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BASIC CHARACTERS OF THE UNITRIANGULAR GROUP (FOR ARBITRARY PRIMES)

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Abstract. Let $U_n(q)$ denote the (upper) unitriangular group of degree n over the finite field $\mathbb{F}_q$ with q elements. In this paper we consider the basic (complex) characters of $U_n(q)$ and we prove that every irreducible (complex) character of $U_n(q)$ is a constituent of a unique basic character. This result extends a previous result which was proved by the author under the assumption $p \geq n$, where p is the characteristic of the field $\mathbb{F}_q$.

Let p be a prime number, let $q = p^e$ $(e \geq 1)$ be a power of p and let $\mathbb{F}_q$ denote the finite field with q elements. Throughout this paper, $U_n(q)$ will denote the unitriangular group of degree n over $\mathbb{F}_q$. This group consists of all unipotent uppertriangular $n \times n$ matrices with coefficients in $\mathbb{F}_q$. We clearly have

$$U_n(q) = 1 + \mathfrak{u}_n(q) = \{1 + a \colon a \in \mathfrak{u}_n(q)\}$$

where $\mathfrak{u}_n(q)$ is the $\mathbb{F}_q$-space consisting of all nilpotent uppertriangular $n \times n$ matrices over $\mathbb{F}_q$. Since $\mathfrak{u}_n(q)$ is the Jacobson radical of the finite dimensional $\mathbb{F}_q$-algebra $\mathbb{F}_q \cdot 1 + \mathfrak{u}_n(q)$, the p-group $U_n(q)$ is an $\mathbb{F}_q$-*algebra group* (in the sense of [10]; see also [8]). Moreover, let $\mathfrak{u}_n(q)^*$ denote the dual $\mathbb{F}_q$-space of $\mathfrak{u}_n(q)$.

For simplicity, we write $\Phi(n) = \{(i,j) \colon 1 \leq i < j \leq n\}$ and we refer to an element of $\Phi(n)$ as a *root* (this abbreviates the standard expression "positive root"). For any root $(i,j) \in \Phi(n)$, let e_{ij} be the (i,j)-*th root vector* of $\mathfrak{u}_n(q)$; by definition, $e_{ij} \in \mathfrak{u}_n(q)$ is the $n \times n$ matrix $e_{ij} = (\delta_{ai}\delta_{bj})_{1 \leq a,b \leq n}$ where δ denotes the usual Kronecker symbol. Then $(e_{ij} \colon (i,j) \in \Phi(n))$ is an $\mathbb{F}_q$-basis of $\mathfrak{u}_n(q)$. On the other hand, for each root $(i,j) \in \Phi(n)$, let $e_{ij}^* \in \mathfrak{u}_n(q)^*$ be defined by $e_{ij}^*(a) = a_{ij}$ for all $a \in \mathfrak{u}_n(q)$ (for an arbitrary matrix x, we will denote by x_{ij} the (i,j)-th coefficient of x). Then $(e_{ij}^* \colon (i,j) \in \Phi(n))$ is an $\mathbb{F}_q$-basis of $\mathfrak{u}_n(q)^*$, dual to the basis $(e_{ij} \colon (i,j) \in \Phi(n))$ of $\mathfrak{u}_n(q)$.

Let $\psi \colon \mathbb{F}_q^+ \to \mathbb{C}$ be an arbitrary non-trivial (complex) character of the additive group $\mathbb{F}_q^+$ of the field $\mathbb{F}_q$ (this character will be kept fixed throughout the paper). For any element $f \in \mathfrak{u}_n(q)^*$, let $\psi_f : \mathfrak{u}_n(q) \to \mathbb{C}$ be the function defined by $\psi_f(a) = \psi(f(a))$ for all $a \in \mathfrak{u}_n(q)$; it is clear that this function is a (linear) character of the additive group $\mathfrak{u}_n(q)^+$ of $\mathfrak{u}_n(q)$ and that the mapping $f \mapsto \psi_f$ defines a one-to-one

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correspondence between $\mathfrak{u}_n(q)^*$ and the set of all irreducible characters of $\mathfrak{u}_n(q)^+$. (Throughout the article, all characters are taken over the complex field.)

The group $U_n(q)$ acts on $\mathfrak{u}_n(q)^*$ via the *coadjoint representation*; by definition, for any $x \in U_n(q)$ and any $f \in \mathfrak{u}_n(q)^*$, the (linear) map $x \cdot f \in \mathfrak{u}_n(q)^*$ is defined by $(x \cdot f)(a) = f(x^{-1}ax)$ for all $a \in \mathfrak{u}_n(q)$. Let $\Omega_n(q)$ denote the set of all $U_n(q)$-orbits of $\mathfrak{u}_n(q)^*$ and let $\mathcal{O} \in \Omega_n(q)$ be arbitrary. We claim that the cardinality $|\mathcal{O}|$ of $\mathcal{O}$ is a power of q^2. To see this, we consider an arbitrary finite dimensional $\mathbb{F}_q$-algebra A (with an identity element), we let $J = J(A)$ be the Jacobson radical of A and we consider the $\mathbb{F}_q$-algebra group $G = 1 + J$ which is associated with J (see [10]; see also [8]). Moreover, let $J^* = \hom_{\mathbb{F}_q}(J, \mathbb{F}_q)$ be the dual $\mathbb{F}_q$-space of J and, for any $f \in J^*$, let $\psi_f \colon J \to \mathbb{C}$ be the map defined by $\psi_f(a) = \psi(f(a))$ for all $a \in J$. As in the case where $G = U_n(q)$, the $\mathbb{F}_q$-algebra group $G = 1 + J$ acts on J^* via the *coadjoint representation*: $(x \cdot f)(a) = f(x^{-1}ax)$ for all $x \in G$, all $f \in J^*$ and all $a \in J$. Let $f \in J^*$ be arbitrary and define $B_f \colon J \times J \to \mathbb{F}_q$ by $B_f(a,b) = f([a,b])$ for all $a, b \in J$ (here $[\cdot,\cdot]$ denotes the standard Lie bracket operation). Then B_f is a skew-symmetric $\mathbb{F}_q$-bilinear form. Let

$$R_f = \{a \in J \colon f([a,b]) = 0 \text{ for all } b \in J\}$$

be the radical of B_f. Then $|J : R_f| = q^m$ where $m = \dim J - \dim R_f$ is even. We have the following result (see [5, Proposition 2.1]).

