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BASIC PROPERTIES OF CRITICAL LOGNORMAL MULTIPLICATIVE CHAOS

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We study one-dimensional exact scaling lognormal multiplicative chaos measures at criticality. Our main results are the determination of the exact asymptotics of the right tail of the distribution of the total mass of the measure, and an almost sure upper bound for the modulus of continuity of the cumulative distribution function of the measure. We also find an almost sure lower bound for the increments of the measure almost everywhere with respect to the measure itself, strong enough to show that the measure is supported on a set of Hausdorff dimension 0.

1. Introduction. Multiplicative chaos is a theory developed by Kahane in the eighties [26, 28, 29]. It deals with multiplicative processes generating martingales, which take values in the cone of nonnegative Radon measures on σ-compact metric spaces. This theory is based on the lognormal multiplicative chaos proposed by Mandelbrot to model turbulence [36], as well as the works previously achieved by Kahane and Peyrière [31] on the simplified model of multiplicative cascades on trees still proposed by Mandelbrot [35, 37], namely the so-called Mandelbrot cascades, which assume no lognormality property. The study of random measures generated by such multiplicative processes also originates from random covering and percolation theory questions (see [6, 22, 26, 27, 29, 30]). When statistically self-similar, as it is the case for limits of Mandelbrot cascades, these measures provide nice illustrations of the so-called multifractal formalism, as well as models in the study of intermittent phenomena beyond turbulence, like the distribution of rare minerals in earth [38] or stock exchange fluctuations in finance [39]. Examples of such measures on $\mathbb{R}^d$ possessing continuous (rather than only discrete for limits of Mandelbrot cascades) scaling properties are some of the Gaussian multiplicative chaos built by Kahane in [28] or the Lévy multiplicative chaos built by Fan in [23], the compound Poisson cascades built by Barral and Mandelbrot [10] and their generalization to the so-called infinitely divisible cascades by Bacry and Muzy in [5].

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Kahane's lognormal multiplicative chaos has been recently revisited and completed in several directions [3, 43, 44]. Also, it is now a central tool in two-dimensional quantum gravity theory since it provides, through the exponential of the Gaussian free field, the random measures used to obtain the first rigorous results in direction to the so-called KPZ formula in works by Duplantier and Sheffield [19, 20], as well as Rhodes and Vargas [42] (see also Benjamini and Schramm [12] for a one-dimensional version in the framework of Mandelbrot multiplicative cascades on $[0, 1]$). Nondegenerate limits of lognormal multiplicative chaos associated with the exponential of the Gaussian free field on the circle have also been used successfully by Astala, Jones, Kupiainen and Saksman in [4] to build random planar curves by conformal welding. The families of Gaussian multiplicative chaos considered in these questions are naturally parameterized by a continuous parameter $\beta \in [0, \beta_c)$. In the application to quantum gravity, β is in bijection with the so-called central charge; in random energy models, it corresponds to the inverse of a temperature; in turbulence, it is a measure of the intermittence; from a purely geometric viewpoint, it is a decreasing function of the Hausdorff dimension of the associated measure in the Euclidean geometry. At the critical temperature, and below it, the limit μ_β of the martingale $\mu_{\beta,t}$ provided by the associated multiplicative process vanishes almost surely. For $\beta > \beta_c$, it is nevertheless possible to give a sense to the corresponding dual KPZ formula [8, 16] by considering measures essentially by subordinating a suitable nondegenerate Gaussian multiplicative chaos to some stable Lévy subordinators; this yields an atomic measure.

At the critical value β_c, one needs new results in multiplicative chaos theory. They were recently obtained by Duplantier, Rhodes, Sheffield and Vargas in [17, 18], inspired by results recently achieved by Aïdékon and Shi in the context of the martingales in the branching random walk [2]. Thus, it is possible to get a nontrivial positive measure at the critical temperature as the limit of the signed measures $-\frac{\mathrm{d}\mu_{\beta,t}}{\mathrm{d}\beta}|_{\beta=\beta_c}$ as $t \to \infty$. Moreover, this measure is continuous. We also mention that like in the context of martingales in the branching random walks [2], the critical measure can be obtained as limit in probability of $\mu_{\beta_c,t}$ properly normalized [18]. During the completion of this paper, corresponding normalization results [34] were obtained also in the case $\beta > \beta_c$. These normalization results are analogous to those known in the branching random walk and random energy models frameworks [11, 33, 45].

This paper is dedicated to the study of some properties of such critical lognormal multiplicative chaos measure. We concentrate on the exactly scale-invariant one-dimensional construction. Our main results are the determination of the asymptotic behavior of the tail of the distribution of the total mass of the measure, a bound for the modulus of continuity of the measure for which the previous tail asymptotic behavior is crucial, and an estimate from below of the measure increments almost everywhere with respect to the measure, which completes the estimation provided by the modulus of continuity and goes beyond the simple fact that the measure has Hausdorff dimension 0; see Theorems 1, 2 and 4 below.

As a motivation to study the exactly scale invariant measure, let us note that the Gaussian field used to construct the exactly scale invariant measure in one dimension is simply the Gaussian free field restricted to a line segment. Thus, the measure can be viewed as a boundary measure of Liouville quantum gravity (see, e.g., [20]) and conjecturally as the boundary measure of random planar maps mapped to the upper half-plane. Moreover, while the results are mainly stated for the one-dimensional exactly scale invariant measure, we expect similar results to hold quite generally for Gaussian multiplicative chaos measures in any dimension. In Section 5, we finish the paper with a discussion of extensions of our results to a higher-dimensional setting.

1.1. *Definitions and notation.* In this section, we fix notation and give the precise definitions of the objects studied in this paper. Formally, the one-dimensional lognormal multiplicative chaos measures μ_β are random measures given by

$$\mu_\beta(\mathrm{d}x) = e^{\beta X(x) - (\beta^2/2)\mathbb{E}X(x)^2}\,\mathrm{d}x, \tag{1}$$

where $(X(x))_{x\in\mathbb{R}}$ is a logarithmically correlated centered Gaussian field, that is, a centered Gaussian process with

$$\mathbb{E}X(x)X(y) \sim \log\frac{1}{|x-y|} \qquad \text{as } |x-y| \to 0.$$

However, the logarithmic singularity of the correlation kernel implies that the realizations of X are not smooth enough to be functions, but must instead be defined as random distributions. To overcome this major technical obstacle, in Kahane's theory of multiplicative chaos one gives a rigorous meaning to the expression (1) by considering nonsingular approximations X_t to the field X, defining the measures $\mu_{\beta,t}$ corresponding to these regularizations and then taking the weak limit of the measures $\mu_{\beta,t}$ as the approximation parameter is taken to infinity. In this way, one completely avoids the problem of defining the exponential of a distribution.

We mainly concentrate on the *exactly scale invariant* construction. This scaling property, to be defined below, is central to the proof of Theorem 1. The one-dimensional exactly scale invariant construction is most easily understood through the following geometric construction, originally due to Bacry and Muzy [5].

Let λ denote the hyperbolic area measure on the upper half-plane, that is,

$$\lambda(A) = \int_A \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \qquad \text{for all } A \subset \mathbb{R}\times\mathbb{R}^+.$$

For $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$, let $\mathcal{C}_t(x)$ denote the set

$$\mathcal{C}_t(x) = \{(x',y') | y' > \max(2|x'-x|, e^{-t}), |x'-x| < \tfrac{1}{2}\}$$

and for a compact interval $I \subset \mathbb{R}$ of length less than or equal to 1, denote

$$\mathcal{C}_t(I) = \bigcap_{x \in I} \mathcal{C}_t(x).$$

Note that for $t \geq \log 1/|I|$ we have $\mathcal{C}_t(I) = \mathcal{C}_{\log 1/|I|}(I)$. Next, let W denote the white noise on $\mathbb{R} \times \mathbb{R}^+$ with control measure λ. We consider W a random real function on the Borel sets of $\mathbb{R} \times \mathbb{R}^+$ with finite λ-measure characterized by the following properties: for all disjoint Borel sets $A, B \subset \mathbb{R} \times \mathbb{R}^+$ such that $\lambda(A), \lambda(B) < \infty$:

(1) $W(A)$ is a centered Gaussian random variable with variance $\lambda(A)$,
(2) the random variables $W(A)$ and $W(B)$ are independent, and
(3) almost surely we have $W(A \cup B) = W(A) + W(B)$.

Define

$$X_t(x) = W\big(\mathcal{C}_t(x)\big) \qquad \text{for all } x \in \mathbb{R}, t \in [0, \infty).$$

For a fixed $t > 0$, the covariance structure of the process $(X_t(x))_{x \in \mathbb{R}}$ can be computed to be

$$\mathbb{E}X_t(x)X_t(y) = \begin{cases} t + 1 - e^t|x-y|, & |x-y| < e^{-t}, \\ \log \dfrac{1}{|x-y|}, & e^{-t} \leq |x-y| \leq 1, \\ 0, & 1 < |x-y|. \end{cases}$$

For any interval $I \subset \mathbb{R}$ of length less than or equal to 1 and $x \in I$, we denote

$$X_t(I) = W\big(\mathcal{C}_t(I)\big) \quad \text{and} \quad X_t^I(x) = W\big(\mathcal{C}_t(x) \setminus \mathcal{C}_t(I)\big)$$

to obtain the decomposition $X_t(x) = X_t(I) + X_t^I(x)$, where $X_t(I)$ is independent of the process $(X_t^I(x))_{x \in I}$. Since $\mathcal{C}_t(I) = \mathcal{C}_{\log 1/|I|}(I)$ for $t \geq \log 1/|I|$, we denote $X(I) := X_{\log 1/|I|}(I)$. Owing to the geometry of the construction, the field $(X_t(x))$ satisfies the following scale invariance property: for all intervals $I \subset \mathbb{R}$ and $e^{-t} < |I| < 1$, we have

$$(2) \qquad \big(X_t(x)\big)_{x \in I} = \big(X_t(I) + X_t^I(x)\big)_{x \in I} \stackrel{d}{=} \big(X_t(I) + X'_{t - \log|I|}\big(x/|I|\big)\big)_{x \in I},$$

where X' is an independent realization of the field X. For the reader's convenience, we give the geometric explanation for this scaling property in the Appendix.

For $\beta \in (0, \sqrt{2})$, we construct the measures $\mu_{\beta,t}$ on the unit interval by setting

$$(3) \qquad \mu_{\beta,t}(I) = \int_I e^{\beta X_t(x) - (\beta^2/2)\mathbb{E}X_t(x)^2} \, \mathrm{d}x$$

for all intervals $I \subset [0, 1]$. This construction fits into the framework of Kahane's theory of multiplicative chaos [26], which implies that almost surely the limit $\mu_\beta = \lim_{t \to \infty} \mu_{\beta,t}$ exists in the sense of weak convergence of measures and that the limit measure satisfies $\mu_\beta(I) > 0$ for all intervals $I \subset [0, 1]$. The scaling property (2) implies that the measures μ_β are exactly scale invariant, especially that

$$\mu_\beta(I) \overset{d}{=} |I| e^{\beta X(I) - (\beta^2/2)\mathbb{E}X(I)^2} \mu'_\beta([0, 1]) \qquad \text{for all intervals } I \subset [0, 1], \tag{4}$$

where μ'_β is an independent realization of μ_β and $X(I)$, defined as above, is a centered Gaussian random variable of variance $\log \frac{1}{|I|}$.

Kahane's work also implies that the corresponding construction for $\beta \geq \sqrt{2}$ results in degenerate limit measures, that is, the limit measure will be almost surely null. However, the exact scaling relation above makes sense for all $\beta > 0$. It has recently been shown by Duplantier, Rhodes, Sheffield and Vargas [17] that by defining for each interval $I \subset [0, 1]$

$$\begin{aligned} \mu_{\sqrt{2},t}(I) &= -\frac{\mathrm{d}}{\mathrm{d}\beta}\bigg|_{\beta=\sqrt{2}} \mu_{\beta,t}(I) \\ &= \int_I \bigl(\sqrt{2}(t+1) - X_t(x)\bigr) e^{\sqrt{2}X_t(x) - \mathbb{E}X_t(x)^2} \, \mathrm{d}x \end{aligned} \tag{5}$$

one obtains a nondegenerate almost sure weak limit $\mu_{\sqrt{2}} = \lim_{t\to\infty} \mu_{\sqrt{2},t}$ for which $\mu_{\sqrt{2}}(I) > 0$ almost surely for all intervals $I \subset [0, 1]$. As in the case of branching random walks (or equivalently, multiplicative cascades), this derivative turns out to be the correct replacement for the measures (3) in the case $\beta = \sqrt{2}$, at the very least in the sense that $\mu_{\sqrt{2}}$ is nontrivial and turns out to satisfy the exact scaling property: as detailed in the Appendix, we have especially

$$\mu_{\sqrt{2}}(I) \overset{d}{=} |I| e^{\sqrt{2}X(I) - \mathbb{E}X(I)^2} \mu'_{\sqrt{2}}([0, 1]) \qquad \text{for all intervals } I \subset [0, 1]. \tag{6}$$

In defining the lognormal multiplicative chaos measure for the critical parameter value $\beta = \sqrt{2}$, the peculiar normalizing factor $(\sqrt{2}(t+1) - X_t(x))$ may also be replaced by a normalization that is deterministic and also independent of x. Inspired by the arguments of Aïdékon and Shi [2] in the case of branching random walks, Duplantier, Rhodes, Sheffield and Vargas [18] recently proved that there exists a deterministic constant $c > 0$ such that for every interval $I \subset [0, 1]$ one has

$$\sqrt{t} \int_I e^{\sqrt{2}X_t(x) - \mathbb{E}X_t(x)^2} \, \mathrm{d}x \to c\mu_{\sqrt{2}}(I) \qquad \text{in probability as } t \to \infty. \tag{7}$$

Before moving on to the statements of our results on the fine properties of $\mu_{\sqrt{2}}$, we make a final comment on the scale invariance properties of multiplicative chaos measures. In [17] and [18], the authors deal primarily with a slightly different construction, the $\star$-*scale invariant* lognormal multiplicative chaos measures. In terms of the geometric construction presented here, a $\star$-scale invariant random measure is obtained by replacing the field $(X_t(x))$ in (3), (5) or (7) by the field $(X_t(x) - X_0(x))$. Since we will make use of this construction in the proof of Theorem 2, we have included details on $\star$-scale invariance in the Appendix. However, as also noted in the papers themselves, the proofs of the convergence results in [17] and [18] are insensitive to these differences.

1.2. *Main results*. We will make use of the following result of Duplantier, Rhodes, Sheffield and Vargas [18], which is a corollary of the deterministic normalization (7).

THEOREM A. *For all* $h \in (0, 1)$, $\mathbb{E}(\mu_{\sqrt{2}}([0, 1])^h) < \infty$.

The first of our main theorems is a strengthening of this result, and analogous to the theorem of Buraczewski [14] on the fixed points of the smoothing transform.

THEOREM 1. *The tail probability of* $\mu_{\sqrt{2}}$ *has the asymptotic behavior*

$$\lim_{\lambda\to\infty} \lambda \mathbb{P}\big(\mu_{\sqrt{2}}([0, 1]) > \lambda\big) = c_1,$$

where the constant is given explicitly by

$$c_1 = \frac{2}{\log 2} \mathbb{E}\mu_{\sqrt{2}}([0, 1/2]) \log\left(1 + \frac{\mu_{\sqrt{2}}([1/2, 1])}{\mu_{\sqrt{2}}([0, 1/2])}\right) < \infty.$$

This theorem allows one to get detailed information on the geometric properties of the measure $\mu_{\sqrt{2}}$. The following result is analogous to our earlier result [9] on multiplicative cascades.

THEOREM 2. *For any interval* $I \subset [0, 1]$ *and* $\gamma < \frac{1}{2}$,

$$\mu_{\sqrt{2}}(I) \le C(\omega)\big(\log\big(1 + |I|^{-1}\big)\big)^{-\gamma}, \tag{8}$$

where $C(\omega) > 0$ *is an almost surely finite random constant*.

The proof of this theorem is inspired by the earlier result, but as the correlations of the field X in the construction of $\mu_{\sqrt{2}}$ are much more intricate than in the branching random walk underlying the cascade measures, more involved arguments are needed.

REMARK 3. We note that this result gives another proof for the result of [17] stating that almost surely, $\mu_{\sqrt{2}}$ has no atoms.

We also get a bound on the appropriate Hausdorff gauge function to measure the size of the smallest Borel sets fully supporting $\mu_{\sqrt{2}}$. We have the following result.

THEOREM 4. *Denote* $f_\alpha(n) = \exp(-\sqrt{6\log 2}\sqrt{n(\log n + \alpha \log\log n)})$ *for* $\alpha > \frac{1}{3}$. *Almost surely*,

$$\mu_{\sqrt{2}}\big(\big\{x : \mu_{\sqrt{2}}\big(I_n(x)\big) \ge f_\alpha(n) \text{ for all but finitely many } n\big\}\big) = \mu_{\sqrt{2}}([0, 1]),$$

where $I_n(x) \subset [0, 1]$ *is the dyadic interval of length* 2^{-n} *containing* x.

The proof uses large deviations estimates exploiting both the exact scaling property of $\mu_{\sqrt{2}}$ and the tail probabilities given by Theorem 1. This theorem implies the weaker claim that almost surely there exists a set of full $\mu_{\sqrt{2}}$-measure that has Hausdorff dimension 0, a fact that we state as Corollary 24.

For the log-normal critical Mandelbrot measure μ on trees, we establish in [9] that $\mu(\{x : \mu(I_n(x)) \le \psi(n) \text{ for all but finitely many } n\}) = \mu([0, 1])$, for all functions $\psi(n) = n^{-k}$, $k \ge 1$. In particular, the modulus of continuity (shown to be optimal) does not capture the measure increments behavior μ-almost everywhere—indeed, this is something one would expect of any multifractal measure. The proof exploits fine information about the renormalization theory for the low temperature measures $\mu_{\beta,n}$. Establishing this result in the present setting remains a challenge, as does proving the optimality of the bound provided by Theorem 2.

