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Base change for semiorthogonal decompositions

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Abstract

Let X be an algebraic variety over a base scheme S and $\phi : T \to S$ a base change. Given an admissible subcategory $\mathcal{A}$ in $\mathcal{D}^b(X)$, the bounded derived category of coherent sheaves on X, we construct under some technical conditions an admissible subcategory $\mathcal{A}_T$ in $\mathcal{D}^b(X \times_S T)$, called the base change of $\mathcal{A}$, in such a way that the following base change theorem holds: if a semiorthogonal decomposition of $\mathcal{D}^b(X)$ is given, then the base changes of its components form a semiorthogonal decomposition of $\mathcal{D}^b(X \times_S T)$. As an intermediate step, we construct a compatible system of semiorthogonal decompositions of the unbounded derived category of quasicoherent sheaves on X and of the category of perfect complexes on X. As an application, we prove that the projection functors of a semiorthogonal decomposition are kernel functors.

1. Introduction

An important approach to non-commutative algebraic geometry is to consider triangulated categories with good properties as substitutes for non-commutative varieties. Given such a category, we consider it as the bounded derived category of coherent sheaves on a would-be variety and try to do some geometry. Note, however, that even the simplest geometric functors between derived categories often do not preserve boundedness or coherence; the pullback functor preserves boundedness only if the corresponding morphism has finite Tor-dimension and the pushforward functor preserves coherence only if the corresponding map is proper. So, to do non-commutative geometry we need some unbounded and quasicoherent versions of the triangulated categories under consideration. One goal of this paper is the following: given a good triangulated category $\mathcal{A}$ (considered as a bounded derived category of coherent sheaves), to define a category $\mathcal{A}_{\mathrm{qc}}$, a substitute for the unbounded derived category of quasicoherent sheaves and a category $\mathcal{A}^-$, a substitute for the bounded above derived category of coherent sheaves.

A straightforward approach to construct $\mathcal{A}_{\mathrm{qc}}$ would be just to consider the closure of $\mathcal{A}$ under colimits. However, it is not clear how to define a triangulated structure there. So, instead, we assume that the category $\mathcal{A}$ is given as an admissible subcategory in $\mathcal{D}^b(X)$, the bounded derived category of coherent sheaves on some algebraic variety X, and consider the minimal triangulated subcategory $\hat{\mathcal{A}} \subset \mathcal{D}_{\mathrm{qc}}(X)$ containing $\mathcal{A}$ and closed under arbitrary direct sums. Defined in this way, the category $\hat{\mathcal{A}}$ inherits a triangulated structure automatically, but there arises a question of dependence of $\hat{\mathcal{A}}$ on the choice of the variety X and of the embedding $\mathcal{A} \to \mathcal{D}^b(X)$. We prove that it is actually independent of these choices under some technical condition.

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Another, and in fact the most important, goal of the paper is to define a base change for triangulated categories. Assume that S is an algebraic variety and $\mathcal{A}$ is a good triangulated category over S (which can be understood, for example, as that $\mathcal{A}$ is a module category over the tensor triangulated category $\mathcal{D}^{\mathsf{perf}}(S)$ of perfect complexes on S). Given a base change $\phi : T \to S$, we would like to define a triangulated category $\mathcal{A}_T$ over T to be considered as the base change of $\mathcal{A}$. Again, an abstract approach is too complicated, so we assume that $\mathcal{A}$ is given as an S-linear admissible subcategory in $\mathcal{D}^b(X)$ (S-linear means closed under tensoring with pullbacks of perfect complexes on S), where X is an algebraic variety over S, and construct $\mathcal{A}_T$ as a certain triangulated subcategory in $\mathcal{D}^b(X \times_S T)$. Once again there arises an issue of dependence on the chosen embedding $\mathcal{A} \to \mathcal{D}^b(X)$, and again we show that the result is independent of the choice.

The most important technical notion used in the paper is that of a semiorthogonal decomposition. Actually, we start not with an admissible subcategory $\mathcal{A} \subset \mathcal{D}^b(X)$ but with a semiorthogonal decomposition $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m \rangle$. Then we consider a chain of triangulated categories $\mathcal{D}^{\mathsf{perf}}(X) \subset \mathcal{D}^b(X) \subset \mathcal{D}^-(X) \subset \mathcal{D}_{\mathrm{qc}}(X)$ (here $\mathcal{D}^-(X)$ is the derived category of bounded above complexes with coherent cohomology) and ask whether there exist semiorthogonal decompositions of these categories compatible with the initial decomposition. It turns out that the categories $\mathcal{A}_i^{\mathsf{perf}} = \mathcal{A}_i \cap \mathcal{D}^{\mathsf{perf}}(X)$ always give a semiorthogonal decomposition of $\mathcal{D}^{\mathsf{perf}}(X)$, while the categories $\hat{\mathcal{A}}_i$ (the minimal triangulated subcategories of $\mathcal{D}_{\mathrm{qc}}(X)$ containing $\mathcal{A}_i^{\mathsf{perf}}$ and closed under arbitrary direct sums) and $\mathcal{A}_i^- = \hat{\mathcal{A}}_i \cap \mathcal{D}^-(X)$ always form semiorthogonal decompositions of $\mathcal{D}_{\mathrm{qc}}(X)$ and $\mathcal{D}^-(X)$, respectively. However, for compatibility of the last two decompositions with the initial decomposition of $\mathcal{D}^b(X)$, we need a technical condition to be satisfied, namely the right cohomological amplitude of the projection functors of the initial decomposition should be finite (this condition holds automatically if X is smooth).

Similarly, in a situation of a base change we start with a semiorthogonal decomposition of $\mathcal{D}^b(X)$. However, here we need some additional assumptions from the very beginning. First of all, the decomposition of $\mathcal{D}^b(X)$ should be S-linear and, second, the base change $\phi : T \to S$ should be faithful for the projection $f : X \to S$. The latter condition more or less by definition (see [Kuz06]) is equivalent to the base change isomorphism $f_*\phi^* \cong \phi^* f_*$, where the projections of $X_T = X \times_S T$ to X and T by an abuse of notation are denoted by ϕ and f, respectively.

The semiorthogonal decomposition of $\mathcal{D}^b(X_T)$ is constructed in several steps. First, we consider the semiorthogonal decomposition of $\mathcal{D}^{\mathsf{perf}}(X)$ constructed above. Then we define the subcategory $\mathcal{A}_{iT}^p$ of $\mathcal{D}^{\mathsf{perf}}(X_T)$ to be the closed under direct summands triangulated subcategory generated by objects of the form $\phi^* F \otimes f^* G$ with $F \in \mathcal{A}_i^{\mathsf{perf}}$ and $G \in \mathcal{D}^{\mathsf{perf}}(T)$. It turns out that by acting in this way we always obtain a semiorthogonal decomposition of $\mathcal{D}^{\mathsf{perf}}(X_T)$. Further, we define the category $\hat{\mathcal{A}}_{iT}$ to be the minimal triangulated subcategory of $\mathcal{D}_{\mathrm{qc}}(X_T)$ containing $\mathcal{A}_{iT}^p$ and closed under arbitrary direct sums, and $\mathcal{A}_{iT}^- = \hat{\mathcal{A}}_{iT} \cap \mathcal{D}^-(X_T)$. Thus, we obtain semiorthogonal decompositions of $\mathcal{D}_{\mathrm{qc}}(X_T)$ and $\mathcal{D}^-(X_T)$. Finally, we consider subcategories $\mathcal{A}_{iT} = \mathcal{A}_{iT}^- \cap \mathcal{D}^b(X_T) \subset \mathcal{D}^b(X_T)$. But, to prove that they form a semiorthogonal decomposition, we again need the assumption of finiteness of cohomological amplitude of the projection functors of the initial semiorthogonal decomposition of $\mathcal{D}^b(X)$. We prove that the projection functors of the obtained decomposition of $\mathcal{D}^b(X_T)$ also have finite cohomological amplitude.

We show that the constructed semiorthogonal decompositions of $\mathcal{D}_{\mathrm{qc}}(X)$ and $\mathcal{D}_{\mathrm{qc}}(X_T)$ are compatible with respect to the pushforward and the pullback functors via the projection morphism $\phi : X_T \to X$. It follows that the semiorthogonal decompositions of $\mathcal{D}^b(X)$ and $\mathcal{D}^b(X_T)$ are compatible with respect to ϕ_* whenever ϕ is proper, and with respect to ϕ^* whenever ϕ has finite Tor-dimension.

It should be mentioned that the seemingly too complicated procedure of constructing $\mathcal{A}_{iT}$ is probably inevitable. The straightforward approach of taking for $\mathcal{A}_{iT}$ the subcategory of $\mathcal{D}^b(X_T)$ generated by objects of the form $\phi^*F \otimes f^*G$ with $F \in \mathcal{A}_i$ and $G \in \mathcal{D}^b(T)$ does not give the desired result even when both ϕ and f have finite Tor-dimension. Indeed, assume that $\mathcal{A}_i = \mathcal{D}^b(X)$ and X is smooth. Then $\mathcal{D}^b(X) = \mathcal{D}^{\mathsf{perf}}(X)$ and it is clear that the defined in this way subcategory of $\mathcal{D}^b(X_T)$ is just the category of perfect complexes $\mathcal{D}^{\mathsf{perf}}(X_T)$, not the whole $\mathcal{D}^b(X_T)$ as one would wish. So, one definitely needs to add something to this category to obtain the right answer. It seems that to add all colimits and then to intersect with $\mathcal{D}^b(X_T)$ is the simplest possible solution. And, considering perfect complexes as an intermediate step both removes many technical problems and gives additional information.

As an application of the obtained results, we prove the following. Assume that $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ is a semiorthogonal decomposition the projection functors of which have finite cohomological amplitude. We prove then that these functors are isomorphic to kernel functors Φ_{K_i} given by some explicit kernels $K_i \in \mathcal{D}^b(X \times X)$. In particular, if $\mathcal{A} \subset \mathcal{D}^b(X)$ is an admissible subcategory and the projection functor to $\mathcal{A}$ has finite cohomological amplitude, then it is isomorphic to a kernel functor. In a special case, when $\mathcal{A} \cong \mathcal{D}^b(Y)$ for a smooth projective variety Y, this follows from Orlov's theorem on representability of fully faithful functors [Orl97]. Indeed, in this case the embedding functor $\mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ as well as its adjoint are given by appropriate kernels on $X \times Y$, so the projection functor is given by the convolution of these kernels. Thus, our result can be considered as a generalization of Orlov's theorem.

The paper is organized as follows. In § 2, we recall the main technical notions used in the paper; semiorthogonal decompositions, cohomological amplitude, homotopy colimits etc. We also discuss several notions and facts related to approximation of unbounded quasicoherent complexes by perfect ones. In § 3, we investigate when a semiorthogonal decomposition of a triangulated category $\mathcal{T}'$ induces a semiorthogonal decomposition of its full triangulated subcategory $\mathcal{T} \subset \mathcal{T}'$. In § 4, we construct extensions of a semiorthogonal decomposition of $\mathcal{D}^b(X)$ to $\mathcal{D}^{\mathsf{perf}}(X) \subset \mathcal{D}^-(X) \subset \mathcal{D}_{\mathrm{qc}}(X)$. In § 5, we define the base change for an admissible subcategory and prove the faithful base change theorem. In § 6, we show that extensions $\hat{\mathcal{A}}$, $\mathcal{A}^-$ and the base change $\mathcal{A}_T$ of $\mathcal{A}$ do not depend on the choice of X and of the embedding $\mathcal{A} \to \mathcal{D}^b(X)$ involved in the definitions. In § 7, we prove that the projection functors of a semiorthogonal decomposition can be represented as kernel functors.

2. Preliminaries

2.1 Notation

All algebraic varieties are assumed to be quasiprojective.

For an algebraic variety X, we denote by $\mathcal{D}^b(X)$ the bounded derived category of coherent sheaves on X, by $\mathcal{D}^-(X)$ the bounded above derived category of coherent sheaves on X and by $\mathcal{D}_{\mathrm{qc}}(X)$ the unbounded derived category of quasicoherent sheaves on X. Recall that an object $F \in \mathcal{D}_{\mathrm{qc}}(X)$ is a perfect complex if it is locally quasi-isomorphic to a bounded complex of locally free sheaves of finite rank. Recall that perfect complexes are precisely compact objects in $\mathcal{D}_{\mathrm{qc}}(X)$, i.e. if P is perfect, then

$$\mathsf{Hom}\left(P, \bigoplus_\alpha F_\alpha\right) \cong \bigoplus_\alpha \mathsf{Hom}(P, F_\alpha)$$

for any system $F_\alpha \in \mathcal{D}_{\mathrm{qc}}(X)$. We denote by $\mathcal{D}^{\mathsf{perf}}(X)$ the full subcategory of $\mathcal{D}_{\mathrm{qc}}(X)$ consisting of perfect complexes. Note that $\mathcal{D}^{\mathsf{perf}}(X)$ is a triangulated subcategory in $\mathcal{D}^b(X)$. Given an object $F \in \mathcal{D}_{\mathrm{qc}}(X)$, we denote by $\mathcal{H}^i(F)$ the ith cohomology sheaf of F.

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For $F, G \in \mathcal{D}_{qc}(X)$, we denote by $\mathsf{R}\mathcal{H}om(F, G)$ the local $\mathsf{R}\mathcal{H}om$-complex and by $F \otimes G$ the derived tensor product. Similarly, for a map $f : X \to Y$, we denote by $f_* : \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(Y)$ the derived pushforward functor and by $f^* : \mathcal{D}_{qc}(Y) \to \mathcal{D}_{qc}(X)$ the derived pullback functor. We refer to [KS06] for the definition of these functors. We also denote by $f^! : \mathcal{D}_{qc}(Y) \to \mathcal{D}_{qc}(X)$ the right adjoint functor of f_* (usually it is referred to as the twisted pullback functor). It exists by [Nee96] (see also [KS06]). If the morphism f has finite Tor-dimension then $f^!(F) \cong f^*(F) \otimes \omega_{X/Y}[\dim X - \dim Y]$, again by [Nee96].

Given a class $\mathcal{E}$ of objects in a triangulated category $\mathcal{T}$, we denote by $\langle \mathcal{E} \rangle$ the minimal strictly full triangulated subcategory in $\mathcal{T}$ containing all objects in $\mathcal{E}$ and closed under taking direct summands. We say that $\mathcal{E}$ generates $\mathcal{T}$ if $\mathcal{T} = \langle \mathcal{E} \rangle$.

2.2 Semiorthogonal decompositions

Given a class $\mathcal{E}$ of objects in a triangulated category $\mathcal{T}$, we denote the right and the left orthogonal to $\mathcal{E}$ by

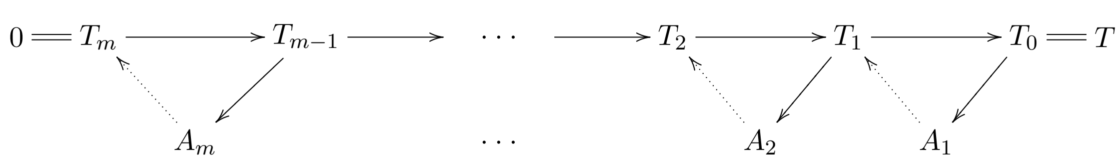
$$\mathcal{E}^\perp = \{T \in \mathcal{T} \mid \mathsf{Hom}(E[k], T) = 0 \text{ for all } E \in \mathcal{E} \text{ and all } k \in \mathbb{Z}\},$$
$$^\perp\mathcal{E} = \{T \in \mathcal{T} \mid \mathsf{Hom}(T, E[k]) = 0 \text{ for all } E \in \mathcal{E} \text{ and all } k \in \mathbb{Z}\}.$$

It is clear that both $\mathcal{E}^\perp$ and $^\perp\mathcal{E}$ are triangulated subcategories in $\mathcal{T}$ closed under taking direct summands. The classes $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{T}$ are called semiorthogonal if $\mathcal{E}_1 \subset \mathcal{E}_2^\perp$ or, equivalently, $\mathcal{E}_2 \subset {}^\perp\mathcal{E}_1$.