Lemma 1. *Let $f \in J^*$ be arbitrary and let $C_G(f)$ be the centralizer of f in G. Then $C_G(f) = 1 + R_f$ (hence, R_f is a multiplicatively closed $\mathbb{F}_q$-subspace of J). In particular, if $\mathcal{O} \subseteq J^*$ is the G-orbit which contains f, then $|\mathcal{O}| = |J : R_f|$ is a power of q^2.*

Proof. Let $x \in G$ be arbitrary. Then $x \in C_G(f)$ if and only if $f(x^{-1}bx) = f(b)$ for all $b \in J$. Hence $x \in C_G(f)$ if and only if $f(bx) = f(xb)$ for all $b \in J$. Let $a = x - 1 \in J$. Then it is clear that $f(xb) - f(bx) = f([a,b])$ for all $b \in J$, hence $x \in C_G(f)$ if and only if $a \in R_f$. Thus, $C_G(f) = 1 + R_f$ and so $|\mathcal{O}| = |G : C_G(f)| = |J : R_f|$ is a power of q^2. □

With the notation as above, let $\mathcal{O} \subseteq J^*$ be an arbitrary G-orbit and let $\phi_{\mathcal{O}} \colon G \to \mathbb{C}$ be the function defined by

$$\phi_{\mathcal{O}}(1 + a) = \frac{1}{\sqrt{|\mathcal{O}|}} \sum_{f \in \mathcal{O}} \psi_f(a) \tag{1}$$

for all $a \in J$. It is clear that $\phi_{\mathcal{O}}$ is a class function of G and that $\phi_{\mathcal{O}}(1) = \sqrt{|\mathcal{O}|}$. We have the following result (see [5, Proposition 2.2]).

Proposition 1. *Let $\Omega(G)$ be the set of all G-orbits on J^*. Then $\{\phi_{\mathcal{O}} \colon \mathcal{O} \in \Omega(G)\}$ is an orthonormal basis for the $\mathbb{C}$-space* $\mathrm{cf}(G)$ *consisting of all class functions on G. In particular, we have $\langle \phi_{\mathcal{O}}, \phi_{\mathcal{O}'} \rangle_G = \delta_{\mathcal{O},\mathcal{O}'}$ for all $\mathcal{O}, \mathcal{O}' \in \Omega_n(q)$. (For any finite group G, we will denote by $\langle \cdot,\cdot \rangle_G$ the Frobenius scalar product between class functions defined on G.)*

Proof. Let $\mathcal{O}, \mathcal{O}' \in \Omega(G)$ be arbitrary. Then, since $|G| = |J|$, we easily deduce that

$$\langle \phi_{\mathcal{O}}, \phi_{\mathcal{O}'} \rangle_G = \frac{1}{\sqrt{|\mathcal{O}|}\sqrt{|\mathcal{O}'|}} \sum_{f \in \mathcal{O}} \sum_{f' \in \mathcal{O}'} \langle \psi_f, \psi_{f'} \rangle_{J^+}$$

where J^+ denotes the (abelian) additive group of J. Now, the mapping $f \mapsto \psi_f$ defines a one-to-one correspondence between J^* and the set of all irreducible characters of J^+. Therefore, we obtain $\langle \phi_{\mathcal{O}}, \phi_{\mathcal{O}'} \rangle_G = \delta_{\mathcal{O},\mathcal{O}'}$ as required. To conclude the proof, we claim that $|\Omega(G)|$ equals the class number k_G of G (we recall that $k_G = \dim_{\mathbb{C}} \mathrm{cf}(G)$; see, for example, [9, Corollary 2.7 and Theorem 2.8]). First, we observe that k_G is the number of G-orbits on J for the *adjoint action*: $x \cdot a = xax^{-1}$ for all $x \in G$ and all $a \in J$. Let θ be the permutation character of G on J (see [9] for the definition). Then, by [9, Corollary 5.15], $k_G = \langle \theta, 1_G \rangle_G$. Moreover, by definition, we have $\theta(x) = |\{a \in J \colon x \cdot a = a\}|$ for all $x \in G$. On the other hand, let $\mathrm{Irr}(J^+)$ denote the set consisting of all irreducible characters of J^+ and consider the action of G on $\mathrm{Irr}(J^+)$ given by $x \cdot \psi_f = \psi_{x \cdot f}$ for all $x \in G$ and all $f \in J^*$. We clearly have $(x \cdot \psi_f)(x \cdot a) = \psi_f(a)$ for all $x \in G$, all $f \in J^*$ and all $a \in J$. It follows from Brauer's Theorem (see [9, Theorem 6.32]) that $\theta(x) = |\{f \in J^* \colon x \cdot \psi_f = \psi_f\}|$ for all $x \in G$. Therefore, θ is also the permutation character of G on $\mathrm{Irr}(J^+)$ and so $\langle \theta, 1_G \rangle_G = |\Omega(G)|$. The claim follows and the proof is complete. □

In general, the class functions $\phi_{\mathcal{O}}$, for $\mathcal{O} \in \Omega(G)$, are not characters (see [11]). However, in the case where $G = U_n(q)$, there are some (important) examples where they are, in fact, (irreducible) characters of $U_n(q)$. A particular (and very special) family consists of the *elementary characters* of $U_n(q)$ which are defined as follows (see [1] for an equivalent definition in the case where $p \geq n$). Let $(i,j) \in \Phi(n)$ be any root and let $\alpha \in \mathbb{F}_q$ be any non-zero element. (Throughout the paper, we will denote by $\mathbb{F}_q^{\#}$ the subset of $\mathbb{F}_q$ consisting of all non-zero elements.) Let $\mathcal{O}_{ij}(\alpha) \in \Omega_n(q)$ be the $U_n(q)$-orbit which contains the element $\alpha e_{ij}^* \in \mathfrak{u}_n(q)^*$ and let $\xi_{ij}(\alpha)$ denote the class function $\phi_{\mathcal{O}_{ij}(\alpha)}$ which corresponds to $\mathcal{O}_{ij}(\alpha)$. We shall see that this class function is, in fact, a character (hence, an irreducible character) of $U_n(q)$ and this will follow once we prove that $\xi_{ij}(\alpha)$ is induced from a character (in fact, from a linear character) of a certain subgroup of $U_n(q)$. We start by proving an auxiliary general result (see Proposition 2 below).

Let A, $J = J(A)$ and $G = 1 + J$ be as before. Let H be a subgroup of G and suppose that there exists an $\mathbb{F}_q$-subspace U of J such that $H = 1+U$; following [10], we refer to such a subgroup as an *algebra subgroup* of G. Then U is multiplicatively closed (because H is a subgroup) and, in fact, U is the Jacobson radical of the $\mathbb{F}_q$-algebra $\mathbb{F}_q \cdot 1 + U$. Thus, H is an $\mathbb{F}_q$-algebra group and so the set $\Omega(H)$ of coadjoint H-orbits and the class functions $\phi_{\mathcal{O}_0}$, for $\mathcal{O}_0 \in \Omega(H)$, are defined as in the case of G. Let $\pi \colon J^* \to U^*$ be the natural projection; by definition, for any $f \in J^*$, $\pi(f) \in U^*$ is the restriction of f to U. Then, for each $\mathcal{O} \in \Omega(G)$, the image $\pi(\mathcal{O}) \subseteq U^*$ is clearly H-invariant, hence it is a disjoint union of H-orbits; we will denote by $\Omega_{\mathcal{O}}(H)$ the set of all $\mathcal{O}_0 \in \Omega(H)$ such that $\mathcal{O}_0 \subseteq \pi(\mathcal{O})$. We have the following result (a more detailed discussion can be found in the expository paper [5]).

