Research Article

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Basic inequalities for statistical submanifolds in Golden-like statistical manifolds

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Abstract: In this paper, we introduce and study Golden-like statistical manifolds. We obtain some basic inequalities for curvature invariants of statistical submanifolds in Golden-like statistical manifolds. Also, in support of our definition, we provide a couple of examples.

Keywords: Chen invariant, Casorati curvature, Golden manifold, statistical manifold

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1 Introduction

The comparison relationships between the intrinsic and extrinsic invariants are the basic problems in submanifold theory. In [1], Chen introduce some curvature invariants and for their usages, he derived optimal relationships between the intrinsic invariants (Chen invariants) and the extrinsic invariants, which become later an active and fruitful area of research (see, for instance, [1–3]).

On the other hand, the notion of Casorati curvature (extrinsic invariant) for the surfaces was originally introduced in 1890 (see [4]). The Casorati curvature gives a better intuition of the curvature compared to the Gaussian curvature. The Gaussian curvature of a developable surface is zero. Thus, Casorati put forward the notion of Casorati curvature of a surface defined as $C = 1/2(1/\kappa_1^2 + 1/\kappa_2^2)$. For example, for developable surfaces (say, cylinder), the Gaussian curvature vanishes, while the Casorti curvature C surely does not vanish. The Casorati curvature of a submanifold in a Riemannian manifold is defined as the normalized square length of the second fundamental form [5].

In the past decade, various geometers attracted toward the study of Chen-type comparison relationships between the Casorati curvature and the intrinsic invariants. For some references in this direction we refer to [6–12]. The submanifolds with equality case in the Chen-type inequalities are called ideal submanifolds and the name ideal is motivated by the fact that these submanifolds inherit the least possible tension from the ambient manifold (see [13]).

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In 1985, Amari introduced the notion of statistical manifolds via information geometry (see [14]). Statistical manifolds are endowed with a pair of dual torsion-free connections. This is analogous to conjugate connections in affine geometry (see [15]). The dual connections are not metric, thus it is very tough to give a notion of sectional curvature using the canonical definitions of Riemannian geometry. In [16], Opozda gave the definition of sectional curvature tensor on a statistical manifold. While studying the geometric properties of a submanifold, a very important problem is to obtain sharp relations between the intrinsic and the extrinsic invariants, and a vast number of such relations are revealed by certain inequalities. For example, let M be a surface in Euclidean 3-space, we know the Euler inequality: $K \leq |H|^2$, where H is the mean curvature (extrinsic property) and K is the Gaussian curvature (intrinsic property). The equality holds at points where M is congruent to an open piece of a plane or a sphere (umbilical points). Chen [17] obtained the same inequality for submanifolds of real space forms. Then in [18], Chen obtained the Chen-Ricci inequality, which is a sharp relation between the squared mean curvature and the Ricci curvature of a Riemannian submanifold of a real space form.

In recent years, statistical manifolds have been studied very actively. In [19], Takano studied statistical manifolds with almost complex and almost contact structure. In 2015, Vîlcu and Vîlcu [20] studied statistical manifolds with quaternionic settings and proposed several open problems. While answering one of those open problems, Aquib [21] obtained some of the curvature properties of submanifolds and a couple of inequalities for totally real statistical submanifolds of quaternionic Kaehler-like statistical space forms. In 2019, Chen et al. derived a Chen first inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature [22]. In the same year, following the same paper of Chen et al., Atimur et al. [23] obtained Chen-type inequalities for statistical submanifolds of Kaehler-like statistical manifolds. Very recently, in 2020, Decu et al. obtained inequalities for the Casorati curvature of statistical manifolds in holomorphic statistical manifolds of constant holomorphic curvature [24]. For some of the recent works, we refer to [15, 19, 25–28].

Motivated by the aforementioned studies, we define Golden-like statistical manifolds and obtain certain interesting inequalities. The structure of this paper is as follows. In Section 2, we first give the definition of Golden-like statistical manifolds. We also construct an example for the Golden-like statistical manifolds. In the next section, we obtain the main inequalities. We also prove the results for their equality cases.

2 Golden-like statistical manifold

Let M be a smooth manifold. A $(1,1)$ tensor field T on M is said to be polynomial structure if T satisfies an algebraic equation [29,30]

$$P(x) = x^n + b_n x^{n-1} + \cdots + b_2 x + b_1 I = 0,$$

where I is the $(1,1)$ identity tensor field and $T^{n-1}(q), T^{n-2}(q), \ldots, T(q), I$ are linearly independent at every point $q \in M$. The polynomial $P(x)$ is called the structure polynomial. For $P(x) = x^2 + I$ and $P(x) = x^2 - I$, we obtain an almost complex structure and an almost product structure, respectively. It has to be noted here that the existence of almost complex structure implies the even dimensions of the manifold. For $P(x) = x^2$, we obtain the notion of an almost tangent structure.

Definition 1. [29,31,32] Let (M, g) be the a semi-Riemannian manifold and let ϕ be the $(1,1)$ tensor field on M satisfying the following equation:

$$\phi^2 = \phi + I.$$

Then the tensor field ϕ is called a Golden structure on M. If the Riemannian metric g is ϕ compatible, the (M, g, ϕ) is called a Golden semi-Riemannian manifold.

For ϕ compatible metric g, we have the following:

$$g(\phi X, Y) = g(X, \phi Y), \tag{2.1}$$

$$g(\phi X, \phi Y) = g(\phi^2 X, Y) = g(\phi X, Y) + g(X, Y), \quad X, Y \in \Gamma(TM). \tag{2.2}$$

A remarkable fact about Golden structures is its appearance in pairs, i.e., if ϕ is Golden structure, the $\hat{\phi} = I - \phi$ is also a Golden structure. But same is the case with almost tangent (R and $-R$) and almost complex structure (J and $-J$). So it is natural to ask the connection between Golden and product structures.

Let M be a Riemannian manifold. Denote a torsion-free affine connection by ∇. The triple (M, ∇, g) is called a statistical manifold if ∇g is symmetric. We define another affine connection ∇^* by

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(\nabla^*_X Z, Y) \tag{2.3}$$

for vector fields E, F, and G on M. The affine connection ∇^* is called conjugate (or dual) to ∇ with respect to g. The affine connection ∇^* is torsion-free, $\nabla^* g$ is symmetric and satisfies $\nabla^0 = \frac{\nabla + \nabla^*}{2}$. Clearly, the triple (M, ∇^*, g) is statistical. We denote by R and R^* the curvature tensors on M with respect to the affine connection ∇ and its conjugate ∇^*, respectively. Also the curvature tensor field R^0 associated with the ∇^0 is called Riemannian curvature tensor. Then we find

$$g(R(X, Y)Z, W) = -g(Z, R^*(X, Y)W)$$

for vector fields X, Y, Z, and W on M, where $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$.

