Base Station Location Optimization for Minimal Energy Consumption in Wireless Networks

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***Abstract*—This paper studies the combined problem of base station location and optimal power allocation, in order to optimize the energy efficiency of a cellular wireless network. Recent work has suggested that moving from a network of a small number of high power macrocells to a larger number of smaller microcells may improve the energy efficiency of the network. This paper investigates techniques to optimize the number of base stations and their locations, in order to minimize energy consumption. An important contribution of the paper is that it takes into account non-uniform user distributions across the coverage area, which is likely to be encountered in practice. The problem is solved using approaches from optimization theory that deal with the facility location problem. Stochastic programming techniques are used to deal with the expected user distributions. An example scenario is presented to illustrate how the technique works and the potential performance gains that can be achieved.**

I. Introduction

There is currently great interest in energy efficient wireless communications systems from the cellular industry. The strong focus on improving data rates for broadband wireless access in third generation mobile systems has led to a situation where cellular base stations consume a significant proportion of the total energy budget for telecommunication networks. Figures from cellular operators and from other companies, e.g. [1], [2],[3] [4], suggest that they are the biggest sources of energy consumption in current networks. A typical UK-based network consumes $40-50$ MW [5], which is not a big portion of the total UK energy budget. However, data volumes in wireless networks are predicted to grow between $100-1000$ times in the next ten years, which would lead to enormous increases in energy budgets and costs. Therefore, the issue of how to create energy efficient "green radio" networks is becoming one of the most important topics in shaping future wireless networks. More specifically, how to reduce the energy consumption of the base stations in the network is of primary interest.

There are a variety of short term measures that can be implemented to improve energy efficiency. However, in the long term cellular operators need to understand how to make their networks as energy efficient as possible, investigating many different potential network architectures. For some time, smaller cell concepts such as microcells or picocells which use lower power transmissions and smaller cell regions than traditional macro-cells have been studied. Recently, the concept of femtocells has also become practical [6]. Unlike a conventional cell in which a base station normally serves more than one hundred users, a femtocell base station normally serves a few users in a home or office environment. Apart from offering better quality of service (QoS), a microcell or femtocell might have a potential advantage of reducing the energy consumption in the network since the base stations are much closer to the users in this scenario.

However, employing a microcell or femtocell requires installing many more base stations. The sources of base station energy consumption can be broadly divided into two major parts (see section 2.4 in [1]): (1) The generation of radio frequency (RF) signals, especially the power amplifier, and (2) central equipment. Between these two sources, the energy consumption by RF generation plays a major role and counts for $60\%-75\%$ of the total energy consumption, and can be effectively reduced by decreasing the transceiver power. Upgrading the hardware (such as materials) can reduce the energy consumption of the central equipment [4]. Nevertheless, when evaluating the energy efficiency of microcells or femtocells, there is an tradeoff between the savings in transceiver energy and the additional energy costs for the extra central equipment required in the larger number of base stations. In order to find the optimal tradeoff and minimize the total energy consumption, one needs to carefully consider the number and locations of base stations, given different users' distributions and densities in different parts of the network.

This paper focuses on developing optimization tools that can effectively determine the number and locations of the base stations for minimizing the overall long term energy consumption of the network, while ensuring that a minimal level of QoS in the network is always satisfied. We consider time-division-multiple-access (TDMA) and a simple base station cooperation protocol. It is shown that the optimization tool is a combination of facility location optimization and stochastic programming.

Past works on optimization of base stations or connecting points have focused more on coverage, transmission efficiency or user demands [9], [10], [11], [12]. Furthermore, many of these works focus purely on upper layer instead of physical layer and medium access control issues. Some of the previous works have considered only static networks instead of mobile networks [7], [8]. In [13], numerical results were given to reveal the relationship between the number of micro-cells and network energy consumption, but only a uniform user distribution was assumed and there was no optimization on the specific location of the micro-cells. To the best of the authors' knowledge, there has been no work focusing on optimizing the base station pattern for minimal physical layer energy consumption of cellular networks considering non-uniform user distributions.