Lemma 2.1. *If classes $\mathcal{E}_1$ and $\mathcal{E}_2$ are semiorthogonal, then the subcategories $\langle \mathcal{E}_1 \rangle$ and $\langle \mathcal{E}_2 \rangle$ are semiorthogonal as well.*

Proof. We have $\mathcal{E}_1 \subset \mathcal{E}_2^\perp$; hence $\langle \mathcal{E}_1 \rangle \subset \mathcal{E}_2^\perp$, hence $\mathcal{E}_2 \subset {}^\perp\langle \mathcal{E}_1 \rangle$ and hence $\langle \mathcal{E}_2 \rangle \subset {}^\perp\langle \mathcal{E}_1 \rangle$. □

Definition 2.2 [BK89, BO95, BO02]. A semiorthogonal decomposition of a triangulated category $\mathcal{T}$ is a sequence of full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_m$ in $\mathcal{T}$ such that $\mathcal{A}_i \subset \mathcal{A}_j^\perp$ for $i < j$ and, for every object $T \in \mathcal{T}$, there exists a chain of morphisms $0 = T_m \to T_{m-1} \to \cdots \to T_1 \to T_0 = T$ such that the cone of the morphism $T_k \to T_{k-1}$ is contained in $\mathcal{A}_k$ for each $k = 1, 2, \dots, m$. In other words, there exists a diagram



(1)

where all triangles are distinguished (dashed arrows have degree one) and $A_k \in \mathcal{A}_k$.

Thus, every object $T \in \mathcal{T}$ admits a decreasing 'filtration' with factors in $\mathcal{A}_1, \dots, \mathcal{A}_m$, respectively.

Lemma 2.3. *If $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ is a semiorthogonal decomposition and $T \in \mathcal{T}$, then the diagram (1) for T is unique and functorial (for any morphism $T \to T'$, there exists a unique collection of morphisms $T_i \to T'_i$, $A_i \to A'_i$ combining into a morphism of diagram (1) for T into diagram (1) for T').*

Proof. Note that $T_1 \in \langle \mathcal{A}_2, \dots, \mathcal{A}_m \rangle$ by (1). It follows from the semiorthogonality that we have $\mathsf{Hom}(T_1, A'_1[k]) = 0$ for all $k \in \mathbb{Z}$. Therefore, any map $T_0 = T \to T' = T'_0$ extends in a unique way to a map of the triangle $T_1 \to T_0 \to A_1$ into the triangle $T'_1 \to T'_0 \to A'_1$. In particular, we obtain a map $T_1 \to T'_1$ as well as a map $A_1 \to A'_1$ and proceed by induction. □

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We denote by $\alpha_k : \mathcal{T} \to \mathcal{T}$ the functor $T \mapsto A_k$. We call α_k the kth projection functor of the semiorthogonal decomposition.

Definition 2.4 [Bon89, BK89]. A full triangulated subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$ is called right admissible if, for the inclusion functor $i : \mathcal{A} \to \mathcal{T}$, there is a right adjoint $i^! : \mathcal{T} \to \mathcal{A}$, and left admissible if there is a left adjoint $i^* : \mathcal{T} \to \mathcal{A}$. Subcategory $\mathcal{A}$ is called admissible if it is both right and left admissible.

Lemma 2.5 [Bon89]. *If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition, then $\mathcal{A}$ is left admissible and $\mathcal{B}$ is right admissible. Conversely, if $\mathcal{A} \subset \mathcal{T}$ is left admissible, then $\mathcal{T} = \langle \mathcal{A}, {}^\perp\mathcal{A} \rangle$ is a semiorthogonal decomposition and, if $\mathcal{B} \subset \mathcal{T}$ is right admissible, then $\mathcal{T} = \langle \mathcal{B}^\perp, \mathcal{B} \rangle$ is a semiorthogonal decomposition.*

Definition 2.6. We will say that a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ is a strong semiorthogonal decomposition if, for each k, the category $\mathcal{A}_k$ is admissible in $\langle \mathcal{A}_k, \dots, \mathcal{A}_m \rangle$.

Note that $\mathcal{A}_k$ is left admissible in $\langle \mathcal{A}_k, \dots, \mathcal{A}_m \rangle$, by Lemma 2.5. So, the additional condition in the definition is the right admissibility. Note also that if $\mathcal{A}_k$ is right admissible in $\mathcal{T}$, then it is also admissible in $\langle \mathcal{A}_k, \dots, \mathcal{A}_m \rangle$ (thus, a semiorthogonal decomposition with admissible components is a strong semiorthogonal decomposition) and that in the case when $\mathcal{T} = \mathcal{D}^b(X)$ with X being smooth and projective any semiorthogonal decomposition is strong.

2.3 *S*-linearity

Let $f : X \to S$ be a morphism of algebraic varieties. A triangulated subcategory $\mathcal{A} \subset \mathcal{D}_{\mathrm{qc}}(X)$ is called S-linear (see [Kuz06]) if it is stable with respect to tensoring by pullbacks of perfect complexes on S. In other words, if $A \otimes f^*F \in \mathcal{A}$ for any $A \in \mathcal{A}$, then $F \in \mathcal{D}^{\mathsf{perf}}(S)$.

Lemma 2.7. *A pair of S-linear subcategories $\mathcal{A}, \mathcal{B} \subset \mathcal{D}_{\mathrm{qc}}(X)$ is semiorthogonal if and only if the equality $f_* \mathsf{R}\mathcal{H}om(B, A) = 0$ holds for any $A \in \mathcal{A}$, $B \in \mathcal{B}$.*

Proof. First we note that for any object $0 \neq G \in \mathcal{D}_{\mathrm{qc}}(S)$ there exists a non-zero map $P \to G$ from a perfect complex $P \in \mathcal{D}^{\mathsf{perf}}(S)$. Indeed, represent G by a complex of quasicoherent sheaves and assume that $\mathcal{H}^i(G) \neq 0$. Let $Z^i = \mathsf{Ker}(G^i \to G^{i+1})$, so that we have an epimorphism $Z^i \to \mathcal{H}^i(G)$. It is clear that there exist a locally free sheaf P of finite rank and a map $P \to Z^i$ such that the composition $P \to Z^i \to \mathcal{H}^i(G)$ is non-zero. Then the composition $P \to Z^i \subset G^i$ induces the required morphism $P[-i] \to G$ (it is non-zero, since the induced morphism of the cohomology $\mathcal{H}^i(P[-i]) = P \to \mathcal{H}^i(G)$ is non-zero).

Further, $\mathsf{RHom}(P, f_* \mathsf{R}\mathcal{H}om(B, A)) \cong \mathsf{RHom}(f^*P, \mathsf{R}\mathcal{H}om(B, A)) \cong \mathsf{RHom}(B \otimes f^*P, A)$ for any $P \in \mathcal{D}^{\mathsf{perf}}(S)$. So, if $\mathcal{A}$ and $\mathcal{B}$ are semiorthogonal, then $\mathsf{RHom}(B \otimes f^*P, A) = 0$, since $\mathcal{B}$ is S-linear and the above observation shows that $f_*\mathsf{R}\mathcal{H}om(B, A) = 0$. The inverse is evident. □

Let $f : X \to S$ and $g : Y \to S$ be algebraic morphisms, and assume that $\mathcal{A} \subset \mathcal{D}_{\mathrm{qc}}(X)$, $\mathcal{B} \subset \mathcal{D}_{\mathrm{qc}}(Y)$ are S-linear triangulated subcategories. A functor $\Phi : \mathcal{A} \to \mathcal{B}$ is called S-linear if there is given a functorial isomorphism $\Phi(F \otimes f^*G) \cong \Phi(F) \otimes g^*G$ for all $F \in \mathcal{A}$, $G \in \mathcal{D}^{\mathsf{perf}}(S)$.

Lemma 2.8. *If $\mathcal{T} \subset \mathcal{D}_{\mathrm{qc}}(X)$ is an S-linear triangulated subcategory and $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ is an S-linear semiorthogonal decomposition, then its projection functors $\alpha_i : \mathcal{T} \to \mathcal{T}$ are S-linear.*

Proof. Take any $G \in \mathcal{D}^{\mathsf{perf}}(S)$ and consider the endofunctor of $\mathcal{T}$ given by tensoring with f^*G. It preserves all $\mathcal{A}_i$ and hence, by Lemma 3.1 below, it commutes with the projection functors. This gives the required functorial isomorphism. □

2.4 Faithful base changes

Let $f : X \to S$ and $\phi : T \to S$ be algebraic morphisms. Let $X_T = X \times_T S$ be the fiber product. By an abuse of notation, denote the projections $X_T \to T$ and $X_T \to X$ also by f and ϕ, respectively. It is easy to see that there is a canonical morphism of functors $\phi^* f_* \to f_* \phi^*$. Recall that the cartesian square

$$\begin{array}{ccc} X_T & \xrightarrow{\phi} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\phi} & S \end{array}$$

is called exact (see [Kuz06]) if this morphism of functors is an isomorphism. By [Kuz06], the square is exact if either f or ϕ is flat, and the square is exact if and only if the transposed square is exact.

A map $\phi : T \to S$ considered as a change of base is called faithful for $f : X \to S$ (see [Kuz06]) if the corresponding cartesian square is exact. Thus, any change of base is faithful for a flat f and similarly a flat change of base is faithful for any f.

2.5 Truncations

Given a complex $C^\bullet$, its stupid truncations are defined as

$$(\sigma^{\leqslant m} C)^n = \begin{cases} C^n & \text{if } n \leqslant m, \\ 0 & \text{if } n > m, \end{cases} \quad \text{and} \quad (\sigma^{\geqslant m} C)^n = \begin{cases} C^n & \text{if } n \geqslant m, \\ 0 & \text{if } n < m. \end{cases}$$

It is clear that $\sigma^{\geqslant m} C \to C \to \sigma^{\leqslant m-1} C$ is a distinguished triangle in the derived category. The advantage of the stupid truncations which we will use subsequently in the paper is that when applied to a complex of locally free sheaves (a perfect complex) they produce a perfect complex as well.

Similarly, the canonical truncations (also known as smart truncations) are defined as

$$(\tau^{\leqslant m} C)^n = \begin{cases} C^n & \text{if } n < m, \\ \mathsf{Ker}(C^m \xrightarrow{d} C^{m+1}) & \text{if } n = m, \\ 0 & \text{if } n > m, \end{cases} \quad (\tau^{\geqslant m} C)^n = \begin{cases} C^n & \text{if } n > m, \\ \mathsf{Coker}(C^{m-1} \xrightarrow{d} C^m) & \text{if } n = m, \\ 0 & \text{if } n < m. \end{cases}$$

Again, in the derived category we have a distinguished triangle $\tau^{\leqslant m} C \to C \to \tau^{\geqslant m+1} C$. The advantage of the canonical truncations is that they descend to functors on the derived category. Note also that

$$\mathcal{H}^n(\tau^{\leqslant m} C) \cong \begin{cases} \mathcal{H}^n(C) & \text{if } n \leqslant m, \\ 0 & \text{if } n > m, \end{cases} \quad \text{and} \quad \mathcal{H}^n(\tau^{\geqslant m} C) \cong \begin{cases} \mathcal{H}^n(C) & \text{if } n \geqslant m, \\ 0 & \text{if } n < m. \end{cases}$$

2.6 Cohomological amplitude

Let $\mathcal{D}_{\mathrm{qc}}^{[p,q]}(X)$ denote the full subcategory of $\mathcal{D}_{\mathrm{qc}}(X)$ consisting of all complexes $F \in \mathcal{D}_{\mathrm{qc}}(X)$ with $\mathcal{H}^i(F) = 0$ for $i \notin [p, q]$. Let $\mathcal{T} \subset \mathcal{D}_{\mathrm{qc}}(X)$ be a triangulated subcategory. We say that (a, b) is the cohomological amplitude of a triangulated functor $\Phi : \mathcal{T} \to \mathcal{D}_{\mathrm{qc}}(Y)$ if

$$\Phi(\mathcal{T} \cap \mathcal{D}_{\mathrm{qc}}^{[p,q]}(X)) \subset \mathcal{D}_{\mathrm{qc}}^{[p+a,q+b]}(Y)$$

for all $p, q \in \mathbb{Z}$. In particular, we say that Φ has finite left (respectively right) cohomological amplitude if $a > -\infty$ (respectively $b < \infty$). If both a and b are finite, we say that Φ has finite cohomological amplitude.

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Lemma 2.9 [Kuz08]. *Every exact functor* $\Phi : \mathcal{D}^{\mathsf{perf}}(X) \to \mathcal{D}^b_{\mathrm{qc}}(Y)$ *has finite cohomological amplitude.*

Actually, it was shown in [Kuz08, Proposition 2.5] that one can replace $D^b_{\mathrm{qc}}(Y)$ by any triangulated category with bounded t-structure.

Let X and Y be algebraic varieties. Consider the product $X \times Y$ and denote the projections by $p : X \times Y \to X$ and $q : X \times Y \to Y$. Recall (see [Kuz06, 10.39]) that by definition an object $K \in \mathcal{D}^b(X \times Y)$ has finite Tor-amplitude over X if the functor $F \mapsto K \otimes p^*F$ has finite cohomological amplitude. Similarly, an object $K \in \mathcal{D}^b(X \times Y)$ has finite Ext-amplitude over Y if the functor $F \mapsto \mathsf{R}\mathcal{H}om(K, q^!F)$ has finite cohomological amplitude.

Lemma 2.10. *If an object* $K \in \mathcal{D}^b(X \times Y)$ *has finite* Tor-*amplitude over* X*, then the functor* $\Phi_K(F) = q_*(K \otimes p^*F)$ *has finite cohomological amplitude. Similarly, if* $K \in \mathcal{D}^b(X \times Y)$ *has finite* Ext-*amplitude over* Y*, then the functor* $\Phi^!_K(G) = q_*\mathsf{R}\mathcal{H}om(K, q^!G)$ *has finite cohomological amplitude.*

Proof. It suffices to note that the pushforward functor has finite cohomological amplitude (it is equal to $(0, d)$, where d is the maximum of the dimensions of the fibers). □

2.7 Homotopy colimits

Recall (see [BN93]) the definition of homotopy colimits in triangulated categories. Let $F_1 \to F_2 \to F_3 \to \cdots$ be a sequence of objects of a triangulated category having countable direct sums. Its homotopy colimit, $\mathsf{hocolim}\, F_i$, is defined as a cone of the canonical morphism $\bigoplus F_i \xrightarrow{\mathsf{id}-\mathsf{shift}} \bigoplus F_i$, where shift denotes the map $\bigoplus F_i \to \bigoplus F_i$ defined on F_i as the composition $F_i \to F_{i+1} \subset \bigoplus F_j$. Thus, we have a distinguished triangle

$$\bigoplus F_i \xrightarrow{\mathsf{id}-\mathsf{shift}} \bigoplus F_i \longrightarrow \mathsf{hocolim}\, F_i .$$

In what follows, we only consider homotopy colimits over the set of positive integers. Colimits over other partially ordered sets are not considered at all.

Lemma 2.11. *If a functor* Φ *commutes with countable direct sums, that is, the canonical morphism* $\bigoplus_i \Phi(F_i) \to \Phi(\bigoplus_i F_i)$ *is an isomorphism, then* Φ *commutes with homotopy colimits in the sense that there is a non-canonical isomorphism* $\mathsf{hocolim}\, \Phi(F_i) \cong \Phi(\mathsf{hocolim}\, F_i)$*. In particular, homotopy colimits commute with tensor products, pullbacks and pushforwards.*

Proof. By the assumptions, we have a diagram

$$\begin{array}{ccccc} \bigoplus_i \Phi(F_i) & \xrightarrow{\mathsf{id}-\mathsf{shift}} & \bigoplus_i \Phi(F_i) & \longrightarrow & \mathsf{hocolim}\, \Phi(F_i) \\ \downarrow \cong & & \downarrow \cong & & \\ \Phi(\bigoplus_i F_i) & \xrightarrow{\mathsf{id}-\mathsf{shift}} & \Phi(\bigoplus_i F_i) & \longrightarrow & \Phi(\mathsf{hocolim}\, F_i) \end{array}$$

which is evidently commutative. It follows that it can be extended by an isomorphism of third vertices $\mathsf{hocolim}\, \Phi(F_i) \cong \Phi(\mathsf{hocolim}\, F_i)$. For the second claim, we use the fact that countable direct sums commute with tensor products, pullbacks (evident) and pushforwards [BV03, 3.3.4]. □

Remark 2.12. Note that by [BV03, 3.3.4] tensor products, pullbacks and pushforwards commute with arbitrary direct sums (not only with countable ones). We will use subsequently this fact.