In general, the dual connections are not metric, one cannot define the sectional curvature in statistical environment as in the case of semi-Riemannian geometry. Thus, Opozda proposed two notions of sectional curvature on statistical manifolds (see [16,33]).

Let M be a statistical manifold and π a plane section in TM with orthonormal basis $\{X, Y\}$, then the sectional K-curvature is defined in [16] as

$$K(\pi) = \frac{1}{2}[g(R(X, Y)Y, X) + g(R^*(X, Y)Y, X) - g(R^0(X, Y)Y, X)].$$

Definition 2. Let (M, g, ϕ) be a Golden semi-Riemannian manifold endowed with a tensor field ϕ^* of type (1,1) satisfying

$$g(\phi X, Y) = g(X, \phi^* Y) \tag{2.4}$$

for vector fields X and Y. In view of (2.4), we easily derive

$$(\phi^*)^2 X = \phi^* X + X, \tag{2.5}$$

$$g(\phi X, \phi^* Y) = g(\phi X, Y) + g(X, Y). \tag{2.6}$$

Then (M, g, ϕ) is called Golden-like statistical manifold.

According to (2.5) and (2.6), the tensor fields $\phi + \phi^*$ and $\phi - \phi^*$ are symmetric and skew symmetric with respect to g, respectively. The equations (2.4), (2.5), and (2.6) imply the following proposition.

Proposition 1. *(M, g, ϕ) is a Golden-like statistical manifold if and only if it is (M, g, ϕ^*).*

We remark that if we choose $\phi = \phi^*$ in a Golden-like statistical manifold, then we have a Golden semi-Riemannian manifold.

We first present an example of a Golden-Riemannian manifold.

Example 1. [34] Consider the Euclidean 6-space $\mathbb{R}^6$ with standard coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$. Let ϕ be an $(1, 1)$ tensor field on $\mathbb{R}^6$ defined by

$$\phi(x_1, x_2, x_3, x_4, x_5, x_6) = (\psi x_1, \psi x_2, \psi x_3, (1 - \psi)x_4, (1 - \psi)x_5, (1 - \psi)x_6)$$

for any vector field $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6$, where $\psi = \frac{1+\sqrt{5}}{2}$ and $1 - \psi = \frac{1-\sqrt{5}}{2}$ are the roots of the equation $x^2 = x + 1$. Then we obtain

$$\begin{aligned}\phi^2(x_1, x_2, x_3, x_4, x_5, x_6) &= (\psi^2 x_1, \psi^2 x_2, \psi^2 x_3, (1-\psi)^2 x_4, (1-\psi)^2 x_5, (1-\psi)^2 x_6)\\ &= (\psi x_1, \psi x_2, \psi x_3, (1-\psi)x_4, (1-\psi)x_5, (1-\psi)x_6) + (x_1, x_2, x_3, x_4, x_5, x_6).\end{aligned}$$

Thus, we have $\phi^2 = \phi + I$. Moreover, we can easily see that standard metric $\langle\ ,\ \rangle$ on $\mathbb{R}^6$ is ϕ compatible. Hence, $(\mathbb{R}^6, \langle\ ,\ \rangle, \phi)$ is a Golden Riemannian manifold.

Next, we construct an example of a Golden-like statistical manifold in the following example.

Example 2. Consider the semi-Euclidean space $\mathbb{R}^3_1$ with standard coordinates (x_1, x_2, x_3) and the semi-Riemannian metric g with the signature $(-, +, +)$. Let ϕ be an $(1, 1)$ tensor field on $\mathbb{R}^3_1$ defined by

$$\phi(x_1, x_2, x_3) = \frac{1}{2}(x_1 + \sqrt{5}x_2, x_2 + \sqrt{5}x_1, 2\psi x_3)$$

for any vector field $(x_1, x_2, x_3) \in \mathbb{R}^3_1$, where $\psi = \frac{1+\sqrt{5}}{2}$ is the Golden mean. Then we obtain $\phi^2 = \phi + I$, this implies that ϕ is a Golden structure on $\mathbb{R}^3_1$.

Now we define an $(1, 1)$ tensor field ϕ^* on $\mathbb{R}^3_1$ by

$$\phi^*(x_1, x_2, x_3) = \frac{1}{2}(x_1 - \sqrt{5}x_2, x_2 - \sqrt{5}x_1, 2\psi x_3).$$

Thus, we have $\phi^{*2} = \phi^* + I$. Moreover, we have the equation (2.4). Hence, $(\mathbb{R}^3_1, g, \phi)$ is a Golden-like simplified statistical manifold.

Now we give a generalized example of the above example.

Example 3. Let $\mathbb{R}_n$ be a $(2n + m)$-dimensional affine space with the coordinate system $(x_1, \cdots, x_n, y_1, \dots, y_n, z_1, \dots, z_m)$. Assume we define a semi-Riemannian metric g with the signature $(\underbrace{-, \dots, -}_{n-times}, \underbrace{+, \dots, +}_{(n+m)-times})$ and the tensor field ϕ as follows:

$$\phi = \frac{1}{2}\begin{bmatrix} \delta_{ij} & \sqrt{5}\delta_{ij} & 0 \\ \sqrt{5}\delta_{ij} & \delta_{ij} & 0 \\ 0 & 0 & \psi \end{bmatrix},$$

where ψ is the Golden mean. Then ϕ is golden structure on $\mathbb{R}^{2n+m}_n$. Moreover, if the conjugate tensor field ϕ^* is defined as

$$\phi^* = \frac{1}{2}\begin{bmatrix} \delta_{ij} & -\sqrt{5}\delta_{ij} & 0 \\ \sqrt{5}\delta_{ij} & -\delta_{ij} & 0 \\ 0 & 0 & \psi \end{bmatrix}.$$

Then we can easily see that $(\mathbb{R}^{2n+m}_n, g, \phi)$ and $(\mathbb{R}^{2n+m}_n, g, \phi^*)$ are Golden-like statistical manifolds. Also, this verifies Proposition 1.