The remainder of this paper is organised as follows. Section II describes the system model used in this paper. Section III describes the algorithms and optimization tools that are used in this paper to solve the power consumption problem described above. Section IV then presents example results that show the validity of the methods proposed here. Finally, Section V presents conclusions to the paper.

II. System Model

Consider a conventional urban or rural cellular network, in which all mobile users are served by a base station. The problem to be solved is to place several (smaller) base stations with much lower transceiver power budget in the same region to reduce the overall use of energy. This can be achieved by allowing collaboration among the new base stations to serve the users cooperatively. More specifically, we assume a TDMA Protocol in which one user is served in each time slot. Therefore, we assume no interference among the users. We consider only the power consumption in downlink transmission, since this dominates the RF power consumption of a typical base station. For simplicity, we ignore the small scale fading (e.g., Rayleigh fading) or lognormal shadowing terms and consider only the pathloss terms when modeling the physical channel. However, these effects can easily be included without changing the operation of the algorithms in this paper. We assume that the base station knows the forward channel state information (CSI) so that the transmit powers are optimally allocated among the base stations when serving a user.

A. Power and QoS Constraint

We assume that on average there are N users in the region and we plan to locate M base stations. The receive power at each user i ($i = 1, ..., N$) is required to meet the minimal value P_i, which depends on the QoS requirements of the user. In this paper we define the minimal QoS for each user i as the minimal data rate V_i required by that user. Assume the receive noise at the receiver has unit variance. Under the assumption that each link can operate at a data rate approaching the Shannon bound on capacity, the power P_i needs to meet the following inequality:

$$\frac{1}{N} \log\left(1 + P_i\right) \geq V_i. \qquad (1)$$

The division by N denotes the fact that the bandwidth is shared equally among the N users according to the TDMA protocol. The base station j transmits at a power p_{ij} to user i. On assuming that the path loss between base station j and user i is D_{ij}, the receive power at user i can be expressed as

$$P_i = \sum_{j=1}^{M} p_{ij} D_{ij}. \qquad (2)$$

Each base station has a transmission power constraint

$$p_{ij} \leq P_{up}, \qquad (3)$$

due to power amplifier limitations.

Apart from transmitting the message, each base station will need additional power for communicating back to the core network, commonly termed the backhaul. We assume the power required for this operation of the base station is a constant P_c, unlike its variable RF transmit power.

B. User Distribution

We divided the region into R smaller areas. We then assume that the calling users in each area are uniformly distributed. Since we care about the long term effect of the system, we consider samples across large time scales (e.g., days) instead of small time scales (e.g., minutes). Thus we assume there is no correlation among the samples. We pick L different times (t_1, t_2,..,t_L) of a day. In each area r ($r = 1, .., R$), we make an observation of the user pattern, denoted by $O_q^{t_l,r}$, at time t_l ($l = 1, .., L$) in each day q ($q = 1, .., Q$). We assume that the number of users in area r observed at time t_l in day q, denoted by $n_q^{t_l,r}$, follows a uniform distribution between $N_{t_l,r}^{low}$ and $N_{t_l,r}^{high}$ across day q. The user patterns $O_q^{t_l,r}$ are assumed to be independent across q, t_l and r.

C. Problem Formulation

The problem is to find an optimal number of base stations and the position of each base station in this region so that the long term total energy consumption is minimized. This problem is similar to the facility location problem [14]. The facility location problem has been studied for several years in the context of location decision-making. The problem is understood as finding the optimal way of allocating facilities that minimizes the cost of installation ($i.e$, fixed cost) and the unitary cost of transporting "products" from these facilities to each customer ($i.e$, variable costs). Therefore, the relation with the power efficiency problem studied in this paper is straightforward.