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Now assume that the triangulated category under consideration is the unbounded derived category $\mathcal{D}(\mathcal{A})$, where $\mathcal{A}$ is an abelian category with exact countable colimits.

Lemma 2.13. *If $F_1 \to F_2 \to F_3 \to \cdots$ is a direct system of complexes in $\mathcal{A}$ and F is the complex obtained by taking termwise colimits of the above direct system, then* $\mathsf{hocolim}\, F_i \cong F$.

Proof. Consider the sequence of complexes $\bigoplus F_i \xrightarrow{\mathsf{id}-\mathsf{shift}} \bigoplus F_i \longrightarrow F$. Since $\mathcal{A}$ has exact colimits, the sequence is termwise exact. Therefore, F is isomorphic to the cone of the map $\bigoplus F_i \xrightarrow{\mathsf{id}-\mathsf{shift}} \bigoplus F_i$. ☐

Lemma 2.14 [BN93]. *If $\{F_i\}$ is a direct system in $\mathcal{D}(\mathcal{A})$, then we have* $\mathcal{H}^n(\mathsf{hocolim}\, F_i) \cong \varinjlim \mathcal{H}^n(F_i)$.

Proof. The long exact sequence of cohomology sheaves of the triangle defining $\mathsf{hocolim}\, F_i$ gives

$$\cdots \to \bigoplus_i \mathcal{H}^n(F_i) \to \bigoplus_i \mathcal{H}^n(F_i) \to \mathcal{H}^n(\mathsf{hocolim}\, F_i) \to \bigoplus_i \mathcal{H}^{n+1}(F_i) \to \bigoplus_i \mathcal{H}^{n+1}(F_i) \to \cdots .$$

Since the category $\mathcal{A}$ has exact colimits, the last map in the above sequence is injective. It follows that $\mathcal{H}^n(\mathsf{hocolim}\, F_i) \cong \mathsf{Coker}(\bigoplus_i \mathcal{H}^n(F_i) \to \bigoplus_i \mathcal{H}^n(F_i)) \cong \varinjlim \mathcal{H}^n(F_i)$, the last isomorphism being the definition of the colimit. ☐

Lemma 2.15. *If $\{F_i\}$ is a direct system and there is given a morphism of this direct system to F, then there exists a map $\mathsf{hocolim}\, F_i \to F$ compatible with the maps $F_i \to F$. Moreover, if $\varinjlim \mathcal{H}^t(F_i) = \mathcal{H}^t(F)$ for each $t \in \mathbb{Z}$, then* $\mathsf{hocolim}\, F_i \cong F$.

Proof. We have a canonical map $\oplus F_i \to F$. Its composition with $\bigoplus F_i \xrightarrow{\mathsf{id}-\mathsf{shift}} \bigoplus F_i$ vanishes, since the map is induced by a map of the direct system $\{F_i\}$ to F. Hence, it can be factored through a map $\mathsf{hocolim}\, F_i \to F$. On the tth cohomology, it gives the map $\mathcal{H}^t(\mathsf{hocolim}\, F_i) = \varinjlim \mathcal{H}^t(F_i) \to \mathcal{H}^t(F)$ induced by the map of the direct system $\{\mathcal{H}^t(F_i)\}$ to $\mathcal{H}^t(F)$. If it is an isomorphism for all t, then the map $\mathsf{hocolim}\, F_i \to F$ is a quasi-isomorphism. ☐

2.8 Approximation

We say that a direct system $\{F_i\}$ in $\mathcal{D}(\mathcal{A})$ approximates $F \in \mathcal{D}(\mathcal{A})$ if there is given a morphism from the direct system to F such that for any $n \geqslant 0$ the map $\tau^{\leqslant n}\tau^{\geqslant -n} F_k \to \tau^{\leqslant n}\tau^{\geqslant -n} F$ is an isomorphism for $k \gg 0$. The following is an immediate corollary of Lemma 2.15.

Lemma 2.16. *If a direct system $\{F_i\}$ approximates F in $\mathcal{D}(\mathcal{A})$, then* $\mathsf{hocolim}\, F_i \cong F$.

Recall (see [Kuz08]) that a direct system $\{F_i\}$ in $\mathcal{D}(\mathcal{A})$ is said to be stabilizing in finite degrees if for any $n \in \mathbb{Z}$ the map $\tau^{\geqslant n} F_i \to \tau^{\geqslant n} F_{i+1}$ is an isomorphism for $i \gg 0$.

Let $\mathcal{B} \subset \mathcal{A}$ be an abelian subcategory and let $\mathcal{D}^-_{\mathcal{B}}(\mathcal{A})$ denote the full subcategory in $\mathcal{D}^-(\mathcal{A})$, the bounded above derived category of $\mathcal{A}$, consisting of all objects with cohomology in $\mathcal{B}$.

Lemma 2.17. *If a direct system $\{F_i\}$ in $\mathcal{D}^-_{\mathcal{B}}(\mathcal{A})$ stabilizes in finite degrees, then we have an inclusion* $\mathsf{hocolim}\, F_i \in \mathcal{D}^-_{\mathcal{B}}(\mathcal{A})$.

Proof. This follows immediately from Lemma 2.14. ☐

The following easy lemma shows that every object of $\mathcal{D}^-(X)$ can be approximated by a stabilizing in finite degrees direct system of perfect complexes. This fact will be used subsequently in the paper.

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Lemma 2.18. *For every $F \in \mathcal{D}^-(X)$, there is a stabilizing in finite degrees direct system of perfect complexes $F_k \in \mathcal{D}^{\mathsf{perf}}(X)$ which approximates F. In particular,* $\mathsf{hocolim}\, F_k \cong F$.

Proof. Choose a locally free resolution for F and denote by F_k its stupid truncation at degree $-k$. Then F_k is a perfect complex and the F_k form a stabilizing in finite degrees direct system. Moreover, for any $n \in \mathbb{Z}$, we have $\tau^{\geqslant -n} F_k \cong \tau^{\geqslant -n} F$ for $k \gg 0$; hence, F_k approximates F. By Lemma 2.16, we have $F \cong \mathsf{hocolim}\, F_k$. □

We are also interested in approximation of arbitrary unbounded quasicoherent complexes. Certainly, arbitrary objects of $\mathcal{D}_{\mathrm{qc}}(X)$ cannot be represented as homotopy colimits of perfect complexes. There is however the following implicit approximation result.

Lemma 2.19. *The minimal full triangulated subcategory of $\mathcal{D}_{\mathrm{qc}}(X)$ closed under arbitrary direct sums and containing $\mathcal{D}^{\mathsf{perf}}(X)$ is $\mathcal{D}_{\mathrm{qc}}(X)$.*

Proof. Let $\mathcal{R} \subset \mathcal{D}_{\mathrm{qc}}(X)$ be the minimal full triangulated subcategory closed under arbitrary direct sums and containing $\mathcal{D}^{\mathsf{perf}}(X)$. By the Bousfield localization theorem (see [Nee92, Lemma 1.7]), there is a semiorthogonal decomposition $\mathcal{D}_{\mathrm{qc}}(X) = \langle \mathcal{R}^\perp, \mathcal{R} \rangle$ (the category $\mathcal{R}^\perp$ is the category of $\mathcal{R}$-local objects). But, $\mathcal{R}^\perp \subset (\mathcal{D}^{\mathsf{perf}}(X))^\perp$ and the latter category is zero (e.g. by the argument in the proof of Lemma 2.7); hence, $\mathcal{R} = \mathcal{D}_{\mathrm{qc}}(X)$. □

We conclude this section with the following simple result, which will be used later.

Lemma 2.20. *Let $\phi : Y \to X$ be a quasiprojective morphism and assume that L is a line bundle on Y ample over X. If $F \in \mathcal{D}^{[p,q]}(Y)$, then, for any $k \gg 0$, there is a direct system G_m in $\mathcal{D}^{[p,q]}(X)$ such that $\phi_*(F \otimes L^k) \cong$* $\mathsf{hocolim}\, G_m$.

Proof. Taking the smart truncations of F at p and q, we can assume that F is a complex such that $F^t = 0$ unless $t \in [p, q]$. Since L is ample over X for $k \gg 0$, the higher direct images of $F^t \otimes L^k$ vanish; hence, $\phi_*(F \otimes L^k)$ is isomorphic to the complex

$$\cdots \to 0 \xrightarrow{d} R_0\phi_*(F^p \otimes L^k) \xrightarrow{d} \cdots \xrightarrow{d} R_0\phi_*(F^q \otimes L^k) \xrightarrow{d} 0 \to \cdots .$$

Since ϕ is quasiprojective, each sheaf $R_0\phi_*(F^t \otimes L^k)$ is a quasicoherent sheaf which can be represented as a countable union of coherent subsheaves. Choose such a representation $R_0\phi_*(F^t \otimes L^k) = \cup C_i^t$ and take

$$G_m^t = \bigcup_{i \leqslant m} C_i^t + d\left(\bigcup_{i \leqslant m} C_i^{t-1}\right).$$

Then it is clear that the G_m form a direct system of complexes the termwise colimit of which is the above complex. Hence, $\phi_*(F \otimes L^k) \cong \mathsf{hocolim}\, G_m$, by Lemma 2.13. □

3. Inducing a semiorthogonal decomposition

Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories and assume that we are given semiorthogonal decompositions $\mathcal{T} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$ and $\mathcal{T}' = \langle \mathcal{A}'_1, \ldots, \mathcal{A}'_m \rangle$. A triangulated functor $\Phi : \mathcal{T} \to \mathcal{T}'$ is compatible with the semiorthogonal decompositions if $\Phi(\mathcal{A}_i) \subset \mathcal{A}'_i$ for all $1 \leqslant i \leqslant m$.

Let $\alpha_i : \mathcal{T} \to \mathcal{T}$ and $\alpha'_i : \mathcal{T}' \to \mathcal{T}'$ be the projection functors of the semiorthogonal decompositions.

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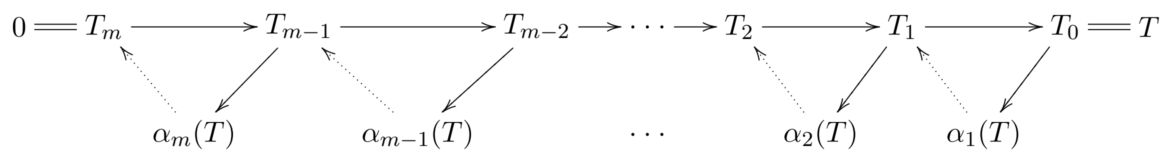
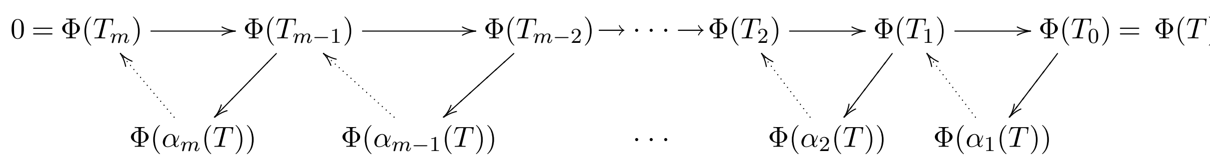
Lemma 3.1. *If the functor Φ is compatible with the semiorthogonal decompositions, then it commutes with the projection functors, that is, we have an isomorphism of functors $\Phi \circ \alpha_i \cong \alpha'_i \circ \Phi$.*

Proof. Take any $T \in \mathcal{T}$ and let



be the filtration of T with factors in $\mathcal{A}_i$. Applying the functor Φ, we obtain a diagram.



Since $\Phi(\alpha_i(T)) \in \Phi(\mathcal{A}_i) \subset \mathcal{A}'_i$, we see that this diagram gives the filtration of $\Phi(T)$ with factors in $\mathcal{A}'_i$; hence, we get isomorphisms $\Phi(\alpha_i(T)) \cong \alpha'_i(\Phi(T))$. Since such filtration is functorial, by Lemma 2.3, the obtained isomorphisms are functorial as well. □

Lemma 3.2. *Assume that $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ and $\mathcal{T} = \langle \mathcal{A}'_1, \dots, \mathcal{A}'_m \rangle$ are semiorthogonal decompositions such that $\mathcal{A}'_i \subset \mathcal{A}_i$ for all $1 \leqslant i \leqslant m$. Then $\mathcal{A}'_i = \mathcal{A}_i$ for all i.*

Proof. The identity functor $\mathcal{T} \to \mathcal{T}$ is compatible with these semiorthogonal decompositions; hence, their projection functors are isomorphic, by Lemma 3.1. In particular, for any i and any $A \in \mathcal{A}_i$, we have $A \cong \alpha_i(A) \cong \alpha'_i(A) \in \mathcal{A}'_i$, where α_i and α'_i are the projection functors; hence, $\mathcal{A}_i \subset \mathcal{A}'_i$. □

Lemma 3.3. *If $\Phi : \mathcal{T} \to \mathcal{T}'$ is a fully faithful functor and $\mathcal{T}' = \langle \mathcal{A}'_1, \dots, \mathcal{A}'_m \rangle$ is a semiorthogonal decomposition, then there exists at most one semiorthogonal decomposition of $\mathcal{T}$ compatible with Φ. This decomposition is given by $\mathcal{A}_i = \Phi^{-1}(\mathcal{A}'_i)$.*

Proof. Let $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ be a semiorthogonal decomposition compatible with Φ. Then we have $\mathcal{A}_i \subset \Phi^{-1}(\mathcal{A}'_i)$. On the other hand, let $A \in \Phi^{-1}(\mathcal{A}'_i)$. Then $\alpha'_j(\Phi(A)) = 0$ for all $j \neq i$. Hence, by Lemma 3.1, we have $\Phi(\alpha_j(A)) = 0$ for all $j \neq i$. But, since Φ is fully faithful, it follows that $\alpha_j(A) = 0$ for all $j \neq i$, so $A \in \mathcal{A}_i$. Thus, we are forced to have $\mathcal{A}_i = \Phi^{-1}(\mathcal{A}'_i)$. □

In general, the collection of subcategories $\mathcal{A}_i = \Phi^{-1}(\mathcal{A}'_i)$ does not give a semiorthogonal decomposition. Actually, it is easy to see that this collection is semiorthogonal (by faithfulness of Φ); however, it can be not full. The simplest example is the functor $\Phi : \mathcal{D}^b(\mathsf{k}) \to \mathcal{D}^b(\mathbb{P}^1)$ which takes k to $\mathcal{O}_{\mathbb{P}^1}$. If one considers the semiorthogonal decomposition $\mathcal{D}^b(\mathbb{P}^1) = \langle \mathcal{A}'_1, \mathcal{A}'_2 \rangle$ with $\mathcal{A}'_i = \langle \mathcal{O}_{\mathbb{P}^1}(i) \rangle$, then $\Phi^{-1}(\mathcal{A}'_i) = 0$ for $i = 1, 2$.

Nevertheless, if the subcategories $\mathcal{A}_i = \Phi^{-1}(\mathcal{A}'_i)$ form a semiorthogonal decomposition of $\mathcal{T}$, we will say that this decomposition is induced on $\mathcal{T}$ by the semiorthogonal decomposition of $\mathcal{T}'$ via Φ.