Proposition 2. *Let G be an arbitrary (finite) $\mathbb{F}_q$-algebra group and let H be an algebra subgroup of G. Let $\mathcal{O} \in \Omega(G)$ and let ϕ denote the class function $\phi_{\mathcal{O}} \in \mathrm{cf}(G)$. Then*

$$\phi_H = \sum_{\mathcal{O}_0 \in \Omega_{\mathcal{O}}(H)} n_{\mathcal{O}_0} \phi_{\mathcal{O}_0} \tag{2}$$

where, for each $\mathcal{O}_0 \in \Omega_{\mathcal{O}}(H)$, the multiplicity $n_{\mathcal{O}_0} = \langle \phi_H, \phi_{\mathcal{O}_0} \rangle_H$ is a positive integer.

Proof. By Proposition 1, we know that ϕ_H is a $\mathbb{C}$-linear combination of the class functions $\phi_{\mathcal{O}_0}$ for $\mathcal{O}_0 \in \Omega(H)$. Let $\mathcal{O}_0 \in \Omega(H)$ be arbitrary and let $\phi_0 = \phi_{\mathcal{O}_0}$. Then from the definitions it is easy to deduce that

$$\begin{aligned}\langle \phi_H, \phi_0 \rangle_H &= \frac{1}{\sqrt{|\mathcal{O}|}\sqrt{|\mathcal{O}_0|}} \sum_{f \in \mathcal{O}} \sum_{f_0 \in \mathcal{O}_0} \langle \psi_{\pi(f)}, \psi_{f_0} \rangle_{U^+} \\ &= \frac{1}{\sqrt{|\mathcal{O}|}\sqrt{|\mathcal{O}_0|}} \sum_{f_0 \in \mathcal{O}_0} |\mathcal{O} \cap \pi^{-1}(f_0)|\end{aligned}$$

where U, J and $\pi\colon J^* \to U^*$ are as above. Let $f_0 \in \mathcal{O}_0$ be arbitrary. Then, since $\pi^{-1}(x \cdot f_0) = x \cdot \pi^{-1}(f_0)$ for all $x \in H$, we conclude that

$$\langle \phi_H, \phi_0 \rangle_H = \frac{\sqrt{|\mathcal{O}_0|}|\mathcal{O} \cap \pi^{-1}(f_0)|}{\sqrt{|\mathcal{O}|}}. \tag{3}$$

It follows that $\langle \phi_H, \phi_0 \rangle_H \neq 0$ if and only if $f_0 \in \pi(\mathcal{O})$ and, since $\pi(\mathcal{O}) \subseteq U^*$ is H-invariant, this is equivalent to saying that $\mathcal{O}_0 \subseteq \pi(\mathcal{O})$. Therefore, we obtain the linear combination (2) with non-zero rational coefficients. In order to prove that these coefficients are integers, we proceed by induction on $|G : H|$.

First, let us assume that $|G : H| = q$ (we note that $|G : H| = |J : U|$ is always a power of q). In this case, U is a maximal $\mathbb{F}_q$-subspace of J (in fact, $\dim U = \dim J - 1$) and we have $J^2 \subseteq U$ (otherwise, $U + J^2 = J$ and this implies that $U = J$; see [10, Lemma 3.1]). It follows that U is an ideal of J and so $H = 1 + U$ is a normal subgroup of G. Therefore, all the H-orbits in $\Omega_{\mathcal{O}}(H)$ have equal cardinality and, since $|G : H| = q$, we must have $|\Omega_{\mathcal{O}}(H)| \leq q$. Moreover, G acts transitively on $\pi(\mathcal{O})$ and so, given any $f_0 \in \pi(\mathcal{O})$, we conclude that

$$|\mathcal{O} \cap \pi^{-1}(x \cdot f_0)| = |\mathcal{O} \cap \pi^{-1}(f_0)|$$

for all $x \in G$ (because $\pi^{-1}(x \cdot f_0) = x \cdot \pi^{-1}(f_0)$ for all $x \in G$). Let $\mathcal{O}_0 \in \Omega_{\mathcal{O}}(H)$ and $f_0 \in \mathcal{O}_0$ be arbitrary. Then, by (2) and (3), we deduce that $\sqrt{|\mathcal{O}|}\phi(1) = |\Omega_{\mathcal{O}}(H)||\mathcal{O} \cap \pi^{-1}(f_0)|\sqrt{|\mathcal{O}_0|}\phi_{\mathcal{O}_0}(1)$ and so

$$|\mathcal{O}| = |\Omega_{\mathcal{O}}(H)||\mathcal{O} \cap \pi^{-1}(f_0)||\mathcal{O}_0|$$

(because $\phi(1) = \sqrt{|\mathcal{O}|}$ and $\phi_{\mathcal{O}_0}(1) = \sqrt{|\mathcal{O}_0|}$). Since $|\mathcal{O}|$ and $|\mathcal{O}_0|$ are powers of q^2 and since $|\mathcal{O} \cap \pi^{-1}(f_0)| \leq q$, we conclude that either $|\Omega_{\mathcal{O}}(H)| = |\mathcal{O} \cap \pi^{-1}(f_0)| = 1$ (hence, $|\mathcal{O}| = |\mathcal{O}_0|$) or $|\Omega_{\mathcal{O}}(H)| = |\mathcal{O} \cap \pi^{-1}(f_0)| = q$ (hence, $|\mathcal{O}| = q^2|\mathcal{O}_0|$). In both cases, (3) implies that $\langle \phi_H, \phi_{\mathcal{O}_0} \rangle_H = 1$ and this completes the proof in the case where $|G : H| = q$.

Now, assume that $|G : H| > q$ and let V be an $\mathbb{F}_q$-subspace of J containing U and such that $\dim J = \dim V + 1$. Then we have $J^2 \subseteq V$ (by [10, Lemma 3.1]) and so V is multiplicatively closed. Therefore, $K = 1 + V$ is an algebra subgroup of G and $|G : K| = |J : V| = q$. By the first step of the induction, we know that $\phi_K = \phi_{\mathcal{O}_1} + \cdots + \phi_{\mathcal{O}_k}$ where either $k = 1$ or $k = q$, and where $\mathcal{O}_1, \ldots, \mathcal{O}_k$ are all the distinct K-orbits in $\Omega_{\mathcal{O}}(K)$. Let $\mathcal{O}_0 \in \Omega_{\mathcal{O}}(H)$ be arbitrary and let $\phi_0 = \phi_{\mathcal{O}_0}$. Then $\langle \phi_H, \phi_0 \rangle_H = \langle (\phi_{\mathcal{O}_1})_H, \phi_0 \rangle_H + \cdots + \langle (\phi_{\mathcal{O}_k})_H, \phi_0 \rangle_H$ and the result follows immediately (by induction, because $|K : H| = q^{-1}|G : H| < |G : H|$). □

Now, we can prove the following result. (A different approach, using Clifford's theory (see, for example, [7, Theorems 11.5 and 11.8]), can be found in the paper [12]).