Let $(M = M_p(c_p) \times M_q(c_q), g, \phi)$ be a Golden product space form. Then the Riemannian curvature tensor R of M is given by [32]:

$$\begin{aligned}R(X, Y)Z = &\left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y\}\\ &+ \left(-\frac{(1-\psi)c_p + \psi c_q}{4}\right)\{g(\phi Y, Z)X - g(\phi X, Z)Y + g(Y, Z)\phi X - g(X, Z)\phi Y\},\end{aligned} \tag{2.7}$$

where M_p and M_q are space forms with constant sectional curvatures c_p and c_q, respectively. We can obtain the curvature tensor R^* with respect to dual connection just by replacing ϕ by ϕ^*.

Let M^n be statistical submanifold of (N^m, g, ϕ). The Gauss and Weingarten formulae are

$$\nabla_X Y = \nabla_X Y + \sigma(X, Y), \qquad \nabla_X \xi = -A_\xi X + \nabla_X^\perp \xi$$
$$\nabla_X^* Y = \nabla_X^* Y + \sigma^*(X, Y), \qquad \nabla_X^* \xi = -A_\xi^* X + \nabla_X^{*\perp} \xi$$

for all $X, Y \in TM$ and $\xi \in T^\perp M$, respectively. Moreover, we have the following equations:

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z)$$
$$g(\sigma(X, Y), \xi) = g(A_\xi^* X, Y), \qquad g(\sigma^*(X, Y), \xi) = g(A_\xi X, Y)$$
$$Xg(\xi, \eta) = g(\nabla_X^\perp \xi, \eta) + g(\xi, \nabla_X^{*\perp} \eta).$$

The mean curvature vector fields for an orthonormal tangent frame $\{e_1, e_2, \dots, e_n\}$ and a normal frame $\{e_{n+1}, \dots, e_m\}$, respectively, are defined as

$$H = \frac{1}{n}\sum_{i=1}^{n} \sigma(e_i, e_i) = \frac{1}{n}\sum_{\gamma=n+1}^{m}\left(\sum_{i=1}^{n}\sigma_{ii}^{\gamma}\right)\xi_\gamma, \qquad \sigma_{ij}^{\gamma} = g(\sigma(e_i, e_j), e_\gamma)$$

and

$$H^* = \frac{1}{n}\sum_{i=1}^{n} \sigma^*(e_i, e_i) = \frac{1}{n}\sum_{\gamma=n+1}^{m}\left(\sum_{i=1}^{n}\sigma_{ii}^{*\gamma}\right)\xi_\gamma, \qquad \sigma_{ij}^{*\gamma} = g(\sigma^*(e_i, e_j), e_\gamma)$$

for $1 \le i, j \le n$, and $1 \le l \le m$. Moreover, we have $2h^0 = h + h^*$ and $2H^0 = H + H^*$, where the second fundamental form h^0 and the mean curvature H^0 are calculated with respect to Levi-Civita connection ∇^0 on M.

The squared mean curvatures are defined as

$$\|H\|^2 = \frac{1}{n^2}\sum_{\gamma=n+1}^{m}\left(\sum_{i=1}^{n}\sigma_{ii}^{\gamma}\right)^2, \qquad \|H^*\|^2 = \frac{1}{n^2}\sum_{\gamma=n+1}^{m}\left(\sum_{i=1}^{n}\sigma_{ii}^{*\gamma}\right)^2.$$

The Casorati curvatures are defined as

$$C = \frac{1}{n}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n}(\sigma_{ij}^{\gamma})^2, \qquad C^* = \frac{1}{n}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n}(\sigma_{ij}^{*\gamma})^2.$$

If we suppose that $\mathcal{W}$ is a d-dimensional subspace of TM, $d \ge 2$, and $\{e_1, e_2, \dots, e_d\}$ is an orthonormal basis of $\mathcal{W}$, then the scalar curvature of the d-plane section is given as

$$\tau(\mathcal{W}) = \sum_{1 \le u < v \le d} K(e_u \wedge e_v),$$

and the normalized scalar curvature ρ is defined as

$$\rho = \frac{2\tau}{s(s-1)}.$$

Also, the Casorati curvature of the subspace $\mathcal{W}$ is given by

$$C(\mathcal{W}) = \frac{1}{d}\sum_{\gamma=r+1}^{m}\sum_{i,j=1}^{d}(\sigma_{ij}^{\gamma})^2, \qquad C^*(\mathcal{W}) = \frac{1}{d}\sum_{\gamma=r+1}^{m}\sum_{i,j=1}^{d}(\sigma_{ij}^{*\gamma})^2.$$

A point $x \in M$ is called as quasi-umbilical point, if at x there exist $m - n$ mutually orthogonal unit normal vectors e_i, $i \in \{n+1, \dots, m\}$ in a way the shape operators with respect to all vectors e_i have an eigenvalue with multiplicity $n - 1$ and for each e_i the distinguished eigen vector is the same.

The normalized δ-Casorati curvatures $\delta_c(n-1)$ and $\widehat{\delta}_c(n-1)$ of the submanifold M^s are, respectively, given by

$$[\delta_c(n-1)]_x = \frac{1}{2}C_x + \frac{n+1}{2s}\inf\{C(\mathcal{W})|\mathcal{W} \text{ a hyperplane of } T_xM\}$$

and

$$[\widehat{\delta}_c(n-1)]_x = 2C_x - \frac{2n-1}{2n}\sup\{C(\mathcal{W})|\mathcal{W} \text{ a hyperplane of } T_xM\}.$$

In [5], Decu et al. generalized the notion of normalized δ-Casorati curvature to the generalized normalized δ-Casorati curvatures $\delta_C(k; n-1)$ and $\widehat{\delta}_C(k; n-1)$. For a submanifold M^n and for any positive real number $k \neq n(n-1)$, the generalized normalized δ-Casorati curvature is given by:

$$[\delta_C(k;\, n-1)]_x = kC_x + \frac{(n-1)(n+k)(n^2-n-k)}{kn}\inf\{C(\mathcal{W})|\mathcal{W} \text{ a hyperplane of } T_xM\},$$

if $0 < k < n^2 - n$, and

$$[\widehat{\delta}_C(k;\, n-1)]_x = kC_x - \frac{(n-1)(n+k)(k-n^2+n)}{kn}\sup\{C(\mathcal{W})|\mathcal{W} \text{ a hyperplane of } T_xM\},$$

if $k > n^2 - n$.