Lemma 3.4. *Let $\Phi : \mathcal{T} \to \mathcal{T}'$ be a fully faithful functor and $\mathcal{T}' = \langle \mathcal{A}'_1, \dots, \mathcal{A}'_m \rangle$ a semiorthogonal decomposition. It induces a semiorthogonal decomposition on $\mathcal{T}$ if and only if the image of Φ is stable under the projection functors of the semiorthogonal decomposition of $\mathcal{T}'$.*

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Proof. The 'only if' part follows immediately from Lemma 3.1. For the 'if' part, we only have to prove that every object T of $\mathcal{T}$ can be decomposed with respect to the collection of subcategories $\mathcal{A}_i = \Phi^{-1}(\mathcal{A}'_i)$. So, let $T' = \Phi(T)$ and let $0 = T'_m \to T'_{m-1} \to \cdots \to T'_1 \to T'_0 = T'$ be its filtration with factors in $\mathcal{A}'_i$. Note that the factors are given by $\alpha'_i(T') \cong \alpha'_i(\Phi(T))$. Since the image of Φ is stable under α'_i, it follows that $\alpha'_i(T') \cong \Phi(A_i)$ for some objects $A_i \in \mathcal{A}_i$. Let us check that these are the components of T. To do this, we have to construct a filtration $0 = T_m \to T_{m-1} \to \cdots \to T_1 \to T_0 = T$ such that its factors are isomorphic to A_i. We do it inductively. First of all, we put $T_0 = T$. Now assume that T_i is constructed in such a way that $\Phi(T_i) \cong T'_i$. Then we compose this isomorphism with the map $T'_i \to \alpha'_i(T') \cong \Phi(A_i)$. Since Φ is fully faithful, the resulting map comes from a map $T_i \to A_i$ in $\mathcal{T}$. We take T_{i+1} to be the cone of this morphism shifted by -1. Applying to the triangle $T_{i+1} \to T_i \to A_i$ the functor Φ, we conclude that $\Phi(T_{i+1}) \cong T'_{i+1}$. Applying this procedure m times, we construct T_m. Note that $\Phi(T_m) \cong T'_m = 0$. Since Φ is fully faithful, it follows that $T_m = 0$, so the desired filtration of T is constructed. □

Lemma 3.5. *Let $\Phi : \mathcal{T} \to \mathcal{T}'$ be a fully faithful embedding, and assume that $\mathcal{T}' = \langle \mathcal{A}'_1, \dots, \mathcal{A}'_m \rangle$ and $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ are semiorthogonal decompositions compatible with Φ. Let $\Psi' : \mathcal{T}' \to \mathcal{T}'$ be an endofunctor, such that $\mathcal{T}$ and all $\mathcal{A}'_i$ are stable under Ψ'. Then every $\mathcal{A}_i$ is also stable under Ψ'.*

Proof. Since $\mathcal{T}$ is stable under Ψ' and Φ is fully faithful, the restriction of Ψ' to $\mathcal{T}$ defines an endofunctor $\Psi : \mathcal{T} \to \mathcal{T}$, such that $\Phi \circ \Psi = \Psi' \circ \Phi$. Since $\mathcal{A}_i = \Phi^{-1}(\mathcal{A}'_i)$, we have to check that $\Phi(\Psi(\mathcal{A}_i)) \subset \mathcal{A}'_i$. But, $\Phi(\Psi(\mathcal{A}_i)) = \Psi'(\Phi(\mathcal{A}_i)) \subset \Psi'(\mathcal{A}'_i) \subset \mathcal{A}'_i$, since $\mathcal{A}'_i$ is Ψ'-stable. □

4. Extensions of a semiorthogonal decomposition

Let X be an algebraic variety and assume that we are given a semiorthogonal decomposition of $\mathcal{D}^b(X)$. In this section, we construct a compatible system of semiorthogonal decompositions of the categories $\mathcal{D}^{\mathsf{perf}}(X) \subset \mathcal{D}^-(X) \subset \mathcal{D}_{\mathrm{qc}}(X)$.

4.1 Perfect complexes

First of all, we note that any strong semiorthogonal decomposition (see Definition 2.6) of $\mathcal{D}^b(X)$ induces a semiorthogonal decomposition of the category of perfect complexes.

Proposition 4.1. *Let $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ be a strong semiorthogonal decomposition. Then there is a unique semiorthogonal decomposition of the category $\mathcal{D}^{\mathsf{perf}}(X)$ compatible with the natural embedding $\mathcal{D}^{\mathsf{perf}}(X) \to \mathcal{D}^b(X)$.*

Proof. The existence of a semiorthogonal decomposition of $\mathcal{D}^{\mathsf{perf}}(X)$ compatible with that of $\mathcal{D}^b(X)$ follows from [Orl06, 1.10 and 1.11]. Moreover, it follows from Lemma 3.3 that the components of this decomposition are given by

$$\mathcal{A}_i^{\mathsf{perf}} = \mathcal{A}_i \cap \mathcal{D}^{\mathsf{perf}}(X) \tag{2}$$

and that the decomposition is unique. □

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4.2 Unbounded quasicoherent complexes

Now we are going to show that any (not necessarily strong) semiorthogonal decomposition of $\mathcal{D}^{\mathsf{perf}}(X)$ induces a semiorthogonal decomposition of the unbounded derived category of quasicoherent sheaves $\mathcal{D}_{\mathrm{qc}}(X)$.

Proposition 4.2. *Let $\mathcal{D}^{\mathsf{perf}}(X) = \langle \mathcal{A}_1^{\mathsf{perf}}, \ldots, \mathcal{A}_m^{\mathsf{perf}} \rangle$ be a semiorthogonal decomposition. Then there is a unique semiorthogonal decomposition $\mathcal{D}_{\mathrm{qc}}(X) = \langle \hat{\mathcal{A}}_1, \ldots, \hat{\mathcal{A}}_m \rangle$ compatible with the natural embedding $\mathcal{D}^{\mathsf{perf}}(X) \to \mathcal{D}_{\mathrm{qc}}(X)$ and with closed under arbitrary direct sums components. The projection functors $\hat{\alpha}_i$ of this decomposition commute with direct sums and homotopy colimits.*

Moreover, if the initial decomposition of the category $\mathcal{D}^{\mathsf{perf}}(X)$ is induced by a semiorthogonal decomposition $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$ of $\mathcal{D}^b(X)$, the projection functors of which have finite right cohomological amplitude, then the obtained decomposition of $\mathcal{D}_{\mathrm{qc}}(X)$ is compatible with the natural embedding $\mathcal{D}^b(X) \to \mathcal{D}_{\mathrm{qc}}(X)$ as well.

Proof. Define the subcategory $\hat{\mathcal{A}}_i \subset \mathcal{D}_{\mathrm{qc}}(X)$ to be the subcategory of $\mathcal{D}_{\mathrm{qc}}(X)$ obtained by iterated addition of cones to the closure of $\mathcal{A}_i^{\mathsf{perf}}$ in $\mathcal{D}_{\mathrm{qc}}(X)$ under all direct sums. Let us check that the categories $\hat{\mathcal{A}}_i$ form a semiorthogonal decomposition of $\mathcal{D}_{\mathrm{qc}}(X)$. First of all, if $j > i$, $A_j^l \in \mathcal{A}_j^{\mathsf{perf}}$, $A_i^k \in \mathcal{A}_i^{\mathsf{perf}}$, then

$$\mathsf{Hom}\left(\bigoplus_l A_j^l, \bigoplus_k A_i^k\right) \cong \prod_l \mathsf{Hom}\left(A_j^l, \bigoplus_k A_i^k\right) \cong \prod_l \bigoplus_k \mathsf{Hom}(A_j^l, A_i^k) = 0$$

(in the second isomorphism, we used the fact that the A_j^l are perfect complexes, and hence compact objects of $\mathcal{D}_{\mathrm{qc}}(X)$). Addition of cones does not spoil semiorthogonality (see Lemma 2.1); hence, the collection of subcategories $\hat{\mathcal{A}}_1, \ldots, \hat{\mathcal{A}}_m$ is semiorthogonal. Note also that a direct sum of cones is a cone of direct sums by [KS06, 10.1.19], so $\hat{\mathcal{A}}_i$ is a closed under all direct sums triangulated subcategory of $\mathcal{D}_{\mathrm{qc}}(X)$.

Now consider the triangulated subcategory $\langle \hat{\mathcal{A}}_1, \ldots, \hat{\mathcal{A}}_m \rangle$ generated in $\mathcal{D}_{\mathrm{qc}}(X)$ by the subcategories $\hat{\mathcal{A}}_1, \ldots, \hat{\mathcal{A}}_m$. It is clear that it is a triangulated subcategory of $\mathcal{D}_{\mathrm{qc}}(X)$ closed under all direct sums. Moreover, it contains $\langle \mathcal{A}_1^{\mathsf{perf}}, \ldots, \mathcal{A}_m^{\mathsf{perf}} \rangle = \mathcal{D}^{\mathsf{perf}}(X)$. Hence, it coincides with $\mathcal{D}_{\mathrm{qc}}(X)$, by Lemma 2.19. This means that $\langle \hat{\mathcal{A}}_1, \ldots, \hat{\mathcal{A}}_m \rangle = \mathcal{D}_{\mathrm{qc}}(X)$. The uniqueness of such a semiorthogonal decomposition is evident, by Lemma 3.2.

The compatibility with the embedding $\mathcal{D}^{\mathsf{perf}}(X) \to \mathcal{D}_{\mathrm{qc}}(X)$ and closedness under arbitrary direct sums are evident. Commutativity of $\hat{\alpha}_i$ with arbitrary direct sums follows immediately and for homotopy colimits we apply Lemma 2.11.

Further, to check that the constructed semiorthogonal decomposition of $\mathcal{D}_{\mathrm{qc}}(X)$ is compatible with the semiorthogonal decomposition of $\mathcal{D}^b(X)$, we have to check that for any $A \in \mathcal{A}_i \subset \mathcal{D}^b(X)$ we have $\hat{\alpha}_i(A) \cong A$. Let α_i be the projection functors of the semiorthogonal decomposition of $\mathcal{D}^b(X)$. Choose a locally free resolution $P^\bullet \to A$, and consider the stupid truncation of the complex $P^\bullet$ at degree $-n$, so that we have a distinguished triangle

$$\sigma^{\geqslant -n} P^\bullet \to A \to \sigma^{\leqslant -n-1} P^\bullet.$$

Note that the direct system $\sigma^{\geqslant -n} P^\bullet$ approximates A in the sense of paragraph 2.8; hence, by Lemma 2.16, we have an isomorphism $\mathsf{hocolim}\,(\sigma^{\geqslant -n} P^\bullet) \cong A$. Therefore,

$$\hat{\alpha}_i(A) \cong \hat{\alpha}_i(\mathsf{hocolim}(\sigma^{\geqslant -n} P^\bullet)) \cong \mathsf{hocolim}\; \hat{\alpha}_i(\sigma^{\geqslant -n} P^\bullet) \cong \mathsf{hocolim}\; \alpha_i(\sigma^{\geqslant -n} P^\bullet)$$

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(the last isomorphism is due to the fact that $\sigma^{\geqslant -n}P^\bullet$ is a perfect complex, and hence contained in $\mathcal{D}^b(X)$). So, it suffices to check that $\mathsf{hocolim}\ \alpha_i(\sigma^{\geqslant -n}P^\bullet) \cong A$. Indeed, applying α_i to the above triangle, we obtain

$$\alpha_i(\sigma^{\geqslant -n}P^\bullet) \to A \to \alpha_i(\sigma^{\leqslant -n-1}P^\bullet),$$

since $\alpha_i(A) = A$. Let (a_i, b_i) be the cohomological amplitude of the functor α_i. Since the truncation $\sigma^{\leqslant -n-1}P^\bullet \in \mathcal{D}^{\leqslant -n-1}(X)$ is bounded, we have $\alpha_i(\sigma^{\leqslant -n-1}P^\bullet) \in \mathcal{D}^{\leqslant -n-1+b_i}(X)$; hence, $\alpha_i(\sigma^{\geqslant -n}P^\bullet)$ approximates A and so $\mathsf{hocolim}\ \alpha_i(\sigma^{\geqslant -n}P^\bullet) \cong A$. □

4.3 Bounded above coherent complexes

The next step is the following.

Proposition 4.3. *Let $\mathcal{D}^{\mathsf{perf}}(X) = \langle \mathcal{A}_1^{\mathsf{perf}}, \dots, \mathcal{A}_m^{\mathsf{perf}} \rangle$ be a semiorthogonal decomposition. Then there is a unique semiorthogonal decomposition of $\mathcal{D}^-(X)$ compatible with this decomposition of $\mathcal{D}^{\mathsf{perf}}(X)$ and with the decomposition of $\mathcal{D}_{\mathrm{qc}}(X)$ constructed in Proposition 4.2 with respect to the natural embeddings $\mathcal{D}^{\mathsf{perf}}(X) \to \mathcal{D}^-(X) \to \mathcal{D}_{\mathrm{qc}}(X)$. Its components are closed under homotopy colimits of stabilizing in finite degrees direct systems.*

Proof. We have to check that $\mathcal{D}^-(X)$ is stable under the projection functors $\hat{\alpha}_i$. Then, by Lemma 3.4, it would follow that the subcategories

$$\mathcal{A}_i^- = \hat{\mathcal{A}}_i \cap \mathcal{D}^-(X) \tag{3}$$

give a semiorthogonal decomposition, which is evidently compatible with those of $\mathcal{D}^{\mathsf{perf}}(X)$ and $\mathcal{D}_{\mathrm{qc}}(X)$. So, we take any $F \in \mathcal{D}^-(X)$. By Lemma 2.18, there exists a stabilizing in finite degrees direct system of perfect complexes F_k such that $F \cong \mathsf{hocolim}\ F_k$. It follows that

$$\hat{\alpha}_i(F) \cong \hat{\alpha}_i(\mathsf{hocolim}\ F_k) \cong \mathsf{hocolim}\ \alpha_i(F_k)$$

(the second isomorphism follows from Proposition 4.2). But, by Lemma 2.9, the projection functors $\alpha_i : \mathcal{D}^{\mathsf{perf}}(X) \to \mathcal{D}^{\mathsf{perf}}(X)$ have finite cohomological amplitude; hence, the direct system $\alpha_i(F_k)$ also stabilizes in finite degrees and so it follows from Lemma 2.17 that $\mathsf{hocolim}\ \alpha_i(F_k) \in \mathcal{D}^-(X)$.

The last claim is clear, since both $\hat{\mathcal{A}}_i$ and $\mathcal{D}^-(X)$ are closed under homotopy colimits of stabilizing in finite degrees direct systems. □

4.4 *S*-linearity

Assume that X is a scheme over S, that is, we are given a map $f : X \to S$. Recall that any strong semiorthogonal decomposition of $\mathcal{D}^b(X)$ by Proposition 4.1 induces a compatible semiorthogonal decomposition of $\mathcal{D}^{\mathsf{perf}}(X)$, which in its turn by Propositions 4.2 and 4.3 induces compatible semiorthogonal decompositions of $\mathcal{D}_{\mathrm{qc}}(X)$ and $\mathcal{D}^-(X)$.

Lemma 4.4. *If the initial semiorthogonal decomposition of the category $\mathcal{D}^b(X)$ is S-linear, then the induced semiorthogonal decomposition of $\mathcal{D}^{\mathsf{perf}}(X)$ is S-linear. Similarly, if the semiorthogonal decomposition of the category $\mathcal{D}^{\mathsf{perf}}(X)$ is S-linear, then the induced semiorthogonal decompositions of $\mathcal{D}_{\mathrm{qc}}(X)$ and $\mathcal{D}^-(X)$ are S-linear as well.*

Proof. Take any $G \in \mathcal{D}^{\mathsf{perf}}(S)$. Then $\Psi_G(H) := H \otimes f^*G$ is an endofunctor of $\mathcal{D}_{\mathrm{qc}}(X)$ which preserves $\mathcal{D}^-(X)$, $\mathcal{D}^b(X)$ and $\mathcal{D}^{\mathsf{perf}}(X)$ as well as the initial semiorthogonal decomposition. It follows from Lemma 3.5 that the semiorthogonal decomposition (2) of $\mathcal{D}^{\mathsf{perf}}(X)$ is stable under Ψ_G. Now let us check that each component $\hat{\mathcal{A}}_i$ of the semiorthogonal decomposition of $\mathcal{D}_{\mathrm{qc}}(X)$ is stable under Ψ_G. Indeed, by definition, $\hat{\mathcal{A}}_i$ is the smallest triangulated subcategory of $\mathcal{D}_{\mathrm{qc}}(X)$ containing $\mathcal{A}_i^{\mathsf{perf}}$ and closed under arbitrary direct sums. But, the functor Ψ_G commutes with direct sums (see [BV03, 3.3.4]) and is exact, which implies the claim. Again applying Lemma 3.5, we conclude that the semiorthogonal decomposition (3) of $\mathcal{D}^-(X)$ is also stable under Ψ_G. Since this is true for all $G \in \mathcal{D}^{\mathsf{perf}}(S)$, we see that all these decompositions are S-linear. □

Actually, for the components of semiorthogonal decompositions of $\mathcal{D}_{\mathrm{qc}}(X)$ and $\mathcal{D}^-(X)$, we have a stronger result.