Lemma 2. *Let $(i, j) \in \Phi(n)$ and let $\alpha \in \mathbb{F}_q^{\#}$. Then the class function $\xi_{ij}(\alpha)$ is an irreducible character of $U_n(q)$. Moreover, let $U_{ij}(q)$ be the subgroup of $U_n(q)$*

consisting of all matrices $x \in U_n(q)$ *which satisfy* $x_{ik} = 0$ *for all* $i < k < j$, *and let* $\lambda_{ij}(\alpha)\colon U_{ij}(q) \to \mathbb{C}$ *be the function defined by* $\lambda_{ij}(\alpha)(x) = \psi(\alpha x_{ij})$ *for all* $x \in U_{ij}(q)$. *Then* $\lambda_{ij}(\alpha)$ *is a linear character of* $U_{ij}(q)$ *and* $\xi_{ij}(\alpha) = \lambda_{ij}(\alpha)^{U_n(q)}$ *is induced by this linear character.*

Proof. By Proposition 1, we know that $\langle \xi_{ij}(\alpha), \xi_{ij}(\alpha)\rangle_{U_n(q)} = 1$. Therefore, to prove that $\xi_{ij}(\alpha)$ is an irreducible character of $U_n(q)$ it is enough to show that it is, in fact, a character (we note that $\xi_{ij}(\alpha)(1) = \sqrt{|\mathcal{O}_{ij}(\alpha)|}$ is a positive integer) and this will follow by the second part of the lemma. The first assertion is clear: since ψ is a linear character of $\mathbb{F}_q^+$, the function $\lambda_{ij}(\alpha)\colon U_{ij}(q) \to \mathbb{C}^\#$ is a homomorphism of (multiplicative) groups. For the second assertion, by Proposition 1, we know that $\lambda_{ij}(\alpha)^{U_n(q)}$ is a $\mathbb{C}$-linear combination of the class functions $\phi_{\mathcal{O}}$ for $\mathcal{O} \in \Omega_n(q)$; and, by Proposition 2 and Frobenius reciprocity, we know that the coefficients of this linear combination are non-negative integers. Now, let $\mathfrak{u}_{ij}(q)$ be the $\mathbb{F}_q$-subspace of $\mathfrak{u}_n(q)$ consisting of all matrices $x-1$ with $x \in U_{ij}(q)$ (hence, $U_{ij}(q) = 1+\mathfrak{u}_{ij}(q)$), let $f = \alpha e_{ij}^* \in \mathfrak{u}_n(q)^*$ and let $f_0 \in \mathfrak{u}_{ij}(q)^*$ be the restriction of f to $\mathfrak{u}_{ij}(q)$. Then, since $f(ab) = 0$ for all $a, b \in \mathfrak{u}_{ij}(q)$, $\{f_0\}$ is a single $U_{ij}(q)$-orbit on $\mathfrak{u}_{ij}(q)^*$. Moreover, by definition, $\lambda_{ij}(\alpha)$ is the class function which corresponds to this $U_{ij}(q)$-orbit (in the sense of (1)). Since $f \in \mathcal{O}_{ij}(q)$ and since $\xi_{ij}(\alpha) = \phi_{\mathcal{O}_{ij}(\alpha)}$, we conclude that

$$\langle \lambda_{ij}(\alpha)^{U_n(q)}, \xi_{ij}(\alpha)\rangle_{U_n(q)} = \langle \lambda_{ij}(\alpha), \xi_{ij}(\alpha)_{U_{ij}(q)}\rangle_{U_{ij}(q)} \neq 0$$

(by Frobenius reciprocity and Proposition 2). Finally, in order to conclude the proof, it is enough to show that $|\mathcal{O}_{ij}(\alpha)| = q^{2(j-i-1)}$ (because $\lambda_{ij}(\alpha)^{U_n(q)}(1) = |U_n(q) : U_{ij}(q)| = q^{j-i-1}$, because $\xi_{ij}(\alpha)(1) = \sqrt{|\mathcal{O}_{ij}(\alpha)|}$ and because $\lambda_{ij}(\alpha)^{U_n(q)}$ is a $\mathbb{Z}$-linear combination with non-negative coefficients of the class functions $\phi_{\mathcal{O}}$, for $\mathcal{O} \in \Omega_n(q)$). However, it is easy to see that the centralizer $C_{U_n(q)}(f)$ of f in $U_n(q)$ consists of all matrices $x \in U_n(q)$ which satisfy $x_{ik} = x_{kj} = 0$ for all $i < k < j$. Therefore, $|\mathcal{O}_{ij}(\alpha)| = |U_n(q) : C_{U_n(q)}(f)| = q^{2(j-i-1)}$, as required. □

In the notation of the previous lemma, we will refer to the irreducible character $\xi_{ij}(\alpha)$ of $U_n(q)$ as the (i,j)-th *elementary character* of $U_n(q)$ associated with α.

We are now able to define the basic characters of $U_n(q)$. To start with, a subset $D \subseteq \Phi(n)$ is called a *basic subset* if $|D \cap \{(i,j)\colon i < j \leq n\}| \leq 1$ for all $1 \leq i < n$, and if $|D \cap \{(i,j)\colon 1 \leq i < j\}| \leq 1$ for all $1 < j \leq n$. In particular, the empty set is a basic subset of $\Phi(n)$. Given an arbitrary non-empty basic subset D of $\Phi(n)$ and an arbitrary map $\varphi\colon D \to \mathbb{F}_q^\#$, we define the *basic character* $\xi_D(\varphi)$ of $U_n(q)$ to be the product (of elementary characters)

$$\xi_D(\varphi) = \prod_{(i,j)\in D} \xi_{ij}(\alpha_{ij}) \tag{4}$$

where $\alpha_{ij} = \varphi(i,j)$ for $(i,j) \in D$. For our purposes, it is convenient to consider the trivial character $1_{U_n(q)}$ of $U_n(q)$ as the basic character $\xi_D(\varphi)$ corresponding to the empty subset of $\Phi(n)$ and to the empty function $\varphi\colon D \to \mathbb{F}_q^\#$.

The main goal of this paper is to extend [1, Theorem 1] to all prime numbers p. In fact, we will prove the following result.

Theorem 1. *Let* χ *be an irreducible character of* $U_n(q)$. *Then* χ *is a constituent of a unique basic character of* $U_n(q)$*; in other words, there exists a unique basic subset* D *of* $\Phi(n)$ *and a unique map* $\varphi\colon D \to \mathbb{F}_q^\#$ *such that* χ *is a constituent of* $\xi_D(\varphi)$.