2. Tail probabilities. The proof of Theorem 1 follows the same idea as the earlier closely related results of Durrett and Liggett [21], Guivarc'h [25], Liu [32], Buraczewski [13, 14] and Barral and Jin [7]: one uses the smoothing transform (or in the case of multiplicative chaos, a similar distributional equation with more dependencies) to derive a Poisson equation satisfied by the quantity one is interested in, and then analyzes the behavior of the solutions of the Poisson equation at infinity. A key point in the derivations of the Poisson equations in all these proofs is the use of an alternate probability measure (the Peyrière probability), the idea of which goes back to the seminal paper of Kahane and Peyrière [31]. While using quite different kinds of methods, we also point out the result of Fyodorov and Bouchaud ([24]), where in the specific case of the Gaussian free field restricted to the unit circle, an explicit probability distribution for $\mu_\beta([0, 1])$ is obtained (though nonrigorously).

We also note that our form of the tail is related to the freezing transition scenario: it is believed (see, e.g., [15]) that a freezing transition occurs in essentially any logarithmically correlated random energy model and one universal feature of these models is that at the critical point, the Laplace transform should be of the form $1 - \mathbb{E}(\exp(-e^{-\beta_c x}\mu_{\beta_c}([0, 1]))) \asymp x e^{-\beta_c x}$ as $x \to \infty$. This is consistent with the tail $\mathbb{P}(\mu_{\beta_c}([0, 1]) > x) \asymp x^{-1}$ being universal.

In this section, we denote $\mu := \mu_{\sqrt{2}}$ and $Y := \mu([0, 1])$. The variable Y may be written as the fixed point of a "nonindependent smoothing transform" as follows:

$$
\begin{aligned}
Y &= \mu\bigl([0,1]\bigr) \\
&= \mu\bigl([0,1/2]\bigr) + \mu\bigl([1/2,1]\bigr) \\
&=: \tfrac{1}{2} e^{\sqrt{2}X([0,1/2]) - \mathbb{E}X([0,1/2])^2} Y_0 + \tfrac{1}{2} e^{\sqrt{2}X([1/2,1]) - \mathbb{E}X([1/2,1])^2} Y_1 \\
&=: W_0 Y_0 + W_1 Y_1.
\end{aligned}
\tag{9}
$$

Note that this requires that $\mu(\{\frac{1}{2}\}) = 0$ almost surely. To see this, simply note that by the scaling relation (6) and Theorem A we have, for any given $h \in (0, 1)$,

$\mathbb{E}\mu([\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon])^h \to 0$ as $\varepsilon \to 0$. By the exact scale invariance property (6) of μ we have $Y_0 \perp W_0$, $Y_1 \perp W_1$ and $Y_0 \overset{d}{=} Y \overset{d}{=} Y_1$. Note, however, that Y_0 is not independent of either Y_1 or W_1.

We then define the version of Peyrière probability that is most convenient for our needs.

DEFINITION 5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space on which μ is defined and define a probability space $(\Omega \times \{0,1\}, \mathcal{F} \times \sigma(\{0,1\}), \mathbb{Q})$ by setting

$$\mathbb{E}_{\mathbb{Q}} f(\omega, j) = \mathbb{E}\big(W_0(\omega) f(\omega, 0) + W_1(\omega) f(\omega, 1)\big)$$

for all bounded $\mathcal{F} \times \sigma(\{0,1\})$-measurable functions $f : \Omega \times \{0,1\}$. Define the random variables $\widetilde{Y}$, $\widetilde{W}$ and $\widetilde{B}$ on this probability space by setting

$$\widetilde{Y}(\omega, j) = Y_j(\omega), \qquad \widetilde{W}(\omega, j) = W_j(\omega)$$

and

$$\widetilde{B}(\omega, j) = \begin{cases} W_1(\omega) Y_1(\omega), & j = 0, \\ W_0(\omega) Y_0(\omega), & j = 1. \end{cases}$$

For an intuitive idea of what the measure $\mathbb{Q}$ does, consider the random probability measure on $[0,1]$ defined by $\frac{\mu(\mathrm{d}x)}{\mu([0,1])}$. Then W_0 can be seen as the (random) probability that a point sampled according to this measure is in $[0, \frac{1}{2}]$ and similarly for W_1. So $\mathbb{Q}$ can be seen as a probability distribution that is obtained by weighting with the information of which half of $[0,1]$ a point sampled according to μ is in.

We state the essential properties of the variables defined above as the following lemma.

LEMMA 6. *The following statements hold*:

(1) $\widetilde{W}$ *and* $\widetilde{Y}$ *are independent.*
(2) $\widetilde{Y}$ (*under* $\mathbb{Q}$) *has the same law as* Y (*under* $\mathbb{P}$).
(3) $-\log \widetilde{W}$ *is a centered Gaussian with variance* $2\log 2$.

PROOF. Let $f, g : \mathbb{R} \mapsto \mathbb{R}$ be bounded and continuous. By direct computation and the independences $W_0 \perp Y_0$ and $W_1 \perp Y_1$,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} f(\widetilde{W}) g(\widetilde{Y}) &= \mathbb{E}\big(W_0 f(W_0) g(Y_0) + W_1 f(W_1) g(Y_1)\big) \\ &= 2\big(\mathbb{E} W_0 f(W_0)\big)\big(\mathbb{E} g(Y_0)\big). \end{aligned}$$

By taking $g \equiv 1$, we see that $\mathbb{E}_{\mathbb{Q}} f(\widetilde{W}) = 2\mathbb{E} W_0 f(W_0)$, and taking $f \equiv 1$ yields $\mathbb{E}_{\mathbb{Q}} g(\widetilde{Y}) = \mathbb{E} g(Y_0) = \mathbb{E} g(Y)$. Thus, (2) holds, and moreover,

$$\mathbb{E}_{\mathbb{Q}} f(\widetilde{W}) g(\widetilde{Y}) = \mathbb{E}_{\mathbb{Q}} f(\widetilde{W}) \mathbb{E}_{\mathbb{Q}} g(\widetilde{Y}),$$

which means that $\widetilde{W}$ and $\widetilde{Y}$ are independent as claimed in (1).

The law of $-\log \widetilde{W}$ is easy to identify by computing the moment generating function. Since $W_0 \stackrel{d}{=} W_1 \stackrel{d}{=} \frac{1}{2}e^{\sqrt{2\log 2}N-\log 2}$, where N is a standard Gaussian,

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}} e^{t(-\log \widetilde{W})} &= \mathbb{E}_{\mathbb{Q}} \widetilde{W}^{-t} = \mathbb{E}\big(W_0^{1-t} + W_1^{1-t}\big) \\
&= 2\mathbb{E}2^{-(1-t)} e^{(1-t)\sqrt{2\log 2}N-(1-t)\log 2} = 2^{2t-1+(1-t)^2} \\
&= e^{t^2 \log 2}.
\end{aligned}
$$

□

Define the measure ν on the positive real axis by setting

$$
\int f \,\mathrm{d}\nu = \mathbb{E}Yf(Y) = \mathbb{E}_{\mathbb{Q}}\widetilde{Y} f(\widetilde{Y}) \tag{10}
$$

for all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}$ with compact support. The asymptotics of this measure will be determined through the functions

$$
F_{\alpha,\beta}(x) = \nu\big((\alpha e^x, \beta e^x]\big) \qquad \text{for } 0 < \alpha < \beta.
$$

In terms of $F_{\alpha,\beta}$, the statement of Theorem 1 is essentially equivalent to the following proposition.

PROPOSITION 7. *Let $F_{\alpha,\beta}$ be defined by ν as above. Then*

$$
\lim_{x\to\infty} F_{\alpha,\beta}(x) = c_1 \log \frac{\beta}{\alpha},
$$

where

$$
c_1 = \frac{2}{\log 2}\mathbb{E}\mu\big([0, 1/2]\big) \log\left(1 + \frac{\mu([1/2, 1])}{\mu([0, 1/2])}\right) < \infty.
$$

The first step toward the proof of the proposition above is deriving the Poisson equation satisfied by $F_{\alpha,\beta}$. Let τ denote the law of $-\log \widetilde{W}$. By using the independence of $\widetilde{W}$ and $\widetilde{Y}$, we get

$$
\begin{aligned}
\tau * F_{\alpha,\beta}(x) &= \int_{\mathbb{R}} \mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{\widetilde{Y}\in(\alpha e^{x+y}, \beta e^{x+y}]\}} \tau(\mathrm{d}y) \\
&= \mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{\widetilde{W}\widetilde{Y}\in(\alpha e^x, \beta e^x]\}} \\
&= F_{\alpha,\beta}(x) + \mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{\widetilde{W}\widetilde{Y}\in(\alpha e^x, \beta e^x]\}} - \mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{\widetilde{Y}\in(\alpha e^x, \beta e^x]\}},
\end{aligned}
$$

where the convolution of the measure τ with a function $F : \mathbb{R} \to \mathbb{R}$ is defined by

$$
\tau * F(x) = \int_{\mathbb{R}} F(x + y)\tau(\mathrm{d}y) = \int_{\mathbb{R}} F(x - y)\tau(\mathrm{d}y).
$$

By using part (2) of Lemma 6, the distributional equation (9) and the definitions of the variables $\widetilde{W}$, $\widetilde{Y}$ and $\widetilde{B}$, the term $\mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{\widetilde{Y}\in(\alpha e^x,\beta e^x]\}}$ above may be expressed as

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{\widetilde{Y}\in(\alpha e^x,\beta e^x]\}}\\
&\qquad = \mathbb{E}Y\mathbf{1}_{\{Y\in(\alpha e^x,\beta e^x]\}}\\
&\qquad = \mathbb{E}\big(W_0Y_0\mathbf{1}_{\{\alpha e^x - W_1Y_1 < W_0Y_0 \le \beta e^x - W_1Y_1\}} + W_1Y_1\mathbf{1}_{\{\alpha e^x - W_0Y_0 < W_1Y_1 \le \beta e^x - W_0Y_0\}}\big)\\
&\qquad = \mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{\alpha e^x - \widetilde{B} < \widetilde{W}\widetilde{Y} \le \beta e^x - \widetilde{B}\}}.
\end{aligned}$$

The previous computations imply that $F_{\alpha,\beta}$ satisfies the Poisson equation

$$F_{\alpha,\beta}(x) - \tau * F_{\alpha,\beta}(x) = \psi_{\alpha,\beta}(x) \tag{11}$$

with the function $\psi_{\alpha,\beta}$ given by

$$\psi_{\alpha,\beta}(x) = \mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{\alpha e^x - \widetilde{B} < \widetilde{W}\widetilde{Y} \le \beta e^x - \widetilde{B}\}} - \mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{\alpha e^x < \widetilde{W}\widetilde{Y} \le \beta e^x\}}. \tag{12}$$

The desired results on the solutions of this Poisson equation at infinity could be achieved almost exactly in the same way as in Buraczewski [13], that is, by building on the general theory developed by Port and Stone [40]. The following proposition is originally due to Buraczewski, but we prefer to give it a self-contained proof of independent interest that uses only elementary Fourier analysis.

PROPOSITION 8. *Let ν be a locally finite* (*nonnegative*) *Borel measure on* $[0,\infty)$ *that grows at most polynomially in the sense that there exist* $\gamma, C > 0$ *such that*

$$\nu\big((0,x]\big) \le C(1+x)^{\gamma} \qquad \text{for all } x \ge 0.$$

Define the functions

$$F_{\alpha,\beta}(x) = \nu\big((\alpha e^x, \beta e^x]\big) \qquad \text{for all } 0 < \alpha < \beta$$

and assume that for each α, β *the function* $\psi_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ *is a bounded and continuous function indexed by the parameters* α *and* β *that satisfies*

$$\int_{-\infty}^{\infty} (1+|x|)^2 \big|\psi_{\alpha,\beta}(x)\big| \, \mathrm{d}x < \infty$$

and

$$\int_{-\infty}^{\infty} \psi_{\alpha,\beta}(x) \, \mathrm{d}x = 0.$$

Denote

$$C_{\alpha,\beta} = \int_{-\infty}^{\infty} x\psi_{\alpha,\beta}(x) \, \mathrm{d}x$$

and assume that the map $(\alpha,\beta) \mapsto C_{\alpha,\beta}$ *is continuous. Finally, let* τ *be a Gaussian measure on* $\mathbb{R}$, *that is,* τ *is the law of a centered Gaussian random variable with variance* $\sigma^2 > 0$.

Then, if $F_{\alpha,\beta}$ satisfies the Poisson equation

$$F_{\alpha,\beta} - \tau * F_{\alpha,\beta} = \psi_{\alpha,\beta},$$

it has the asymptotics

$$\lim_{x\to\infty} F_{\alpha,\beta}(x) = \frac{2}{\sigma^2} C_{\alpha,\beta}.$$

We split our proof of this proposition into two lemmas and a finalizing convolution argument.

LEMMA 9. *Let the function $F:\mathbb{R}\to\mathbb{R}$ be bounded from below and satisfy*

$$\lim_{x\to-\infty} F(x) = 0.$$

Assume also that F grows at most exponentially at infinity[3] *and solves the Poisson equation*

$$F - \tau * F = \psi, \tag{13}$$

where $\tau \sim N(0,\sigma^2)$ is as in Proposition 8 and $\psi:\mathbb{R}\to\mathbb{R}$ satisfies

$$\int_{-\infty}^{\infty} (1+|x|)^2 \big|\psi(x)\big|\,\mathrm{d}x < \infty, \qquad \int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x = 0 \quad \textit{and} \quad \lim_{x\to\pm\infty} \psi(x) = 0.$$

Then F has the asymptotics

$$\lim_{x\to\infty} F(x) = \frac{2}{\sigma^2}\int_{-\infty}^{\infty} x\psi(x)\,\mathrm{d}x. \tag{14}$$

PROOF. To shorten the notation, we denote $\int_{-\infty}^{\infty} x\psi(x)\,\mathrm{d}x = A$.

We start by proving that equation (13) has some bounded solution F_1 that has the desired asymptotics

$$\lim_{x\to-\infty} F_1(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} F_1(x) = \frac{2}{\sigma^2}A. \tag{15}$$

We first consider the case $A = 0$. Then our assumptions imply that the Fourier transform of ψ satisfies [our convention for the Fourier transform of $\psi \in L^1(\mathbb{R})$ is $\widehat{\psi}(\xi) = \int_{\mathbb{R}} e^{-ix\xi}\psi(x)\,\mathrm{d}x$]

$$\widehat{\psi} \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \quad \text{and} \quad \widehat{\psi}(0) = \widehat{\psi}'(0) = 0.$$

As $1 - \widehat{\tau}(\xi) = 1 - \exp(-\sigma^2\xi^2/2)$ is smooth with zero of order 2 at the origin we may directly define F_1 in the distribution sense through

$$\widehat{F}_1(\xi) := \frac{\widehat{\psi}(\xi)}{1 - e^{-\sigma^2\xi^2/2}} = \widehat{\psi}(\xi) + \left(\frac{\widehat{\psi}(\xi)}{\xi^2}\right)\left(\frac{\xi^2 e^{-\sigma^2\xi^2/2}}{1 - e^{-\sigma^2\xi^2/2}}\right) =: \widehat{\psi}(\xi) + \widehat{F}_2(\xi).$$

[3]The assumption of exponential growth is used only to ensure that the convolution with τ is well defined.

Since obviously $\widehat{F_2} \in L^1$, we have $\lim_{x\to\pm\infty} F_2(x) = 0$ by the Riemann–Lebesgue lemma, and the same follows for F_1 by the assumption on ψ.

In order to construct a solution F_1 in the general case $A \neq 0$, first define $\psi_0 = F_0 - \tau * F_0$ with $F_0 = \chi_{(0,\infty)}$, with χ referring to the indicator function of a set. Directly from the definition we see that $\psi_0 \in L^\infty(\mathbb{R})$ and that ψ_0 decays exponentially as $x \to \pm\infty$, so that it satisfies the moment conditions of the lemma, and moreover $\widehat{\psi_0} \in C^\infty$. Also, $\widehat{F_0}(\xi) = \pi\delta_0 - i\xi^{-1}$ (here ξ^{-1} is understood as a principal value distribution). Since $1 - \widehat{\tau}(\xi) = \sigma^2\xi^2/2 + O(\xi^3)$, we see that

$$\widehat{\psi_0}(\xi) = -\frac{i\sigma^2}{2}\xi + O(\xi^2)$$

at the origin. Hence,

$$\int_{-\infty}^{\infty} \psi_0(x)\,\mathrm{d}x = \widehat{\psi_0}(0) = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} x\psi_0(x)\,\mathrm{d}x = i\widehat{\psi}_0'(0) = \frac{\sigma^2}{2}.$$

Thus, in the case $A \neq 0$ we define F_1 by finding the solution for the Poisson equation (13) with the right-hand side $\widetilde{\psi} = \psi - \frac{2}{\sigma^2}A\psi_0$ and then adding $\frac{2}{\sigma^2}AF_0$. At this point, it is clear that the solution obtained this way is bounded and has the desired behavior at $\pm\infty$.