Lemma 4.5. *If $\mathcal{D}^-(X) = \langle \mathcal{A}_i^-, \dots, \mathcal{A}_m^- \rangle$ is an S-linear semiorthogonal decomposition with components closed under homotopy colimits of stabilizing in finite degrees direct systems, then $\mathcal{A}_i^- \otimes f^*\mathcal{D}^-(S) \subset \mathcal{A}_i^-$. Similarly, if $\mathcal{D}_{\mathrm{qc}}(X) = \langle \hat{\mathcal{A}}_i, \dots, \hat{\mathcal{A}}_m \rangle$ is an S-linear semiorthogonal decomposition with components closed under arbitrary direct sums, then $\hat{\mathcal{A}}_i \otimes f^*\mathcal{D}_{\mathrm{qc}}(S) \subset \hat{\mathcal{A}}_i$.*

Proof. Take any G in $\mathcal{D}^-(S)$. Applying Lemma 2.18, choose a stabilizing in finite degrees direct system of perfect complexes G_k approximating G so that $G \cong \mathsf{hocolim}\, G_k$. Then, for any $F \in \mathcal{A}_i^-$, we have $F \otimes f^*G \cong F \otimes f^*(\mathsf{hocolim}\, G_k) \cong \mathsf{hocolim}(F \otimes f^*G_k)$. Since the functors $\otimes$ and f^* are right exact, it follows that the direct system $F \otimes f^*G_k$ stabilizes in finite degrees. Hence, its homotopy colimit belongs to $\mathcal{A}_i^-$, since $\mathcal{A}_i^-$ is S-linear and closed under homotopy colimits of stabilizing in finite degrees direct systems.

For the second claim, recall that, by Lemma 2.19, the category $\mathcal{D}_{\mathrm{qc}}(S)$ can be obtained by iterated addition of cones to the closure of $\mathcal{D}^{\mathsf{perf}}(S)$ under arbitrary direct sums. Further, we know by S-linearity of $\hat{\mathcal{A}}_i$ that $\hat{\mathcal{A}}_i \otimes f^*G \subset \hat{\mathcal{A}}_i$ for any perfect G. Since f^* and $\otimes$ commute with direct sums, it follows that the same is true for G being an arbitrary direct sum of perfect complexes. Finally, since f^* and $\otimes$ are exact and $\hat{\mathcal{A}}_i$ is triangulated, the same embedding holds for arbitrary G. □

5. Change of a base

Let $f : X \to S$ be an algebraic map. Consider a base change $\phi : T \to S$ and denote by $X_T = X \times_S T$ the fiber product. Denote the projections $X_T \to T$ and $X_T \to X$ by f and ϕ, respectively, so that we have a cartesian diagram.

$$\begin{array}{ccc} X_T & \xrightarrow{\phi} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\phi} & S \end{array} \tag{4}$$

Throughout this section, we assume that the base change ϕ is faithful for $f : X \to S$ (see paragraph 2.4 for the definition).

5.1 Base change for perfect complexes

Let $\mathcal{D}^{\mathsf{perf}}(X) = \langle \mathcal{A}_1^{\mathsf{perf}}, \dots, \mathcal{A}_m^{\mathsf{perf}} \rangle$ be an S-linear semiorthogonal decomposition. Let $\mathcal{A}_{iT}^p$ denote the minimal triangulated subcategory of $\mathcal{D}^{\mathsf{perf}}(X_T)$ closed under taking direct summands and containing all objects of the form $\phi^*F \otimes f^*G$ with $F \in \mathcal{A}_i^{\mathsf{perf}}$, $G \in \mathcal{D}^{\mathsf{perf}}(T)$:

$$\mathcal{A}_{iT}^p = \langle \phi^*\mathcal{A}_i^{\mathsf{perf}} \otimes f^*\mathcal{D}^{\mathsf{perf}}(T) \rangle. \tag{5}$$

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Note that the subcategory $\mathcal{A}^p_{iT} \subset \mathcal{D}^{\mathsf{perf}}(X_T)$ is T-linear, since the class $\phi^*\mathcal{A}^{\mathsf{perf}}_i \otimes f^*\mathcal{D}^{\mathsf{perf}}(T)$ generating it is T-linear, and the process of adding cones and direct summands preserves T-linearity.

Proposition 5.1. *We have $\mathcal{D}^{\mathsf{perf}}(X_T) = \langle \mathcal{A}^p_{1T}, \dots, \mathcal{A}^p_{mT}\rangle$, a T-linear semiorthogonal decomposition compatible with the functor $\phi^* : \mathcal{D}^{\mathsf{perf}}(X) \to \mathcal{D}^{\mathsf{perf}}(X_T)$.*

Proof. Because of Lemmas 2.7 and 2.1, to verify semiorthogonality, it suffices to check that $f_*\mathsf{R}\mathcal{H}om(\phi^*F_i \otimes f^*G, \phi^*F_j \otimes f^*G') = 0$ for any $F_i \in \mathcal{A}^{\mathsf{perf}}_i$, $F_j \in \mathcal{A}^{\mathsf{perf}}_j$ and any $G, G' \in \mathcal{D}^{\mathsf{perf}}(T)$ if $i > j$. But,

$$\begin{aligned} f_*\mathsf{R}\mathcal{H}om(\phi^*F_i \otimes f^*G, \phi^*F_j \otimes f^*G') &\cong f_*\phi^*\,\mathsf{R}\mathcal{H}om(F_i, F_j) \otimes G^* \otimes G' \\ &\cong \phi^* f_*\mathsf{R}\mathcal{H}om(F_i, F_j) \otimes G^* \otimes G' = 0 \end{aligned}$$

(for the first isomorphism, we use perfectness of F_i, F_j, G and G', for the second, we use faithfulness of the base change ϕ and, for the third, we use S-linearity of the initial semiorthogonal decomposition of $D^{\mathsf{perf}}(X)$ and Lemma 2.7 for it).

It remains to check that the subcategories $\mathcal{A}^p_{iT}$ generate $\mathcal{D}^{\mathsf{perf}}(X_T)$. Indeed, take any object $H \in \mathcal{D}^{\mathsf{perf}}(X_T)$. Then, by Lemma 5.2 below, it can be obtained by consecutively taking cones and direct summands starting from the collection of objects $\phi^*F^t \otimes f^*G^t$, where $F^t \in \mathcal{D}^{\mathsf{perf}}(X)$, $G^t \in \mathcal{D}^{\mathsf{perf}}(T)$ and $t = 1, \dots, N$. On the other hand, every object F^t can be decomposed with respect to the semiorthogonal decomposition $\mathcal{D}^{\mathsf{perf}}(X) = \langle \mathcal{A}^{\mathsf{perf}}_1, \dots, \mathcal{A}^{\mathsf{perf}}_m\rangle$; in other words, it can be obtained by consecutively taking cones from a collection of objects $A^t_i \in \mathcal{A}^{\mathsf{perf}}_i$, $i = 1, \dots, m$. It follows that H can be obtained by consecutively taking cones and direct summands starting from the collection of objects $\phi^*A^t_i \otimes f^*G^t$, and it remains to note that $\phi^*A^t_i \otimes f^*G^t \in \mathcal{A}^p_{iT}$ by definition.

The second claim follows immediately from (5). □

Lemma 5.2. *The category $\mathcal{D}^{\mathsf{perf}}(X_T)$ coincides with the minimal triangulated subcategory of the category $\mathcal{D}_{\mathrm{qc}}(X)$ closed under taking direct summands and containing the class of objects $\phi^*\mathcal{D}^{\mathsf{perf}}(X) \otimes f^*\mathcal{D}^{\mathsf{perf}}(T) := \{\phi^*F \otimes f^*G \mid F \in \mathcal{D}^{\mathsf{perf}}(X), G \in \mathcal{D}^{\mathsf{perf}}(T)\}$.*

Proof. Take any object $H \in \mathcal{D}^{\mathsf{perf}}(X)$ and construct a locally free resolution $P^\bullet \to H$ in which all sheaves P^k have the form $P^k \cong \phi^*F \otimes f^*G$, where F and G are locally free sheaves on X and T, respectively (this can be done, since ϕ is quasiprojective). Then its stupid truncation $\sigma^{\geqslant n}(P^\bullet) \in \langle \phi^*\mathcal{D}^{\mathsf{perf}}(X) \otimes f^*\mathcal{D}^{\mathsf{perf}}(T)\rangle$ for all n, and for $n \ll 0$ the object H is a direct summand of $\sigma^{\geqslant n}(P^\bullet)$. Indeed, since H is a perfect complex, it is quasi-isomorphic to a bounded complex of locally free sheaves of finite rank. Assume that this complex is bounded from the left by degree $l \in \mathbb{Z}$. Take $n \leqslant l - \dim X$ and consider the triangle

$$\sigma^{\geqslant n}P^\bullet \to P^\bullet \to \sigma^{\leqslant n-1}P^\bullet.$$

Note that, since $P^\bullet$ is quasi-isomorphic to H and H is quasi-isomorphic to a complex of locally free sheaves supported in degrees $\geqslant l$, it follows that the complex computing $\mathcal{E}xt^i(P^\bullet, \sigma^{\leqslant n-1}P^\bullet)$ is supported in degrees $\leqslant n - 1 - l$. The hypercohomology spectral sequence then shows that $\mathsf{Ext}^i(P^\bullet, \sigma^{\leqslant n-1}P^\bullet) = 0$ for $i > n - 1 - l + \dim X$. But, $n - 1 - l + \dim X \leqslant -1$ for $n \leqslant l - \dim X$; hence, $\mathsf{Hom}(P^\bullet, \sigma^{\leqslant n-1}P^\bullet) = 0$. In particular, the above triangle splits; hence, $P^\bullet$ is a direct summand of $\sigma^{\geqslant n}P^\bullet$ and we are done, since $P^\bullet$ is quasi-isomorphic to H. □

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5.2 Base change for unbounded quasicoherent complexes

We start with an S-linear semiorthogonal decomposition $\mathcal{D}^{\mathsf{perf}}(X) = \langle \mathcal{A}_1^{\mathsf{perf}}, \dots, \mathcal{A}_m^{\mathsf{perf}} \rangle$. Let $\mathcal{D}^{\mathsf{perf}}(X_T) = \langle \mathcal{A}_{1T}^p, \dots, \mathcal{A}_{mT}^p \rangle$ be the T-linear semiorthogonal decomposition constructed in Proposition 5.1. Then, applying Proposition 4.2, we construct semiorthogonal decompositions $\mathcal{D}_{\mathrm{qc}}(X) = \langle \hat{\mathcal{A}}_1, \dots, \hat{\mathcal{A}}_m \rangle$ and $\mathcal{D}_{\mathrm{qc}}(X_T) = \langle \hat{\mathcal{A}}_{1T}, \dots, \hat{\mathcal{A}}_{mT} \rangle$. By Lemma 4.4, these decompositions are S- and T-linear.

Proposition 5.3. *The functors $\phi_* : \mathcal{D}_{\mathrm{qc}}(X_T) \to \mathcal{D}_{\mathrm{qc}}(X)$ and $\phi^* : \mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(X_T)$ are compatible with the above semiorthogonal decompositions. Moreover,*

$$\hat{\mathcal{A}}_{iT} = \{H \in \mathcal{D}_{\mathrm{qc}}(X_T) \mid \phi_*(H \otimes f^*G) \in \hat{\mathcal{A}}_i \text{ for all } G \in \mathcal{D}^{\mathsf{perf}}(T)\}. \tag{6}$$

Proof. Recall that both $\hat{\mathcal{A}}_i$ and $\hat{\mathcal{A}}_{iT}$ are obtained from $\mathcal{A}_i^{\mathsf{perf}}$ and $\mathcal{A}_{iT}^p$ by addition of arbitrary direct sums and iterated addition of cones and both are closed under arbitrary direct sums triangulated categories. Since both ϕ_* and ϕ^* commute with arbitrary direct sums and are exact, it suffices to check that $\phi^*(\mathcal{A}_i^{\mathsf{perf}}) \subset \hat{\mathcal{A}}_{iT}$ and that $\phi_*(\mathcal{A}_{iT}^p) \subset \hat{\mathcal{A}}_i$. The first is evident by definition of $\hat{\mathcal{A}}_{iT}$. For the second, take any $F \in \mathcal{A}_i^{\mathsf{perf}}$, $G \in \mathcal{D}^{\mathsf{perf}}(T)$. Then $\phi_*(\phi^*F \otimes f^*G) \cong F \otimes \phi_* f^*G \cong F \otimes f^*\phi_*G$. But, $F \otimes f^*\phi_*G \in \hat{\mathcal{A}}_i$, by Lemma 4.5.

To prove (6), we note that the left-hand side is contained in the right-hand side by the T-linearity of $\hat{\mathcal{A}}_{iT}$ and compatibility with ϕ_*. Conversely, assume that H is in the right-hand side but not in $\hat{\mathcal{A}}_{iT}$, so that we have $\hat{\alpha}_{jT}(H) \neq 0$ for some $j \neq i$. Since the semiorthogonal decomposition $\langle \hat{\mathcal{A}}_{1T}, \dots, \hat{\mathcal{A}}_{mT} \rangle$ is T-linear, the functors $\hat{\alpha}_{jT}$ are T-linear, by Lemma 2.8; hence, $\hat{\alpha}_{jT}(H \otimes f^*L^k) \cong \hat{\alpha}_{jT}(H) \otimes f^*L^k$ for any line bundle L on T and any $k \in \mathbb{Z}$. By Lemma 5.4 below, we have $\mathsf{hocolim}\, \phi_*(\hat{\alpha}_{jT}(H) \otimes f^*L^{k_i}) \neq 0$ for some sequence $L^{k_1} \to L^{k_2} \to L^{k_3} \to \cdots$ if L is ample over S. It remains to note that

$$\hat{\alpha}_j(\mathsf{hocolim}\, \phi_*(H \otimes f^*L^{k_i})) \cong \mathsf{hocolim}\, \hat{\alpha}_j(\phi_*(H \otimes f^*L^{k_i})) \cong \mathsf{hocolim}\, \phi_*(\hat{\alpha}_{jT}(H \otimes f^*L^{k_i})) \neq 0$$

(the first isomorphism is by Proposition 4.2 and the second is by Lemma 3.1). This means that $\mathsf{hocolim}\, \phi_*(H \otimes f^*L^{k_i}) \notin \hat{\mathcal{A}}_i$; hence, $\phi_*(H \otimes f^*L^k) \notin \hat{\mathcal{A}}_i$ for some $k \in \mathbb{Z}$, since $\hat{\mathcal{A}}_i$ is closed under homotopy colimits. So, H is not in the right-hand side of (6), a contradiction. $\square$

Lemma 5.4. *Let $\phi : Y \to X$ be a quasiprojective morphism and let L be a line bundle on Y ample over X. Take any $F \in \mathcal{D}_{\mathrm{qc}}(Y)$. Then $F \in \mathcal{D}_{\mathrm{qc}}^{[p,q]}(Y)$ if and only if for any sequence of maps $L^{k_1} \to L^{k_2} \to L^{k_3} \to \cdots$ with $k_i \to \infty$ we have $\mathsf{hocolim}\, \phi_*(F \otimes L^{k_i}) \in \mathcal{D}_{\mathrm{qc}}^{[p,q]}(X)$. In particular, $F = 0$ if and only if for any sequence $L^{k_1} \to L^{k_2} \to L^{k_3} \to \cdots$ with $k_i \to \infty$ we have $\mathsf{hocolim}\, \phi_*(F \otimes L^{k_i}) = 0$.*

Proof. As ϕ is quasiprojective, we can represent ϕ as $\pi_1 \circ j_1$, where $j_1 : Y \to \overline{Y}$ is an open embedding and $\pi_1 : \overline{Y} \to X$ is a projective morphism. Furthermore, any open embedding $j_1 : Y \to \overline{Y}$ can be represented as a composition of an affine open embedding $j : Y \to \widetilde{Y}$ and of a projective morphism $\pi_2 : \widetilde{Y} \to \overline{Y}$ (we take for $\widetilde{Y}$ the blowup of the ideal of the closed subset $\overline{Y} \backslash Y$ in Y). Put $\pi = \pi_1 \circ \pi_2$. Thus, $\phi = \pi \circ j$, where j is an affine open embedding and π is projective. Since j is an affine open embedding, the functors j_* and j^* are exact and $j^*j_* \cong \mathsf{id}$; hence, we have $F \in \mathcal{D}_{\mathrm{qc}}^{[p,q]}(Y)$ if and only if $j_*F \in \mathcal{D}_{\mathrm{qc}}^{[p,q]}(\widetilde{Y})$. Thus, the claim of the lemma reduces to the case when ϕ is projective.