The generalized normalized δ-Casorati curvatures $\delta_C(k : n-1)$ and $\hat{\delta}_C(k : n-1)$ are generalizations of normalized δ-Casorati curvatures $\delta_C(n-1)$ and $\hat{\delta}_C(n-1)$. In fact, we have the following relations (see [5]):

$$\left[\delta_C\left(\frac{n(n-1)}{2};\, n-1\right)\right]_x = n(n-1)[\delta_C(n-1)]_x, \tag{2.8}$$

$$[\hat{\delta}_C(2n(n-1);\, n-1)]_x = n(n-1)[\hat{\delta}_C(n-1)]_x. \tag{2.9}$$

In the same way, the dual Casorati curvatures are obtained just by replacing δ and δ^* and C by C^*.

Now, we state the following fundamental results on statistical manifolds.

Proposition 2. [20] *Let M be statistical submanifold of (M, g, ϕ). Let R and R^* be the Riemannian curvature tensors on M for ∇ and ∇^*, respectively. Then we have the following.*

$$g(R(X,Y)Z,W) = g(R(X,Y)Z,W) + g(\sigma(X,Z),\sigma^*(Y,W)) - g(\sigma^*(X,W),\sigma(Y,Z)),$$
$$g(R^*(X,Y)Z,W) = g(R^*(X,Y)Z,W) + g(\sigma^*(X,Z),\sigma(Y,W)) - g(\sigma(X,W),\sigma^*(Y,Z)),$$
$$g(R^\perp(X,Y)\xi,\eta) = g(R(X,Y)\xi,\eta) + g([A^*_\xi, A_\eta]X, Y),$$
$$g(R^{*\perp}(X,Y)\xi,\eta) = g(R^*(X,Y)\xi,\eta) + g([A_\xi, A^*_\eta]X, Y),$$

where $[A_\xi, A^*_\eta] = A_\xi A^*_\eta - A^*_\eta A_\xi$ and $[A^*_\xi, A_\eta] = A^*_\xi A_\eta - A_\eta A^*_\xi$, for $X, Y, Z, W \in TM$ and $\xi, \eta \in T^\perp M$.

Now, we state two important lemmas which we use to prove the main results in the upcoming sections.

Lemma 1. *Let $n \geq 3$ be an integer and $a_1, a_2, \ldots, a_n$ are n real numbers. Then, we have*

$$\sum_{1\leq i<j\leq n}^{n} a_i a_j - a_1 a_2 \leq \frac{n-2}{2(n-2)}\left(\sum_{i=1}^{n} a_i\right)^2.$$

Moreover, the equality holds if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

The optimization techniques have a pivotal role in improving inequalities involving Chen invariants. Oprea [35] applied the constrained extremum problem to prove Chen-Ricci inequalities for Lagrangian submanifolds of complex space forms. In the characterization of our main result, we will use the following lemma.

Let M be a Riemannian submanifold in a Riemannian manifold $(\hat{M}, g)$ and $y : \hat{M} \to \mathbb{R}$ be a differentiable function. If we have the constrained extremum problem

$$\min_{x \in M}[y(x)]. \tag{2.10}$$

Then we have the following lemma.

Lemma 2. [35] *If* $x_0 \in M$ *is a solution of the problem* (2.10), *then*
(1) $(\text{grad } y)(x_0) \in T^{\perp}_{x_0}M$;
(2) *The bilinear form* $\Omega : T_{x_0}M \times T_{x_0}M \to \mathbb{R}$ *defined by*

$$\Omega(X, Y) = \text{Hess}_y(X, Y) + g(\sigma(X, Y), (\text{grad } y)(x_0))$$

is positive semi-definite, where σ *is the second fundamental form of* M *in* $\hat{M}$ *and* grad y *is the gradient of* y.

In principle, the bilinear form Ω is $\text{Hess}_{y|M}(x_0)$. Therefore, if Ω is positive semi-definite on M, then the critical points of $y|M$, which coincide with the points where grad y is normal to M, are global optimal solutions of the problem (2.6) (for instance see [36, Remark 3.2]).

3 Main inequalities

Let π be a two plane spanned by $\{e_1, e_2\}$ and denote $g(\phi e_1, e_1)g(\phi_2, e_2) = \Psi(\pi)$. Also, as in [37], for an orthonormal basis $\{e_1, e_2\}$ of two-plane section, we denote $\Theta(\pi) = g(\phi e_1, e_2)g(\phi^* e_1, e_2)$, where $\Theta(\pi)$ is a real number in $[0, 1]$.

Theorem 1. *Let* (N, g, ϕ) *be a Golden-like statistical manifold of dimension* m *and* M *be its statistical submanifold of dimension* n. *Then, we have the following*:

$$\begin{aligned}
&(\tau - K(\pi)) - (\tau_0 - K_0(\pi))\\
&\geq \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[n(n-2) + \text{tr}^2(\phi) - \text{tr}(\phi^*)] + \left(-\frac{(1-\psi)c_p + \psi c_q}{4}\right)2(n-1)\text{tr}(\phi)\\
&\quad + \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[1 + \Psi(\pi) + \Theta(\pi)] - \frac{n^2(n-2)}{4(n-1)}[\|H\|^2 + \|H^*\|^2] + 2\hat{K}_0(\pi) - 2\hat{\tau}_0.
\end{aligned}$$

Proof. Let $\{e_1, e_2, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_m\}$ be the orthonormal frames of TM and $T^{\perp}M$, respectively.

The scalar curvature corresponding to the sectional K-curvature is

$$\tau = \frac{1}{2}\sum_{1\leq i<j\leq n}[g(R(e_i, e_j)e_j, e_i) + g(R^*(e_i, e_j)e_j, e_i) - 2g(R^0(e_i, e_j)e_j, e_i)].$$

Using (2.7) and Gauss equation for R and R^* and doing some simple calculations, we obtain

$$\begin{aligned}
\tau &= \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[n(n-1) + \text{tr}^2(\phi) - \text{tr}(\phi^{*2})] + \left(-\frac{(1-\psi)c_p + \psi c_q}{4}\right)2(n-1)\text{tr}(\phi) - \tau_0\\
&\quad + \frac{1}{2}\sum_{\gamma=n+1}^{m}\sum_{1\leq i<j\leq n}[\sigma^{*\gamma}_{ii}\sigma^{*\gamma}_{jj} + \sigma^{\gamma}_{ii}\sigma^{*\gamma}_{jj} - 2\sigma^{*\gamma}_{ij}\sigma^{\gamma}_{ij}].
\end{aligned}$$