Let us finally assume that F and ψ are as in the theorem. Let F_1 be the bounded solution of (13) constructed above, so that F_1 satisfies the conclusion of the theorem. It is enough to verify that $H := F - F_1$ is constant since then $H \equiv 0$ by considering the limit at $-\infty$. Now H is bounded from below and satisfies the homogeneous Poisson equation

$$H = \tau * H. \tag{16}$$

The claim follows from Lemma 10 below. $\square$

LEMMA 10. *Let H solve the homogeneous Poisson equation* (16) *and assume that it is bounded from below and has at most exponential growth at $\pm\infty$. Then H is constant.*

PROOF. By adding a constant, we may without loss of generality assume that $H \geq 0$. Let $u(x,t)$ $(x \in \mathbb{R},\ t \geq 0)$ denote the heat extension of H to the upper half-plane, explicitly given by

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(y-x)^2/(2t)} H(y)\,\mathrm{d}y.$$

By assumption, u is periodic in t: denoting $t_0 = \sigma^2$, $u(x, t + t_0) = u(x,t)$ for all $t \geq 0$. Define the function v in the upper-half plane by setting

$$v(x,t) := \int_0^{t_0} u(x, t+s)\,\mathrm{d}s.$$

Then v solves the heat equation and, by the periodicity of u, it is constant in t. Thus, it is harmonic in x, that is, a linear function $v(x,t) = ax + b$. Here, $a = 0$ by the nonnegativity of u, whence v is constant. This shows that there is a constant C independent of x so that $\int_{t_0/2}^{t_0} u(x,s)\,\mathrm{d}s < C$. Especially, for each x there is $t_1 = t_1(x) \in (t_0/2, t_0)$ so that $u(x,t_1) \le 2C/t_0$. The heat kernel $(2\pi t)^{-1/2} e^{-x^2/2t}$ can be bounded from below on $x \in [-1,1]$ uniformly in $t \in (t_0/2, t_0)$, whence again using the nonnegativity of H we deduce that $\int_{x-1}^{x+1} H(y)\,\mathrm{d}y \le C'$ for all $x \in \mathbb{R}$. As we combine this information with the fact that $H = \tau * H$ it follows that H is bounded. Then the equation

$$\bigl(1 - e^{-\xi^2/2}\bigr)\widehat{H}(\xi) = 0,$$

interpreted in the sense of distributions, shows that $\widehat{H} = c_1\delta_0 + c_2\delta_0'$, that is, H is linear. By nonnegativity, we finally deduce that H is constant. □

REMARK 11. As pointed out by one of the referees, these types of results often have more probabilistic proofs as well. For example, let us sketch one for the previous lemma. Consider again $H \ge 0$ and note that the condition $H = \tau * H$ means that $(H(S_n))_{n\ge 0}$ is a martingale for the Gaussian random walk (S_n) with increments distributed according to τ. Since H is nonnegative, the martingale converges to some nonnegative random variable, say $\mathcal{H}$. On the other hand, (S_n) is neighborhood recurrent, so for any $\epsilon > 0$ and $x \in \mathbb{R}$ we can find a subsequence n_k such that $n_k \to \infty$ as $k \to \infty$ and $|S_{n_k} - x| < \epsilon$ for all k. Now, the fact that $H = \tau * H$ together with the growth condition assumed of H implies that H is a smooth function. Thus, for any given x we have $\mathcal{H} = \lim_{k\to\infty} H(S_{n_k}) = H(x)$ and, therefore, H is constant.

We finish the proof of Proposition 8 by deducing it from Lemma 9 by a convolution argument analogous to the one of Buraczewski [13].

PROOF OF PROPOSITION 8. Let $\phi \ge 0$ be an arbitrary symmetric smooth test function with $\operatorname{supp}\phi \subset [-1,1]$ and $\int_R \phi = 1$. Given any locally integrable $g : \mathbb{R} \to \mathbb{R}$, let g_ε denote the convolution $g_\varepsilon = g * \varepsilon^{-1}\phi(\varepsilon^{-1}\cdot)$. By convolving the Poisson equation, we obtain [writing, e.g., $(F_{\alpha,\beta})_\varepsilon = F_{\alpha,\beta,\varepsilon}$] for any $0 \le \alpha < \beta$ and $\varepsilon > 0$

$$F_{\alpha,\beta,\varepsilon} = \tau * F_{\alpha,\beta,\varepsilon} + \psi_{\alpha,\beta,\varepsilon}.$$

By the continuity of $\psi_{\alpha,\beta,\varepsilon}$ and integrability of $\psi_{\alpha,\beta}$, we have $\lim_{x\to\pm\infty} \psi_{\alpha,\beta,\varepsilon} = 0$. From Lemma 9, we thus obtain, for each $\varepsilon > 0$, the asymptotics

$$\lim_{x\to\pm\infty} F_{\alpha,\beta,\varepsilon}(x) = \frac{2}{\sigma^2} C_{\alpha,\beta}, \tag{17}$$

since

$$\int_{\mathbb{R}} x\psi_{\alpha,\beta,\varepsilon}(x)\,\mathrm{d}x = i\widehat{\psi}'_{\alpha,\beta,\varepsilon}(0) = i\widehat{\psi}'_{\alpha,\beta}(0) = \int_{\mathbb{R}} x\psi_{\alpha,\beta}(x) = C_{\alpha,\beta}.$$

In order to remove the ε from (17), let $k \in (1, (\beta/\alpha)^{1/2})$ be given and observe that by the definition of $F_{\alpha,\beta}$ as a measure of an interval we have $F_{k\alpha,k^{-1}\beta}(x) \le F_{\alpha,\beta,\varepsilon}(x)$ for all x as soon as ε is small enough. Hence, we obtain from (17) that

$$\limsup_{x\to\infty} F_{\alpha,\beta}(x) \le \frac{2}{\sigma^2} C_{k^{-1}\alpha,k\beta}.$$

By letting $k \to 1^+$ and recalling the assumption of the continuity of $(\alpha,\beta) \mapsto C_{\alpha,\beta}$ it follows that $\limsup_{x\to\infty} F_{\alpha,\beta}(x) \le 2C_{\alpha,\beta}/\sigma^2$. The converse direction $\liminf_{x\to\infty} F_{\alpha,\beta}(x) \ge 2C_{\alpha,\beta}/\sigma$ is obtained analogously by starting from the inequality $F_{k^{-1}\alpha,k\beta}(x) \le F_{\alpha,\beta,\varepsilon}(x)$. $\square$

The proof of Proposition 7 has now essentially been reduced to checking that the Poisson equation (11) with $F_{\alpha,\beta}$ determined by ν as in (10) and $\psi_{\alpha,\beta}$ given by (12) satisfies the assumptions of Proposition 8.

PROOF OF PROPOSITION 7. We first check that the measure ν satisfies $\nu((0,x]) \le C(1+x)^\gamma$ for some $C, \gamma > 0$. This is clear from the definition and Theorem A: for any $\gamma \in (0,1)$,

$$\nu\big((0,x]\big) = \mathbb{E}Y\mathbf{1}_{\{Y\in(0,x]\}} \le x^\gamma \mathbb{E}Y^{1-\gamma}\mathbf{1}_{\{Y\in(0,x]\}} \le \mathbb{E}Y^{1-\gamma}x^\gamma.$$

To check the integrability conditions on $\psi_{\alpha,\beta}$, we define the functions

$$\psi_\alpha(x) = \mathbb{E}_\mathbb{Q}\widetilde{Y}\mathbf{1}_{\{\alpha e^x - \widetilde{B} < \widetilde{W}\widetilde{Y} \le \alpha e^x\}} \quad \text{and} \quad \psi_\beta(x) = \mathbb{E}_\mathbb{Q}\widetilde{Y}\mathbf{1}_{\{\beta e^x - \widetilde{B} < \widetilde{W}\widetilde{Y} \le \beta e^x\}}.$$

By this definition,

$$\psi_{\alpha,\beta}(x) = \mathbb{E}_\mathbb{Q}\widetilde{Y}\mathbf{1}_{\{\alpha e^x - \widetilde{B} < \widetilde{W}\widetilde{Y} \le \beta e^x - \widetilde{B}\}} - \mathbb{E}_\mathbb{Q}\widetilde{Y}\mathbf{1}_{\widetilde{W}\widetilde{Y}\in(\alpha e^x, \beta e^x]} = \psi_\alpha(x) - \psi_\beta(x).$$

Since the functions ψ_α and ψ_β are positive, to check the integrability conditions of Theorem 8 on the functions $\psi_{\alpha,\beta}$ it is sufficient to show that

$$\int_{-1}^{1} \psi_\alpha(x)\,\mathrm{d}x < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} x^2\psi_\alpha(x)\,\mathrm{d}x < \infty \qquad \text{for all } \alpha > 0.$$

In our situation, the first condition is clear, since $\mathbb{E}_\mathbb{Q}\widetilde{W}^{-1} < \infty$. For the second condition, some computation and a separate lemma is needed. We write

$$\mathbf{1}_{\{\alpha e^x - \widetilde{B} < \widetilde{W}\widetilde{Y} \le \alpha e^x\}} = \mathbf{1}_{\{(\widetilde{W}\widetilde{Y})/\alpha \le t < (\widetilde{W}\widetilde{Y}+\widetilde{B})/\alpha\}} \qquad \text{for } t = e^x$$

and use the integral

$$\int_a^b \frac{\log t}{t}\,\mathrm{d}t = \frac{1}{2}\left(\log\frac{b}{a}\right)(\log ab) \qquad \text{for } 0 < a < b$$

to compute

$$\begin{aligned}
\int_0^\infty & x^2\psi_\alpha(x)\,\mathrm{d}x \\
&= \mathbb{E}_{\mathbb{Q}}\widetilde{Y}\int_0^\infty x^2\mathbf{1}_{\{\alpha e^x-\widetilde{B}<\widetilde{W}\widetilde{Y}\le\alpha e^x\}}\,\mathrm{d}x \\
&= \mathbb{E}_{\mathbb{Q}}\widetilde{Y}\int_1^\infty \mathbf{1}_{\{(\widetilde{W}\widetilde{Y})/\alpha\le t<(\widetilde{W}\widetilde{Y}+\widetilde{B})/\alpha\}}\frac{(\log t)^2}{t}\,\mathrm{d}t \\
&= \mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{(\widetilde{W}\widetilde{Y})/\alpha>1\}}\int_{(\widetilde{W}\widetilde{Y})/\alpha}^{(\widetilde{W}\widetilde{Y}+\widetilde{B})/\alpha}\frac{(\log t)^2}{t}\,\mathrm{d}t \\
&\quad+\mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{(\widetilde{W}\widetilde{Y})/\alpha<1<(\widetilde{W}\widetilde{Y}+\widetilde{B})/\alpha\}}\int_1^{(\widetilde{W}\widetilde{Y}+\widetilde{B})/\alpha}\frac{(\log t)^2}{t}\,\mathrm{d}t \\
&\le \frac{1}{2}\mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{(\widetilde{W}\widetilde{Y})/\alpha>1\}}\log\Big(1+\frac{\widetilde{B}}{\widetilde{W}\widetilde{Y}}\Big)\log\Big(\frac{\widetilde{W}\widetilde{Y}}{\alpha}\cdot\frac{\widetilde{W}\widetilde{Y}+\widetilde{B}}{\alpha}\Big)\log\Big(\frac{\widetilde{W}\widetilde{Y}+\widetilde{B}}{\alpha}\Big) \\
&\quad+\frac{1}{2}\mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{(\widetilde{W}\widetilde{Y})/\alpha<1<(\widetilde{W}\widetilde{Y}+\widetilde{B})/\alpha\}}\Big(\log\Big(\frac{\widetilde{W}\widetilde{Y}+\widetilde{B}}{\alpha}\Big)\Big)^2\log\Big(\frac{\widetilde{W}\widetilde{Y}+\widetilde{B}}{\alpha}\Big) \\
&=: I_1+I_2,
\end{aligned}$$

and similarly by the change of variables $s=e^{-x}$ we get

$$\begin{aligned}
\int_{-\infty}^0 & x^2\psi_\alpha(x)\,\mathrm{d}x \\
&= \mathbb{E}_{\mathbb{Q}}\widetilde{Y}\int_{-\infty}^0 x^2\mathbf{1}_{\{\alpha e^x-\widetilde{B}<\widetilde{W}\widetilde{Y}\le\alpha e^x\}}\,\mathrm{d}x \\
&= \mathbb{E}_{\mathbb{Q}}\widetilde{Y}\int_1^\infty \mathbf{1}_{\{\alpha/s-\widetilde{B}<\widetilde{W}\widetilde{Y}\le\alpha/s\}}\frac{(\log s)^2}{s}\,\mathrm{d}s \\
&\le \frac{1}{2}\mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{(\widetilde{W}\widetilde{Y}+\widetilde{B})/\alpha<1\}}\log\Big(1+\frac{\widetilde{B}}{\widetilde{W}\widetilde{Y}}\Big)\log\Big(\frac{\alpha}{\widetilde{W}\widetilde{Y}}\cdot\frac{\alpha}{\widetilde{W}\widetilde{Y}+\widetilde{B}}\Big)\log\Big(\frac{\alpha}{\widetilde{W}\widetilde{Y}}\Big) \\
&\quad+\frac{1}{2}\mathbb{E}_{\mathbb{Q}}\widetilde{Y}\mathbf{1}_{\{(\widetilde{W}\widetilde{Y})/\alpha<1<(\widetilde{W}\widetilde{Y}+\widetilde{B})/\alpha\}}\Big(\log\Big(\frac{\widetilde{W}\widetilde{Y}}{\alpha}\Big)\Big)^2\log\Big(\frac{\alpha}{\widetilde{W}\widetilde{Y}}\Big) \\
&=: I_3+I_4.
\end{aligned}$$

To show that $I_1<\infty$, we use the crude estimate $\log(1+x)\le C_p x^p$, valid for all $p>0$ for sufficiently large constant $C_p>0$ depending only on p, to get

$$\begin{aligned}
I_1 \le \frac{C_{p_1}C_{p_2}C_{p_3}}{\alpha^{p_1}\alpha^{p_2}}\mathbb{E}\Big(&\mu([0,1/2])\Big(\frac{\mu([1/2,1])}{\mu([0,1/2])}\Big)^{p_1} \\
&\times\big(\mu([0,1/2])^{p_2}+\mu([0,1])^{p_2}\big)\mu([0,1])^{p_3}\Big).
\end{aligned}$$

In Lemma 13 below, we show that for any $0 < h < 1$ we have

$$\mathbb{E}\mu\big([0,1/2]\big)^h \mu\big([1/2,1]\big)^h < \infty.$$

By choosing $p_1, p_2, p_3 > 0$ such that $0 < 1 - p_1 + p_2 + p_3 < 1$ and $p_1 + p_2 + p_3 < 1$, this implies the finiteness of I_1. For I_2 one may estimate $\frac{\widetilde{W}\widetilde{Y}+\widetilde{B}}{\alpha} \le 1 + \frac{\widetilde{B}}{\widetilde{W}\widetilde{Y}}$ and proceed as in the case of I_1. In estimating I_3, one may write $\frac{\alpha}{\widetilde{W}\widetilde{Y}+\widetilde{B}} < \frac{\alpha}{\widetilde{W}\widetilde{Y}}$ and proceed as before, and the finiteness of I_4 follows the same route.

In order to apply Proposition 8, we still need to show that

$$\int_{-\infty}^{\infty} \psi_{\alpha,\beta}(x)\,\mathrm{d}x = 0$$

and compute the value of the integral

$$C_{\alpha,\beta} = \int_{-\infty}^{\infty} x\psi_{\alpha,\beta}(x)\,\mathrm{d}x.$$

The first integral follows immediately from the integrability of ψ_α and the fact that

$$\psi_{\alpha,\beta}(x) = \psi_\alpha(x) - \psi_\beta(x) = \psi_\alpha(x) - \psi_\alpha\left(x + \log\frac{\alpha}{\beta}\right).$$

The value of $C_{\alpha,\beta}$ can be calculated by using the change of variables $x = e^t$ as above to obtain

$$\int_{-\infty}^{\infty} x\psi_\alpha(x)\,\mathrm{d}x = \frac{1}{2}\mathbb{E}_{\mathbb{Q}}\widetilde{Y}\log\left(1 + \frac{\widetilde{B}}{\widetilde{W}\widetilde{Y}}\right)\log\frac{\widetilde{W}\widetilde{Y}(\widetilde{W}\widetilde{Y}+\widetilde{B})}{\alpha^2},$$

which implies

$$C_{\alpha,\beta} = \int_{-\infty}^{\infty} x\big(\psi_\alpha(x) - \psi_\beta(x)\big)\,\mathrm{d}x = \mathbb{E}_{\mathbb{Q}}\widetilde{Y}\log\left(1 + \frac{\widetilde{B}}{\widetilde{W}\widetilde{Y}}\right)\log\frac{\beta}{\alpha}.$$

Proposition 8 now gives the desired asymptotics

$$\begin{aligned} F_{\alpha,\beta}(x) &\overset{x\to\infty}{\longrightarrow} \frac{2\mathbb{E}_{\mathbb{Q}}\widetilde{Y}\log(1+\widetilde{B}/(\widetilde{W}\widetilde{Y}))}{2\log 2}\log\frac{\beta}{\alpha} \\ &= \frac{2}{\log 2}\mathbb{E}\mu\big([0,1/2]\big)\log\left(1 + \frac{\mu([1/2,1])}{\mu([0,1/2])}\right)\log\frac{\beta}{\alpha} \end{aligned}$$

for all $0 < \alpha < \beta$. $\square$

Before moving on to the final step of the proof of Theorem 1, we complete the proof of Proposition 7 by proving Lemma 13.