The proof of this theorem splits into two parts. First, we prove that the basic characters of $U_n(q)$ are pairwise orthogonal (see [1, Proposition 2] for the case where $p \geq n$) and, secondly, we obtain a decomposition of the regular character ρ of $U_n(q)$ as a sum of basic characters (see [3, Theorem 1] and [4, Corollary 7.1] for the case where $p \geq n$). For the first part of the proof, we start by proving the following general result.

Proposition 3. *Let G be an $\mathbb{F}_q$-algebra group and let $\mathcal{O}_1, \dots, \mathcal{O}_t \in \Omega(G)$. For each $i \leq i \leq t$, let ϕ_i denote the class function $\phi_{\mathcal{O}_i} \in \mathrm{cf}(G)$. Then, for any $\mathcal{O} \in \Omega(G)$, the class function $\phi_{\mathcal{O}}$ is a constituent of the product $\phi_1 \cdots \phi_t$ if and only if $\mathcal{O} \subseteq \mathcal{O}_1 + \cdots + \mathcal{O}_t$; moreover, the scalar product $\langle \phi_{\mathcal{O}}, \phi_1 \cdots \phi_t \rangle_G$ is a non-negative integer.*

Proof. Let $G^t = G \times \cdots \times G$ be the direct product of t copies of G. Let A be a finite dimensional $\mathbb{F}_q$-algebra such that $G = 1 + J$ where $J = J(A)$ is the Jacobson radical of A. Then G^t is the $\mathbb{F}_q$-algebra group associated with the Jacobson radical $J^t = J \times \cdots \times J$ (t copies) of the $\mathbb{F}_q$-algebra $\mathbb{F}_q \cdot 1 + J^t$. Moreover, G can be naturally identified with the diagonal subgroup $G_d = \{(x, \dots, x) \colon x \in G\}$ of G^t. Then the class function $\phi_1 \cdots \phi_t$ of G is naturally identified with the restriction $(\phi_1 \times \cdots \times \phi_t)_{G_d}$ to G_d of the class function $\phi_1 \times \cdots \times \phi_t$ of G^t. It is clear that this class function is associated (by the rule (1)) with the G^t-orbit $\mathcal{O}_1 \times \cdots \times \mathcal{O}_t \in \Omega(G^t)$; we note that the dual space $(J^t)^*$ of J^t is naturally isomorphic to $(J^*)^t = J^* \times \cdots \times J^*$ (t copies). Let J_d be the diagonal $\mathbb{F}_q$-subspace of J^t and let $\pi \colon (J^*)^t \to (J_d)^*$ be the natural projection. Since G_d is an algebra subgroup of G^t (because J_d is multiplicatively closed), we may apply Proposition 2: given $\mathcal{O} \in \Omega(G)$, we have $\langle \phi_{\mathcal{O}}, \phi_1 \cdots \phi_t \rangle_G \neq 0$ if and only if $\mathcal{O} \subseteq \pi(\mathcal{O}_1 \times \cdots \times \mathcal{O}_t)$; moreover, that scalar product is a non-negative integer. The result follows because $\pi(\mathcal{O}_1 \times \cdots \times \mathcal{O}_t) = \mathcal{O}_1 + \cdots + \mathcal{O}_t$. □

We now apply this result to an arbitrary basic character of $U_n(q)$. Let D be a non-empty basic subset of $\Phi(n)$ and let $\varphi \colon D \to \mathbb{F}_q^{\#}$ be a map. Following [2], we denote by $\mathcal{O}_D(\varphi)$ the *basic subvariety* of $\mathfrak{u}_n(q)^*$ associated with the pair (D, φ):

$$\mathcal{O}_D(\varphi) = \sum_{(i,j) \in D} \mathcal{O}_{ij}(\alpha_{ij}) \tag{5}$$

where $\alpha_{ij} = \varphi(i,j)$ for all $(i,j) \in D$. For convenience, we extend this definition to the case where D is the empty subset of $\Phi(n)$: we consider φ to be the empty function and we define $\mathcal{O}_D(\varphi) = \{0\}$. Then the following result is an obvious consequence of the previous proposition.

Corollary 1. *Let D be a basic subset of $\Phi(n)$ and let $\varphi \colon D \to \mathbb{F}_q^{\#}$ be a map. Let $\mathcal{O} \in \Omega_n(q)$ be arbitrary. Then the class function $\phi_{\mathcal{O}}$ of $U_n(q)$ is a constituent of the basic character $\xi_D(\varphi)$ if and only if $\mathcal{O} \subseteq \mathcal{O}_D(\varphi)$. Moreover, the scalar product $\langle \phi_{\mathcal{O}}, \xi_D(\varphi) \rangle_{U_n(q)}$ is a non-negative integer.*

By [2, Theorem 1 and Eq. (12)], the dual space $\mathfrak{u}_n(q)^*$ decomposes as the disjoint union

$$\mathfrak{u}_n(q)^* = \bigcup_{D,\varphi} \mathcal{O}_D(\varphi) \tag{6}$$

where the union is over all basic subsets D of $\Phi(n)$ and all maps $\varphi \colon D \to \mathbb{F}_q^{\#}$. This decomposition allows us to establish the orthogonality relations for the basic characters (see [1, Proposition 2] for the case where $p \geq n$). For simplicity, given an arbitrary basic subset D of $\Phi(n)$ and an arbitrary map $\varphi\colon D \to \mathbb{F}_q^{\#}$, we will denote by $\Omega_D(\varphi)$ the set of all $U_n(q)$-orbits $\mathcal{O} \in \Omega_n(q)$ such that $\mathcal{O} \subseteq \mathcal{O}_D(\varphi)$.

Proposition 4. *Let D and D' be basic subsets of $\Phi(n)$ and let $\varphi\colon D \to \mathbb{F}_q^{\#}$ and $\varphi'\colon D' \to \mathbb{F}_q^{\#}$ be maps. Then $\langle \xi_D(\varphi), \xi_{D'}(\varphi')\rangle_{U_n(q)} \neq 0$ if and only if $D = D'$ and $\varphi = \varphi'$.*

Proof. By Corollary 1 (and also by Proposition 2), we have

$$\xi_D(\varphi) = \sum_{\mathcal{O} \in \Omega_D(\varphi)} n_{\mathcal{O}} \phi_{\mathcal{O}}$$

where $n_{\mathcal{O}}$, for $\mathcal{O} \in \Omega_D(\varphi)$, is a positive integer. It follows that

$$\langle \xi_D(\varphi), \xi_{D'}(\varphi')\rangle_{U_n(q)} = \sum_{\mathcal{O} \in \Omega_D(\varphi)} n_{\mathcal{O}} \langle \phi_{\mathcal{O}}, \xi_{D'}(\varphi')\rangle_{U_n(q)}$$

and so $\langle \xi_D(\varphi), \xi_{D'}(\varphi')\rangle_{U_n(q)} \neq 0$ if and only if $\langle \phi_{\mathcal{O}}, \xi_{D'}(\varphi')\rangle_{U_n(q)} \neq 0$ for some $\mathcal{O} \in \Omega_D(\phi)$ (because, for any $\mathcal{O} \in \Omega_D(\varphi)$, $n_{\mathcal{O}}$ is a positive integer and $\langle \phi_{\mathcal{O}}, \xi_{D'}(\varphi')\rangle_{U_n(q)}$ is a non-negative integer). By Corollary 1, we conclude that $\langle \xi_D(\varphi), \xi_{D'}(\varphi')\rangle_{U_n(q)} \neq 0$ if and only if $\Omega_D(\varphi) \cap \Omega_{D'}(\varphi') \neq \emptyset$. The result follows by (6). □

The following result will be useful to decompose the regular character of $U_n(q)$ as a sum of basic characters (which will, of course, imply (together with the previous proposition) our main result).