So, assume that ϕ is projective. For any non-zero coherent sheaf H on X, we know that $\mathcal{H}^t(\phi_*(H \otimes L^k))$ is zero for $t \neq 0$ and $k \gg 0$. Therefore, for any quasicoherent sheaf H on X, we have $\varinjlim \mathcal{H}^t(\phi_*(H \otimes L^{k_i})) = 0$ for $t \neq 0$ if $k_i \to \infty$. So, the hypercohomology spectral sequence and Lemma 2.14 imply that

$$\mathcal{H}^t(\mathsf{hocolim}\, \phi_*(F \otimes L^{k_i})) \cong \varinjlim \mathcal{H}^0(\phi_*(\mathcal{H}^t(F) \otimes L^{k_i})).$$

It follows immediately that $F \in \mathcal{D}_{\mathrm{qc}}^{[p,q]}(Y)$ implies that $\mathsf{hocolim}\, \phi_*(F \otimes L^{k_i}) \in \mathcal{D}_{\mathrm{qc}}^{[p,q]}(X)$. As for the other implication, it suffices to check that for any quasicoherent sheaf $H \neq 0$ on Y there exists a sequence of maps $L^{k_1} \to L^{k_2} \to L^{k_3} \to \cdots$ with $k_i \to \infty$ such that $\varinjlim \mathcal{H}^0(\phi_*(H \otimes L^{k_i})) \neq 0$. Since tensoring with a line bundle and the colimit are exact functors on the abelian category $\mathsf{Q}coh(X)$, while $\mathcal{H}^0\phi_*$ is left exact, it follows that it suffices to prove the above for any non-zero subsheaf of H. Thus, we can assume that H is coherent. Then, using ampleness of L, we can find m and a section s of L^m such that the map $H \to H \otimes L^m$ given by s is an embedding. Now consider the sequence $L^m \to L^{2m} \to L^{3m} \to \cdots$ with all maps given by s. Then all the maps in the sequence $\mathcal{H}^0(\phi_*(H \otimes L^m)) \to \mathcal{H}^0(\phi_*(H \otimes L^{2m})) \to \mathcal{H}^0(\phi_*(H \otimes L^{3m})) \to \cdots$ are embeddings. Moreover, $\mathcal{H}^0(\phi_*(H \otimes L^{im})) \neq 0$ for $i \gg 0$. Hence, the limit is non-zero and we are done. □

5.3 Base change for bounded above coherent complexes

As above, we start with an S-linear semiorthogonal decomposition $\mathcal{D}^{\mathsf{perf}}(X) = \langle \mathcal{A}_1^{\mathsf{perf}}, \ldots, \mathcal{A}_m^{\mathsf{perf}} \rangle$. Let $\mathcal{D}^{\mathsf{perf}}(X_T) = \langle \mathcal{A}_{1T}^p, \ldots, \mathcal{A}_{mT}^p \rangle$ be the T-linear semiorthogonal decomposition constructed in Proposition 5.1. Let $\mathcal{D}_{\mathrm{qc}}(X) = \langle \hat{\mathcal{A}}_1, \ldots, \hat{\mathcal{A}}_m \rangle$ and $\mathcal{D}_{\mathrm{qc}}(X_T) = \langle \hat{\mathcal{A}}_{1T}, \ldots, \hat{\mathcal{A}}_{mT} \rangle$ be the S- and T-linear semiorthogonal decompositions constructed in Proposition 4.2 from the above decompositions of $\mathcal{D}^{\mathsf{perf}}(X)$ and $\mathcal{D}^{\mathsf{perf}}(X_T)$, respectively. Finally, let $\mathcal{D}^-(X) = \langle \mathcal{A}_1^-, \ldots, \mathcal{A}_m^- \rangle$ and $\mathcal{D}^-(X_T) = \langle \mathcal{A}_{1T}^-, \ldots, \mathcal{A}_{mT}^- \rangle$ be the S- and T-linear semiorthogonal decompositions constructed in Proposition 4.3.

Lemma 5.5. *The functors $\phi_* : \mathcal{D}^-(X_T) \to \mathcal{D}_{\mathrm{qc}}(X)$ and $\phi^* : \mathcal{D}^-(X) \to \mathcal{D}^-(X_T)$ are compatible with the above semiorthogonal decompositions.*

Proof. This follows immediately from Proposition 5.3, since, by definition, $\mathcal{A}_i^- = \hat{\mathcal{A}}_i \cap \mathcal{D}^-(X)$ and $\mathcal{A}_{iT}^- = \hat{\mathcal{A}}_{iT} \cap \mathcal{D}^-(X_T)$. □

5.4 Base change for bounded coherent complexes

This time we start with an S-linear strong semiorthogonal decomposition $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$. Let $\mathcal{D}^{\mathsf{perf}}(X) = \langle \mathcal{A}_1^{\mathsf{perf}}, \ldots, \mathcal{A}_m^{\mathsf{perf}} \rangle$ be the induced S-linear semiorthogonal decomposition of the category $\mathcal{D}^{\mathsf{perf}}(X)$. Further, consider the category $\mathcal{D}^{\mathsf{perf}}(X_T)$ and its T-linear semiorthogonal decomposition $\mathcal{D}^{\mathsf{perf}}(X_T) = \langle \mathcal{A}_{1T}^p, \ldots, \mathcal{A}_{mT}^p \rangle$ of Proposition 5.1, and let $\mathcal{D}_{\mathrm{qc}}(X) = \langle \hat{\mathcal{A}}_1, \ldots, \hat{\mathcal{A}}_m \rangle$ and $\mathcal{D}_{\mathrm{qc}}(X_T) = \langle \hat{\mathcal{A}}_{1T}, \ldots, \hat{\mathcal{A}}_{mT} \rangle$ be the S- and T-linear semiorthogonal decompositions constructed in Proposition 4.2 from the above decompositions of $\mathcal{D}^{\mathsf{perf}}(X)$ and $\mathcal{D}^{\mathsf{perf}}(X_T)$, respectively. Further, let $\mathcal{D}^-(X) = \langle \mathcal{A}_1^-, \ldots, \mathcal{A}_m^- \rangle$ and $\mathcal{D}^-(X_T) = \langle \mathcal{A}_{1T}^-, \ldots, \mathcal{A}_{mT}^- \rangle$ be the S- and T-linear semiorthogonal decompositions constructed in Proposition 4.3. Finally, we define

$$\mathcal{A}_{iT} = \mathcal{A}_{iT}^- \cap \mathcal{D}^b(X_T). \tag{7}$$

Theorem 5.6. *Let $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$ be an S-linear strong semiorthogonal decomposition the projection functors of which have finite cohomological amplitude and assume that the base change ϕ is faithful for f. Then the subcategories $\mathcal{A}_{iT} \subset \mathcal{D}^b(X_T)$ defined in (7) form a T-linear semiorthogonal decomposition $\mathcal{D}^b(X_T) = \langle \mathcal{A}_{1T}, \ldots, \mathcal{A}_{mT} \rangle$. The projection functors of*

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this semiorthogonal decomposition have the same cohomological amplitude as the projection functors of the initial semiorthogonal decomposition. Moreover, the functors $\phi_* : \mathcal{D}^b(X_T) \to \mathcal{D}_{\mathrm{qc}}(X)$ *and* $\phi^* : \mathcal{D}^b(X) \to \mathcal{D}^-(X_T)$ *are compatible with the semiorthogonal decompositions of* $\mathcal{D}_{\mathrm{qc}}(X)$ *and* $\mathcal{D}^-(X_T)$*, respectively.*

Proof. Take any $H \in \mathcal{D}^{[p,q]}(X_T)$. We have to check that $\alpha_{iT}^-(H)$ is bounded. Let (a_i, b_i) be the cohomological amplitude of α_i. Let us show that $\alpha_{iT}^-(H) \in \mathcal{D}_{\mathrm{qc}}^{[p+a_i,q+b_i]}(X_T)$. This will prove both that the categories $\mathcal{A}_{iT}$ form a semiorthogonal decomposition of $\mathcal{D}^b(X_T)$ and that the cohomological amplitude of the projection functors is the same as that of α_i. Using Lemma 5.4, we see that it suffices to check that for $k \gg 0$ we have $\phi_*(\alpha_{iT}^-(H) \otimes L^k) \in \mathcal{D}_{\mathrm{qc}}^{[p+a_i,q+b_i]}(X)$, where L is a line bundle on X_T ample over X. We can take $L = f^*M$, where M is a line bundle on T ample over S. Note that $\phi_*(\alpha_{iT}^-(H) \otimes f^*M^k) \cong \phi_*(\alpha_{iT}^-(H \otimes f^*M^k)) \cong \hat{\alpha}_i(\phi_*(H \otimes f^*M^k))$, by Lemma 5.5. Further, note that, by Lemma 2.20, for $k \gg 0$ we have $\phi_*(H \otimes f^*M^k) \cong \mathsf{hocolim}\, G_m$ for a certain direct system G_m with $G_m \in \mathcal{D}^{[p,q]}(X)$. Therefore,

$$\hat{\alpha}_i(\phi_*(H \otimes f^*M^k)) = \hat{\alpha}_i(\mathsf{hocolim}\, G_m) \cong \mathsf{hocolim}\, \alpha_i(G_m),$$

since the functor $\hat{\alpha}_i$ commutes with homotopy colimits and the G_m are bounded. Finally, we have $\alpha_i(G_m) \in \mathcal{D}^{[p+a_i,q+b_i]}(X)$; hence, $\mathsf{hocolim}\, \alpha_i(G_m) \in \mathcal{D}_{\mathrm{qc}}^{[p+a_i,q+b_i]}(X)$, by Lemma 2.14, which means that $\hat{\alpha}_i(\phi_*(H \otimes f^*M^k)) \in \mathcal{D}_{\mathrm{qc}}^{[p+a_i,q+b_i]}(X)$, as was required.

Finally, it remains to check that the subcategories (7) are T-linear, and also that $\phi_*(\mathcal{A}_{iT}) \subset \hat{\mathcal{A}}_i$ and $\phi^*(\mathcal{A}_i) \in \mathcal{A}_{iT}^-$. The first is clear since $\mathcal{A}_{iT}^-$ is T-linear and the other two claims follow from Lemma 5.5. □

The semiorthogonal decomposition of $\mathcal{D}^b(X_T)$ constructed in Theorem 5.6 will be referred to as the induced decomposition of $\mathcal{D}^b(X_T)$ with respect to the base change ϕ. Note that the definition of its component $\mathcal{A}_{iT}$ depends only on $\mathcal{A}_i$ (i.e. does not depend on the choice of a semiorthogonal decomposition containing $\mathcal{A}_i$ as a component). Indeed, spelling out (6), (3) and (7), we obtain the following.

COROLLARY 5.7. *If* $\mathcal{A} \subset \mathcal{D}^b(X)$ *is an* S*-linear admissible subcategory such that the corresponding projection functor has finite cohomological amplitude and* $\phi : T \to S$ *is a quasiprojective base change faithful for* $f : X \to S$*, then the category*

$$\mathcal{A}_T = \{F \in \mathcal{D}^b(X_T) \mid \phi_*(F \otimes f^*G) \in \hat{\mathcal{A}} \text{ for all } G \in \mathcal{D}^{\mathsf{perf}}(T)\} \tag{8}$$

(where $\hat{\mathcal{A}}$ *is the minimal closed under arbitrary direct sums triangulated subcategory of* $\mathcal{D}_{\mathrm{qc}}(X)$ *containing* $\mathcal{A}$*) is a* T*-linear admissible subcategory in* $\mathcal{D}^b(X_T)$ *such that the corresponding projection functor has finite cohomological amplitude. Moreover, we have* $\phi^*(\mathcal{A}) \subset \mathcal{A}_T$ *if* ϕ *has finite* Tor*-dimension and* $\phi_*(\mathcal{A}_T) \subset \mathcal{A}$ *if* ϕ *is projective.*

5.5 Exterior product of semiorthogonal decompositions

Now assume that we are given two algebraic varieties over the same base, say $f : X \to S$ and $g : Y \to S$, and S-linear strong semiorthogonal decompositions of their derived categories $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$ and $\mathcal{D}^b(Y) = \langle \mathcal{B}_1, \ldots, \mathcal{B}_n \rangle$. Assume that their projection functors have

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finite cohomological amplitude. Assume also that the cartesian square

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & S \end{array} \tag{9}$$

is exact, so that g is a faithful base change for f and f is a faithful base change for g. Applying Theorem 5.6, we obtain a pair of semiorthogonal decompositions of $\mathcal{D}^b(X \times_S Y)$:

$$\mathcal{D}^b(X \times_S Y) = \langle \mathcal{A}_{1Y}, \dots, \mathcal{A}_{mY} \rangle \quad \text{and} \quad \mathcal{D}^b(X \times_S Y) = \langle \mathcal{B}_{1X}, \dots, \mathcal{B}_{nX} \rangle.$$

Let

$$\mathcal{A}_i \boxtimes_S \mathcal{B}_j := \mathcal{A}_{iY} \cap \mathcal{B}_{jX} \subset \mathcal{D}^b(X \times_S Y). \tag{10}$$

We call the category $\mathcal{A}_i \boxtimes_S \mathcal{B}_j$ the exterior product (over S) of $\mathcal{A}_i$ and $\mathcal{B}_j$.

Consider any complete order on the set $\{(i,j)\}_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$ extending the natural partial order.