LEMMA 12. *For any $h \in (0,1)$ and any pair of intervals $I_1, I_2 \subset [0,1]$ such that $d(I_1, I_2) > 0$,*

$$\mathbb{E}\big(\mu(I_1)^h \mu(I_2)^h\big) < \infty.$$

PROOF. We use Kahane's convexity inequality, to be given as Proposition 19, and the definition (7) of the critical measure as the limit of

$$\mu_t(\mathrm{d}x) = \sqrt{t} e^{\sqrt{2} X_t(x) - \mathbb{E} X_t(x)^2} \, \mathrm{d}x$$

as $t \to \infty$. Write the product $\mu_t(I_1)\mu_t(I_2)$ as

$$\mu_t(I_1)\mu_t(I_2) = t \int_{I_1} \mathrm{d}x \int_{I_2} \mathrm{d}y \, e^{\sqrt{2}(X_t(x)+X_t(y)) - \mathbb{E} X_t(x)^2 - \mathbb{E} X_t(y)^2}$$

and consider the Gaussian fields $Z_t(x, y) = X_t(x) + X_t(y)$ and $\widetilde{Z}_t(x, y) = X_t(x) + \widetilde{X}_t(y)$ indexed by $I_1 \times I_2$, where $\widetilde{X}_t$ is an independent realization of the field X_t. The covariance kernel of $\widetilde{Z}_t$ is clearly dominated by the covariance kernel of Z_t, so Proposition 19 gives the inequality

$$\begin{aligned}
&\mathbb{E}\biggl(t \int_{I_1} \mathrm{d}x \int_{I_2} \mathrm{d}y \, e^{\sqrt{2} Z_t(x,y) - \mathbb{E} Z_t(x,y)^2}\biggr)^h \\
&\quad \le \mathbb{E}\biggl(t \int_{I_1} \mathrm{d}x \int_{I_2} \mathrm{d}y \, e^{\sqrt{2} \widetilde{Z}_t(x,y) - \mathbb{E} \widetilde{Z}_t(x,y)^2}\biggr)^h \\
&\quad = \mathbb{E}\biggl(\sqrt{t} \int_{I_1} e^{\sqrt{2} X_t(x) - \mathbb{E} X_t(x)^2} \, \mathrm{d}x\biggr)^h \mathbb{E}\biggl(\sqrt{t} \int_{I_2} e^{\sqrt{2} \widetilde{X}_t(x) - \mathbb{E} \widetilde{X}_t(x)^2} \, \mathrm{d}x\biggr)^h \\
&\quad = \mathbb{E}\bigl(\mu_t(I_1)^h\bigr) \mathbb{E}\bigl(\mu_t(I_2)^h\bigr) \\
&\quad < \infty.
\end{aligned}$$

By expanding the variance $\mathbb{E} Z_t(x, y)^2 = \mathbb{E} X_t(x)^2 + \mathbb{E} X_t(y)^2 + 2\mathbb{E} X_t(x) X_t(y)$ we note that the first expression may be estimated from below by

$$e^{-2 \sup_{x \in I_1, y \in I_2} \mathbb{E} X_t(x) X_t(y)} \mathbb{E}\bigl(t \mu_t(I_1)\mu_t(I_2)\bigr)^h.$$

Since the intervals I_1 and I_2 are separated by a positive distance, the supremum in the exponent stays bounded as $t \to \infty$, which proves the claim. $\square$

LEMMA 13. *For any* $h \in (0, 1)$,

$$\mathbb{E}\bigl(\mu([0, 1/2])\mu([1/2, 1])\bigr)^h < \infty.$$

PROOF. Fix $h \in (0, 1)$. For every $k \in \mathbb{N}$, let $J_k = [1/2 - 2^{-k}, 1/2 + 2^{-k}]$. Denote the left and right half of J_k by J_k^0 and J_k^1, and the right and left halves of J_k^0 (and J_k^1) by J_k^{00} and J_k^{01} (J_k^{10} and J_k^{11}). Define the sets A_k by

$$A_k = \bigl(J_k^{00} \times J_k^{11}\bigr) \cup \bigl(J_k^{00} \times J_k^{10}\bigr) \cup \bigl(J_k^{01} \times J_k^{11}\bigr).$$

Write

$$Z = \mu\bigl([0, 1/2]\bigr)\mu\bigl([1/2, 1]\bigr) = \int_{[0,1/2]} \mu(\mathrm{d}x) \int_{[1/2,1]} \mu(\mathrm{d}y)$$

and define the random variables

$$Z_k = \int_{[0,1/2]} \mu(\mathrm{d}x) \int_{[1/2,1]} \mu(\mathrm{d}y) \chi_{A_k}(x,y)$$
$$= \mu(J_k^{00})\mu(J_k^{11}) + \mu(J_k^{00})\mu(J_k^{10}) + \mu(J_k^{01})\mu(J_k^{11})$$

for $k \in \mathbb{N}$. It is clear that

$$Z = \sum_{k=1}^{\infty} Z_k$$

and thus by the subadditivity of $x \mapsto x^h$

$$\mathbb{E}Z^h \le \sum_{k=1}^{\infty} \mathbb{E}Z_k^h.$$

By the exact scaling property of the construction, the measure μ satisfies

$$(18) \qquad \big(\mu\big(J_k^{\sigma_1\sigma_2}\big)\big)_{\sigma_1,\sigma_2\in\{0,1\}} = 2^{-k+1} e^{\sqrt{2}X(J_k) - \mathbb{E}X(J_k)^2} \big(\mu'\big(J_1^{\sigma_1\sigma_2}\big)\big)_{\sigma_1,\sigma_2\in\{0,1\}},$$

where $X(J_k) = W(\mathcal{C}(J_k))$ is a centered Gaussian random variable with variance $\lambda(\mathcal{C}(J_k)) = (k-1)\log 2$ and μ' is random measure independent of $X(J_k)$ that has the same distribution as μ. But this implies that

$$Z_k \stackrel{d}{=} 2^{-2k+2} e^{2\sqrt{2}X(J_k) - 2\mathbb{E}X(J_k)^2} Z_1',$$

where $Z_1' \stackrel{d}{=} Z_1$ is a random variable independent of Z_1. Since

$$2^{(-2k+2)h} \mathbb{E} e^{2\sqrt{2}hX(J_k) - 2h\mathbb{E}X(J_k)^2} = 2^{(-2k+2)h} 2^{(4h^2-2h)(k-1)} = 2^{4(h^2-h)(k-1)}$$

and $\mathbb{E}Z_1^h$ is finite by Lemma 12, we have

$$\mathbb{E}Z^h \le \mathbb{E}Z_1^h \sum_{k=1}^{\infty} 2^{4(h^2-h)(k-1)} < \infty. \qquad \square$$

REMARK 14. While it can be seen from the proof of Proposition 7, we emphasize that the finiteness of c_1 follows from this lemma: simply use the elementary inequality $\log(1+x) \le \sqrt{x}$ for $x \ge 0$ to bound c_1 by a term proportional to $\mathbb{E}(\mu([0,\frac{1}{2}])^{1/2}\mu([\frac{1}{2},1])^{1/2})$.

PROOF OF THEOREM 1. We will show that for any $r > 1$ there exists a λ_r such that

$$\mathbb{P}(Y > \lambda) \le \frac{c_1}{\lambda} r \qquad \text{for all } \lambda \ge \lambda_r.$$

The verification of the lower bound is similar and is left to the reader.

Let $r > 1$ and fix $q > 1$ so that $\frac{q \log q}{q-1} < \sqrt{r}$. By Proposition 7, there exists a λ_r such that

$$F_{1,q}(x) \le c_1 \sqrt{r} \log q \qquad \text{for all } x \ge \log \lambda_r,$$

where we have defined $F_{1,q}(x) = \mathbb{E}(Y \mathbf{1}_{\{Y \in (e^x, qe^x]\}})$. We now have for $\lambda \ge \lambda_r$

$$\begin{aligned} \mathbb{P}(Y > \lambda) &= \sum_{k=0}^{\infty} \mathbb{P}\big(Y \in \big(\lambda q^k, \lambda q^{k+1}\big]\big) \\ &\le \frac{1}{\lambda} \sum_{k=0}^{\infty} q^{-k} F_{1,q}(k \log q + \log \lambda) \\ &\le \frac{1}{\lambda} \sum_{k=0}^{\infty} q^{-k} c_1 \sqrt{r} \log q = \frac{c_1}{\lambda} \sqrt{r} \frac{q \log q}{q-1} \le \frac{c_1}{\lambda} r, \end{aligned}$$

as was to be shown. □

3. Modulus of continuity.

3.1. *Outline of the proof.* In this section, we prove Theorem 2. Our plan of attack is to follow the arguments carried out in [9] in the case of multiplicative cascades. However, the delicate dependence structure of multiplicative chaos calls for nontrivial modifications. Let us briefly sketch the main steps in the case of multiplicative cascades to see what the main structure of the proof will be and what kind of modifications we shall need.

The main part of the proof in the situation for cascades was showing that if we write $(I_\sigma)_{\sigma \in \{0,1\}^n}$ for the dyadic subintervals of $[0, 1]$ of length 2^{-n} and μ for the critical measure, then for any $\epsilon > 0$ there exists a $C_\epsilon > 0$ such that for $\gamma \in (0, \frac{1}{2})$, $\mathbb{P}(\max_{\sigma \in \{0,1\}^n} \mu(I_\sigma) \ge n^{-\gamma}) \le C_\epsilon n^{(1-\epsilon)(\gamma-(1/2))}$. The corresponding result for the modulus of continuity then follows from this through a Borel–Cantelli argument.

To get a hold of this estimate, one uses the scaling relation $(\mu(I_\sigma))_\sigma \stackrel{d}{=} (e^{X_\sigma} Y^{(\sigma)})_\sigma$, where $Y^{(\sigma)}$ are i.i.d. copies of $\mu([0, 1])$ which are also independent of the random variables $(X_\sigma)_\sigma$. By using the scaling relation, conditioning on (X_σ) and the tail estimate $\mathbb{P}(Y^{(\sigma)} \ge \lambda) \approx C\lambda^{-1}$ (along with some technical details to justify the approximations used)

$$\begin{aligned} \mathbb{P}\Big(\max_{\sigma \in \{0,1\}^n} \mu(I_\sigma) < n^{-\gamma}\Big) &= \mathbb{E}\Big(\prod_{\sigma \in \{0,1\}^n} \big(1 - \mathbb{P}\big(Y^{(\sigma)} \ge n^{-\gamma} e^{-X_\sigma} | (X_\sigma)\big)\big)\Big) \\ &\approx \mathbb{E}\Big(\prod_{\sigma \in \{0,1\}^n} \big(1 - C n^{\gamma} e^{X_\sigma}\big)\Big) \\ &\approx \mathbb{E} e^{-C n^{\gamma} \sum_{\sigma \in \{0,1\}^n} e^{X_\sigma}}. \end{aligned}$$

The last term we can write as $\phi_n(Cn^{\gamma-(1/2)})$, where ϕ_n is the Laplace transform of the correctly normalized total mass: $\phi_n(t) = \mathbb{E}e^{-tS_n}$, where $S_n = \sqrt{n}\sum_{\sigma\in\{0,1\}^n} e^{X_\sigma}$. One can then prove that for any fixed $q \in (0,1)$, $\sup_n \mathbb{E}(S_n^q) < \infty$. Using this and Markov's inequality, one can show that for $q < 1$, $1-\phi_n(t) \leq C_q t^q$ from which one concludes that

$$
\begin{aligned}
\mathbb{P}\Big(\max_{\sigma\in\{0,1\}^n} \mu(I_\sigma) \geq n^{-\gamma}\Big) &\leq 1-\phi_n\big(Cn^{\gamma-(1/2)}\big) \\
&\leq C_\epsilon n^{(1-\epsilon)(\gamma-(1/2))}.
\end{aligned} \tag{19}
$$

While this sketch swept a lot of the technical details under the rug, it still forms the back bone of the proof and one can see some of the difficulties that will be present in the case of multiplicative chaos. Let us consider some of the differences we can expect to be present in the current context. First of all, if we manage to prove the same estimate for the maximum of the measure of dyadic intervals, the Borel–Cantelli argument will go through in a similar manner. The first major difference is the scaling relation. For the exactly scale invariant critical measure, one has a similar distributional relation: $(\mu_{\sqrt{2}}(I_\sigma))_\sigma \stackrel{d}{=} (e^{X_\sigma}\mu^{(\sigma)}([0,1]))_\sigma$, but the difference is that we have nontrivial correlations—$\mu^{(\sigma)}$ are not independent from each other and they may depend on some of the X_σ as well. To remedy this, we consider instead of $\mu_{\sqrt{2}}$ another random measure which is absolutely continuous with respect to $\mu_{\sqrt{2}}$ which possesses nice scaling properties, nice decorrelation properties as well as a nicely behaving Radon–Nikodym derivative with respect to $\mu_{\sqrt{2}}$. Moreover, one gets similar asymptotic behavior for the tail of the measure of the unit interval for this measure as well.

The next step of the proof is to use scaling, independence and tail behavior to obtain a similar estimate in terms of a Laplace transform and some errors due to approximations. This step of the proof requires a fair amount of technical details which are even more involved than in the multiplicative cascade setup, but philosophically similar. Finally, we are left with estimating moments of the correctly normalized approximation to the critical measure. This can be done by using Gaussian comparison inequalities and the result from multiplicative cascades.

3.2. *Tools for the proof.* Let us now collect some of the tools we shall need for the proof. First of all, we shall consider modifications of the field X and the measure $\mu_{\sqrt{2}}$ for which we still have a similar result for the tail.

LEMMA 15. *Assume that we can write $\mu_{\sqrt{2}}(\mathrm{d}x) = e^{Z(x)}\nu(\mathrm{d}x)$, for some random measure $\nu(\mathrm{d}x)$ and random Gaussian field Z which is independent of ν and $\min_{x\in[0,1]} Z(x) > 0$ with positive probability, then there exists a constant C such that $\mathbb{P}(\nu([0,\alpha]) > \lambda) \leq C\alpha\lambda^{-1}$.*

PROOF. Plugging in the definitions,

$$\begin{aligned}\mathbb{P}(\mu_{\sqrt{2}}([0,\alpha]) > \lambda) &= \mathbb{P}\Big(\int_0^\alpha e^{Z(x)}\nu(\mathrm{d}x) > \lambda\Big)\\ &\geq \mathbb{P}\big(e^{\min_{x\in[0,1]} Z(x)}\nu([0,\alpha]) > \lambda\big)\\ &\geq \mathbb{P}\big(e^{\min_{x\in[0,1]} Z(x)} > 1, \nu([0,\alpha]) > \lambda\big)\\ &= \mathbb{P}\Big(\min_{x\in[0,1]} Z(x) > 0\Big)\mathbb{P}\big(\nu([0,\alpha]) > \lambda\big).\end{aligned}$$

On the other hand, by scaling

$$\mathbb{P}(\mu_{\sqrt{2}}([0,\alpha]) > \lambda) = \mathbb{P}\big(\alpha e^{X_\alpha - (1/2)\mathbb{E}(X_\alpha^2)}\mu_{\sqrt{2}}([0,1]) > \lambda\big), \tag{20}$$

where X_α is a centered Gaussian independent of $\mu_{\sqrt{2}}([0,1])$. Conditioning on X_α and using the tail estimate for $\mu_{\sqrt{2}}([0,1])$

$$\mathbb{P}\big(\alpha e^{X_\alpha - (1/2)\mathbb{E}(X_\alpha^2)}\mu_{\sqrt{2}}([0,1]) > \lambda\big) \leq C\alpha\lambda^{-1}. \tag{21}$$

Collecting everything gives the desired result. $\square$

REMARK 16. While the class of measures ν covered by this result is rather limited (due to the fact that the result was easy to prove and sufficient for our needs concerning the modulus of continuity), we believe that such a result for the tail should hold quite generally for critical Gaussian multiplicative chaos measures.

We next note that the regular variation with exponent -1 of the tail is robust under linear combinations of copies of random variables:

LEMMA 17. *Let $X \geq 0$ satisfy $\mathbb{P}(X > \lambda) \leq \frac{A}{\lambda}$ for $\lambda > 0$.*
Let X_j, $j \in \{1,\dots,N\}$ be (possibly dependent) random variables with the same distribution as X and let $a_j \geq 0$ for $j \in \{1,\dots,N\}$. Then

$$\mathbb{P}\Bigg(\sum_{j=1}^N a_j X_j > \lambda\Bigg) \leq \frac{C\cdot A\log(N+1)(\sum_{j=1}^N a_j)}{\lambda} \qquad \textit{for all } \lambda > 0,$$

with a universal (in particular, independent of A) constant $C < \infty$.