Proposition 5. *Let $\mathcal{O} \in \Omega_n(q)$ be arbitrary. Then there exists a unique basic subset D of $\Phi(n)$ and a unique map $\varphi\colon D \to \mathbb{F}_q^{\#}$ such that the class function $\phi_{\mathcal{O}}$ of $U_n(q)$ is a constituent (with non-zero integer multiplicity) of $\xi_D(\varphi)$.*

Proof. By (6), there exists a unique basic subset D of $\Phi(n)$ and a unique map $\varphi\colon D \to \mathbb{F}_q^{\#}$ such that $\mathcal{O} \subseteq \mathcal{O}_D(\varphi)$. The result follows easily using Corollary 1. □

Next, we consider the decomposition of the regular character of $U_n(q)$ as a sum of basic characters. In fact, we will prove the following result (for the case where $p \geq n$, see [3, Theorem 1] and [4, Corollary 7.2]).

Theorem 2. *Let ρ denote the regular character of $U_n(q)$. Then*

$$\rho = \sum_{D,\varphi} \frac{q^{s(D)}}{\xi_D(\varphi)(1)}\, \xi_D(\varphi)$$

where the sum is over all basic subsets D of $\Phi(n)$ and all maps $\varphi\colon D \to \mathbb{F}_q^{\#}$ and where, for any basic subset D of $\Phi(n)$, $s(D)$ denotes the cardinality of the subset $S(D) = \bigcup_{(i,j)\in D} \{(i,k),(k,j)\colon i<k<j\}$ of $\Phi(n)$.

For some steps of the proof of this result, we will refer to [4]. We start by introducing some notation. Let D be an arbitrary non-empty basic subset of $\Phi(n)$, let $\varphi\colon D \to \mathbb{F}_q^{\#}$ be an arbitrary map and, for any $(i,j) \in D$, let $\alpha_{ij} = \varphi(i,j)$. Let $e_D(\varphi)$ denote the element

$$e_D(\varphi) = \sum_{(i,j)\in D} \alpha_{ij} e_{ij} \in \mathfrak{u}_n(q)$$

and let

$$\mathfrak{o}_D(\varphi) = \left\{ x e_D(\varphi) y^{-1} \colon x, y \in U_n(q) \right\} \subseteq \mathfrak{u}_n(q)$$

be the $(U_n(q) \times U_n(q))$-orbit on $\mathfrak{u}_n(q)$ for the action defined by $(x, y) \cdot a = xay^{-1}$ for all $x, y \in U_n(q)$ and all $a \in \mathfrak{u}_n(q)$. Moreover, let

$$\mathcal{K}_D(\varphi) = 1 + \mathfrak{o}_D(\varphi) \subseteq U_n(q).$$

On the other hand, for each root $(i, j) \in \Phi(n)$, let $S'(i, j)$ be the subset of $\Phi(n)$ which consists of all roots (i, k), for $j < k \leq n$, and (k, j), for $1 \leq k < i$ of $\Phi(n)$. Then, for any basic subset D of $\Phi(n)$, we define the subsets $S'(D) = \bigcup_{(i,j)\in D} S'(i, j)$ and $R'(D) = \Phi(n) - S'(D)$ of $\Phi(n)$. Moreover, for each root $(i, j) \in \Phi(n)$, we denote by $D(i, j)$ the subset of $\Phi(n)$ which consists of all roots $(k, l) \in D$ with $i < k < l < j$.

The following result is precisely [4, Proposition 5.1] (see also [4, Proposition 5.2]).

Lemma 3. *Let D be a basic subset of $\Phi(n)$, let $\varphi \colon D \to K^{\#}$ be a map and let $x \in \mathcal{K}_D(\varphi)$. Then, for any $(i, j) \in \Phi(n)$ and any $\alpha \in \mathbb{F}_q^{\#}$, we have*

$$\xi_{ij}(\alpha)(x) = \begin{cases} q^{d(i,j)}, & \text{if } (i,j) \in R'(D) - D, \\ q^{d(i,j)} \psi(\alpha\beta), & \text{if } (i,j) \in D, \\ 0, & \text{otherwise,} \end{cases}$$

where $d(i, j) = (j - i - 1) - |D(i, j)|$ and where $\beta = \varphi(i, j)$ whenever $(i, j) \in D$. In particular, if $x_D(\varphi)$ denotes the element $x_D(\varphi) = 1 + e_D(\varphi) \in \mathcal{K}_D(\varphi)$, we have

$$\xi_{ij}(\alpha)(x) = \xi_{ij}(\alpha)(x_{D,\varphi})$$

for all $x \in \mathcal{K}_D(\varphi)$.

Using this lemma (and the definition of the basic characters), we easily deduce the following result (which is precisely the statement of [4, Theorem 5.1]).

Theorem 3. *Let D and D' be basic subsets of $\Phi(n)$, let $\varphi \colon D \to \mathbb{F}_q^{\#}$ and $\varphi' \colon D' \to \mathbb{F}_q^{\#}$ be maps and let $x \in \mathcal{K}_{D'}(\varphi')$. Then*

$$\xi_D(\varphi)(x) = \begin{cases} q^{e(D,D')} \psi_D(\varphi)(e_{D'}(\varphi')), & \text{if } D \subseteq R'(D'), \\ 0, & \text{otherwise,} \end{cases}$$

where $e(D, D') = |S_c(D) - S'(D')| = |S_c(D) \cap R'(D')|$, where $S_c(D)$ is the union of all the subsets $\{(k, j) \in \Phi(n) \colon i < k < j\} \subseteq \Phi(n)$ for $(i, j) \in D$, and where $\psi_D(\varphi) \colon \mathfrak{u}_n(q)^+ \to \mathbb{C}$ is the (linear) character of the additive group $\mathfrak{u}_n(q)^+$ of $\mathfrak{u}_n(q)$ defined by

$$\psi_D(\varphi)(a) = \prod_{(i,j)\in D} \psi(\varphi(i, j) a_{ij})$$

for all $a \in \mathfrak{u}_n(q)$.

We now prove the following result (see [1, Corollary 5] for the case where $p \geq n$).

