \end{aligned}$$

So

$$\begin{aligned} a &\leq \bigwedge_{r\in I}(\prod_{\alpha\in J}\tau_\alpha)(\sigma_r(\mu_A)) \\ &= \omega(\prod_{\alpha\in J}\tau_\alpha)(\mu_A). \end{aligned}$$

Similarly, we have

$$a \leq \omega(\prod_{\alpha\in J}\tau_\alpha)(\underline{1}-\gamma_A).$$

Hence $a \leq \mu(\mathrm{I}\omega(\prod_{\alpha\in J}\tau_\alpha)(A))$. By the arbitrariness of a, we have $\mu((\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(A)) \leq \mu(\mathrm{I}\omega(\prod_{\alpha\in J}\tau_\alpha)(A))$.

On the other hand, for $\forall\ a < \mu(\mathrm{I}\omega(\prod_{\alpha\in J}\tau_\alpha)(A))$, we have

$$a < \omega(\prod_{\alpha\in J}\tau_\alpha)(\mu_A) = \bigwedge_{r\in I}(\prod_{\alpha\in J}\tau_\alpha)(\sigma_r(\mu_A))$$

and

$$a < \omega(\prod_{\alpha\in J}\tau_\alpha)(\underline{1}-\gamma_A).$$

Then for all $r\in I$, we have

$$a < (\prod_{\alpha\in J}\tau_\alpha)(\sigma_r(\mu_A)).$$

Thus there exists $\{U^a_{j,r}\}_{j\in K}\subseteq X$ satisfies $\bigvee_{j\in K}U^a_{j,r}=\sigma_r(\mu_A)$, and for all $j\in K$, there exists $\{A^a_{\lambda,j,r}\}_{\lambda\in E}$, where E is an finite index set, such that $\bigwedge_{\lambda\in E}A^a_{\lambda,j,r}=U^a_{j,r}$. For all $\lambda\in E$, there exists $\alpha(\lambda)\in J, W_\alpha\in\zeta^{X_\alpha}$, such that $P^{\leftarrow}_\alpha(W_\alpha)=A^a_{\lambda,j,r}$. By Lemma 4.5 we have

$$\begin{aligned} a < \tau_\alpha(W_\alpha) &= \mu_{\mathrm{I}\omega(\tau_\alpha)}(\langle 1_{W_\alpha}, 1_{W^c_\alpha}\rangle) \\ &\leq \mu(\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(P^{\leftarrow}_\alpha(\langle 1_{W_\alpha}, 1_{W^c_\alpha}\rangle)) \\ &= \mu(\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(\langle 1_{P^{\leftarrow}_\alpha(W_\alpha)}, 1_{P^{\leftarrow}_\alpha(W^c_\alpha)}\rangle) \\ &= \mu(\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(\langle 1_{A^a_{\lambda,j,r}}, 1_{(A^a_{\lambda,j,r})^c}\rangle) \\ &\leq \mu(\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(\langle \bigwedge_{\lambda\in E}1_{A^a_{\lambda,j,r}}, \bigvee_{\lambda\in E}1_{(A^a_{\lambda,j,r})^c}\rangle) \\ &= \mu(\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(\langle 1_{\bigwedge_{\lambda\in E}A^a_{\lambda,j,r}}, 1_{\bigvee_{\lambda\in E}(A^a_{\lambda,j,r})^c}\rangle) \\ &= \mu(\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(\langle 1_{U^a_{j,r}}, 1_{(U^a_{j,r})^c}\rangle). \end{aligned}$$

Then

$$\begin{aligned} a &\leq \mu(\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(\langle 1_{\bigvee_{j\in K} U^a_{j,r}}, 1_{(\bigvee_{j\in K} U^a_{j,r})^c}\rangle) \\ &= \mu(\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(\langle 1_{\sigma_r(\mu_A)}, 1_{(\sigma_r(\mu_A))^c}\rangle). \end{aligned}$$

By Lemma 4.6 we have

$$\begin{aligned} a &\leq \bigwedge_{r\in I} \mu(\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(\langle 1_{\sigma_r(\mu_A)}, 1_{(\sigma_r(\mu_A))^c}\rangle) \\ &\leq \mu((\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A)). \end{aligned}$$

Then

$$\mu((\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A)) \geq \mu(\mathrm{I}\omega(\prod_{\alpha\in J} \tau_\alpha)(A)).$$

Hence

$$\mu((\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A)) = \mu(\mathrm{I}\omega(\prod_{\alpha\in J} \tau_\alpha)(A)).$$

Then

$$\gamma((\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A)) = \gamma(\mathrm{I}\omega(\prod_{\alpha\in J} \tau_\alpha)(A)).$$

Therefore,

$$(\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A) = \mathrm{I}\omega(\prod_{\alpha\in J} \tau_\alpha)(A).$$

□

5. Further remarks

As we have shown, the notions of the base and subbase in intuitionistic I-fuzzy topological spaces are introduced in this paper, and some important applications of them are obtained. Specially, we also use the concept of subbase to study the product of intuitionistic I-fuzzy topological spaces. In addition, we have proved that the functor Iω preserves the product.

There are two categories in our paper, the one is the category **FYTS** of fuzzifying topological spaces, and the other is the category **IFTS** of intuitionistic I-fuzzy topological spaces. It is easy to find that Iω is the functor from **FYTS** to **IFTS**. We discussed the property of the functor Iω in Theorem 4.7. A direction worthy of further study is to discuss the the properties of the functor Iω in detail. Moreover, we hope to point out that another continuation of this paper is to deal with other topological properties of intuitionistic I-fuzzy topological spaces.

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