PROOF. We may assume that $\sum_{j=1}^N a_j = 1$ since the statement scales in the right way. Fix $t \in (0,1)$ and observe first that for all positive $y_1,\dots,y_N$ one has the subadditivity inequality

$$\Bigg(\sum_{j=1}^N a_j y_j\Bigg)^t \leq \sum_{j=1}^N a_j^t y_j^t.$$

Fix $\lambda > 0$. The above holds if we set $y_j = (x_j - \lambda)_+$, where we denote the positive part by $y_+ := \max(0, y)$ and let, for now, the numbers $(x_j)_{1\le j\le N}$ be arbitrary reals. We obtain using $\sum_{j=1}^N a_j = 1$ (and Jensen) that

$$\left(\sum_{j=1}^N a_j x_j - \lambda\right)_+^t \le \left(\sum_{j=1}^N a_j (x_j - \lambda)_+\right)^t \le \sum_{j=1}^N a_j^t (x_j - \lambda)_+^t,$$

or, in other words,

$$\phi\left(\sum_{j=1}^N a_j x_j\right) \le \sum_{j=1}^N a_j^t \phi(x_j),$$

where $\phi(x) := (x - \lambda)_+^t$. Especially, we have

$$\mathbb{E}\phi\left(\sum_{j=1}^N a_j X_j\right) \le \mathbb{E}\phi(X) \sum_{j=1}^N a_j^t. \tag{22}$$

The right-hand side can be estimated as follows:

$$\begin{aligned}
\mathbb{E}\phi(X) &= \int_0^\infty \phi'(u)\mathbb{P}(X > u)\,\mathrm{d}u \\
&\le A \int_\lambda^\infty t(u-\lambda)^{t-1} u^{-1}\,\mathrm{d}u \\
&= At \int_0^\lambda \frac{y^{t-1}\,\mathrm{d}y}{y+\lambda} + At \int_\lambda^\infty \frac{y^{t-1}\,\mathrm{d}y}{y+\lambda} \\
&\le At\lambda^{-1} \int_0^\lambda y^{t-1}\,\mathrm{d}y + At \int_\lambda^\infty y^{t-2}\,\mathrm{d}y \\
&= A(1-t)^{-1}\lambda^{t-1}.
\end{aligned} \tag{23}$$

From Markov's inequality and (22), we thus obtain

$$\phi(2\lambda)\cdot \mathbb{P}\left(\sum_{j=1}^N a_j X_j > 2\lambda\right) \le \mathbb{E}\phi\left(\sum_{j=1}^N a_j X_j\right) \le \left(\sum_{j=1}^N a_j^t\right)\mathbb{E}\phi(X),$$

and by combining with (23)

$$\mathbb{P}\left(\sum_{j=1}^N a_j X_j > 2\lambda\right) \le \frac{A}{\lambda}\left(\frac{1}{1-t}\sum_{j=1}^N a_j^t\right). \tag{24}$$

Finally, choosing $t = t_0 := 1 - 1/\log N$ (for $N \ge 3$) we get

$$\left(\frac{1}{1-t_0}\sum_{j=1}^N a_j^{t_0}\right) \le \left(\frac{N^{1-t_0}}{1-t_0}\right)\left(\sum_{j=1}^N a_j\right)^{t_0} = \frac{N^{1-t_0}}{1-t_0} = e\log N,$$

and then (24) yields the stated result. $\square$

REMARK 18. The above result is essentially optimal: choose $\Omega = [0, 1)$, that is, the one-dimensional torus with the Lebesgue measure. Let

$$X_0(\omega) = \frac{N}{k} \qquad \text{for } \omega \in \big[(k-1)/N, k/N\big), k = 1, 2, \dots, N.$$

Then $\mathbb{P}(X > \lambda) < 1/\lambda$. Define the random variables X_j, $j = 1, \dots, N$ with the formula $X_j(\omega) = X_0(\omega + (j-1)/N)$, which is well defined since we are now in the torus. Then each X_j has the same tail as X_0. However, the average $X := (1/N)\sum_{j=1}^N X_j$ is the constant variable: $X(\omega) = \sum_{j=1}^N j^{-1} \geq \log N$ for all $\omega \in \Omega$. We thus have $\mathbb{P}(X \geq \log N) = 1$.

For comparing the present setting with that of multiplicative cascades, we shall make use of Kahane's convexity inequalities [26].

PROPOSITION 19. *Let $G : [0, \infty) \to \mathbb{R}$ be a concave function such that $|G(x)| \leq C(1 + x^\alpha)$ for some positive constants C and α. Let $A \subset \mathbb{R}^d$ be a Borel set, ρ be a Radon measure on A and $(X_r)_{r\in A}$ and $(Y_r)_{r\in A}$ be two continuous and centered Gaussian processes on A such that the covariance kernels satisfy $k_X(u, v) \leq k_Y(u, v)$ for all $u, v \in A$. Then*

$$\mathbb{E}G\left(\int_A e^{X_r - (1/2)\mathbb{E}(X_r^2)}\rho(\mathrm{d}r)\right) \geq \mathbb{E}G\left(\int_A e^{Y_r - (1/2)\mathbb{E}(Y_r^2)}\rho(\mathrm{d}r)\right).$$

To apply this inequality, we construct a Gaussian field on $[0, 1]$ for which the moments of the corresponding measure can be calculated and for which we have a covariance structure that allows comparing with more correlated situations (such a comparison is also used in [17] to prove that the limit of the total mass martingale associated to nonrenormalized critical chaos measures vanishes almost surely).

The Gaussian field we shall employ is essentially a Gaussian branching random walk. Let us associate to the collection $\{I_\sigma\}$ of dyadic subintervals of $[0, 1]$ an i.i.d. collection of standard Gaussian random variables $\{V_\sigma\}$. Let us write $\Sigma_k = \{0, 1\}^k$ and define the field

$$U_n(x) = \sum_{k=1}^n \sum_{\sigma\in\Sigma_k : x\in I_\sigma} V_\sigma. \tag{25}$$

The covariance of U_n is given by

$$\begin{aligned}
\mathbb{E}\big(U_n(x)U_n(y)\big) &= \sum_{k,k'=1}^n \sum_{\sigma\in\Sigma_k \sigma'\in\Sigma_{k'} : x\in I_\sigma, y\in I_{\sigma'}} \mathbb{E}(V_\sigma V_{\sigma'}) \\
&= \sum_{k,k'=1}^n \sum_{\sigma\in\Sigma_k, \sigma'\in\Sigma_{k'} : x\in I_\sigma, y\in I_{\sigma'}} \mathbf{1}(\sigma = \sigma') \\
&= \sum_{k=1}^n \sum_{\sigma\in\Sigma_k : x,y\in I_\sigma} 1.
\end{aligned}$$

For comparison with other fields, we note that to have a $\sigma \in \Sigma_k$ such that $x, y \in I_\sigma$, we must have $|x-y| \leq 2^{-k}$ and we see that

$$\mathbb{E}\big(U_n(x)U_n(y)\big) \leq \sum_{k=1}^{(-\log|x-y|/\log 2)\wedge n} 1 = \frac{-\log|x-y|}{\log 2} \wedge n.$$

Our last technical lemma is a version of the Borell–Tsirelson–Ibragimov–Sudakov inequality [1], Theorem 2.1.1. For our purposes, we need a version which relates the tail probability of the supremum of a Gaussian process on an interval both to the size of the interval and to the modulus of continuity of the covariance of the process in a quantitative manner.

LEMMA 20. *Let $I \subset \mathbb{R}$ be a bounded interval and $L > 0$. Let $(Y(x))_{x\in I}$ be an arbitrary centered Gaussian process on I such that $\mathbb{E}|Y(x) - Y(y)|^2 \leq L|x-y|$ for all $x, y \in I$, and further suppose there is some* (*deterministic*) *$x_0 \in I$ for which $Y(x_0) = 0$ almost surely. Then, for any $\varepsilon > 0$, there exists an absolute constant $c_\varepsilon > 0$* (*i.e., the choice of c_ε depends only on ε*) *such that for all $s > 0$*

$$\mathbb{P}\Big(\sup_{x\in I} Y(x) > s\Big) \leq c_\varepsilon e^{-s^2/((2+\varepsilon)|I|L)}. \tag{26}$$

PROOF. By considering the scaled process $\frac{1}{\sqrt{|I|L}} Y(|I|\cdot)$ instead of $Y(\cdot)$ we may without loss of generality reduce to the case $|I| = L = 1$. Since $\mathbb{E}Y(x_0)^2 = 0$, this normalization also implies that $\sigma_Y^2 := \sup_{x\in I} \mathbb{E}Y(x)^2 \leq 1$. The Borell–TIS inequality then states that for $s > 0$ we have

$$\mathbb{P}\Big(\sup_{x\in I} Y(x) - \mathbb{E}\sup_{x\in I} Y(x) > s\Big) \leq e^{-s^2/(2\sigma_Y^2)} \leq e^{-s^2/2}. \tag{27}$$

Then consider the Gaussian process $X(x) = B_x - B_{x_0}$, where $(B_x)_{x\in I}$ is a one-dimensional Brownian motion. Clearly, $(X(x))_{x\in I}$ satisfies the assumptions of the lemma, and moreover,

$$\mathbb{E}\big|Y(x) - Y(y)\big|^2 \leq |x-y| = \mathbb{E}\big|X(x) - X(y)\big|^2$$

for all $x, y \in I$. By the Sudakov–Fernique inequality ([1], Theorem 2.2.3), we then have

$$\mathbb{E}\sup_{x\in I} Y(x) \leq \mathbb{E}\sup_{x\in I} X(x) \leq M < \infty$$

for some absolute constant $M > 0$. In (27), for $s > M$ this implies

$$\mathbb{P}\Big(\sup_{x\in I} Y(x) > s\Big) \leq e^{-(s-M)^2/2}.$$

Since the choice of M does not depend on the parameters of the process $(Y(x))_{x\in I}$, it clear that for any $\varepsilon > 0$ there exists an absolute constant $c_\varepsilon > 0$ for which (26) holds. $\square$

REMARK 21. The statement of the lemma generalizes to processes on bounded domains $U \subset \mathbb{R}^d$ for $d \geq 2$ simply by replacing the length $|I|$ of the interval I by the diameter $\operatorname{diam}(U)$ of U. The only difference in the proof is that instead of one-dimensional Brownian motion one compares the arbitrary process to Lévy's Brownian motion on $\mathbb{R}^d$, that is, the Gaussian process $(X(x))_{x\in\mathbb{R}^d}$ with $\mathbb{E}X(x)X(y) = \frac{1}{2}(|x| + |y| - |x-y|)$; we refer to [27] for a proof that this function is indeed a covariance kernel.

We are ready to proceed to the main proof.

3.3. *Main results for the modulus of continuity.* Let $((X_t(x))_{x\in\mathbb{R}})_{t\geq 0}$ be the exactly scale invariant Gaussian field on $\mathbb{R}$ as before and define the Gaussian field $((Y_t(x))_{x\in\mathbb{R}})_{t\geq 0}$ by setting

$$Y_t(x) = W\big(\mathcal{C}_t(x) \setminus \mathcal{C}_0(x)\big) = X_t(x) - X_0(x) \qquad \text{for } x \in \mathbb{R}, t \geq 0.$$

In the proof of Theorem 2, it is convenient to use the characterization (7) of critical lognormal multiplicative chaos. To keep the notation simpler, we normalize the construction by the deterministic constant $c > 0$ in (7). Explicitly, we consider the critical measures associated to the fields X and Y and denote

$$\mu_{\sqrt{2}}(\mathrm{d}x) = \lim_{t\to\infty} \sqrt{t}e^{\sqrt{2}X_t(x)-(t+1)}\,\mathrm{d}x$$

and

$$\nu_{\sqrt{2}}(\mathrm{d}x) = \lim_{t\to\infty} \sqrt{t}e^{\sqrt{2}Y_t(x)-t}\,\mathrm{d}x,$$

where the limits exist in probability in the weak sense. By construction, it is clear that almost surely, the Radon–Nikodym derivative $\frac{\mathrm{d}\mu_{\sqrt{2}}}{\mathrm{d}\nu_{\sqrt{2}}}(x) = e^{\sqrt{2}X_0(x)-1}$ is almost surely positive and uniformly bounded away from 0 and ∞ for all $x \in [0, 1]$ (in particular, the assumptions of Lemma 15 are met), so for the purpose of our result on the modulus of continuity the difference between these two measures is insignificant. The measure $\mu_{\sqrt{2}}$ is exactly scale invariant as before, but in this section we make more use of the measure $\nu_{\sqrt{2}}$ which satisfies the $\star$-*scaling relation*: for every $\epsilon \in (0, 1]$ we have

$$(28) \qquad \big(\nu_{\sqrt{2}}(A)\big)_{A\in\mathcal{B}([0,1])} \stackrel{d}{=} \left(\epsilon \int_A e^{\sqrt{2}Y_{-\log\epsilon}+\log\epsilon}\nu^{\epsilon}_{\sqrt{2}}(\mathrm{d}x)\right)_{A\in\mathcal{B}([0,1])},$$

where $\nu^{\epsilon}_{\sqrt{2}}$ is independent of $Y_{-\log\epsilon}$ and $(\nu^{\epsilon}_{\sqrt{2}}(A))_A \stackrel{d}{=} (\nu_{\sqrt{2}}(\epsilon^{-1}A))_A$. The proof of this scaling relation is recalled in the Appendix. We also stress that $\nu_{\sqrt{2}}$ satisfies the conditions of Lemma 15.

The next lemma contains the key technical estimates that lead to the proof of Theorem 2.

LEMMA 22. *Let us index by $\sigma \in \Sigma_n = \{0,1\}^n$ the dyadic subintervals I_σ of $[0,1]$ of length 2^{-n}. Moreover, write $\Sigma_n^{(e)}$ for the family of even dyadic intervals of length 2^{-n} (i.e., intervals of the form $[(2j)2^{-n},(2j+1)2^{-n})$). Then for $\gamma \in (0,\frac{1}{2})$ and $\epsilon \in (0,1)$ there exists a constant $C = C(\epsilon)$ such that*

$$\mathbb{P}\Big(\max_{\sigma\in\Sigma_n^{(e)}} \nu_{\sqrt{2}}(I_\sigma) \geq n^{-\gamma}\Big) \leq Cn^{(1-\epsilon)(\gamma-(1/2))}. \tag{29}$$

The same holds if we replace $\Sigma_n^{(e)}$ with $\Sigma_n^{(o)}$, the corresponding collection of odd dyadic intervals.

PROOF. The proof is rather lengthy so we shall split it into steps that somewhat parallel the cascade proof.

Step 1: *Using scaling and independence.*

We begin by noting that by specializing the $\star$-scaling relation to dyadics, we get

$$\big(\nu_{\sqrt{2}}(I_\sigma)\big)_{\sigma\in\Sigma_n} \stackrel{d}{=} \left(2^{-n}\int_{I_\sigma} e^{\sqrt{2}Y_{n\log 2}(x)-n\log 2}\nu_{\sqrt{2}}^{(n)}(\mathrm{d}x)\right)_{\sigma\in\Sigma_n}, \tag{30}$$

where $\nu_{\sqrt{2}}^{(n)}$ is independent of $Y_{n\log 2}$ and $(\nu_{\sqrt{2}}^{(n)}(A))_A \stackrel{d}{=} (\nu_{\sqrt{2}}(2^n A))_A$. Since $Y_t(x)$ and $Y_t(y)$ are independent when $|x-y|\geq 1$, $\nu_{\sqrt{2}}(A)$ is independent of $\nu_{\sqrt{2}}(B)$ when $d(A,B)\geq 1$. Thus, the scaling property implies that $(\nu_{\sqrt{2}}^{(n)}\lfloor I_\sigma)_{\sigma\in\Sigma_n^{(e)}}$ is a family of independent random measures (and similarly for the odd intervals)—here $\nu_{\sqrt{2}}^{(n)}\lfloor I_\sigma$ denotes the restriction of $\nu_{\sqrt{2}}^{(n)}$ to I_σ.

Let us write

$$W_{n,\sigma} = 2^{-n}\int_{I_\sigma} e^{\sqrt{2}Y_{n\log 2}(x)-n\log 2}\nu_{\sqrt{2}}^{(n)}(\mathrm{d}x). \tag{31}$$

Using the independence noted above, we see that

$$\begin{aligned}\mathbb{P}\Big(\max_{\sigma\in\Sigma_n^{(e)}} W_{n,\sigma} < n^{-\gamma}\Big) &= \mathbb{E}\prod_{\sigma\in\Sigma_n^{(e)}} \mathbb{P}\big(W_{n,\sigma} < n^{-\gamma}|Y_{n\log 2}\big)\\ &\geq \mathbb{E}\prod_{\sigma\in\Sigma_n}\big(1-\mathbb{P}\big(W_{n,\sigma}\geq n^{-\gamma}|Y_{n\log 2}\big)\big).\end{aligned} \tag{32}$$

Step 2: *Getting to the Laplace transform.*

To estimate $\mathbb{P}(W_{n,\sigma}\geq n^{-\gamma}|Y_{n\log 2})$, we will approximate the integral (31) by a Riemann sum and then make use of Lemma 15. For brevity, we will denote $f_{\sqrt{2}}^{(n)}(\cdot) := e^{\sqrt{2}Y_{n\log 2}(\cdot)-n\log 2}$. Fix $\sigma\in\Sigma_n$ for the moment, let $k\in\mathbb{N}_+$ and divide I_σ into 2^k subintervals $(I_{\sigma,j})_{j=1}^{2^k}$ of equal length. Denote the midpoint of $I_{\sigma,j}$ by $x_{\sigma,j}$.