Theorem 4. *Let D and D' be basic subsets of $\Phi(n)$ and let $\varphi \colon D \to \mathbb{F}_q^{\#}$ and $\varphi' \colon D' \to \mathbb{F}_q^{\#}$ be maps. Then*

$$\langle \xi_D(\varphi), \xi_{D'}(\varphi') \rangle_{U_n(q)} = \begin{cases} 0, & \text{if } (D, \varphi) \neq (D', \varphi'), \\ q^{-s(D)} \xi_D(\varphi)(1)^2, & \text{if } (D, \varphi) = (D', \varphi'). \end{cases}$$

Proof. By Proposition 4, it remains to prove that

$$\langle \xi_D(\varphi), \xi_D(\varphi) \rangle_{U_n(q)} = q^{-s(D)} \xi_D(\varphi)(1)^2.$$

Let $\mathcal{F}$ be the set of all pairs (D', φ') where D' is a basic subset of $\Phi(n)$ satisfying $D \subseteq R'(D')$ and where $\varphi' \colon D' \to \mathbb{F}_q^{\#}$ is a map. Then, using Theorem 3, we deduce that (as usual, we denote by $\overline{z}$ the conjugate of a given complex number $z \in \mathbb{C}$)

$$\begin{aligned}
\langle \xi_D(\varphi), \xi_D(\varphi) \rangle_{U_n(q)} &= \frac{1}{|U_n(q)|} \sum_{x \in U_n(q)} \xi_D(\varphi)(x) \overline{\xi_D(\varphi)(x)} \\
&= \frac{1}{|U_n(q)|} \sum_{(D',\varphi') \in \mathcal{F}} \sum_{x \in \mathcal{K}_{D'}(\varphi')} \xi_D(\varphi)(x) \overline{\xi_D(\varphi)(x)} \\
&= \frac{1}{|U_n(q)|} \sum_{(D',\varphi') \in \mathcal{F}} |\mathcal{K}_{D'}(\varphi')| q^{2e(D,D')} \psi_D(\varphi)(e_{D'}(\varphi')) \overline{\psi_D(\varphi)(e_{D'}(\varphi'))} \\
&= \frac{1}{|U_n(q)|} \sum_{(D',\varphi') \in \mathcal{F}} |\mathcal{K}_{D'}(\varphi')| q^{2e(D,D')} \psi_D(\varphi)(e_{D'}(\varphi')) \psi_D(\varphi)(-e_{D'}(\varphi')) \\
&= \frac{1}{|U_n(q)|} \sum_{(D',\varphi') \in \mathcal{F}} |\mathcal{K}_{D'}(\varphi')| q^{2e(D,D')}.
\end{aligned}$$

Now, by [4, Proposition 4.1], we have $|\mathcal{K}_{D'}(\varphi')| = q^{|S'(D')|}$ and so

$$\begin{aligned}
\text{(7)} \qquad \langle \xi_D(\varphi), \xi_D(\varphi) \rangle_{U_n(q)} &= \sum_{(D',\varphi') \in \mathcal{F}} q^{-|R'(D')|} q^{2e(D,D')} \\
&= \sum_{D' \in \mathcal{B}} (q-1)^{|D'|} \, q^{2e(D,D') - |R'(D')|}
\end{aligned}$$

where $\mathcal{B}$ is the set of all basic subsets D' of $\Phi(n)$ which satisfy $D \subseteq R'(D')$. Finally, by Proposition 6 (see below), we have

$$\sum_{D' \in \mathcal{B}} (q-1)^{|D'|} \, q^{2e(D,D') - |R'(D')|} = q^{2\ell(D) - s(D)}$$

where $\ell(D) = \sum_{(i,j) \in D} (j - i - 1)$. The proof is complete because $\xi_D(\varphi)(1) = q^{\ell(D)}$. □

The following result was used at the end of the previous proof.

Proposition 6. *Let D be a basic subset of $\Phi(n)$, let $\mathcal{B}$ be the set of all basic subsets D' of $\Phi(n)$ which satisfy $D \subseteq R'(D')$ and let $\ell(D) = \sum_{(i,j) \in D} (j - i - 1)$. Then the identity*

$$\sum_{D' \in \mathcal{B}} (t-1)^{|D'|} t^{e(D,D') - |R'(D)|} = t^{2\ell(D) - s(D)}$$

holds in the polynomial ring $\mathbb{Z}[t]$ in one indeterminate t over $\mathbb{Z}$.

Proof. Let $p \geq n$ be a prime and consider the group $U_n(q)$ for an arbitrary power q of p. Then, by [1, Corollary 5], we have

$$\langle \xi_D(\varphi), \xi_D(\varphi) \rangle_{U_n(q)} = q^{-s(D)} \xi_D(\varphi)(1)^2 = q^{2\ell(D) - s(D)}.$$

As we have deduced in the previous proof, we have

$$\langle \xi_D(\varphi), \xi_D(\varphi) \rangle_{U_n(q)} = \sum_{D' \in \mathcal{B}} (q-1)^{|D'|} q^{e(D,D') - |R'(D')|}$$

and so

$$\sum_{D' \in \mathcal{B}} (q-1)^{|D'|} q^{e(D,D') - |R'(D')|} = q^{2\ell(D) - s(D)}.$$

It follows that the polynomial

$$t^{2\ell(D) - s(D)} - \sum_{D' \in \mathcal{B}} (t-1)^{|D'|} t^{e(D,D') - |R'(D')|} \in \mathbb{Z}[t]$$

has an infinite number of roots, hence it must be the zero polynomial. The result follows. □

Using Theorem 4, we can prove the following orthogonality relations where, for arbitrary basic subsets D and D' of $\Phi(n)$ and for arbitrary maps $\varphi \colon D \to \mathbb{F}_q^{\#}$ and $\varphi' \colon D' \to \mathbb{F}_q^{\#}$, we denote by $\xi_{D,\varphi}^{D',\varphi'} \in \mathbb{C}$ the constant value of the basic character $\xi_D(\varphi)$ on the subset $\mathcal{K}_{D'}(\varphi')$; hence, by Theorem 3, we have $\xi_{D,\varphi}^{D',\varphi'} = \xi_D(\varphi)(x_{D'}(\varphi'))$. (The proof of this theorem is the same as that of [4, Theorem 7.1] and so we omit it.)

Theorem 5. *Let D' and D'' be basic subsets of $\Phi(n)$ and let $\varphi' \colon D' \to \mathbb{F}_q^{\#}$ and $\varphi'' \colon D'' \to \mathbb{F}_q^{\#}$ be maps. Then*

$$\sum_{D,\varphi} \frac{q^{s(D)}}{\xi_D(\varphi)(1)^2}\, \xi_{D,\varphi}^{D',\varphi'} \overline{\xi_{D,\varphi}^{D'',\varphi''}} = \begin{cases} 0, & \text{if } (D',\varphi') \neq (D'',\varphi''), \\ q^{n(n-1)/2 - |S'(D')|}, & \text{if } (D',\varphi') = (D'',\varphi''), \end{cases}$$

where the sum is over all basic subsets D of $\Phi(n)$ and all maps $\varphi \colon D \to \mathbb{F}_q^{\#}$.