Finding the optimal points for the locations of the base stations is extremely complex. A suboptimal but much simplified approach is to predefine the M potential locations and to choose a subset of these locations to build base stations such that the total power consumption of the network is minimized. Following this approach, we define M binary variables x_j ($j = 1, ..., M$). A base station j will be deployed if $x_j = 1$, otherwise $x_j = 0$. Since a base station will only consume energy only if it is deployed, we introduce the following constraint:

$$p_{ij} \leq P_{up} x_j. \tag{4}$$

For simplicity we assume that the QoS requirements for all users are the same, i.e., $V_i \equiv V$ for $i = 1, ..., N$. Combining (1) and (2) and changing the measures to a linear form, we obtain:

$$\sum_{j=1}^{M} \frac{p_{ij}}{L_{ij}} \geq P, \tag{5}$$

where L_{ij} represents the distance attenuation with $i = 1, ..., N$ and $j = 1, ..., M$. Thus, the optimization problem for a particular pattern $O_q^{t_l,r}$ can be presented in the following mathematical form:

$$\begin{aligned}
\min \quad & P_c \sum_{j=1}^{M} x_j + \sum_{i=1}^{N} \sum_{j=1}^{M} p_{ij} && (6)\\
\text{s.t.} \quad & p_{ij} \leq P_{up} x_j, \ \forall i \in \Omega_N, \forall j \in \Omega_M && (7)\\
& \sum_{j=1}^{M} \frac{p_{ij}}{L_{ij}} \geq P, \ \forall i \in \Omega_N && (8)\\
& x_j \in \{0, 1\}, \ \forall j \in \Omega_M && (9)\\
& p_{ij} \geq 0, \ \forall i \in \Omega_N, \forall j \in \Omega_M && (10)
\end{aligned}$$

where, $\Omega_M = \{1, .., M\}$ and $\Omega_N = \{1, .., N\}$

The approach presented on this paper is to determine the base station allocation that minimizes the expected power consumption over some user patterns $O_q^{t_l,r}$ ($q = 1, .., Q$ and $l = 1, ..., L$) for a given area r ($r = 1, .., R$). Our approach is to average the power values over K distinct user location scenarios, which is also termed a stochastic model under certainty. The result is as follows:

$$\begin{aligned}
\min \quad & \sum_{k=1}^{K} \pi^k \left(P_c \sum_{j=1}^{M} x_j + \sum_{i=1}^{N_k} \sum_{j=1}^{M} p_{ij}^k \right) && (11)\\
\text{s.t.} \quad & \sum_{j=1}^{M} \frac{p_{ij}^k}{L_{ij}^k} \geq P^k, \quad \forall k \in \Omega_K, \forall i \in \Omega_{Nk} && (12)\\
& p_{ij}^k \leq P_{up} x_j, \quad \forall k \in \Omega_K, \forall i \in \Omega_{Nk}, \forall j \in \Omega_M && (13)\\
& x_j \in \{0, 1\}, \quad \forall j \in \Omega_M && (14)\\
& p_{ij}^k \geq 0, \quad \forall k \in \Omega_K, \forall i \in \Omega_{Nk}, \forall j \in \Omega_M && (15)
\end{aligned}$$

where K is the number of scenarios ($i.e$, patterns) considered in the analysis for a given area r and $\Omega_K = \{1, .., K\}$. N_k represents the number of users at scenario k and therefore, P^k also depends on k. At each scenario we observe a particular user distribution within the region. Hence, we can represent the distance attenuation between user i and base station j as L_{ij}^k. Since we would like to serve each user at least at a minimum QoS rate, p_{ij}^k will also depend on each scenario. Notice that x_j will remain the same over all scenarios so we will get a long term solution for the base station allocation but a particular one (recourse action) for the transmitting policy. The scalar π^k is the probability of the occurrence of scenario k. The practical behaviour of any numerical algorithm strongly depends on the conditioning of the problem. We have observed in particular that although constraint (13) was expressed in dB scale it was always very badly scaled and this challenged the optimization algorithm. We have adopted an ad-hoc scaling of this constraint in order to reduce the magnitude of the coefficients which appear in it. Namely, we divided this constraint by a factor of 10^{10}. This has not changed the mathematical model but it has improved the performance of the algorithm and the accuracy of optimal solution.