Theorem 5.8. *The exterior product subcategories $\mathcal{A}_i \boxtimes_S \mathcal{B}_j \subset \mathcal{D}^b(X \times_S Y)$ form a semiorthogonal decomposition of the category $\mathcal{D}^b(X \times_S Y)$:*

$$\mathcal{D}^b(X \times_S Y) = \langle \mathcal{A}_i \boxtimes_S \mathcal{B}_j \rangle_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}.$$

Moreover, we have the following semiorthogonal decompositions:

$$\mathcal{A}_{iY} = \langle \mathcal{A}_i \boxtimes_S \mathcal{B}_1, \dots, \mathcal{A}_i \boxtimes_S \mathcal{B}_n \rangle \quad \text{and} \quad \mathcal{B}_{jX} = \langle \mathcal{A}_1 \boxtimes_S \mathcal{B}_j, \dots, \mathcal{A}_m \boxtimes_S \mathcal{B}_j \rangle.$$

Proof. Let $\mathcal{C}^p_{ij} = \langle p^* \mathcal{A}_i^{\mathsf{perf}} \otimes q^* \mathcal{B}_j^{\mathsf{perf}} \rangle \subset \mathcal{D}^{\mathsf{perf}}(X \times_S Y)$ be the minimal triangulated subcategory of $\mathcal{D}^{\mathsf{perf}}(X \times_S Y)$ closed under taking direct summands and containing objects of the form $p^*A \otimes q^*B$ with $A \in \mathcal{A}_i^{\mathsf{perf}}, B \in \mathcal{B}_j^{\mathsf{perf}}$. The arguments of Proposition 5.1 show that $\mathcal{D}^{\mathsf{perf}}(X \times_S Y) = \langle \mathcal{C}^p_{ij} \rangle_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$ is a semiorthogonal decomposition. Moreover, it is clear from the construction that we have semiorthogonal decompositions

$$\mathcal{A}^p_{iY} = \langle \mathcal{C}^p_{i1}, \dots, \mathcal{C}^p_{in} \rangle \quad \text{and} \quad \mathcal{B}^p_{jX} = \langle \mathcal{C}^p_{1j}, \dots, \mathcal{C}^p_{mj} \rangle.$$

Extending these decompositions to $\mathcal{D}_{\mathrm{qc}}(X \times_S Y)$ as in Proposition 4.2, we obtain semiorthogonal decompositions $\mathcal{D}_{\mathrm{qc}}(X \times_S Y) = \langle \hat{\mathcal{C}}_{ij} \rangle_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$ as well as

$$\hat{\mathcal{A}}_{iY} = \langle \hat{\mathcal{C}}_{i1}, \dots, \hat{\mathcal{C}}_{in} \rangle \quad \text{and} \quad \hat{\mathcal{B}}_{jX} = \langle \hat{\mathcal{C}}_{1j}, \dots, \hat{\mathcal{C}}_{mj} \rangle,$$

where $\hat{\mathcal{C}}_{ij}$ is obtained from $\mathcal{C}_{ij}$ by addition of arbitrary direct sums and iterated addition of cones. Finally, intersecting with $\mathcal{D}^b(X \times_S Y)$, we obtain semiorthogonal decompositions $\mathcal{D}^b(X \times_S Y) = \langle \mathcal{C}_{ij} \rangle_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$ as well as

$$\mathcal{A}_{iY} = \langle \mathcal{C}_{i1}, \dots, \mathcal{C}_{in} \rangle \quad \text{and} \quad \mathcal{B}_{jX} = \langle \mathcal{C}_{1j}, \dots, \mathcal{C}_{mj} \rangle,$$

where $\mathcal{C}_{ij} = \hat{\mathcal{C}}_{ij} \cap \mathcal{D}^b(X \times Y)$. So, it remains to check that $\mathcal{C}_{ij} = \mathcal{A}_{iY} \cap \mathcal{B}_{jX}$. Since $\mathcal{C}_{ij} \subset \mathcal{A}_{iY} \cap \mathcal{B}_{jX}$ by construction, it suffices to check only the other inclusion. Indeed, we have

$$\mathcal{B}_{jX} = {}^\perp\langle \mathcal{B}_{1X}, \dots, \mathcal{B}_{j-1,X} \rangle \cap \langle \mathcal{B}_{j+1,X}, \dots, \mathcal{B}_{nX} \rangle^\perp = {}^\perp\langle \mathcal{C}_{it} \rangle_{1 \leqslant i \leqslant m, 1 \leqslant t \leqslant j-1} \cap \langle \mathcal{C}_{it} \rangle^\perp_{1 \leqslant i \leqslant m, j+1 \leqslant t \leqslant n};$$

hence,

$$\mathcal{A}_{iY} \cap \mathcal{B}_{jX} \subset \mathcal{A}_{iY} \cap {}^\perp\langle \mathcal{C}_{it} \rangle_{1 \leqslant t \leqslant j-1} \cap \langle \mathcal{C}_{it} \rangle^\perp_{j+1 \leqslant t \leqslant n} = \mathcal{C}_{ij},$$

which is precisely what we need. ∎

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5.6 Products

If S is a point, then any semiorthogonal decomposition of $\mathcal{D}^b(X)$ is S-linear. Moreover, any base change $T \to S$ is flat, hence faithful for $f: X \to S$, and $X \times_S T = X \times T$ is the product. Thus, given a semiorthogonal decomposition of $\mathcal{D}^b(X)$, we can construct a compatible semiorthogonal decomposition of the bounded derived category of the product of X with any quasiprojective variety. Explicitly, applying Theorem 5.6, we obtain the following.

COROLLARY 5.9. *Let $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ be a strong semiorthogonal decomposition the projection functors of which have finite cohomological amplitude. Let Y be a quasiprojective variety. Then the subcategories*

$$\mathcal{A}_{iY} = \{F \in \mathcal{D}^b(X \times Y) \mid p_*(F \otimes q^* G) \in \hat{\mathcal{A}}_i \text{ for any } G \in \mathcal{D}^{\mathsf{perf}}(Y)\},$$

where $p: X \times Y \to X$ and $q: X \times Y \to Y$ are the projections, and $\hat{\mathcal{A}}_i$ is obtained from $\mathcal{A}_i$ by addition of arbitrary direct sums and iterated addition of cones, form a Y-linear semiorthogonal decomposition $\mathcal{D}^b(X \times Y) = \langle \mathcal{A}_{1Y}, \dots, \mathcal{A}_{mY} \rangle$. The projection functors of this semiorthogonal decomposition also have finite cohomological amplitude. The functors $p_*: \mathcal{D}^b(X \times Y) \to \mathcal{D}_{\mathrm{qc}}(X)$ and $p^*: \mathcal{D}^b(X) \to \mathcal{D}^b(X \times Y)$ are compatible with the semiorthogonal decompositions of $\mathcal{D}_{\mathrm{qc}}(X)$ and $\mathcal{D}^b(X)$, respectively.

Similarly, Theorem 5.8 gives the following.

COROLLARY 5.10. *Let $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ and $\mathcal{D}^b(Y) = \langle \mathcal{B}_1, \dots, \mathcal{B}_n \rangle$ be strong semiorthogonal decompositions with projection functors of finite cohomological amplitude. Then there is a semiorthogonal decomposition*

$$\mathcal{D}^b(X \times Y) = \langle \mathcal{A}_i \boxtimes \mathcal{B}_j \rangle_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n},$$

where $\mathcal{A}_i \boxtimes \mathcal{B}_j = \mathcal{A}_{iY} \cap \mathcal{B}_{jX}$. Moreover, we have semiorthogonal decompositions

$$\mathcal{A}_{iY} = \langle \mathcal{A}_i \boxtimes \mathcal{B}_1, \dots, \mathcal{A}_i \boxtimes \mathcal{B}_n \rangle \quad \text{and} \quad \mathcal{B}_{jX} = \langle \mathcal{A}_1 \boxtimes \mathcal{B}_j, \dots, \mathcal{A}_m \boxtimes \mathcal{B}_j \rangle.$$

6. Correctness

The goal of this section is to show that the extensions $\mathcal{A}^{\mathsf{perf}}$, $\hat{\mathcal{A}}$, $\mathcal{A}^-$ of a triangulated category $\mathcal{A}$ and its base change $\mathcal{A}_T$ under a base change $T \to S$ (if $\mathcal{A}$ is S-linear) do not depend on a choice of an embedding $\mathcal{A} \to \mathcal{D}^b(X)$. The most important technical notion for this section is that of a splitting functor.

6.1 Splitting functors

An exact functor $\Phi: \mathcal{T} \to \mathcal{T}'$ is called right splitting if $\mathsf{Ker}\,\Phi$ is a right admissible subcategory in $\mathcal{T}$, the restriction of Φ to $(\mathsf{Ker}\,\Phi)^\perp$ is fully faithful and $\mathsf{Im}\,\Phi$ is right admissible in $\mathcal{T}'$ (note that $\mathsf{Im}\,\Phi = \mathsf{Im}(\Phi_{|(\mathsf{Ker}\,\Phi)^\perp})$ is a triangulated subcategory of $\mathcal{T}'$). For more information on splitting functors, see [Kuz07]. Here we will need only the following.

LEMMA 6.1 [Kuz07]. *Let $\Phi: \mathcal{T} \to \mathcal{T}'$ be an exact functor. Then the following conditions are equivalent:*

(1) *Φ is right splitting;*

(2) *Φ has a right adjoint functor $\Phi^!$ and the composition of the canonical morphism of functors $\mathsf{id}_{\mathcal{T}} \to \Phi^!\Phi$ with Φ gives an isomorphism $\Phi \cong \Phi\Phi^!\Phi$;*

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(3) *Φ has a right adjoint functor $\Phi^!$, there are semiorthogonal decompositions*

$$\mathcal{T} = \langle \mathsf{Im}\ \Phi^!, \mathsf{Ker}\ \Phi \rangle, \quad \mathcal{T}' = \langle \mathsf{Ker}\ \Phi^!, \mathsf{Im}\ \Phi \rangle$$

and the functors Φ and $\Phi^!$ give quasi-inverse equivalences $\mathsf{Im}\ \Phi^! \cong \mathsf{Im}\ \Phi$;

(4) *there exist a full triangulated left admissible subcategory $\alpha : \mathcal{A} \subset \mathcal{T}$, a full triangulated right admissible subcategory $\mathcal{B} \subset \mathcal{T}'$ and an equivalence $\xi : \mathcal{A} \to \mathcal{B}$ such that $\Phi = \beta \circ \xi \circ \alpha^*$, $\Phi^! = \alpha \circ \xi^{-1} \circ \beta^!$.*

There is an analogous notion of left splitting functors, which enjoy a similar set of properties. However, we will not need this notion.

6.2 Extensions

Let X be a quasiprojective variety. Let $\alpha : \mathcal{A} \to \mathcal{D}^b(X)$ and $\beta : \mathcal{B} \to \mathcal{D}^b(Y)$ be admissible subcategories, and $\xi : \mathcal{A} \to \mathcal{B}$ an equivalence. Consider the corresponding right splitting functor $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$,

$$\Phi = \beta \circ \xi \circ \alpha^*.$$

We assume also that Φ is geometric, meaning that it is isomorphic to a kernel functor

$$\Phi_{\mathcal{E}} : \mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(Y), \quad \Phi_{\mathcal{E}}(F) = q_*(p^*F \otimes \mathcal{E})$$

with a kernel $\mathcal{E} \in \mathcal{D}^-(X \times Y)$. Here $p : X \times Y \to X$ and $q : X \times Y \to Y$ are the projections. Note that the right adjoint functor $\Phi^!_{\mathcal{E}}$ of $\Phi_{\mathcal{E}}$ is given by the formula

$$\Phi^!_{\mathcal{E}} : \mathcal{D}_{\mathrm{qc}}(Y) \to \mathcal{D}_{\mathrm{qc}}(X), \quad \Phi^!_{\mathcal{E}}(G) = p_* \mathsf{R}\mathcal{H}om(\mathcal{E}, q^! F).$$

It follows in particular that $\Phi_{\mathcal{E}}$ commutes with direct sums. Indeed,

$$\begin{aligned}\mathsf{Hom}\Big(\Phi_{\mathcal{E}}\Big(\bigoplus F_i\Big), G\Big) &\cong \mathsf{Hom}\Big(\bigoplus F_i, \Phi^!_{\mathcal{E}}(G)\Big) \cong \prod \mathsf{Hom}(F_i, \Phi^!_{\mathcal{E}}(G)) \\ &\cong \prod \mathsf{Hom}(\Phi_{\mathcal{E}}(F_i), G) \cong \mathsf{Hom}\Big(\bigoplus \Phi_{\mathcal{E}}(F_i), G\Big)\end{aligned}$$

implies that $\Phi_{\mathcal{E}}(\bigoplus F_i) \cong \bigoplus \Phi_{\mathcal{E}}(F_i)$.

Recall that if $\mathcal{E} \in \mathcal{D}^b(X \times Y)$ has finite Tor-amplitude over X, finite Ext-amplitude over Y and $\mathsf{supp}\,\mathcal{E}$ is projective over both X and Y, then $\Phi_{\mathcal{E}}$ takes $\mathcal{D}^b(X)$ to $\mathcal{D}^b(Y)$ and $\Phi^!_{\mathcal{E}}$ takes $\mathcal{D}^b(Y)$ to $\mathcal{D}^b(X)$ by [Kuz06].

Theorem 6.2. *Assume that an object $\mathcal{E} \in \mathcal{D}^b(X \times Y)$ has finite Tor-amplitude over X, finite Ext-amplitude over Y and $\mathsf{supp}\,\mathcal{E}$ is projective over both X and Y. Assume also that the restriction of the functor $\Phi_{\mathcal{E}} : \mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(Y)$ to $\mathcal{D}^b(X)$ is a right splitting functor giving an equivalence of subcategories $\mathcal{A} \subset \mathcal{D}^b(X)$ and $\mathcal{B} \subset \mathcal{D}^b(Y)$. Then the functor $\Phi_{\mathcal{E}} : \mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(Y)$ and its restriction to $\mathcal{D}^-(X)$ are right splitting functors giving equivalences $\hat{\mathcal{A}} \cong \hat{\mathcal{B}}$ and $\mathcal{A}^- \cong \mathcal{B}^-$.*

Proof. As we already mentioned above, the functor $\Phi_{\mathcal{E}}$ commutes with direct sums. Let us check that $\Phi^!_{\mathcal{E}}$ also commutes with direct sums. To do this, we choose a closed embedding $i : X \to X'$ with X' being smooth and consider the functor $i_*\Phi^!_{\mathcal{E}}$ instead. Since i_* is a conservative functor commuting with direct sums, it suffices to check that $i_*\Phi^!_{\mathcal{E}}$ commutes with direct sums. But, it is clear that $i_*\Phi^!_{\mathcal{E}} \cong \Phi^!_{(i \times \mathsf{id}_Y)_*\mathcal{E}}$, so, from the whole beginning, we can assume that X is smooth. Then the projection $X \times Y \to Y$ is smooth; hence, $q^!(F) \cong q^*(F) \otimes \omega_X[\dim X]$ evidently commutes with direct sums. Further, $\mathcal{E}$ is a perfect complex, by [Kuz06, 10.46]; hence, the

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functor $\mathsf{RHom}(\mathcal{E}, -)$ commutes with direct sums. Finally, the functor p_* commutes with direct sums, by [BV03, 3.3.4]. Thus, $\Phi^!_{\mathcal{E}}$ commutes with direct sums.

Further, since the functors $\Phi_{\mathcal{E}}$ and $\Phi^!_{\mathcal{E}}$ commute with direct sums, they commute with homotopy colimits, by Lemma 2.11. Now, if $F \in \mathcal{D}^-(X)$, then, by Lemma 2.18, there exists a stabilizing in finite degrees direct system of perfect complexes $F_k \in \mathcal{D}^b(X)$ such that $F \cong \mathsf{hocolim}\, F_k$. Therefore, $\Phi_{\mathcal{E}}(F) \cong \Phi_{\mathcal{E}}(\mathsf{hocolim}\, F_k) \cong \mathsf{hocolim}\, \Phi_{\mathcal{E}}(F_k)$. But, the functor $\Phi_{\mathcal{E}}$ has finite cohomological amplitude, by Lemma 2.10. Therefore, the direct system $\Phi_{\mathcal{E}}(F_k) \in \mathcal{D}^b(Y)$ stabilizes in finite degrees; hence, $\mathsf{hocolim}\, \Phi_{\mathcal{E}}(F_k) \in \mathcal{D}^-(Y)$, by Lemma 2.17. Thus, $\Phi_{\mathcal{E}}$ takes $\mathcal{D}^-(X)$ to $\mathcal{D}^-(Y)$. The same argument shows that $\Phi^!_{\mathcal{E}}$ takes $\mathcal{D}^-(Y)$ to $\mathcal{D}^-(X)$.

To check that $\Phi_{\mathcal{E}}$ is right splitting on $\mathcal{D}_{\mathrm{qc}}(X)$, we have to check that applying $\Phi_{\mathcal{E}}$ to the canonical morphism of functors $\mathsf{id} \to \Phi^!_{\mathcal{E}}\Phi_{\mathcal{E}}$ gives an isomorphism $\Phi_{\mathcal{E}} \cong \Phi_{\mathcal{E}}\Phi^!_{\mathcal{E}}\Phi_{\mathcal{E}}$. Consider the full subcategory $\mathcal{T} \subset \mathcal{D}_{\mathrm{qc}}(X)$ consisting of all objects $F \in \mathcal{D}_{\mathrm{qc}}(X)$ for which $\Phi_{\mathcal{E}}(F) \cong \Phi_{\mathcal{E}}\Phi^!_{\mathcal{E}}\Phi_{\mathcal{E}}(F)$ in $\mathcal{D}_{\mathrm{qc}}(Y)$. We want to show that $\mathcal{T} = \mathcal{D}_{\mathrm{qc}}(X)$. Note that $\mathcal{D}^b(X) \subset \mathcal{T}$ by the conditions; hence, $\mathcal{D}^{\mathsf{perf}}(X) \subset \mathcal{T}$. Moreover, since $\Phi_{\mathcal{E}}$ and $\Phi^!_{\mathcal{E}}$ commute with direct sums, $\mathcal{T}$ is closed under arbitrary direct sums. Finally, since $\Phi_{\mathcal{E}}$ and $\Phi^!_{\mathcal{E}}$ are exact, $\mathcal{T}$ is triangulated. So, by Lemma 2.19, we have $\mathcal{T} = \mathcal{D}_{\mathrm{qc}}(X)$.

