Basic Semigroup Theory

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Abstract

In these lectures we discuss and explain the basic theory of continuous one-parameter semigroups from two different points of view. The semigroups can be considered as providing an abstract framework for the solution of evolution equations which will be described at greater length in the lectures of Ecker and Urbas or as providing the basic elements of the functional calculus to be developed in the lectures of Albrecht, Duong and McIntosh.

1 Exponentiation

1.1 The exponential function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

plays a central role in functional analysis. It provides the formal solution of the evolution equation

$$\frac{df_t}{dt} + H f_t = 0 \tag{1}$$

since one then has

$$f_t = e^{-tH} f_0 \quad . \tag{2}$$

Moreover, it gives the key algorithm

$$\hat{f}(p) = \int_{\mathbf{R}} dx \, e^{-px} f(x) \tag{3}$$

for the Laplace–Fourier transform in harmonic analysis. Much of the material in the introductory lectures is devoted to explaining and analyzing equations (1), (2) and (3).

1.2 If (1) is interpreted as a numerical equation, i.e., if $f_t \in \mathbf{R}$ (or $\mathbf{C}$) and $H \in \mathbf{R}$ (or $\mathbf{C}$), then indeed one has

$$f_t = e^{-tH} f_0 = \sum_{n \geq 0} \frac{(-tH)^n}{n!} f_0 \quad . \tag{4}$$

1.3 If (1) is a matrix equation, i.e., if $f_t = (f_{1,t}, \ldots, f_{n,t})$ and H is an $n \times n$-matrix one can again use (4) to define the solution. The series is norm convergent and

$$\|f_t\| \leq e^{|t|\,\|H\|} \|f_0\| \quad .$$

Moreover,

$$\|f_t - f_0\| \leq \sum_{n \geq 1} \frac{(|t|\,\|H\|)^n}{n!} \|f_0\|$$

and hence

$$\lim_{t \to 0} \|f_t - f_0\| = 0 \quad . \tag{5}$$

1.4 If H is a self-adjoint (symmetric) matrix there is an alternative method of expressing the solution. Then

$$H = U^{-1} M U \tag{6}$$

where U is a unitary matrix and M is the real diagonal matrix with the eigenvalues of H $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ as its diagonal entries. Thus M multiplies the i-th component of a vector by λ_i and

$$e^{-tH} = U^{-1} e^{-tM} U \tag{7}$$

where e^{-tM} is the diagonal matrix with entries $e^{-t\lambda_1}, \ldots, e^{-t\lambda_n}$. Thus e^{-tH} acts by a unitary twist U, multiplication by e^{-tM} and then the inverse unitary twist U^{-1}.

1.5 Next suppose f_t is a function (real or complex valued) of a real variable x and $H = -d^2/dx^2$. Thus (1) corresponds to the partial differential equation

$$\frac{\partial f_t}{\partial t}(x) - \frac{\partial^2 f_t}{\partial x^2}(x) = 0 \quad , \tag{8}$$

the heat equation. This equation describes the diffusion of heat in an idealized one-dimensional rod. Now it is less clear how to explain the formal solution (2) of this equation. But one way is by Fourier transformation.

1.6 Let $L_2(\mathbf{R})$ denote the Hilbert space of complex-valued square-integrable functions of one real variable x with the scalar product

$$(f, g) = \int_{\mathbf{R}} dx \, \overline{f(x)} g(x)$$

and the associated norm

$$\|f\|_2 = \left(\int_{\mathbf{R}} dx \, |f(x)|^2 \right)^{1/2} \quad . \tag{9}$$

Then the Fourier transform is a linear map $Uf = \hat{f}$ on L_2 defined such that

$$(Uf)(p) = \hat{f}(p) = (2\pi)^{-1/2} \int_{\mathbf{R}} dx \, e^{-ipx} f(x) \quad . \tag{10}$$

It can be established that this transform has the properties

1. $Uf = \hat{f} \in L_2(\mathbf{R}) \qquad \text{and} \qquad \|Uf\|_2 = \|f\|_2,$

2. $\qquad \widehat{Uf}(x) = f(-x) \quad .$

The first property states that the Fourier map U is isometric. This isometry is usually called the Plancherel formula. The second property states that U has an isometric inverse

$$(U^{-1}\hat{f})(x) = f(x) = (2\pi)^{-1/2} \int_{\mathbf{R}} dp \, e^{ipx} \hat{f}(p) \quad . \tag{11}$$

Therefore Fourier transformation corresponds to a unitary map from L_2 to L_2.

1.7 Fourier transformation is relevant to ordinary differential equations since it 'diagonalizes' the operation of differentiation. Differentiation corresponds to multiplication of the Fourier transform;

$$\frac{df}{dx}(x) = (2\pi)^{-1/2} \int_{\mathbf{R}} dp \, \frac{d}{dx} e^{ipx} \hat{f}(p) = (2\pi)^{-1/2} \int_{\mathbf{R}} dp \, e^{ipx} ip \hat{f}(p) \quad .$$

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More concisely

$$\widehat{\frac{df}{dx}}(p) = ip\hat{f}(p) \tag{12}$$

and differentiation corresponds to multiplication of the Fourier transform by ip.