Let $s > 0$ and define the event $\mathcal{D}_s = \{\sup_{x \in I_{\sigma,j}} |Y_{n\log 2}(x) - Y_{n\log 2}(x_{\sigma,j})| \le s$ for all $j = 1, 2, \ldots, 2^k\}$. We then have on $\mathcal{D}_s$

$$2^n W_{n,\sigma} = \int_{I_\sigma} f^{(n)}_{\sqrt{2}}(x)\nu^{(n)}_{\sqrt{2}}(\mathrm{d}x) \le e^{2\sqrt{2}s} \sum_{j=1}^{2^k} \fint_{I_{\sigma,j}} f^{(n)}_{\sqrt{2}}(x)\,\mathrm{d}x\, \nu^{(n)}_{\sqrt{2}}(I_{\sigma,j}),$$

where $\fint_A f(x)\,\mathrm{d}x := \frac{1}{|A|}\int_A f(x)\,\mathrm{d}x$ is the integral average. Let $\mathcal{F}_n = \sigma(\{Y_t(x) : x \in [0,1], t \le n\log 2\})$. Since $\nu^{(n)}_{\sqrt{2}}(I_{\sigma,j}) \overset{d}{=} \nu^{(n)}_{\sqrt{2}}(I_{\sigma,i})$ for $j \ne i$ and the function $f^{(n)}_{\sqrt{2}}$ is independent of the measure $\nu^{(n)}_{\sqrt{2}}$, Lemmas 17 and 15 imply that on $\mathcal{D}_s$

$$\begin{aligned}
&\mathbb{P}\biggl(\int_{I_\sigma} f^{(n)}_{\sqrt{2}}(x)\nu^{(n)}_{\sqrt{2}}(\mathrm{d}x) > \lambda \Big| \mathcal{F}_n\biggr) \\
&\quad\le \mathbb{P}\Biggl(e^{2\sqrt{2}s} \sum_{j=1}^{2^k} \fint_{I_{\sigma,j}} f^{(n)}_{\sqrt{2}}(x)\,\mathrm{d}x\, \nu^{(n)}_{\sqrt{2}}(I_{\sigma,j}) > \lambda \Big| \mathcal{F}_n\Biggr) \\
&\quad\le Ck2^{-k}\biggl(\frac{e^{2\sqrt{2}s}\sum_{j=1}^{2^k} \fint_{I_{\sigma,j}} f^{(n)}_{\sqrt{2}}(x)\,\mathrm{d}x}{\lambda}\biggr)
\end{aligned}$$

for some constant $C > 0$. Setting $\lambda = n^{-\gamma}2^n$ and combining this inequality with (32) and the inequality $e^{-2x} \le 1 - x$ valid for $x \in [0, 1/2]$, we get

$$\begin{aligned}
&\mathbb{E}\mathbb{P}\Bigl(\max_{\sigma \in \Sigma_n^{(e)}} \nu_{\sqrt{2}}(I_\sigma) > n^{-\gamma} | \mathcal{F}_n\Bigr) \\
&\quad\le 1 - \mathbb{E}\prod_{\sigma \in \Sigma_n} \bigl(1 - \mathbb{P}(W_{n,\sigma} \ge n^{-\gamma} | Y_{n\log 2})\bigr) \\
&\quad\le 1 - \mathbb{E}\exp\biggl(-2Ck2^{-k}e^{2\sqrt{2}s} \sum_{\sigma \in \Sigma_n}\biggl(\frac{\sum_{j=1}^{2^k} \fint_{I_{\sigma,j}} f^{(n)}_{\sqrt{2}}(x)\,\mathrm{d}x}{2^n n^{-\gamma}}\biggr)\biggr) \\
&\qquad + 1 - \mathbb{P}(\mathcal{A}_{n,k,s}) \\
&\quad= 1 - \mathbb{E}\exp\biggl(-2Cke^{2\sqrt{2}s} n^\gamma \int_0^1 f^{(n)}_{\sqrt{2}}(x)\,\mathrm{d}x\biggr) + 1 - \mathbb{P}(\mathcal{A}_{n,k,s}),
\end{aligned}$$

where $\mathcal{A}_{n,k,s}$ is the event

$$\begin{aligned}
\mathcal{A}_{n,k,s} = \biggl\{&\max_{\sigma \in \Sigma_n} Ck2^{-k}\frac{e^{2\sqrt{2}s}\sum_{j=1}^{2^k} \fint_{I_{\sigma,j}} f^{(n)}_{\sqrt{2}}(x)\,\mathrm{d}x}{n^{-\gamma}2^n} < \frac{1}{2}\biggr\} \\
&\cap \Bigl\{\sup_{x \in I_{\sigma,j}} \bigl|Y_{n\log 2}(x) - Y_{n\log 2}(x_{\sigma,j})\bigr| \le s \\
&\qquad \forall j \in \{0, 1, \ldots, 2^k - 1\}\ \forall \sigma \in \Sigma_n\Bigr\}.
\end{aligned}$$

Denoting

$$S_n = n^{1/2} \int_0^1 e^{\sqrt{2} Y_{n\log 2}(x) - n\log 2}\, \mathrm{d}x,$$

we finally get

$$\begin{aligned} \mathbb{P}\Big(\max_{\sigma\in\Sigma_n^{(e)}} \mu_{\sqrt{2}}(I_\sigma) > n^{-\gamma}\Big) &\le 1 - \mathbb{E}\exp\big(-2Ce^{2\sqrt{2}s} k n^{(\gamma-(1/2))} S_n\big) \\ &\quad + 1 - \mathbb{P}(\mathcal{A}_{n,k,s}). \end{aligned} \tag{33}$$

Step 3: *Controlling the error*.

We then estimate the terms in the inequality above. Denote

$$\begin{aligned} \mathcal{B}_n &= \left\{ \max_{\sigma\in\Sigma_n} Ck2^{-k} \frac{e^{2\sqrt{2}s} \sum_{j=1}^{2^k} ⨍_{I_{\sigma,j}} f_{\sqrt{2}}^{(n)}(x)\,\mathrm{d}x}{n^{-\gamma} 2^n} < \frac{1}{2} \right\} \\ &= \left\{ \max_{\sigma\in\Sigma_n} \int_{I_\sigma} e^{\sqrt{2} Y_{n\log 2}(x) - n\log 2}\, \mathrm{d}x < n^{-\gamma} \big(2Cke^{2\sqrt{2}s}\big)^{-1} \right\} \end{aligned}$$

and

$$\mathcal{B}'_{n,k,s} = \Big\{ \sup_{x\in I_{\sigma,j}} \big|Y_{n\log 2}(x) - Y_{n\log 2}(x_{\sigma,j})\big| \le s \ \forall j \in \{0,1,\dots,2^k-1\}\ \forall \sigma\in\Sigma_n \Big\}$$

so that

$$\mathcal{A}_{n,k,s} = \mathcal{B}_n \cap \mathcal{B}'_{n,k,s} \quad \text{and} \quad 1 - \mathbb{P}(\mathcal{A}_{n,k,s}) \le \big(1 - \mathbb{P}(\mathcal{B}_n)\big) + \big(1 - \mathbb{P}(\mathcal{B}'_{n,k,s})\big).$$

We first estimate the probability of $\mathcal{B}'_{n,k,s}$ not occurring. For all σ and j, the length of $I_{\sigma,j}$ is 2^{-n-k} and $\mathbb{E}|Y_{n\log 2}(x) - Y_{n\log 2}(y)|^2 \le 2^{n+1}|x-y|$, so by Lemma 20 we have, for any $\sigma\in\Sigma_n$ and $j = 1,\dots,2^k$,

$$\mathbb{P}\Big(\sup_{x\in I_{\sigma,j}} \big|Y_{n\log 2}(x) - Y_{n\log 2}(x_{\sigma,j})\big| > s\Big) \le ce^{-2^{k-3}s^2},$$

where $c > 0$ is an absolute constant. It follows that

$$1 - \mathbb{P}\big(\mathcal{B}'_{n,k,s}\big) \le c2^{n+k} e^{-2^{k-3}s^2}.$$

For the choice $s_n \sim \sqrt{\epsilon \log n}$ and $k_n \sim \alpha \log n$, the right-hand side of this estimate is asymptotically equivalent to

$$n^{\alpha\log 2} e^{n\log 2 - (\epsilon/8) n^{\alpha \log 2} \log n},$$

from which we see that in order to have $\sum_{n=1}^\infty (1 - \mathbb{P}(\mathcal{B}'_{n,k_n,s_n})) < \infty$ we may take $\epsilon > 0$ arbitrarily small, but must restrict to $\alpha \ge 1/\log 2$. Taking $\alpha = 1/\log 2$, in (33) these choices give

$$\begin{aligned} &\mathbb{P}\Big(\max_{\sigma\in\Sigma_n^{(e)}} \nu_{\sqrt{2}}(I_\sigma) > n^{-\gamma}\Big) \\ &\quad \le 1 - \mathbb{E}\exp\Big(-2Ce^{2\sqrt{2}\sqrt{\epsilon\log n}}\frac{\log n}{\log 2} n^{(\gamma-(1/2))} S_n\Big) \\ &\qquad + c' n^{-c''\log n} + \big(1-\mathbb{P}(\mathcal{B}_n)\big) \end{aligned} \tag{34}$$

for some constants $c', c'' > 0$ depending on ϵ.

To estimate the probability of $\mathcal{B}_n$, we note that

$$\big\{S_n < n^{(1/2)-\gamma}\big(2Ck_n e^{2\sqrt{2}s_n}\big)^{-1}\big\} \subset \mathcal{B}_n. \tag{35}$$

By Chebyshev's inequality, we then see that for any $q < 1$

$$\begin{aligned} 1-\mathbb{P}(\mathcal{B}_n) &\le \mathbb{P}\big(S_n > \big(2Ck_n e^{2\sqrt{2}s_n}\big)^{-1} n^{((1/2)-\gamma)}\big) \\ &\le \big(2Ck_n e^{2\sqrt{2}s_n}\big)^q \frac{\mathbb{E}(S_n^q)}{n^{((1/2)-\gamma)q}}. \end{aligned}$$

Step 4: *Comparison with cascades*.

If we knew that $\mathbb{E}(S_n^q)$ were uniformly bounded in n for some values of q, we would have a quantitative estimate for the speed at which $\mathbb{P}(\mathcal{B}_n)$ tends to one. For this, we employ Kahane's convexity inequalities, that is, Proposition 19, and comparison with the branching random walk U_n defined in (25).

Note that

$$\begin{aligned} \mathbb{E}\big(U_n(x)U_n(y)\big) &\le \frac{-\log|x-y|}{\log 2} \wedge n \\ &\le \frac{1}{\log 2}\mathbb{E}\big(Y_{n\log 2}(x)Y_{n\log 2}(y)\big) + C, \end{aligned}$$

for some large enough constant C, since the covariance of the field $Y_{n\log 2}$ is given by

$$\begin{aligned} &\mathbb{E}\big(Y_{n\log 2}(x)Y_{n\log 2}(y)\big) \\ &\quad = \begin{cases} -\log|x-y| + |x-y| - 1, & 2^{-n} \le |x-y| \le 1, \\ n\log 2 + |x-y| - 2^n|x-y|, & |x-y| \le 2^{-n}. \end{cases} \end{aligned}$$

Let us thus consider a standard Gaussian variable Z independent of $Y_{n\log 2}$ and define the fields

$$A(x) = \sqrt{2\log 2}\,U_n(x) \quad \text{and} \quad B(x) = \sqrt{2}Y_{n\log 2}(x) + \sqrt{2C\log 2}\,Z.$$

We have $\mathbb{E}(A(x)A(y)) \le \mathbb{E}(B(x)B(y))$ for all x, y. We then apply the convexity inequality to the fields A and B with the convex function $G(x) = n^{q(1/2)}x^q$ for

$q < 1$, to get

$$\begin{aligned}&\mathbb{E}\big(e^{q\sqrt{2C\log 2}Z-qC\log 2}\big)\mathbb{E}\big(S_n^q\big)\\&\quad\le \mathbb{E}\biggl(n^{q(1/2)}\biggl(\int_0^1 e^{\sqrt{2\log 2}U_n(x)-\log 2\mathbb{E}(U_n(x)^2)}\,\mathrm{d}x\biggr)^q\biggr).\end{aligned}$$

Comparing with the notation of [9], we see that the quantity on the right here is simply $\mathbb{E}((n^{1/2}Z_{1,n})^q)$, the qth moment of the total mass of the correctly renormalized critical Mandelbrot cascade measure. As noted in [9], the fact that this is uniformly bounded in n for a fixed $q < 1$ follows from [33, 45]. Thus, $\mathbb{E}(S_n^q)$ is also uniformly bounded in n for $q < 1$. So, recalling that $s_n = \sqrt{\epsilon \log n}$ and $k_n = \frac{1}{\log 2}\log n$, we conclude that for any $\epsilon \in (0,1)$, there are constants $C(\frac{\epsilon}{2})$ and $C(\epsilon)$ so that if we take n large enough, then $1 - \mathbb{P}(\mathcal{B}_n) \le C(\frac{\epsilon}{2})(2Ck_n e^{2\sqrt{2}s_n})^{1-(\epsilon/2)} n^{(1-(\epsilon/2))(\gamma-(1/2))} \le C(\epsilon) n^{(1-\epsilon)(\gamma-(1/2))}$. Thus, by (34) all we are left with is to estimate the Laplace transform of S_n.

We make use of the following formula, valid for all nonnegative random variables X:

$$1 - \mathbb{E}\big(\exp(-\alpha X)\big) = \int_0^\infty \alpha e^{-\alpha t}\mathbb{P}(X \ge t)\,\mathrm{d}t.$$

In this formula, we set $\alpha = 2Ce^{2\sqrt{2}s_n}k_n n^{(\gamma-(1/2))}$ and $X = S_n$. Recalling from the argument above that $\mathbb{E}(S_n^q)$ is uniformly bounded in n for $q < 1$, by Chebyshev's inequality we see that for any $q < 1$

$$\mathbb{P}(S_n \ge t) \le C_q t^{-q}.$$

Making the change of variable $\tau = \alpha t$, we get

$$1 - \mathbb{E}\big(e^{-\alpha S_n}\big) \le C_q \alpha^q \int_0^\infty e^{-\tau}\tau^{-q}\,\mathrm{d}\tau.$$

Recalling again that $s_n = \sqrt{\epsilon \log n}$ and $k_n = \frac{1}{\log 2}\log n$, we see that since the integral converges, we can take q so close to one that we get

$$1 - \mathbb{E}\big(e^{-\alpha S_n}\big) \le C'(\epsilon) n^{(\gamma-(1/2))(1-\epsilon)},$$

which completes the proof of Lemma 22. □

Theorem 2 now follows quickly. We first prove the analogous statement for the measure $\nu_{\sqrt{2}}$.

THEOREM 23. *For any interval $I \subset [0,1]$ and $\gamma < \frac{1}{2}$, almost surely*

$$\nu_{\sqrt{2}}(I) \le C(\omega)\big(\log\big(1 + |I|^{-1}\big)\big)^{-\gamma}, \tag{36}$$

where $C(\omega)$ is an almost surely finite random constant.

PROOF. It is enough to restrict to dyadic subintervals. Pick $\gamma \in (0, \frac{1}{2})$. Let l be an integer so that $l(\gamma - \frac{1}{2}) < -2$. We then have by Lemma 22 that

$$\sum_{k=1}^{\infty} \mathbb{P}\Big(\max_{\sigma \in \Sigma_{k^l}^{(e/o)}} \nu_{\sqrt{2}}(I_\sigma) \geq k^{-l\gamma} \Big) \leq C \sum_{k=1}^{\infty} k^{l((\gamma-(1/2))/2)} < \infty.$$

By Borel–Cantelli,

$$\max_{\sigma \in \Sigma_{k^l}^{(e/o)}} \nu_{\sqrt{2}}(I_\sigma) \leq C(\omega) k^{-l\gamma}$$

for a random (almost surely finite) constant $C(\omega)$. Combining the estimates for even and odd intervals, we get

$$\max_{\sigma \in \Sigma_{k^l}} \nu_{\sqrt{2}}(I_\sigma) \leq C'(\omega) k^{-l\gamma}.$$

We note that $\max_{\sigma \in \Sigma_n} \nu_{\sqrt{2}}(I_\sigma)$ is decreasing in n so for $k^l \leq n \leq (k+1)^l$ we have

$$\max_{\sigma \in \Sigma_n} \nu_{\sqrt{2}}(I_\sigma) \leq \max_{\sigma \in \Sigma_{k^l}} \nu_{\sqrt{2}}(I_\sigma) \leq C'(\omega) k^{-l\gamma} \leq C'(\omega) 2^{l\gamma} n^{-\gamma},$$

which is the desired result. □

PROOF OF THEOREM 2. From the definition of $\nu_{\sqrt{2}}$, we note that for any interval $I \subset [0,1]$

$$\mu_{\sqrt{2}}(I) \leq e^{\sqrt{2}\max_{x\in[0,1]} X_0(x) - 1} \nu_{\sqrt{2}}(I),$$

where $(X_0(x))_{x\in[0,1]}$ is a Gaussian process with a continuous covariance kernel. The quantity $e^{\sqrt{2}\max_{x\in[0,1]} X_0(x)}$ is almost surely finite, so Theorem 23 implies the result. □

4. On the $\mu_{\sqrt{2}}$-almost everywhere local behavior of $\mu_{\sqrt{2}}$. We consider the following question: what can be said of the size of smallest possible sets of full $\mu_{\sqrt{2}}$-measure? This question is partially answered by Theorem 4, which is proven in this section.

Let $f : \mathbb{N} \to \mathbb{R}^+$ be an ultimately nonincreasing function tending to 0 at infinity and consider the sets

$$E_n^f = \{x : \mu_{\sqrt{2}}(I_n(x)) \leq f(n)\}.$$

We will determine a class of functions f for which we have

$$\sum_n \mu_{\sqrt{2}}(E_n^f) < \infty \qquad \text{almost surely.}$$

For a nontrivial result, it is already enough to consider the expectation of the series above. We fix a sequence $(\eta_n)_{n\geq 1}$ taking values in $(0,1)$ and write

$$\mu_{\sqrt{2}}(E_n^f) = \int_0^1 \mathbf{1}_{\{\mu_{\sqrt{2}}(I_n(x))\leq f(n)\}}\mu_{\sqrt{2}}(\mathrm{d}x) = \sum_{\sigma\in\Sigma_n}\mu_{\sqrt{2}}(I_\sigma)\mathbf{1}_{\{\mu_{\sqrt{2}}(I_\sigma)\leq f(n)\}}$$
$$\leq \sum_{\sigma\in\Sigma_n}\mu_{\sqrt{2}}(I_\sigma)\left(\frac{f(n)}{\mu_{\sqrt{2}}(I_\sigma)}\right)^{\eta_n} = \sum_{\sigma\in\Sigma_n}\mu_{\sqrt{2}}(I_\sigma)^{1-\eta_n}f(n)^{\eta_n}.$$

Let $\epsilon_n = -\frac{\log(f(n))}{n}$ take the form $\gamma\sqrt{\frac{\log(n)}{n}+\alpha\frac{\log\log(n)}{n}}$ for $n\geq 3$, where $\alpha>0$ and $\gamma>0$ are to be prescribed. Assume $\eta_n=\lambda\epsilon_n$.