III. Algorithms and Optimization Tools

There are several ways to solve facility location problems [7], [14]. Since we formulate the power minimization problem using a Mixed Integer Programming (MIP) model, we have decided to use the simplex method and the branch and bound algorithm. To do so, we model this problem in software called *FICO™Xpress Optimization Suite 7* [15].

The basic methodology proposed to solve the power minimization problem is characterized in the following steps:

1) Reduce the complexity of the problem by choosing M possible locations within a given region.
2) Define K scenarios with their corresponding values of P^k, L_{ij}^k, and π^k.
3) Solve the problem using any suitable algorithm. In this study we suggest the combined use of the simplex method and the branch and bound algorithm.

An extension of this approach is to re-optimize the base station allocation. This can be performed in the following way. Once we have solved the stochastic problem, we can classify all the users served by each base station in a group. Then, we can calculate the weighted center of gravity among the users of each group and relocate the base stations accordingly. This problem can be solved either using a quadratic model or a linear model, with continuous variables representing the new position of each base station. We may consider the weight as the probability of occurrence of each scenario. This extension is useful only if we have the freedom of placing a base station wherever we want. One other possible extension is that once we have decided the location of the base stations, we can calculate the minimum P_{up} needed for each base station to satisfy the requirements of the users. Doing this we just change the fixed power consumption associated with the central equipment.

IV. Numerical Results

In this section we describe the experimental tests performed to check whether using lower power base stations is a competitive option. A conventional approach is to consider a single base station located in the center of the region. We assume that this base station has enough power to serve all user requirements in the region.

A. The size of the cell and distance attenuation factor

Following [6] and choosing a cell radius equal to 167m, we define a square region of 282.42m each side. Then, we divide this region into 9 smaller squares of equal areas ($i.e., R = 9$). By predefining the total number of users in each area, we generate uniformly distributed random numbers for $x \in [0, 282.42]$ and $y \in [0, 282.42]$. This allows us to position each user and therefore, to calculate the distance, d_{ij}, between each user i and each base station j. Hence, we can calculate the distance attenuation, in a linear form, as $L_{ij} = 10^{3.53} d_{ij}^{3.76}$, where d_{ij} is expressed in meters. For simplicity, we define 13 possible base station locations as is shown in Fig. 1.

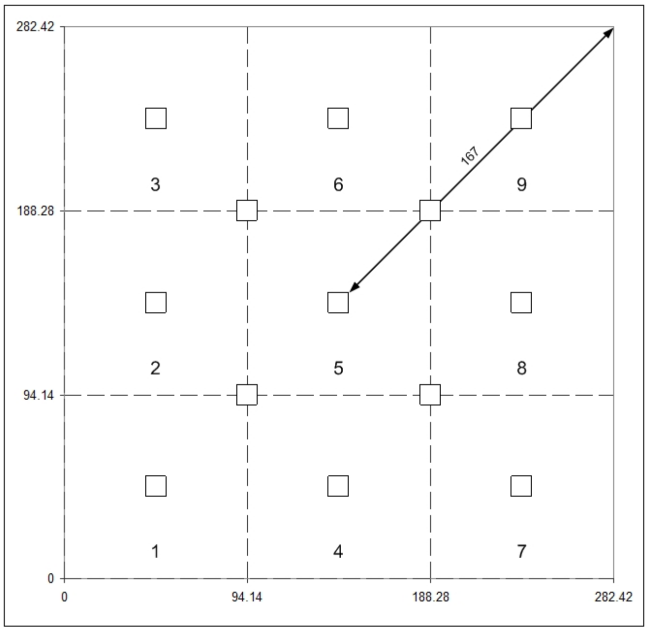

Fig. 1. Predefined locations

B. The values of V and P

In [6], the minimal downlink transmission rate for the emerging Long Term Evolution standard is 1.5bits/s/Hz regardless of the number of users. We assume there are no more than 10 users in a sector and that the minimal QoS for each user is $V = 1.5/10 = 0.15$ bits/s/hz. Employing the capacity equation (1), we will get $P \geq 13.3$ dB if we consider 30 users in the region.