In view of (2.5), we obtain

$$\begin{aligned}
\tau &= \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[n(n-2) + \text{tr}^2(\phi) - \text{tr}(\phi^{*})] + \left(-\frac{(1-\psi)c_p + \psi c_q}{4}\right)2(n-1)\text{tr}(\phi) - \tau_0\\
&\quad + \frac{1}{2}\sum_{\gamma=n+1}^{m}\sum_{1\leq i<j\leq n}[\sigma^{*\gamma}_{ii}\sigma^{*\gamma}_{jj} + \sigma^{\gamma}_{ii}\sigma^{*\gamma}_{jj} - 2\sigma^{*\gamma}_{ij}\sigma^{\gamma}_{ij}],
\end{aligned}$$

which can be written as

$$\tau = \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[n(n-2) + \mathrm{tr}^2(\phi) - \mathrm{tr}(\phi^*)] + \left(-\frac{(1-\psi)c_p + \psi c_q}{4}\right)2(n-1)\mathrm{tr}(\phi)$$
$$- \tau_0 + 2\sum_{\gamma=n+1}^{m}\sum_{1\le i<j\le n}[\sigma_{ii}^{0\gamma}\sigma_{jj}^{0\gamma} - (\sigma_{ij}^{0\gamma})^2] - \frac{1}{2}\sum_{\gamma=n+1}^{m}\sum_{1\le i<j\le n}[\{\sigma_{ii}^{\gamma}\sigma_{jj}^{\gamma} + (\sigma_{ij}^{\gamma})^2\} + \{\sigma_{ii}^{*\gamma}\sigma_{jj}^{*\gamma} - (\sigma_{ij}^{*\gamma})^2\}].$$

By using Gauss equation for the Levi-Civita connection, we have

$$\tau = \tau_0 + \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[n(n-2) + \mathrm{tr}^2(\phi) - \mathrm{tr}(\phi^*)] + \left(-\frac{(1-\psi)c_p + \psi c_q}{4}\right)2(n-1)\mathrm{tr}(\phi) - 2\hat{\tau}_0$$
$$- \frac{1}{2}\sum_{\gamma=n+1}^{m}\sum_{1\le i<j\le n}[\{\sigma_{ii}^{\gamma}\sigma_{jj}^{*\gamma} - (\sigma_{ij}^{\gamma})^2\} + \{\sigma_{ii}^{*\gamma}\sigma_{jj}^{*\gamma} - (\sigma_{ij}^{*\gamma})^2\}]. \tag{3.1}$$

Now, the sectional K-curvature $K(\pi)$ of the plane section π is

$$K(\pi) = \frac{1}{2}[g(R(e_1, e_2)e_2, e_1) + g(R^*(e_1, e_2)e_2, e_1) - 2g(R^0(e_1, e_2)e_2, e_1)]. \tag{3.2}$$

Using (2.7) and Gauss equation for R and R^* and putting the values in (3.2), we obtain

$$K(\pi) = \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[1 + g(\phi e_1, e_1)g(\phi_2, e_2) - g(\phi e_1, e_2)g(\phi e_2, e_1)] - K_0(\pi)$$
$$+ \frac{1}{2}\sum_{\gamma=n+1}^{m}\{[\sigma_{11}^{\gamma}\sigma_{22}^{*\gamma} + \sigma_{11}^{*\gamma}\sigma_{22}^{\gamma} - 2\sigma_{12}^{*\gamma}\sigma_{12}^{\gamma}]\}.$$

Using $\sigma + \sigma^* = 2\sigma^0$, we obtain

$$K(\pi) = \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[1 + g(\phi e_1, e_1)g(\phi_2, e_2) - g(\phi e_1, e_2)g(\phi_2, e_1)]$$
$$- K_0(\pi) + 2\sum_{\gamma=n+1}^{m}[\sigma_{11}^{0\gamma}\sigma_{22}^{0\gamma} - (\sigma_{12}^{0\gamma})^2] - \frac{1}{2}\sum_{\gamma=n+1}^{m}\{[\sigma_{11}^{\gamma}\sigma_{22}^{\gamma} - (\sigma_{12}^{\gamma})^2] + [\sigma_{11}^{*\gamma}\sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2]\}.$$

Using Gauss equation with respect to Levi-Civita connection, we have

$$K(\pi) = K_0(\pi) + \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[1 + g(\phi e_1, e_1)g(\phi_2, e_2) + g(\phi e_1, e_2)g(\phi^* e_1, e_2)]$$
$$- 2\hat{K}_0(\pi) - \frac{1}{2}\sum_{\gamma=n+1}^{m}[\sigma_{11}^{\gamma}\sigma_{22}^{\gamma} - (\sigma_{12}^{\gamma})^2] - \frac{1}{2}\sum_{\gamma=n+1}^{m}[\sigma_{11}^{*\gamma}\sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2].$$

The above equation can be written in the form

$$K(\pi) = K_0(\pi) + \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[1 + \Psi(\pi) + \Theta(\pi)] - 2\hat{K}_0(\pi) - \frac{1}{2}\sum_{\gamma=n+1}^{m}[\sigma_{11}^{\gamma}\sigma_{22}^{\gamma} - (\sigma_{12}^{\gamma})^2]$$
$$- \frac{1}{2}\sum_{\gamma=n+1}^{m}[\sigma_{11}^{*\gamma}\sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2]. \tag{3.3}$$

From (3.1) and (3.3), we have

$$(\tau - K(\pi)) - (\tau_0 - K_0(\pi)) = \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[n(n-2) + \mathrm{tr}^2(\phi) - \mathrm{tr}(\phi^*)]$$
$$+ \left(-\frac{(1-\psi)c_p + \psi c_q}{4}\right)2(n-1)\mathrm{tr}(\phi) + \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[1 + \Psi(\pi) + \Theta(\pi)]$$

$$- \frac{1}{2} \sum_{\gamma=n+1}^{m} [\sigma_{ii}^{\gamma}\sigma_{jj}^{\gamma} - (\sigma_{ij}^{\gamma})^2] - \frac{1}{2} \sum_{\gamma=n+1}^{m} [\sigma_{ii}^{*\gamma}\sigma_{jj}^{*\gamma} - (\sigma_{ij}^{*\gamma})^2]$$
$$+ \frac{1}{2} \sum_{\gamma=n+1}^{m} \sum_{\alpha=1}^{3} \{[\sigma_{11}^{\gamma}\sigma_{22}^{\gamma} - (\sigma_{12}^{\gamma})^2] + [\sigma_{11}^{*\gamma}\sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2]\} + 2\hat{K}_0(\pi) - 2\hat{\tau}_0.$$