Denoting by W_n the nth level lognormal factor

$$W_n \overset{d}{=} \exp(\sqrt{2}X_n - \mathbb{E}X_n^2) \overset{d}{=} \exp\left(\sigma_n N - \frac{\sigma_n^2}{2}\right),$$
$$N\sim N(0,1), \qquad \sigma_n^2 = 2n\log 2,$$

we have, for each $\sigma\in\Sigma_n$, $\mu_{\sqrt{2}}(I_\sigma)\overset{d}{=}2^{-n}W_nY_n$ where Y_n is a copy of Y independent of W_n. Moreover, by Theorem 1 we have $\mathbb{E}(Y^{1-\eta}) = O(\eta^{-1})$ as $\eta\to 0^+$. These remarks yield

$$\mathbb{E}\mu_{\sqrt{2}}(E_n^f) \leq 2^n 2^{-n(1-\eta_n)}\mathbb{E}(W_n^{1-\eta_n})\mathbb{E}(Y^{1-\eta_n})e^{-n\epsilon_n\eta_n}$$
$$\leq Ce^{n(\log(2)\eta_n^2-\epsilon_n\eta_n)-\log(\eta_n)}.$$

A computation yields for $n\geq 3$

$$n\big(\log(2)\eta_n^2-\epsilon_n\eta_n\big)-\log(\eta_n) = \big(c+\tfrac{1}{2}\big)\log(n) + \big(c\alpha-\tfrac{1}{2}\big)\log\log(n)+O(1),$$

where $c=\log(2)\lambda^2\gamma^2-\lambda\gamma^2$. With the order of magnitude chosen for ϵ_n, taking $c=-\frac{3}{2}$ is optimal in view of making $\sum_{n\geq 1}\mathbb{E}\mu_{\sqrt{2}}(E_n^f)$ convergent. This condition requires the equation $\log(2)\lambda^2\gamma^2-\lambda\gamma^2+\frac{3}{2}=0$ to have solutions in λ. This imposes $\gamma\geq\sqrt{6\log(2)}$, hence we choose $\gamma=\sqrt{6\log(2)}$ to minimize ϵ_n. It then turns out that if $-\frac{3}{2}\alpha-\frac{1}{2}<-1$, that is, $\alpha>\frac{1}{3}$, then $\sum_{n\geq 1}\mathbb{E}\mu_{\sqrt{2}}(E_n^f)<\infty$.

Theorem 4 follows from the preceding estimates by an application of the Borel–Cantelli lemma to the measure $\mu_{\sqrt{2}}$. As an application of Theorem 4 we present the following simple corollary.

COROLLARY 24. *Almost surely, there exists a set of Hausdorff dimension* 0 *that has full* $\mu_{\sqrt{2}}$*-measure.*

PROOF. Let

$$E=\{x:\mu_{\sqrt{2}}(I_n(x))\geq f(n)\text{ for all but finitely many }n\},$$

where $f = f_\alpha$ for some $\alpha > \frac{1}{3}$. Since, by Theorem 4, E almost surely has full $\mu_{\sqrt{2}}$-measure, we only need to show that a.s. it has Hausdorff dimension 0.

Let $\{I_\sigma\}_{\sigma \in \Sigma_n^f}$ be the collection of dyadic subintervals of $[0, 1]$ such that $|\sigma| \geq n$ and $\mu_{\sqrt{2}}(I_\sigma) \geq f(|\sigma|)$. Clearly, for any n, $\{I_\sigma\}_{\sigma \in \Sigma_n^f}$ is a cover of E. But for any $s > 0$ and sufficiently large $n \in \mathbb{N}$ we have $2^{-(s/2)|\sigma|} \leq \mu_{\sqrt{2}}(I_\sigma)$ for all $\sigma \in \Sigma_n^f$, so

$$\begin{aligned}\sum_{\sigma \in \Sigma_n^f} |I_\sigma|^s &= \sum_{k \geq n} \sum_{\sigma \in \Sigma_n^f, |\sigma|=k} |I_\sigma|^s = \sum_{k \geq n} \sum_{\sigma \in \Sigma_n^f, |\sigma|=k} |I_\sigma|^{s/2} (2^{-|\sigma|})^{s/2} \\ &\leq \sum_{k \geq n} 2^{-k(s/2)} \sum_{\sigma \in \Sigma_n^f, |\sigma|=k} \mu_{\sqrt{2}}(I_\sigma) \\ &\leq \mu_{\sqrt{2}}([0, 1]) \sum_{k \geq n} 2^{-k(s/2)}.\end{aligned}$$

The last expression tends to 0 as $n \to \infty$. It follows that for any $s > 0$ the set E has zero Hausdorff s-measure, which implies the claim. $\square$

5. Higher dimensions. In this section, we discuss results corresponding to Theorems 1 and 2 in a higher-dimensional setting, that is, for multiplicative chaos measures on $\mathbb{R}^d$ for $d \geq 2$, using similar methods as in the $d = 1$ case. We will focus on the $d = 2$ case. We begin by describing the relevant objects and stating the results, and we will then sketch the minor differences in the proofs. Finally, we will make a remark on the higher-dimensional cases $d \geq 3$.

Formally, a two-dimensional exactly scale invariant lognormal multiplicative chaos measure may be constructed by exponentiating a centered Gaussian field $(X(x))_{x \in \mathbb{R}^2}$ with the covariance $\mathbb{E}X(x)X(y) = \log^+ \frac{r}{|x-y|}$, with $r > 0$. To make a rigorous construction (see [8] Section A.1), one introduces a Gaussian process $(X_t(x))_{x \in \mathbb{R}^2, t \geq 0}$ with covariance:

$$\mathbb{E}(X_t(x)X_s(y)) = \begin{cases} 0, & |x-y| > r, \\ \log \dfrac{r}{|x-y|}, & re^{-t \wedge s} < |x-y| \leq r, \\ t \wedge s + 2\left(1 - \sqrt{\frac{|x-y|}{r}} e^{t \wedge s}\right), & |x-y| \leq re^{-t \wedge s}. \end{cases} \tag{37}$$

It follows from [17, 18] (see Remark 3 in [18] in particular) that a nontrivial critical measure μ exists (the critical point being $\beta_c = 2$) and it can be written as

$$\mu(\mathrm{d}x) = \lim_{t \to \infty} \sqrt{t} e^{2X_t(x) - 2(t+2)} \, \mathrm{d}x, \tag{38}$$

where the limit is taken weakly in probability. The measure can also be constructed through the derivative martingale measure. This measure is exactly scale invariant, that is, for any $\lambda < 1$

$$\big(\mu(\lambda A)\big)_{A\in\mathcal{B}(B_{r/2})} \stackrel{d}{=} \lambda^2 e^{2X_\lambda - 2\mathbb{E}(X_\lambda^2)} \big(\mu(A)\big)_{A\in\mathcal{B}(B_{r/2})},$$

where $B_{r/2}$ is any disk of radius $\frac{r}{2}$, $\mathcal{B}(B_{r/2})$ denotes its Borel subsets and X_λ is a centered Gaussian with variance $\log\frac{1}{\lambda}$ and as in the one-dimensional case, it is independent of $(\mu(A))_{A\in\mathcal{B}(B_{r/2})}$. The parameter r plays the role of a scale parameter. We fix $r = 1$ from now on.

Our proof of Theorem 1 is robust in the sense that in addition to exact scale invariance, very little extra information on the exponentiated field $(X_t(x))$ is used. Indeed, we will prove the following theorem.

THEOREM 25. *Let $Q = [0, a]^2$ with $a \le 1$ and write $Q_1 = [0, \frac{a}{2}]^2$. Then*

$$\lim_{\lambda\to\infty} \lambda \mathbb{P}\big(\mu(Q) > \lambda\big) = c$$

for

$$c = \frac{2}{\log 2}\mathbb{E}\bigg(\mu(Q_1)\log\frac{\mu(Q)}{\mu(Q_1)}\bigg) < \infty.$$

REMARK 26. Using different values of a and r, we obtain upper and lower bounds of similar form for disks (or any other compact set containing an open set) instead of squares. Also, this result can be used to obtain similar bounds for measures other than the exactly scale invariant one (e.g., by controlling the Radon–Nikodym derivative).

For our proof of the modulus of continuity, we needed a further decorrelation property of the family of fields $(X_t(x))$ and the $\star$-scale invariant measure was more convenient than the exactly scale invariant one. We define a corresponding one in two dimensions: consider $Y_t(x) = X_t(x) - X_0(x)$. Again from [17, 18], it follows that

$$\nu(\mathrm{d}x) = \lim_{t\to\infty} \sqrt{t} e^{2Y_t(x)-2t}\,\mathrm{d}x \tag{39}$$

exists when the limit is taken weakly in probability, that the limit is nontrivial and that it has the $\star$-scaling property. The $\star$-scaling property is a consequence of the fact that for $0 < t < t'$, the field Y may be decomposed as $Y_{t'}(x) = Y_t(x) + Y_{t,t'}(x)$, where $Y_{t,t'}$ is a scaled copy of $Y_{t'-t}$ that is sampled independently of Y_t. Especially, $Y_{t,t'}(x)$ is also independent of $Y_{t,t'}(y)$ for $|x - y| \ge e^{-t}$. This decomposition property was crucial and also sufficient for the proof of Theorem 2, so without further comment have the following theorem.

THEOREM 27. *Let $Q = [0, a]^2$ with $a \le 1$ and $\gamma < \frac{1}{2}$. Then*

$$\nu(Q) \le C(\omega)\big(\log(1 + |Q|^{-1})\big)^{-\gamma}$$

for some random, almost surely finite, constant $C(\omega)$.

Again, the result readily extends to other sets besides squares, and also to other measures such as μ.

We now sketch how the proof of Theorem 1 should be adapted in order to prove Theorem 25.

First, a fundamental part of our proof of Theorem 1 was that we were able to write

$$Y = \mu([0,1]) = W_0 Y_0 + W_1 Y_1,$$

where for $i = 1, 2$, $Y_i \stackrel{d}{=} Y$ and $W_i \stackrel{d}{=} \frac{1}{4} e^{\sqrt{2\log 2}N}$, where N is normal, and W_i is independent of Y_i. This decomposition followed from the explicit white noise representation of the field $X_t(x)$ (see the Appendix) which is lacking in dimension two. In the Appendix, we prove the following replacement.

LEMMA 28. *Let $Y = \mu(Q)$ and $Q = [0,a]^2 = \bigcup_{i=1}^4 Q_i$ where Q_i are squares of side $a/2$. By possibly enlarging the probability space where the process $(X_t(x))_{x\in\mathbb{R}^2, t\geq 0}$ is defined, we may write*

$$Y = \sum_{i=1}^{4} \mu(Q_i) = \sum_{i=1}^{4} W_i Y_i,$$

where for each i, $Y_i \stackrel{d}{=} Y$, $W_i \stackrel{d}{=} \frac{1}{16} e^{2\sqrt{\log 2}N}$ with N a standard normal variable, and Y_i is independent of W_i.

With this input, adapting Lemma 13 to the higher-dimensional context turns out to be the only significant task.

PROOF OF THEOREM 25. Using Lemma 28, we may define the Peyrière measure $\mathbb{Q}$ on $\Omega \times \{1,2,3,4\}$ by setting

$$\mathbb{E}_{\mathbb{Q}} f(\omega, j) = \mathbb{E} \sum_{j=1}^{4} W_j(\omega) f(\omega, j),$$

and then we may define the random variables $\widetilde{Y}(\omega, j) = Y_j(\omega)$, $\widetilde{W}(\omega, j) = W_j(\omega)$ and $\widetilde{B}(\omega, j) = \sum_{i\neq j} W_i(\omega) Y_i(\omega)$. From this point on the proof of Theorem 1 may be followed with only cosmetic modifications. Lemma 6 holds true, the measure ν may be defined exactly as in (10) and one obtains the Poisson equation (11). To apply Proposition 8, we only need to check there is an analogue of Lemma 13 in the two-dimensional setup. Note that even though Lemma 13 holds for all $h \in (0,1)$, for the tail result to hold it is sufficient to have the result for $h \in (0, \frac{1}{2} + \varepsilon)$ for some $\varepsilon > 0$. This is proven next as Lemma 29. $\square$

LEMMA 29. *For any $h \in (0, \frac{1}{2} + \frac{1}{2\sqrt{2}})$,*

$$\mathbb{E}\big(\mu(Q_1)^h \mu(Q \setminus Q_1)^h\big) < \infty.$$

PROOF. The idea of the proof of Lemma 13 may be applied, but some differences arise from the fact that the boundary points common to both Q_1 and $Q \setminus Q_1$ are two line segments rather than just one point. We start by noting that Lemma 12 has an analogue in this setting, with exactly the same proof: for two Borel sets $A, B \subset \mathbb{R}^2$ separated by a positive distance, we have $\mathbb{E}(\mu(A)^h \mu(B)^h) < \infty$ for any $h \in (0, 1)$.

By subadditivity, we may estimate

$$\begin{aligned}&\mathbb{E}\big(\mu(Q_1)^h \mu(Q \setminus Q_1)^h\big)\\&\quad \le \mathbb{E}\big(\mu(Q_1)^h \mu(Q_2)^h\big) + \mathbb{E}\big(\mu(Q_1)^h \mu(Q_3)^h\big) + \mathbb{E}\big(\mu(Q_1)^h \mu(Q_4)^h\big).\end{aligned}$$

Suppose that Q_2 and Q_3 are the squares that share a boundary segment with Q_1. Then the first two terms on the right are equal and we need to estimate two different types of terms.

Let us first consider $Q_1 = [0, \frac{a}{2}]^2 =: P_1$ and $Q_4 = [\frac{a}{2}, a]^2 =: R_1$. We then decompose

$$\begin{aligned}P_1 \times R_1 &= \left(\left[\frac{a}{4}, \frac{a}{2}\right]^2 \times \left[\frac{a}{2}, \frac{3a}{4}\right]^2\right) \cup A_1\\&=: (P_2 \times R_2) \cup A_1,\end{aligned}$$

where $A_1 = (P_1 \times R_1) \setminus (P_2 \times R_2)$. We note that $P_2 \times R_2$ is simply a scaled and translated version of $P_1 \times R_1$, so we can repeat this procedure. We obtain

$$P_1 \times R_1 = \left\{\left(\frac{a}{2}, \frac{a}{2}\right)\right\} \cup \bigcup_{k=1}^{\infty} A_k, \tag{40}$$

where P_{k+1} is a square of side length $2^{-k-1}a$ with upper right corner at $(\frac{a}{2}, \frac{a}{2})$ and R_{k+1} is a square of side length $2^{-k-1}a$ with lower left corner at $(\frac{a}{2}, \frac{a}{2})$. Moreover, the A_i are mutually disjoint and disjoint from $P_{k+1} \times R_{k+1}$, and A_k is a scaled and translated version of A_1 with the scale factor 2^{-k+1}. The set A_1 is a finite union of products of two sets with positive distance. Using Lemma 12, we see that $\mathbb{E}((\mu \otimes \mu)(A_1)^h) < \infty$, and by exact scaling we have

$$(\mu \otimes \mu)(A_k) \stackrel{d}{=} 2^{4(-k+1)} e^{4X_k - 4\mathbb{E}X_k^2} (\mu \otimes \mu)(A_1).$$

Thus, by subadditivity, the decomposition (40) yields

$$\mathbb{E}\big(\mu(P_1)^h \mu(R_1)^h\big) \le \mathbb{E}\big((\mu \otimes \mu)(A_1)^h\big) \sum_{k=1}^{\infty} 2^{-4(k-1)h} e^{(8h^2 - 4h)\mathbb{E}(X_k^2)}.$$

Since $\mathbb{E}X_k^2 = k \log 2$, we see that the series converges for any $h \in (0, 1)$. We also made use of the fact that almost surely $(\frac{1}{2}, \frac{1}{2})$ is not an atom.

Consider next the case $P_1 := [0, \frac{a}{2}]^2 = Q_1$ and $R_1 := [\frac{1}{2}, 1] \times [0, \frac{1}{2}] = Q_2$. We may then write

$$P_1 \times R_1 = (P_2^u \times R_2^u) \cup (P_2^u \times R_2^d) \cup (P_2^d \times Q_2^u) \cup (P_2^d \cup Q_2^d) \cup A_1,$$

where P_2^u is the upper half of $[\frac{a}{4}, \frac{a}{2}] \times [0, \frac{a}{2}]$, P_2^d its lower half and similarly for R. The set A_1 is what remains, and again it is a finite union of products of two sets whose distance is positive. The terms corresponding to $P_2^u \times R_2^d$ and $P_2^d \times R_2^u$ are of the form we considered already and the sets $P_2^u \times R_2^u$ and $P_2^d \times R_2^d$ are scaled and translated copies of $P_1 \times R_1$. We repeat this decomposition for $P_2^u \times R_2^u$ and $P_2^d \times R_2^d$ and iterate. At the kth iteration, we have 2^k sets of the form $[0, \frac{a}{2}]^2 \times [\frac{a}{2}, a]^2$ scaled by 2^{-k} and having pairwise disjoint interiors, and also 2^{k-1} copies of A_1 with disjoint interiors, scaled by 2^{-k+1}. Finally, we also have 2^k terms that are scaled and translated copies of $P_1 \times R_1$, which are then further decomposed in the $k+1$th step. Using exact scaling, subadditivity and the fact that the μ-mass of the boundary segments is almost surely zero, we obtain

$$\begin{aligned}\mathbb{E}(\mu(P_1)^h \mu(R_1)^h) &\le C\mathbb{E}((\mu \otimes \mu)(A_1)^h) \sum_{k=1}^{\infty} 2^{k(1-8h+8h^2)} \\ &\quad + C'\mathbb{E}(\mu(Q_1)^h \mu(Q_4)^h) \sum_{k=1}^{\infty} 2^{k(1-8h+8h^2)}.\end{aligned}$$

The series converge for $\frac{1}{2} - \frac{1}{2\sqrt{2}} < h < \frac{1}{2} + \frac{1}{2\sqrt{2}}$, completing the proof of the lemma. $\square$

We close this section by commenting on the case $d \ge 3$. It is known ([41]) that exactly scale invariant multiplicative chaos measures exist in any dimension, but in the known cases, the associated Gaussian field has long range correlations for $d \ge 3$ (i.e., the covariance does not have compact support) and due to this the existence of a nontrivial critical measure is as of yet an open question. This being said, such correlations played no role in our proof of Theorem 1. Indeed, if one could establish the limit (39) the proof of Theorem 25 would also extend to the case $d \ge 3$, with only the combinatorics involved in establishing analogues of Lemma 29 getting slightly more cumbersome.