Since the value of V is determined now we allowed ourselves to extend the program from the previously fixed number of users per scenario to a variable number of users per scenario. For example, we can assume that the total number of users in the region is not always 30 but varies uniformly between 20 and 40. In each sector, the number of users changes as indicated in section II.B.

C. The value of P_{up} and P_c

As now mentioned in the introduction, P_c can be thought of as the power consumption for central equipment while P_{up} largely depends on the power amplifier. From information in section 2.4 of [1], we can observe that the $800W$ power supply for central equipment corresponds to a $2400W$ power supply for the RF equipment (mainly the power amplifier). Thus it is reasonable to assume that $P_{up} = 3 \times P_c$. Since we use lower power base stations, each base station is expected to serve only a few users. Using a uniform distribution of the users, it is reasonable to assume that the coverage of a lower power base station (e.g. for a microcell) is around $1/5 - 1/10$ of that of the conventional base station. To keep the same received SNR and assuming a path loss exponent $r = 3$, it is reasonable to scale the power limit of a microcell base station to around $0.1\% - 1\%$ of the power limit of the conventional base station. The value of P_c can be determined accordingly. In our tests we use $P_{up} = 143$ dB for the conventional base station analysis and we apply a scale factor of 1% to the microcell base station power limit.

Note that these values take into account only the power consumption levels. Other costs such as rental, hardware/software and operation cost are not considered. Note that those can be taken into account as fixed costs and thus if we want to consider them we could add a scaled factor in front of P_c. Our focus is on energy efficiency, therefore we intentionally neglected these additional costs.

D. Comparison between conventional base station and microcell base stations

To illustrate the reductions in the energy consumption that can be achieved using the approach presented in this paper, alternative policies using different scenarios with different probabilities of occurrence have been tested.

In TABLE I we provide details of the user distributions considering 30 users within the region. Notice that **sc2**, **sc7** and **sc8** exploit particular distributions, concentrating users in a few selected areas.

In TABLE II, the first four samples are pure scenarios considering just one user distribution. For samples 5 to 9, we used the stochastic programming approach varying the probability of occurrence of each scenario. The column denoted by **cbs** shows the optimal solution in dB when we use a conventional base station and column **fbs** when we use the stochastic approach with microcell base stations. To see an example of an optimal allocation using microcell base stations, refer to Fig. 2. Column Δ shows the variable power needed using a microcell configuration as a percentage of the variable power used with the conventional base station approach. Even for scenarios in which we concentrate users in $r = 5$ (**sc7**) we achieve considerable reductions in the power consumption for RF generation.

A similar test was performed but now considering $20 \leq N \leq 40$. The number of users per area is illustrated in TABLE III. In these trials we did not choose any particular distribution

TABLE I
DISTRIBUTION OF USERS AMONG THE REGION ($N = 30$)

	number of users			
region	sc2	sc5	sc7	sc8
1	0	5	0	15
2	0	2	0	5
3	0	3	0	0
4	0	3	0	5
5	1	5	30	5
6	0	2	0	0
7	0	2	0	0
8	29	3	0	0
9	0	5	0	0

TABLE II
OPTIMAL POWER REQUIREMENTS FOR DIFFERENT SAMPLES ($N = 30$)

	probability of occurrence (π^k)						
sample	sc2	sc5	sc7	sc8	cbs (dB)	fbs (dB)	Δ
1	1	0	0	0	141.7	126.0	2.5%
2	0	1	0	0	143.8	129.1	1.2%
3	0	0	1	0	138.4	125.6	29.1%
4	0	0	0	1	143.2	127.4	1.9%
5	0.25	0.25	0.25	0.25	142.2	129.1	2.8%
6	0.7	0.1	0.1	0.1	141.9	128.9	3.4%
7	0.1	0.7	0.1	0.1	143.2	129.1	2.0%
8	0.1	0.1	0.7	0.1	140.4	128.8	6.9%
9	0.1	0.1	0.1	0.7	142.9	129.1	2.3%

as we did in previous test. In TABLE IV we compare the optimal allocation given both microcell and conventional base station configurations. The stochastic programming approach gives a reduction of more than 96% of the power consumption on radio frequency generation in all cases.