Using Lemma 1, we can obtain the above equation in simplified form as

$$\begin{aligned}(\tau - K(\pi)) - (\tau_0 - K_0(\pi)) \geq & \left(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \right)[n(n-2) + \operatorname{tr}^2(\phi) - \operatorname{tr}(\phi^*)] \\ & + \left(- \frac{(1-\psi)c_p + \psi c_q}{4} \right) 2(n-1)\operatorname{tr}(\phi) + \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \right)[1 + \Psi(\pi) + \Theta(\pi)] \\ & - \frac{n^2(n-2)}{4(n-1)}[\|H\|^2 + \|H^*\|^2] + 2\hat{K}_0(\pi) - 2\hat{\tau}_0.\end{aligned}$$

This proves our claims. □

Corollary 1. *Let (N, g, ϕ) be a Golden-like statistical manifold of dimension m and M be its totally real statistical submanifold of dimension n. Then, we have the following*

$$(\tau - K(\pi)) - (\tau_0 - K_0(\pi)) \geq \left(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \right)[n(n-2) - 1] - \frac{n^2(n-2)}{4(n-1)}[\|H\|^2 + \|H^*\|^2] + 2\hat{K}_0(\pi) - 2\hat{\tau}_0.$$

Theorem 2. *Let M^n be a statistical submanifold of a Golden-like statistical manifold N^m. Then for the generalized normalized δ-Casorati curvature, we have the following optimal relationships*:

(*i*) *For any real number k, such that $0 < k < n(n-1)$,*

$$\begin{aligned}\rho \leq & \frac{\delta_C^0(k; n-1)}{n(n-1)} + \frac{1}{(n-1)}C^0 - \frac{n}{(n-1)}g(H, H^*) - \frac{2n}{n(n-1)}\|H^0\|^2 \\ & + \frac{1}{n(n-1)}\left(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \right)[n(n-2) + \operatorname{tr}^2(\phi) - \operatorname{tr}(\phi^*)] + \frac{2}{n}\left(- \frac{(1-\psi)c_p + \psi c_q}{4} \right)\operatorname{tr}(\phi),\end{aligned} \tag{3.4}$$

where $\delta_C^0(k; n-1) = \frac{1}{2}[\delta_C(k; n-1) + \delta_C^*(k; n-1)]$.

(*ii*) *For any real number $k > n(n-1)$,*

$$\begin{aligned}\rho \leq & \frac{\hat{\delta}_C^0(k; n-1)}{n(n-1)} + \frac{1}{(n-1)}C^0 - \frac{n}{(n-1)}g(H, H^*) - \frac{2n}{n(n-1)}\|H^0\|^2 \\ & + \frac{1}{n(n-1)}\left(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \right)[n(n-2) + \operatorname{tr}^2(\phi) - \operatorname{tr}(\phi^*)] + \frac{2}{n}\left(- \frac{(1-\psi)c_p + \psi c_q}{4} \right)\operatorname{tr}(\phi),\end{aligned} \tag{3.5}$$

where $\hat{\delta}_C^0(k; n-1) = \frac{1}{2}[\hat{\delta}_C(k; n-1) + \hat{\delta}_C^*(k; n-1)]$.

Proof. Let $p \in M$ and $\{e_1, \ldots, e_n\}$, $\{e_{n+1}, \ldots, e_m\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$, respectively. From Gauss equation, we obtain

$$\begin{aligned}2\tau = & n^2 g(H, H^*) - n \sum_{1 \leq i,j \leq n} g(\sigma^*(e_i, e_j), \sigma(e_i, e_j)) + \left(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \right)[n(n-2) + \operatorname{tr}^2(\phi) - \operatorname{tr}(\phi^*)] \\ & + \left(- \frac{(1-\psi)c_p + \psi c_q}{4} \right) 2(n-1)\operatorname{tr}(\phi).\end{aligned}$$

Denote $H + H^* = 2H^0$ and $C + C^* = 2C^0$. Then the above equation becomes

$$\begin{aligned}2\tau = & \left(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \right)[n(n-2) + \operatorname{tr}^2(\phi) - \operatorname{tr}(\phi^*)] + \left(- \frac{(1-\psi)c_p + \psi c_q}{4} \right) 2(n-1)\operatorname{tr}(\phi) + 2n^2\|H^0\|^2 \\ & - \frac{n^2}{2}(\|H\|^2 + \|H^*\|^2) - 2nC^0 + \frac{n}{2}(C + C^*).\end{aligned}$$

We define a polynomial $\mathcal{P}$ in the components of second fundamental form as:

$$\mathcal{P} = kC^0 + a(k)C^0(\mathcal{W}) + \frac{n}{2}(C + C^*) - \frac{n^2}{2}(\|H\|^2 + \|H^*\|^2) - 2\tau(p) + \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[n(n-2) + \mathrm{tr}^2(\phi) - \mathrm{tr}(\phi^*)] + \left(-\frac{(1-\psi)c_p + \psi c_q}{4}\right)2(n-1)\mathrm{tr}(\phi), \tag{3.6}$$

where $\mathcal{W}$ is a hyperplane in T_pM. Assuming $\mathcal{W}$ is spanned by $\{e_1, \dots, e_{n-1}\}$, we have

$$\mathcal{P} = \sum_{\alpha=n+1}^{m}\left[\frac{2n+k}{n}\sum_{i,j=1}^{n}(\sigma_{ij}^{0\alpha})^2 + a(k)\frac{1}{n-1}\sum_{i,j=1}^{n-1}(\sigma_{ij}^{0\alpha})^2 - 2\left(\sum_{i,j=1}^{n}\sigma_{ij}^{0\alpha}\right)^2\right]. \tag{3.7}$$

Equation (3.7) yields

$$\begin{aligned}\mathcal{P} = \sum_{\alpha=n+1}^{n}\Bigg[&\left(\frac{2(2n+k)}{n} + \frac{2a(k)}{n-1}\right)\sum_{1\le i<j\le n-1}(\sigma_{ij}^{0\alpha})^2 + \left(\frac{2(2n+k)}{n} + \frac{2a(k)}{n-1}\right)\sum_{i=1}^{n-1}(\sigma_{in}^{0\alpha})^2\\ &\left(\frac{2n+k}{n} + \frac{a(k)}{n-1} - 2\right)\sum_{i=1}^{n-1}(\sigma_{ii}^{0\alpha})^2 - 4\sum_{1\le i<j\le n}\sigma_{ii}^{0\alpha}\sigma_{jj}^{0\alpha} + \left(\frac{2n+k}{n} - 2\right)(\sigma_{nn}^{0\alpha})^2\Bigg]\\ \ge \sum_{\alpha=n+1}^{m}\Bigg[&\frac{k(n-1) + a(k)n}{n(n-1)}\sum_{i=1}^{n-1}(\sigma_{ii}^{0\alpha})^2 + \frac{k}{m}(\sigma_{nn}^{0\alpha})^2 - 4\sum_{1\le i<j\le n}\sigma_{ii}^{0\alpha}\sigma_{jj}^{0\alpha}\Bigg].\end{aligned}$$