For the modulus of continuity, the long range correlations, and more specifically the lack of decompositions of the approximating fields with the required decorrelation properties, are more problematic and our proof does not work as it is. On the other hand, in any dimension there exists a $\star$-scale invariant critical measure which does not have long range correlations. Thus, a possible way to proceed is to try to prove the corresponding tail result for this measure.

APPENDIX: SCALE INVARIANCE PROPERTIES

In this section, we give the computations leading to the statements (2) and (6) on the exact scale invariance of the field X and of the measure $\mu_{\sqrt{2}}$. We also discuss the $\star$-scaling relation for the measure $\nu_{\sqrt{2}}$ given in (28), and finally prove Lemma 28.

A.1. Scaling properties for critical one-dimensional measures.

PROPOSITION 30. *The random measure $\mu_{\sqrt{2}}$ satisfies the exact scale invariance property* (6), *that is, for any interval $I \subset [0, 1]$*

$$\mu_{\sqrt{2}} \lfloor I \stackrel{d}{=} |I| e^{\sqrt{2} X(I) - \mathbb{E} X(I)^2} \mu^I_{\sqrt{2}},$$

where $\mu_{\sqrt{2}} \lfloor I$ denotes the restriction of $\mu_{\sqrt{2}}$ onto I and $\mu^I_{\sqrt{2}}$ is a random measure independent of $X(I)$ with the law given by

$$\left(\mu^I_{\sqrt{2}}(A)\right)_{A \in \mathcal{B}(I)} \stackrel{d}{=} \left(\mu_{\sqrt{2}}\left(|I|^{-1} A\right)\right)_{A \in \mathcal{B}(I)}.$$

REMARK. Writing the scaling relation simultaneously for a set $\{I_j\}$ of subintervals of [0, 1], one has

$$\left(\mu_{\sqrt{2}} \lfloor I_j\right)_j \stackrel{d}{=} \left(|I_j| e^{\sqrt{2} X(I_j) - \mathbb{E} X(I_j)^2} \mu^{I_j}_{\sqrt{2}}\right)_j,$$

where the $\mu^{I_j}_{\sqrt{2}}$ are random measures such that for each j,

$$\left(\mu^{I_j}_{\sqrt{2}}(J)\right)_{J \in \mathcal{B}(I_j)} \stackrel{d}{=} \left(\mu_{\sqrt{2}}\left(|I_j|^{-1} J\right)\right)_{J \in \mathcal{B}(I_j)} \quad \text{and} \quad \mu^{I_j}_{\sqrt{2}} \perp \left\{X(A)\right\}_{A \subset \mathcal{C}(I_j)}.$$

However, we stress that for subintervals of the unit interval, for $j \neq k$ the measure $\mu^{I_j}_{\sqrt{2}}$ is not independent either of $\mu^{I_k}_{\sqrt{2}}$ or $X(I_k)$.

PROOF OF PROPOSITION 30. We first show that (2) holds. Consider, for notational convenience, the interval $I = [0, y]$ with $0 < y < 1$. By definition, for $t \geq \log 1/y$ we have

$$\left(X_t(x)\right)_{x \in I} = \left(X(I) + X^I_t(x)\right)_{x \in I}.$$

Therefore, it suffices to check that

$$\left(X^I_t(x)\right)_{x \in I} \stackrel{d}{=} \left(X_{t - \log 1/y}(x/y)\right)_{x \in I}$$

and since the processes are Gaussian, it is enough to consider the covariance structures. Checking that the covariances of the processes are the same is demonstrated in Figure 1.

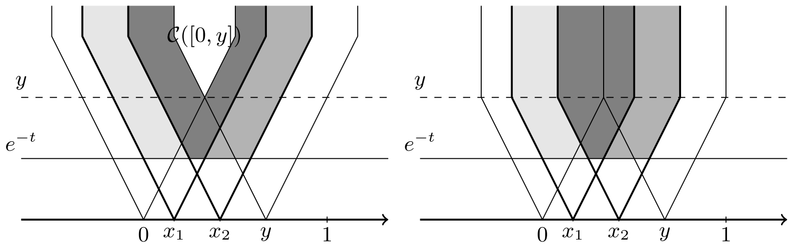

FIG. 1. Left. *The sets* $\mathcal{C}_t(x_1)\setminus\mathcal{C}_t(x_2)$ *and* $\mathcal{C}_t(x_2)\setminus\mathcal{C}_t(x_1)$ *are shaded light gray, while the intersection* $(\mathcal{C}_t(x_1)\cap\mathcal{C}_t(x_2))\setminus\mathcal{C}([0,y])$ *is dark gray. The law of the Gaussian process* $(X_t^I(x))_{x\in[0,y]}$ *is determined by the hyperbolic areas of these sets for all pairs* $(x_1,x_2)\in[0,y]^2$. *The set* $\mathcal{C}([0,y])$, *contained in every* $\mathcal{C}_t(x)$ *for* $x\in[0,y]$, *has been left white*. Right. *Closing the gap left by the set* $\mathcal{C}([0,y])$ *does not affect the hyperbolic areas of any of the shaded regions. Scaling this picture by* $1/y$ *also leaves the hyperbolic areas invariant, giving the distributional equality* $(X_t^I(x))_{x\in I}\overset{d}{=}(X_{t-\log 1/y}(x/y))_{x\in I}$.

Showing the exact scale invariance of $\mu_{\sqrt{2}}$ is now simple, as one only needs to note that the measure-defined analogously to the subcritical measures vanishes: for any intervals $J\subset I\subset[0,1]$ we have

$$\begin{aligned}
\mu_{\sqrt{2}}(J) &= \lim_{t\to\infty}\int_J\big(\sqrt{2}(t+1)-X_t(x)\big)e^{\sqrt{2}X_t(x)-\mathbb{E}X_t(x)^2}\,\mathrm{d}x\\
&= \lim_{t\to\infty}\int_J\big(\sqrt{2}\mathbb{E}X(I)^2-X(I)\big)e^{\sqrt{2}X_t(x)-\mathbb{E}X_t(x)^2}\,\mathrm{d}x\\
&\quad+\lim_{t\to\infty}\int_J\big(\sqrt{2}\big(t+1-\mathbb{E}X(I)^2\big)-X_t^I(x)\big)e^{\sqrt{2}X_t(x)-\mathbb{E}X_t(x)^2}\,\mathrm{d}x\\
&= 0\\
&\quad+e^{\sqrt{2}X(I)-\mathbb{E}X(I)^2}\\
&\qquad\times\lim_{t\to\infty}\int_J\big(\sqrt{2}\big(t+1-\mathbb{E}X(I)^2\big)-X_t^I(x)\big)e^{\sqrt{2}X_t^I(x)-\mathbb{E}X_t^I(x)^2}\,\mathrm{d}x\\
&=: |I|e^{\sqrt{2}X(I)-\mathbb{E}X(I)^2}\mu^I\big(|I|^{-1}J\big),
\end{aligned}$$

where μ^I a random measure with the law of μ and independent of $X(I)$. Note that the measure μ^I defined here depends on the field X only through the processes $(X_t^I(x))_{x\in I}$, $t>0$. This observation implies the statement on the simultaneous scaling relations for a set of intervals $\{I_j\}$. $\square$

We then consider $\star$-scale invariance, as defined in [3], and the measure $\nu_{\sqrt{2}}$ defined for the proof of Theorem 2. A random measure ν on $[0,1]$ is called $\star$-scale invariant on scale $\epsilon\in(0,1]$ if there exist a process $(\omega_\epsilon(x))_{x\in[0,1]}$ and a random measure ν^ϵ that are independent of each other and satisfy

$$\big(\nu(A)\big)_{A\in\mathcal{B}([0,1])} \overset{d}{=} \left(\epsilon\int_A e^{\omega_\epsilon(x)}\,\mathrm{d}\nu^\epsilon(x)\right)_{A\in\mathcal{B}([0,1])}$$

and

$$\big(\nu^\epsilon(A)\big)_{A\in\mathcal{B}([0,1])} \overset{d}{=} \big(\nu(\epsilon^{-1}A)\big)_{A\in\mathcal{B}([0,1])}.$$

The measure

$$\nu_{\sqrt{2}}(\mathrm{d}x) = \lim_{t\to\infty}\sqrt{t}e^{\sqrt{2}Y_t(x)-\mathbb{E}Y_t(x)^2}\,\mathrm{d}x,$$

where $Y_t(x) = X_t(x) - X_0(x) = W(\mathcal{C}_t(x)\setminus\mathcal{C}_0(x))$, is $\star$-scale invariant on every scale $\epsilon \in (0,1]$ with

$$\omega_\epsilon(x) = \sqrt{2}Y_{\log(1/\epsilon)}(x) + \log\epsilon.$$

This can be seen by first deducing the scale invariance property

$$\big(Y_t(x)\big)_{x\in[0,1]} \overset{d}{=} \big(Y_{\log(1/\epsilon)}(x) + Y'_{t-\log(1/\epsilon)}(\epsilon^{-1}x)\big)_{x\in[0,1]}, \tag{41}$$

where Y' is an independent realization of the field Y, from Figure 2 and then performing a computation analogous to the one above for $\mu_{\sqrt{2}}$.

A.2. Joint exact scaling property in two dimensions.

PROOF OF LEMMA 28. For $j = 1,\dots,4$, let $\phi_j : Q \to Q_j$ be the linear maps that map the corners of Q to the corners of Q_j by scaling and translating. We have the following equality in law:

$$\big(X_t(\phi_j(x))\big)_{x\in Q,t\ge\log 2} \overset{d}{=} \big(V + X_{t-\log 2}(x)\big)_{x\in Q,t\ge\log 2}, \tag{42}$$

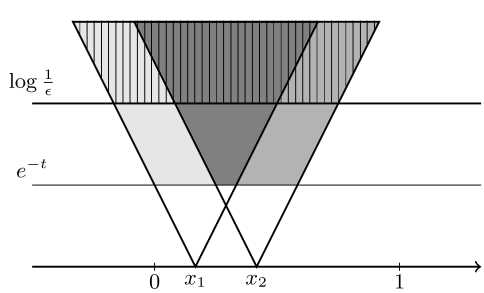

FIG. 2. *The cones $\mathcal{C}_t(x_1)$ and $\mathcal{C}_t(x_2)$ have been shaded gray, with the parts in $\mathcal{C}_{\log(1/\epsilon)}(x_1)$ and $\mathcal{C}_{\log(1/\epsilon)}(x_2)$ highlighted. By scaling the part of the picture below the line $\log\frac{1}{\epsilon}$ by ϵ^{-1} we get the equality of distributions $(Y_t(x) - Y_{\log(1/\epsilon)}(x))_{x\in[0,1]} \overset{d}{=} (Y_{t-\log(1/\epsilon)}(\epsilon^{-1}x))_{x\in[0,1]}$. This immediately implies (41), since the process $(Y_{\log(1/\epsilon)}(x))_{x\in[0,1]}$ is independent of $(Y_t(x) - Y_{\log(1/\epsilon)}(x))_{x\in[0,1]}$.*

where V is a centered Gaussian variable of variance $\log 2$ which is independent of the process $(X_{t-\log 2}(x))_{x\in Q,t\geq \log 2}$. Equation (42) can be readily checked from the form (37) of the covariance.

We would like to show that by possibly extending our probability space we can decompose almost surely

$$X_t\big(\phi_j(x)\big) = V_j + X^{(j)}_{t-\log 2}(x) \qquad \text{for } j=1,2,3,4 \text{ and } x\in Q, \tag{43}$$

where for each $j=1,2,3,4$ the process $(X^{(j)}_{t-\log 2}(x))_{x\in Q,t\geq \log 2}$ has the same law as the process $(X_{t-\log 2}(x))_{x\in Q,t\geq \log 2}$ and is independent of V_j.

As we are interested only in the limit measures, we will need (43) only for t in some sequence tending to ∞. We consider the countable collection of point evaluations given by

$$X_{k\log 2}(x) \qquad \text{where } x\in Q\cap \mathbb{Q}^2, k=1,2,3,\ldots$$

and denote their closed linear span by

$$\mathcal{H} := \overline{\operatorname{span}}\big(X_{k\log 2}(x) : x\in Q\cap\mathbb{Q}^2, k=1,2,\ldots\big).$$

Thus $\mathcal{H}\subset L^2(\Omega,\mathbb{P})$ is a separable (centered) Gaussian Hilbert space. By enlarging our probability space, if needed, we may assume that $(\Omega,\mathcal{F},\mathbb{P})$ supports a centered Gaussian variable V of variance $\log 2$ that is independent of all elements in $\mathcal{H}$. Set

$$\mathcal{H}' := \mathcal{H}\oplus \operatorname{span}(V).$$

Consider the closed subspace

$$\mathcal{G} := \overline{\operatorname{span}}\big(V + X_{k\log 2}(x) : x\in Q\cap\mathbb{Q}^2, k=1,2,\ldots\big)\subset \mathcal{H}'.$$

The dimension of the orthogonal complement of $\mathcal{G}$ in $\mathcal{H}'$ is either 1 or zero since by definition $\overline{\operatorname{span}}(\mathcal{G}\cup\{V\}) = \mathcal{H}'$. Suppose first it is 1 as the latter case is even easier to deal with. Thus, we may write

$$\mathcal{H}' := \mathcal{G}\oplus \operatorname{span}(N),$$

where N is a centered Gaussian vector of variance $\log 2$ independent of all elements in $\mathcal{G}$.

By (42), we have for each $j\in\{1,2,3,4\}$ the equality of joint distributions

$$\big(X_{k\log 2}\big(\phi_j(x)\big)\big)_{k\geq 1, x\in Q\cap\mathbb{Q}^2} \stackrel{d}{=} \big(V + X_{(k-1)\log 2}(x)\big)_{k\geq 1, x\in Q\cap\mathbb{Q}^2}. \tag{44}$$

This allows us to define linear (not necessarily surjective) isometries

$$\Psi_j : \mathcal{H}' = \mathcal{G}\oplus\operatorname{span}(N) \to \mathcal{H}' = \mathcal{H}\oplus \operatorname{span}(V)$$

as follows. First, set, for $k\geq 1$ and $x\in Q\cap\mathbb{Q}^2$

$$\Psi_j\big(V + X_{(k-1)\log 2}(x)\big) = X_{k\log 2}\big(\phi_j(x)\big). \tag{45}$$

By (44) Ψ_j uniquely extends to an isometry $\Psi_j : \mathcal{G} \to \mathcal{H}'$. Then setting

$$\Psi_j(N) = V$$

extends Ψ_j to the whole of $\mathcal{H}'$. Note that in case the dimension of the orthogonal complement of $\mathcal{G}$ in $\mathcal{H}'$ is zero we may omit this last step.

Let us denote

$$V_j := \Psi_j(V),$$
$$X^{(j)}_{k\log 2}(x) := \Psi_j\bigl(X_{k\log 2}(x)\bigr) \qquad \text{for } k \geq 0 \text{ and } x \in Q \cap \mathbb{Q}^2.$$

Since V and $X_{k\log 2}(x))$ are independent and Ψ_j is an isometry then V_j is independent of all the variables $X^{(j)}_{k\log 2}(x)$. (45) then gives

$$X_{k\log 2}\bigl(\phi_j(x)\bigr) = V_j + X^{(j)}_{(k-1)\log 2}(x) \tag{46}$$

for all $k \geq 1$, $x \in Q \cap \mathbb{Q}^2$ and $j = 1, 2, 3, 4$.

Since the covariance (37) is Hölder continuous in x, y we may assume that a.s. $x \to X_{k\log 2}(x)$ is continuous. Since Ψ_j is an isometry the decomposition (46) extends from $x \in Q \cap \mathbb{Q}^2$ to all of Q, almost surely.

Consider now, for $k \geq 1$, the measures

$$\mu_k(\mathrm{d}x) := \sqrt{k\log 2}\, e^{2X_{k\log 2}(x) - 2\mathbb{E}(X_{k\log 2}(x)^2)}\, \mathrm{d}x$$

and for $k \geq 0$ the measures

$$\mu^{(j)}_k(\mathrm{d}x) := \sqrt{(k+1)\log 2}\, e^{2X^{(j)}_{k\log 2}(x) - 2\mathbb{E}(X^{(j)}_{k\log 2}(x)^2)}\, \mathrm{d}x.$$

Using the decomposition (46), we get

$$\mu_k(Q_j) = \tfrac{1}{4} e^{2V_j - 2\log 2} \mu^{(j)}_{k-1}(Q) \tag{47}$$

and defining

$$W_j = \tfrac{1}{16} e^{2V_j}$$

we then get

$$\mu_k(Q) = \sum_{j=1}^{4} W_j \mu^{(j)}_{k-1}(Q). \tag{48}$$

Since $\mu_k \to \mu$ in probability as $k \to \infty$, we infer from (47) that the variables $\mu^{(j)}_{k-1}(Q)$ converge in probability to some random variables Y_j. Since $\mu^{(j)}_{k-1}(Q)$ has the same distribution as $(\frac{k}{k-1})^{1/2}\mu_{k-1}(Q)$, we infer $Y_j \stackrel{d}{=} Y = \mu(Q)$. Hence, taking limit of (48) the desired result follows as $2V_j$ has variance $4\log 2$. $\square$

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