TABLE III
DISTRIBUTION OF USERS AMONG THE REGION ($20 \leq N \leq 40$)

	number of users				
region	sc20	sc25	sc30	sc35	sc40
1	0	4	5	3	3
2	0	1	0	4	5
3	1	5	5	3	2
4	3	3	3	2	4
5	4	2	3	5	5
6	2	1	5	1	7
7	3	3	2	5	6
8	4	5	2	4	4
9	3	1	5	8	4

V. CONCLUSIONS

In order to make cellular wireless networks more energy efficient and sustainable for the future, it is very important to minimize the power required to cover users in a given area. One potential approach to reducing power consumption is to study the trade-off between power consumption and the number of base stations to cover a given area. The optimization criterion includes both terms that are proportional to radio

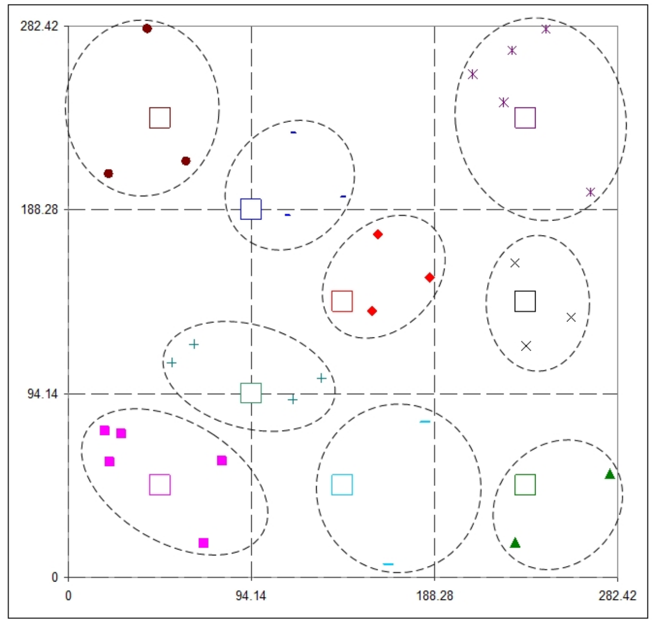

Fig. 2. Optimal allocation for pure sc5 (*i.e.*, sample 2)

TABLE IV
OPTIMAL POWER REQUIREMENTS FOR DIFFERENT SAMPLES
($20 \leq N \leq 40$)

	probability of occurrence (π^k)							
sample	sc20	sc25	sc30	sc35	sc40	cbs (dB)	fbs (dB)	Δ
1	1	0	0	0	0	139.8	126.5	3.8%
2	0	1	0	0	0	142.0	128.0	2.6%
3	0	0	1	0	0	143.5	128.5	2.0%
4	0	0	0	1	0	146.5	130.5	1.2%
5	0	0	0	0	1	148.6	131.9	1.3%
6	0.2	0.2	0.2	0.2	0.2	145.2	130.1	1.4%
7	0.1	0.2	0.4	0.2	0.1	144.7	129.9	1.6%
8	0.05	0.1	0.7	0.1	0.05	144.1	129.6	1.6%
9	0.3	0.3	0.3	0.05	0.05	143.1	129.2	1.6%
10	0.05	0.05	0.3	0.3	0.3	146.4	130.7	1.4%
11	0.3	0.15	0.1	0.15	0.3	145.6	130.3	1.4%

frequency power used by the base stations and also constant power terms that reflect additional power consumption required by base stations, e.g. for communication with the core network of the service provider. This criterion provides a powerful and quite general approach to minimize the energy consumption of the network.