Let y_α be a quadratic form defined by $y_\alpha = \mathbb{R}^n \to \mathbb{R}$ for any $\alpha \in \{n+1, \dots, m\}$,

$$y_\alpha(\sigma_{11}^{0\alpha}, \sigma_{22}^{0\alpha}, \dots, \sigma_{nn}^{0\alpha}) = \sum_{i=1}^{n-1}\frac{k(n-1) + a(k)n}{n(n-1)}(\sigma_{ii}^{0\alpha})^2 + \frac{k}{n}(\sigma_{nn}^{0\alpha})^2 - 4\sum_{1\le i<j\le n}\sigma_{ii}^{0\alpha}\sigma_{jj}^{0\alpha},$$

and the optimization problem

$$\min\{y_\alpha\}$$

subject to $\mathcal{G} : \sigma_{11}^{0\alpha} + \sigma_{22}^{0\alpha} + \ldots + \sigma_{nn}^{0\alpha} = p^\alpha$, where p^α is a real constant.

The partial derivatives of y_α are

$$\begin{cases}\dfrac{\partial y_\alpha}{\partial \sigma_{ii}^{0\alpha}} = 2\dfrac{k(n-1) + a(k)n}{n(n-1)}\sigma_{ii}^{0\alpha} - 4\left(\displaystyle\sum_{l=1}^{n}\sigma_{ll}^{0\alpha} - \sigma_{ii}^{0\alpha}\right) = 0\\ \dfrac{\partial y_\alpha}{\partial \sigma_{nn}^{0\alpha}} = \dfrac{2k}{n}\sigma_{nn}^{0\alpha} - 4\displaystyle\sum_{l=1}^{n-1}\sigma_{ll}^{0\alpha} = 0,\end{cases} \tag{3.8}$$

for every $i \in \{1,\dots,n-1\}$, $\alpha \in \{n+1,\dots,m\}$.

For an optimal solution $(\sigma_{11}^{0\alpha}, \sigma_{22}^{0\alpha}, \dots, \sigma_{nn}^{0\alpha})$ of the problem, the vector grad y_α is normal at $\mathcal{G}$. It is collinear with the vector $(1,\dots,1)$. From (3.8) and $\sum_{i=1}^{n}\sigma_{ii}^{0\alpha} = p^\alpha$ it follows that a critical point of the corresponding problem has the form

$$\begin{cases}\sigma_{ii}^{0\alpha} = \dfrac{2n(n-1)}{(n-1)(2n+k) + na(k)}p^\alpha\\ \sigma_{nn}^{0\alpha} = \dfrac{2n}{2n+k}p^\alpha,\end{cases}$$

for any $i \in \{1,\dots,n-1\}$, $\alpha \in \{n+1,\dots,m\}$.

For $p \in \mathcal{G}$, let $\mathcal{A}$ be a 2-form, $\mathcal{A} : T_p\mathcal{G} \times \mathcal{G} \to \mathbb{R}$ defined by

$$\mathcal{A}(X, Y) = \mathrm{Hess}(y_\alpha)(X, Y) + \langle\sigma'(X, Y), (\mathrm{grad}(y)(p))\rangle,$$

where σ' is the second fundamental form of $\mathcal{G}$ in $\mathbb{R}^n$ and $\langle\cdot, \cdot\rangle$ is the standard inner product on $\mathbb{R}^n$.

Moreover, it is easy to see that the Hessian matrix of y_α has the form

$$\mathrm{Hess}(y_\alpha) = \begin{pmatrix} 2\frac{(n-1)(k+2n)+na(k)}{n(n-1)} & -4 & \cdots & -4 & -4 \\ -4 & 2\frac{(n-1)(k+2n)+na(k)}{n(n-1)} & \cdots & -4 & -4 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -4 & -4 & \cdots & 2\frac{(n-1)(k+2n)+na(k)}{n(n-1)} & -4 \\ -4 & -4 & \cdots & -4 & \frac{2k}{n} \end{pmatrix}.$$

As $\mathcal{G}$ is a totally geodesic hyperplane in $\mathbb{R}^n$, considering a vector $X = (X_1, \ldots, X_n)$ tangent to $\mathcal{G}$ at an arbitrary point x on $\mathcal{G}$, that is, verifying the relation $\sum_{i=1}^{n} = 0$, we obtain

$$\begin{aligned} \mathcal{A}(X,X) &= 2\frac{(n-1)(k+2n)+na(k)}{n(n-1)}\sum_{i=1}^{n-1} X_i^2 + \frac{2k}{n}X_n^2 - 8\sum_{i,j=1}^{n} X_iX_j, \quad i \neq j \\ &= 2\frac{(n-1)(k+2n)+na(k)}{n(n-1)}\sum_{i=1}^{n-1} X_i^2 + \frac{2k}{n}X_n^2 + 4\left(\sum_{i=1}^{n} X_i\right)^2 - 8\sum_{i,j=1}^{n} X_iX_j, \quad i \neq j \\ &= 2\frac{(n-1)(k+2n)+na(k)}{n(n-1)}\sum_{i=1}^{n-1} X_i^2 + \frac{2k}{n}X_n^2 + 4\sum_{i=1}^{n} X_i^2 \geq 0. \end{aligned}$$

Hence, by Lemma 2, the critical point $(\sigma_{11}^{0\alpha}, \sigma_{22}^{0\alpha}, \ldots, \sigma_{nn}^{0\alpha})$ of y_α is the global minimum point of the problem. Moreover, since $y_\alpha(\sigma_{11}^{0\alpha}, \sigma_{22}^{0\alpha}, \ldots, \sigma_{nn}^{0\alpha}) = 0$, we obtain $\mathcal{P} \geq 0$. This implies that

$$\begin{aligned} 2\tau &\leq kC^0 + a(k)C^0(\mathcal{W}) + \frac{n}{2}(C + C^0) - \frac{n^2}{2}(\|H\|^2 + \|H^*\|^2) + \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[n(n-2) + \mathrm{tr}^2(\phi) - \mathrm{tr}(\phi^*)] \\ &\quad + \left(-\frac{(1-\psi)c_p + \psi c_q}{4}\right)2(n-1)\mathrm{tr}(\phi). \end{aligned}$$