Now let us check that $\hat{\mathcal{B}} = \Phi_{\mathcal{E}}(\mathcal{D}_{\mathrm{qc}}(X))$. Indeed, the right-hand side is contained in the left-hand side, by Lemma 2.19, since $\hat{\mathcal{B}}$ is closed under an arbitrary direct sums triangulated subcategory containing $\Phi_{\mathcal{E}}(\mathcal{D}^{\mathsf{perf}}(X)) \subset \Phi_{\mathcal{E}}(\mathcal{D}^b(X)) = \mathcal{B}$. For the other embedding, it suffices to check that $\hat{\mathcal{B}}$ is contained in the full subcategory $\mathcal{T} \subset \mathcal{D}_{\mathrm{qc}}(Y)$ consisting of all objects G such that the canonical morphism $\Phi_{\mathcal{E}}\Phi^!_{\mathcal{E}}(G) \to G$ is an isomorphism. Indeed, $\mathcal{T}$ contains $\mathcal{B}$ by conditions of the proposition. Moreover, it is closed under arbitrary direct sums, since both $\Phi_{\mathcal{E}}$ and $\Phi^!_{\mathcal{E}}$ commute with direct sums, and is triangulated, since both $\Phi_{\mathcal{E}}$ and $\Phi^!_{\mathcal{E}}$ are exact. The same argument shows that $\hat{\mathcal{A}} = \mathsf{Im}\, \Phi^!_{\mathcal{E}}$, so it follows that $\Phi_{\mathcal{E}}$ induces an equivalence $\hat{\mathcal{A}} \cong \hat{\mathcal{B}}$.

Finally, since $\Phi_{\mathcal{E}}$ and $\Phi^!_{\mathcal{E}}$ preserve $\mathcal{D}^-$ and $\mathcal{A}^- = \hat{\mathcal{A}} \cap \mathcal{D}^-(X)$, $\mathcal{B}^- = \hat{\mathcal{B}} \cap \mathcal{D}^-(Y)$, it follows that $\Phi_{\mathcal{E}}$ induces an equivalence $\mathcal{A}^- \cong \mathcal{B}^-$. □

Remark 6.3. One can also check that $\Phi_{\mathcal{E}}$ takes $\mathcal{D}^{\mathsf{perf}}(X)$ to $\mathcal{D}^{\mathsf{perf}}(Y)$ (this follows easily from the fact that $\Phi^!_{\mathcal{E}}$ commutes with direct sums). If it were also known that $\Phi^!_{\mathcal{E}}$ takes $\mathcal{D}^{\mathsf{perf}}(Y)$ to $\mathcal{D}^{\mathsf{perf}}(X)$, then it would follow that $\Phi_{\mathcal{E}}$ induces an equivalence $\mathcal{A}^{\mathsf{perf}} \cong \mathcal{B}^{\mathsf{perf}}$.

6.3 Base change

Now assume that $f : X \to S$ and $g : Y \to S$ are quasiprojective morphisms, $\alpha : \mathcal{A} \to \mathcal{D}^b(X)$ and $\beta : \mathcal{B} \to \mathcal{D}^b(Y)$ are admissible S-linear subcategories and $\xi : \mathcal{A} \to \mathcal{B}$ is an S-linear equivalence. Assume also that $\phi : T \to S$ is a base change faithful for both f and g. Again, consider the corresponding right splitting functor $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$, $\Phi = \beta \circ \xi \circ \alpha^*$. We assume also that Φ is geometrically S-linear, meaning that it is isomorphic to a kernel functor

$$\Phi_{\mathcal{E}} : \mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(Y), \quad \Phi_{\mathcal{E}}(F) = q_*(p^*F \otimes \mathcal{E})$$

with a kernel $\mathcal{E} \in \mathcal{D}^-(X \times_S Y)$ supported on the fiber product of X and Y over S. Here $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ are the projections. Note that the right adjoint functor $\Phi^!_{\mathcal{E}}$ of $\Phi_{\mathcal{E}}$ is given by the formula

$$\Phi^!_{\mathcal{E}} : \mathcal{D}_{\mathrm{qc}}(Y) \to \mathcal{D}_{\mathrm{qc}}(X), \quad \Phi^!_{\mathcal{E}}(G) = p_*\mathsf{R}\mathcal{H}om(\mathcal{E}, q^!F).$$

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Consider the following commutative diagram.

$$\begin{array}{ccccc} X_T & \xleftarrow{p_T} & X_T \times_T Y_T & \xrightarrow{q_T} & Y_T \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ X & \xleftarrow{p} & X \times_S Y & \xrightarrow{q} & Y \end{array}$$

Define the kernel $\mathcal{E}_T := \phi^*\mathcal{E} \in \mathcal{D}^-(X_T \times_T Y_T)$.

Theorem 6.4. *Assume that $\mathcal{E} \in \mathcal{D}^b(X \times_S Y)$ has finite Tor-amplitude over X, finite Ext-amplitude over Y and $\mathsf{supp}\,\mathcal{E}$ is projective over both X and Y. Assume also that the functor $\Phi_{\mathcal{E}} : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is a right splitting functor giving an equivalence of S-linear subcategories $\mathcal{A} \subset \mathcal{D}^b(X)$ and $\mathcal{B} \subset \mathcal{D}^b(Y)$. Then $\Phi_{\mathcal{E}_T} : \mathcal{D}^b(X_T) \to \mathcal{D}^b(Y_T)$ is a right splitting functor inducing an equivalence $\mathcal{A}_T \cong \mathcal{B}_T$.*

Proof. First of all, note that $\mathcal{E}_T$ has finite Tor-amplitude over X_T, finite Ext-amplitude over Y and projective support over both X_T and Y_T, by [Kuz06, 10.47]. Hence, as was mentioned in the proof of Theorem 6.2, the functors $\Phi_{\mathcal{E}}$, $\Phi^!_{\mathcal{E}}$, $\Phi_{\mathcal{E}_T}$ and $\Phi^!_{\mathcal{E}_T}$ commute with direct sums and homotopy colimits.

Moreover, by [Kuz06, 2.4], the functors $\Phi^!_{\mathcal{E}}$ and $\Phi^!_{\mathcal{E}_T}$ are right adjoint to $\Phi_{\mathcal{E}}$ and $\Phi_{\mathcal{E}_T}$, respectively, and all these functors preserve boundedness and coherence. Finally, by [Kuz06, 2.42], there are canonical isomorphisms

$$\begin{aligned} \Phi_{\mathcal{E}_T}\phi^* &= \phi^*\Phi_{\mathcal{E}}, & \Phi_{\mathcal{E}}\phi_* &= \phi_*\Phi_{\mathcal{E}_T}, \\ \Phi^!_{\mathcal{E}_T}\phi^* &= \phi^*\Phi^!_{\mathcal{E}}, & \Phi^!_{\mathcal{E}}\phi_* &= \phi_*\Phi^!_{\mathcal{E}_T}. \end{aligned} \tag{11}$$

Since $\Phi_{\mathcal{E}}$ is right splitting on $\mathcal{D}_{\mathrm{qc}}(X)$ by Theorem 6.2, applying $\Phi_{\mathcal{E}}$ to the canonical morphism of functors $\mathsf{id} \to \Phi^!_{\mathcal{E}}\Phi_{\mathcal{E}}$ gives an isomorphism $\Phi_{\mathcal{E}} \cong \Phi_{\mathcal{E}}\Phi^!_{\mathcal{E}}\Phi_{\mathcal{E}}$. Now take any $H \in \mathcal{D}_{\mathrm{qc}}(X_T)$. We want to show that $\Phi_{\mathcal{E}_T}(H) \cong \Phi_{\mathcal{E}_T}\Phi^!_{\mathcal{E}_T}\Phi_{\mathcal{E}_T}(H)$ in $\mathcal{D}_{\mathrm{qc}}(Y_T)$. By Lemma 5.4, to do this it suffices to check that $\phi_*(\Phi_{\mathcal{E}_T}(H) \otimes g^*L^k) \cong \phi_*(\Phi_{\mathcal{E}_T}\Phi^!_{\mathcal{E}_T}\Phi_{\mathcal{E}_T}(H) \otimes g^*L^k)$ in $\mathcal{D}_{\mathrm{qc}}(Y)$ for an ample over S line bundle L on T and any $k \gg 0$. But,

$$\begin{aligned} \phi_*(\Phi_{\mathcal{E}_T}(H) \otimes g^*L^k) &\cong \phi_*(\Phi_{\mathcal{E}_T}(H \otimes f^*L^k)) \cong \Phi_{\mathcal{E}}(\phi_*(H \otimes f^*L^k)) \\ &\cong \Phi_{\mathcal{E}}\Phi^!_{\mathcal{E}}\Phi_{\mathcal{E}}(\phi_*(H \otimes f^*L^k)) \cong \phi_*(\Phi_{\mathcal{E}_T}\Phi^!_{\mathcal{E}_T}\Phi_{\mathcal{E}_T}(H \otimes f^*L^k)) \\ &\cong \phi_*(\Phi_{\mathcal{E}_T}\Phi^!_{\mathcal{E}_T}\Phi_{\mathcal{E}_T}(H) \otimes g^*L^k). \end{aligned}$$

The first and the fifth isomorphisms are given by T-linearity of the functors $\Phi_{\mathcal{E}_T}$ and $\Phi^!_{\mathcal{E}_T}$, the second and the fourth are given by (11) and the third is because $\Phi_{\mathcal{E}}$ is right splitting. So, we conclude that $\Phi_{\mathcal{E}_T} \cong \Phi_{\mathcal{E}_T}\Phi^!_{\mathcal{E}_T}\Phi_{\mathcal{E}_T}$; hence, $\Phi_{\mathcal{E}_T}$ is a right splitting functor.

Now let us show that $\Phi_{\mathcal{E}_T}(\mathcal{D}_{\mathrm{qc}}(X_T)) = \hat{\mathcal{B}}_T$. Indeed, let $F \in \mathcal{D}_{\mathrm{qc}}(X_T)$. Let G be a perfect complex on T. Then we have

$$\phi_*(\Phi_{\mathcal{E}_T}(F) \otimes g^*G) \cong \phi_*(\Phi_{\mathcal{E}_T}(F \otimes f^*G)) \cong \Phi_{\mathcal{E}}(\phi_*(F \otimes f^*G)) \in \hat{\mathcal{B}};$$

hence, $\Phi_{\mathcal{E}_T}(F) \in \hat{\mathcal{B}}_T$, by (6). Further, since $\Phi_{\mathcal{E}_T}$ is a right splitting T-linear functor commuting with arbitrary direct sums, the category $\Phi_{\mathcal{E}_T}(\mathcal{D}_{\mathrm{qc}}(X_T))$ is a T-linear triangulated subcategory in $\mathcal{D}_{\mathrm{qc}}(Y_T)$ closed under arbitrary direct sums. On the other hand,

$$\phi^*(\mathcal{B}^{\mathsf{perf}}) \subset \phi^*(\mathcal{B}) = \phi^*(\Phi_{\mathcal{E}}(\mathcal{D}^b(X))) = \Phi_{\mathcal{E}_T}(\phi^*(\mathcal{D}^b(X))) \subset \Phi_{\mathcal{E}_T}(\mathcal{D}_{\mathrm{qc}}(X_T)),$$

so it follows from the definition of $\hat{\mathcal{B}}_T$ that $\hat{\mathcal{B}}_T \subset \Phi_{\mathcal{E}_T}(\mathcal{D}_{\mathrm{qc}}(X_T))$. The same argument shows that $\Phi^!_{\mathcal{E}_T}(\mathcal{D}_{\mathrm{qc}}(Y_T)) = \hat{\mathcal{A}}_T$.

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Finally, as we already mentioned, the functors $\Phi_{\mathcal{E}_T}$ and $\Phi^!_{\mathcal{E}_T}$ preserve the bounded category of coherent sheaves and, since $\mathcal{A}_T = \hat{\mathcal{A}}_T \cap \mathcal{D}^b(X_T)$, $\mathcal{B}_T = \hat{\mathcal{B}}_T \cap \mathcal{D}^b(Y_T)$, it follows that $\Phi_{\mathcal{E}_T}$ induces an equivalence $\mathcal{A}_T \cong \mathcal{B}_T$. □

7. Applications

As an application, we deduce that the projection functors of a strong semiorthogonal decomposition are kernel functors.

Theorem 7.1. *Let X be a quasiprojective variety and $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ a strong semiorthogonal decomposition. Let $\alpha_i : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ be the projection functor to the ith component. Assume that each α_i has finite cohomological amplitude. Then for every i there is an object $K_i \in \mathcal{D}^b(X \times X)$ such that $\alpha_i \cong \Phi_{K_i}$.*

Remark 7.2. Note that the condition that the semiorthogonal decomposition is strong is necessary for the projection functors to be representable by kernels. Indeed, every functor isomorphic to Φ_K has a right adjoint functor; hence, if $\alpha_1 \cong \Phi_K$, then α_1 has a right adjoint functor and hence $\mathcal{A}_1$ is right admissible.

Proof. We consider the semiorthogonal decomposition $\mathcal{D}^b(X \times X) = \langle \mathcal{A}_{1X}, \dots, \mathcal{A}_{mX} \rangle$ constructed in Corollary 5.9 and let K_i be the component of $\Delta_* \mathcal{O}_X \in \mathcal{D}^b(X \times X)$ in $\mathcal{A}_{iX}$. Consider the corresponding filtration of $\Delta_* \mathcal{O}_X$.

$$\begin{array}{ccccccccccc} 0 & = & T_m & \longrightarrow & T_{m-1} & \longrightarrow \cdots \longrightarrow & T_1 & \longrightarrow & T_0 & = & \Delta_*\mathcal{O}_X \\ & & \nwarrow & \swarrow & & & \nwarrow & \swarrow & & & \\ & & & K_m & & \cdots & & K_1 & & & \end{array}$$

Take any $F \in \mathcal{D}_{\mathrm{qc}}(X)$, pull it back to $X \times X$ via the projection $p_1 : X \times X \to X$, then tensor it by the above diagram and push it forward to X via the projection $p_2 : X \times X \to X$. We will obtain the following diagram in $\mathcal{D}_{\mathrm{qc}}(X)$.

$$\begin{array}{ccccccccc} 0 = p_{2*}(T_m \otimes p_1^*F) & \to & p_{2*}(T_{m-1} \otimes p_1^*F) & \to \cdots \to & p_{2*}(T_1 \otimes p_1^*F) & \to & p_{2*}(T_0 \otimes p_1^*F) = p_{2*}(\Delta_*\mathcal{O}_X \otimes p_1^*F) \\ \nwarrow & \swarrow & & & \nwarrow & \swarrow & \\ & p_{2*}(K_m \otimes p_1^*F) & & \cdots & & p_{2*}(K_1 \otimes p_1^*F) & \end{array}$$

Note that, by Lemma 4.5, we have $K_i \otimes p_1^*F \in \hat{\mathcal{A}}_{iX}$; hence, $p_{2*}(K_i \otimes p_1^*F) \in \hat{\mathcal{A}}_i$, by Proposition 5.3. On the other hand, $p_{2*}(\Delta_*\mathcal{O}_X \otimes p_1^*F) \cong F$, so we conclude that $p_{2*}(K_i \otimes p_1^*F) \cong \hat{\alpha}_i(F)$. Restricting to $\mathcal{D}^b(X)$ and using Lemma 3.1, we obtain an isomorphism $\Phi_{K_i} \cong \alpha_i$ on $\mathcal{D}^b(X)$. □

This theorem has a relative variant.

Theorem 7.3. *Assume that $f : X \to S$ is a morphism of quasiprojective varieties and let $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ be an S-linear strong semiorthogonal decomposition. Denote the corresponding projection functors by $\alpha_i : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$. Assume that the map f is a faithful base change for itself and each α_i has finite cohomological amplitude. Then for every i there is an object $K_i \in \mathcal{D}^b(X \times_S X)$ such that $\alpha_i \cong \Phi_{K_i}$.*

The proof is analogous. We consider the induced semiorthogonal decomposition of $\mathcal{D}^b(X \times_S X)$ and consider the decomposition of $\Delta_*\mathcal{O}_X$, where this time Δ denotes the diagonal embedding into the fiber product $\Delta : X \to X \times_S X$.

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