The use of a stochastic programming approach using mixed integer programming to model and solve the base station location problem from a power efficiency perspective has been demonstrated in this paper. A key feature of the proposed scheme is that it can provide an optimal allocation considering different user patterns. This allows the algorithm to take account of different traffic distributions in the coverage area at different times of day and even to account for traffic growth projections for the network going into the future. The algorithm selects the optimal base station locations from a finite set of possible choices specified by the algorithm operator. This is a reasonable reflection of reality given that network operators may have access only to certain locations in buildings or streets to deploy their infrastructure.

Results show that the use of this tool can lead to very large power reductions of at least 96% for the example scenario that was considered in the paper. This demonstrates that the proposed technique can find significantly improved base station allocation solutions compared to the baseline case of a macrocell deployment. Some extensions can be included to improve the algorithm performance and this will be the subject of future work in this area.

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REFERENCES

[1] H. Karl, "An overview of energy-efficiency techniques for mobile communication systems," Telecommunication Networks Group, Technical University Berlin, Tech. Rep. TKN-03-XXX, Sept., 2003.

[2] T. Edler and S. Lundberg, "Energy efficiency enhancements in radio access networks," *Ericsson Review*, No. 1, pp. 42-51, 2004

[3] S. Roy, "Energy logic for telecommunications", White paper, Emerson Network Power Ltd., 2008.

[4] S. Vadgama, "Trends in green wireless access," Fujitsu Sci. Tech. J., Vol. 45, No. 4, pp. 404 - 408, Oct., 2009.

[5] "Environment: Progress," Vodafone corporate responsibility report, Mar., 2009, available on $\mathrm{http://www.vodafone.com/static/cr_report09/issues/environment-progress-have.html}$

[6] A. Golaup, M. Mustapha and L.B. Patanapongpibul, "Femtocell access control strategy in UMTS and LTE," *IEEE Commun. Mag.*, vol. 47, no. 9, pp. 117-123, Sept., 2009.

[7] P. V. McGregor and D. Shen, "Network design: An algorithm for the access facility location problem," *IEEE Trans. Commun.*, vol. COM - 25, no. 1, pp. 61-73, Jan. 1977.

[8] Y. Bejerano, "Efficient integration of multihop of multihop wireless and wired networks with QoS constraints," *IEEE/ACM Trans. Networking*, vol. 12, no. 6, pp. 1064-1078, Dec. 2004.

[9] K. Tutschku, "Demand-based radio network planning of cellular mobilecommunication systems," *Proc. INFOCOM '98*, vol. 3, pp. 1054-1061, San Francisco, CA, Mar-Apr, 1998

[10] K. Tutschku and P. Tran-Gia, "Spatial traffic estimation and characterization for mobilecommunication network design", *IEEE J. Sel. Areas Commun.* vol. 16, no. 5, pp. 804-811, Jun. 1998.

[11] J. Robinson, M. Uysal, R. Swaminathan, E. Knightly, "Adding capacity points to a wireless mesh network using local search," *Proc. INFOCOM 2008*, pp. 1247-1255, Apr. 2008.

[12] A.D. Pragad, V. Friderikos, A.H. Aghvami, "Optimal configuration of mobility agents in broadband wireless access networks," *Proc. IEEE GLOBECOM Workshops 2008*, pp.1-6, Nov-Dec, 2008.

[13] F. Richter, A. J. Fehske and G. Fettweis, "Energy efficiency aspects of base station deployment strategies in cellular networks", *Proc. VTC'09 - Fall*, Anchorage, USA, 20 - 23 Sept., 2009.

[14] R.K. Ahuja and T.L. Magnanti and J.B. Orlin, *Network Flows: Theory, Algorithms, and Applications*, Prentice Hall, 1993.

[15] FICO$^{\mathrm{TM}}$, "FICO$^{\mathrm{TM}}$Xpress Optimization Suite 7," for more information: $\mathrm{http://www.fico.com/en/Products/DMTools/Pages/FICO-Xpress-Optimization-Suite.aspx}$