Thus, we have

$$\begin{aligned} \rho &\leq \frac{k}{n(n-1)}C^0 + \frac{(n+k)(n^2-n-k)}{n^2k}C^0(\mathcal{W}) + \frac{1}{2(n-1)}(C + C^*) - \frac{n}{2(n-1)}(\|H\|^2 + \|H^*\|^2) \\ &\quad + \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left[\frac{(n-2)}{(n-1)} + \frac{\mathrm{tr}^2(\phi)}{n(n-1)} - \frac{\mathrm{tr}(\phi^*)}{n(n-1)}\right] + \frac{2}{n}\left(-\frac{(1-\psi)c_p + \psi c_q}{4}\right)\mathrm{tr}(\phi). \end{aligned} \tag{3.9}$$

Now taking the infimum over all tangent hyperplanes $\mathcal{W}$ of T_pM, we obtain

$$\begin{aligned} \rho &\leq \frac{\delta_C(k;\, n-1)}{n(n-1)} + \frac{1}{2(n-1)}(C + C^*) - \frac{n}{2(n-1)}(\|H\|^2 + \|H^*\|^2) \\ &\quad + \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left[\frac{(n-2)}{(n-1)} + \frac{\mathrm{tr}^2(\phi)}{n(n-1)} - \frac{\mathrm{tr}(\phi^*)}{n(n-1)}\right] + \frac{2}{n}\left(-\frac{(1-\psi)c_p + \psi c_q}{4}\right)\mathrm{tr}(\phi). \end{aligned}$$

This gives us the inequality (3.4). Similarly on taking the supremum over all tangent hyperplanes $\mathcal{W}$ of T_pM in (3.9), we obtain the inequality (3.5). □

Corollary 2. *Let M^n be a totally real statistical submanifold of a Golden-like statistical manifold N^m. Then for the generalized normalized δ-Casorati curvature, we have the following optimal relationships*:

(*i*) *For any real number k, such that $0 < k < n(n-1)$,*

$$\rho \leq \frac{\delta_C^0(k;\, n-1)}{n(n-1)} + C^0 - \frac{n}{(n-1)}g(H, H^*) - \frac{2n}{n(n-1)}\|H^0\|^2 + \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left(\frac{n-2}{n-1}\right),$$

where $\delta_C^0(k;\, n-1) = \frac{1}{2}[\delta_C(k;\, n-1) + \delta_C^*(k;\, n-1)]$.

(*ii*) *For any real number* $k > n(n-1)$,

$$\rho \leq \frac{\widehat{\delta}_C^0(k;\, n-1)}{n(n-1)} + \frac{1}{(n-1)}C^0 - \frac{n}{(n-1)}g(H, H^*) - \frac{2n}{n(n-1)}\|H^0\|^2 + \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left(\frac{n-2}{n-1}\right),$$

where $\widehat{\delta}_C^0(k;\, n-1) = \frac{1}{2}[\widehat{\delta}_C(k;\, n-1) + \widehat{\delta}_C^*(k;\, n-1)]$.

3.1 Equality case

The submanifolds enjoying the equality for the Casorati curvature at every point are called Casorati ideal submanifolds (for instance see [38]). In this subsection, we investigate the Casorati ideal submanifolds for (3.4) and (3.5) in terms of their second fundamental form.

Theorem 3. *The Casorati ideal Lagrangian submanifolds for* (3.4) *and* (3.5) *are totally geodesic submanifolds with respect to Levi-Civita connection.*

Proof. First, we find out the critical points of $\mathcal{P}$

$$\sigma^c = (\sigma_{11}^{0n+1}, \sigma_{12}^{0n+1}, \ldots, \sigma_{nn}^{0n+1}, \ldots, \sigma_{11}^{0m}, \ldots, \sigma_{nn}^{0m})$$

as the solutions of the following system of linear homogeneous equations

$$\begin{cases} \dfrac{\partial \mathcal{P}}{\partial \sigma_{ii}^{0\alpha}} = 2\left[\dfrac{2n+k}{k} + \dfrac{(k+n)(n^2-n-k)}{nk} - 2\right]\sigma_{ii}^{0\alpha} - 4\displaystyle\sum_{l=1, l\neq i}^{n} \sigma_{ll}^{0\alpha} = 0, \\ \dfrac{\partial \mathcal{P}}{\partial \sigma_{nn}^{0\alpha}} = 2\dfrac{k}{n}\sigma_{nn}^{0\alpha} - 4\displaystyle\sum_{l=1}^{n-1}\sigma_{ll}^{0\alpha} = 0, \\ \dfrac{\partial \mathcal{P}}{\partial \sigma_{ij}^{0\alpha}} = 4\left[\dfrac{2n+k}{k} + \dfrac{(k+n)(n^2-n-k)}{nk}\right]\sigma_{ij}^{0\alpha} = 0, \quad i \neq j, \\ \dfrac{\partial \mathcal{P}}{\partial \sigma_{in}^{0\alpha}} = 4\left[\dfrac{2n+k}{k} + \dfrac{(k+n)(n^2-n-k)}{nk}\right]\sigma_{in}^{0\alpha} = 0. \end{cases}$$

The critical points satisfy $\sigma_{ij}^{0\alpha} = 0, i, j = \in\{1, \ldots, n\}$ and $\alpha = \{n+1, \ldots, m\}$. Moreover, we know that $\mathcal{P} \geq 0$ and $\mathcal{P}(\sigma^c) = 0$, then the critical point σ^c is a minimum point of $\mathcal{P}$. Consequently, the equality holds in (3.4) and (3.5) if and only if $\sigma_{ij}^{\alpha} + \sigma_{ij}^{*\alpha} = 0$, for $i, j \in \{1, \ldots, n\}$ and $\alpha \in \{n+1, \ldots, m\}$. In other words, the equalities hold identically at all points $p \in M$ if and only if $\sigma + \sigma^* = 0$, where σ and σ^* are the imbedding curvature tensors of the submanifold associated with the dual connection ∇ and ∇^*, respectively. Hence, the equality in (3.4) and (3.5) holds at p if and only if p is totally geodesic point with respect to Levi-Civita connection. ☐

Remark 1. The results for normalized Casorati curvature can be easily obtained by using (2.8) and (2.9) in the inequalities (3.4) and (3.5).

4 Conclusion

In this paper, we introduced and studied Golden-like statistical manifolds. We obtained some basic inequalities for curvature invariants of statistical submanifolds in Golden-like statistical manifolds. Also, in support of our definition, we provided a couple of examples.

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