We are now able to prove Theorem 2.

Proof of Theorem 2. Let $x \in U_n(q)$ be arbitrary. Then $x \in \mathcal{K}_{D'}(\varphi')$ for a unique basic subset D' of $\Phi(n)$ and for a unique map $\varphi' \colon D' \to \mathbb{F}_q^{\#}$. Using the previous theorem, we deduce that

$$\sum_{D,\varphi} \frac{q^{s(D)}}{\xi_D(\varphi)(1)}\, \xi_D(\varphi)(x) = \delta_{x,1} q^{n(n-1)/2}$$

where the sum is over all basic subsets D of $\Phi(n)$ and all maps $\varphi \colon D \to \mathbb{F}_q^{\#}$. Therefore, the sum $\sum_{D,\varphi} \frac{q^{s(D)}}{\xi_D(\varphi)(1)}\, \xi_D(\varphi)$ is the regular character of $U_n(q)$, as required. □

Remark 1. The proof of Theorem 2 can be achieved using the corresponding result for the case where $p \geq n$. In fact, suppose that $p \geq n$. Let D' be an arbitrary basic subset of $\Phi(n)$ and let $\varphi' \colon D' \to \mathbb{F}_q^{\#}$ be an arbitrary map. For simplicity, let us write $x = x_{D'}(\varphi')$ and, for each root $(i,j) \in \Phi(n)$, let $\beta_{ij} \in \mathbb{F}_q$ be the (i,j)-th coefficient of x. Then, using Theorem 3 (or [4, Theorem 5.1]), we deduce that

$$\begin{aligned}\sum_{D,\varphi} \frac{q^{s(D)}}{\xi_D(\varphi)(1)}\, \xi_D(\varphi)(x) &= \sum_{\substack{D,\varphi \\ D\subseteq R'(D')}} \frac{q^{s(D)+e(D,D')}}{\xi_D(\varphi)(1)} \prod_{(i,j)\in D} \psi(\varphi(i,j)\beta_{ij}) \\ &= \sum_{D\subseteq R'(D')} \frac{q^{s(D)+e(D,D')}}{q^{\ell(D)}} \sum_{\varphi} \prod_{(i,j)\in D} \psi(\varphi(i,j)\beta_{ij}) \\ &= \sum_{D\subseteq R'(D')} q^{s(D)+e(D,D')-\ell(D)} \prod_{(i,j)\in D} \sum_{\alpha\in\mathbb{F}_q^{\#}} \psi(\alpha\beta_{ij})\end{aligned}$$

where the sums are over all basic subsets D of $\Phi(n)$ and over all maps $\varphi\colon D \to \mathbb{F}_q^{\#}$ and where, for each basic subset D of $\Phi(n)$, $\ell(D) = \xi_D(\varphi)(1) = \sum_{(i,j)\in D}(j-i-1)$. Now, for each $\alpha \in \mathbb{F}_q$, the map $\psi_\alpha\colon \mathbb{F}_q^{\#} \to \mathbb{C}$, defined by $\psi_\alpha(\beta) = \psi(\alpha\beta)$ for all $\beta \in \mathbb{F}_q$, is a linear character of the additive group $\mathbb{F}_q^{+}$ of $\mathbb{F}_q$. Moreover, the characters ψ_α, for $\alpha \in \mathbb{F}_q$, are all distinct, hence they are all the irreducible characters of $\mathbb{F}_q^{+}$. It follows that the sum $\sum_{\alpha\in\mathbb{F}_q} \psi_\alpha$ is the regular character of $\mathbb{F}_q^{+}$ and so, for an arbitrary $\beta \in \mathbb{F}_q$, we have

$$\sum_{\alpha\in\mathbb{F}_q} \psi_\alpha(\beta) = q\delta_{\beta,0}.$$

Therefore, given an arbitrary root $(i,j) \in \Phi(n)$, we conclude that

$$\sum_{\alpha\in\mathbb{F}_q^{\#}} \psi(\alpha\beta_{ij}) = \begin{cases} q-1, & \text{if } (i,j) \in D', \\ -1, & \text{if } (i,j) \notin D', \end{cases}$$

because (by definition of $x_{D'}(\varphi')$) $\beta_{ij} \neq 0$ if and only if $(i,j) \in D'$. In conclusion, we obtain

$$\sum_{D,\varphi} \frac{q^{s(D)}}{\xi_D(\varphi)(1)}\, \xi_D(\varphi)(x) = \sum_{D\in\mathcal{B}(D')} q^{s(D)+e(D,D')-\ell(D)}(-1)^{|D-D'|}(q-1)^{|D\cap D'|}$$

where the sum on the left-hand side of this equation is over all basic subsets D of $\Phi(n)$ and over all maps $\varphi\colon D \to \mathbb{F}_q^{\#}$ and where $\mathcal{B}(D')$ denotes the set of all basic subsets D of $\Phi(n)$ which satisfy $D \subseteq R'(D')$. By [4, Corollary 7.2], we conclude that the equality

$$\sum_{D\in\mathcal{B}(D')} (-1)^{|D-D'|} q^{s(D)+e(D,D')-\ell(D)}(q-1)^{|D\cap D'|} = q^{n(n-1)/2}\delta_{D',\emptyset} \tag{8}$$

holds whenever q is a power of a prime $p \geq n$. It follows that the identity

$$\sum_{D\in\mathcal{B}(D')} (-1)^{|D-D'|} t^{s(D)+e(D,D')-\ell(D)}(t-1)^{|D\cap D'|} = t^{n(n-1)/2}\delta_{D',\emptyset} \tag{9}$$

holds in the polynomial ring $\mathbb{Z}[t]$, hence the identity (8) holds whenever q is a power of an arbitrary prime p. Since this equality implies (as above) that

$$\sum_{D,\varphi} \frac{q^{s(D)}}{\xi_D(\varphi)(1)}\, \xi_D(\varphi)(x) = q^{n(n-1)/2}\delta_{x,1},$$

Theorem 2 follows at once.

Finally, we prove Theorem 1.

Proof of Theorem 1. Let χ be an arbitrary irreducible character of $U_n(q)$. Then χ is a constituent (with multiplicity $\chi(1)$) of the regular character ρ of $U_n(q)$. By Theorem 2, we conclude that χ is a constituent of at least one basic character of $U_n(q)$ and, by Theorem 4, this basic character must be unique. □

Remark 2. Let D be a basic subset of $\Phi(n)$ and let $\varphi\colon D \to \mathbb{F}_q^{\#}$ be a map. Then, by Theorem 2 and by Theorem 1, we also conclude that

$$\chi(1) = q^{s(D)-\ell(D)}\langle \chi, \xi_D(\varphi)\rangle_{U_n(q)}$$

for any irreducible constituent χ of $\xi_D(\varphi)$.

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