1.8 The heat equation (8) can now be expressed on $L_2(\mathbf{R})$ as the equation

$$\frac{\partial \hat{f}_t}{\partial t}(p) + p^2 \hat{f}_t(p) = 0$$

for the Fourier transform and this has the solution

$$\hat{f}_t(p) = e^{-tp^2} \hat{f}_0(p) \quad .$$

Therefore the solution of (8) can be represented in the operator form

$$f_t = U^{-1} e^{-tM} U f_0 \tag{13}$$

where U is Fourier transformation, M is multiplication by p^2 and U^{-1} is inverse Fourier transformation. Thus H can be exponentiated in a manner analogous to the matrix relation (7). Both relations are a form of the so-called spectral representation.

1.9 The solution (13) of the heat equation can be explicitly written as

$$f_t(x) = (2\pi)^{-1} \int_{\mathbf{R}} dp \int_{\mathbf{R}} dy \, e^{ip(x-y)} e^{-tp^2} f_0(y)$$

$$= \int_{\mathbf{R}} dy \, K_t(x-y) f_0(y) = (K_t * f_0)(y)$$

where

$$K_t(x) = (2\pi)^{-1} \int_{\mathbf{R}} dp \, e^{ipx} e^{-tp^2} = (4\pi t)^{-1/2} e^{-x^2/4t} \quad . \tag{14}$$

(In this calculation we have exchanged the order of integration and this needs some justification. In addition we have evaluated the Gaussian integral explicitly. This requires some computation. Both steps are left as exercises.)

1.10 The conclusion of the foregoing paragraphs can be summarized as follows. If $H = -d^2/dx^2$ operates on $L_2(\mathbf{R})$ then it can be exponentiated by the convolution algorithm

$$e^{-tH} f = K_t * f \tag{15}$$

where K_t is the Gaussian kernel defined by (14). Note that

$$1. \qquad K_t > 0 \quad ,$$

$$2. \qquad K_s * K_t = K_{s+t} \quad ,$$

$$3. \qquad \int_{\mathbf{R}} dx \, K_t(x) = 1 \quad .$$

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The first property is obvious and the second and third follow by straightforward calculation.

Property 1. implies that if $f \geq 0$ then $S_t f = K_t * f > 0$ where we have set $S_t = e^{-tH}$. Property 2. implies that $S_s S_t = S_{s+t}$. Property 3. implies that if $f \in L_2 \cap L_p$, where L_p is the space of functions for which

$$\|f\|_p = \left(\int_{\mathbf{R}} dx\, |f(x)|^p \right)^{1/p} < \infty \quad,$$

then

$$\|S_t f\|_p = \|K_t * f\|_p$$

$$\leq \|K_t\|_1 \|f\|_p = \|f\|_p$$

by use of the Minkowski inequality. Thus the second of the above properties implies that the operators $\{S_t\}_{t \geq 0}$ form a semigroup, $S_s S_t = S_{s+t}$. The first property implies that the semigroup is positive, the S_t map positive functions into positive functions, and it is in fact strictly positive, it maps positive functions into strictly positive functions. The third property implies that the semigroup of operators S can be extended from $L_2 \cap L_p$ to L_p and the extensions are contractive on each of the L_p-spaces,

$$\|S_t\|_{p \to p} = \sup\{\, \|S_t f\|_p \ : \ \|f\|_p \leq 1 \,\} \leq 1$$

for all $p \in [1, \infty]$.

1.11 The Fourier technique can be used to analyze a much wider class of differential and partial differential equations on $L_2(\mathbf{R})$ or $L_2(\mathbf{R}^d)$. For example, one can handle operators

$$H_c = \sum_{n=1}^{m} c_n \frac{d^n}{dx^n} \quad .$$

Formally one obtains

$$S_t^{(c)} f = e^{-tH_c} f = K_t^{(c)} * f$$

where

$$K_t^{(c)}(x) = (2\pi)^{-1} \int_{\mathbf{R}} dp\, e^{ipx} e^{-t\omega_c(p)}$$

with

$$\omega_c(p) = \sum_{n=1}^{m} c_n (ip)^n \quad .$$

This solution can be justified if m is even and $(-1)^{m/2} c_m > 0$ because the integral for $K_t^{(c)}$ is then well-defined. Other cases can be handled by this method but then convergence is more sensitive. The method also extends to higher dimensions, i.e., to partial differential equations on spaces of several variables.

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1.12 The Fourier method is well suited to estimates on the L_2-space but it is more difficult to obtain estimates on the related L_p-spaces. For example, the Plancherel formula gives

$$\|S_t^{(c)} f\|_2 = \left(\int_{\mathbf{R}} dp \, |e^{-t\omega_c(p)} \hat{f}(p)|^2 \right)^{1/2}$$

$$\leq \left(\int_{\mathbf{R}} dp \, |\hat{f}(p)|^2 \right)^{1/2} = \|f\|_2$$

if $\operatorname{Re} \omega_c \geq 0$. Thus

$$\|S_t^{(c)}\|_{2 \to 2} \leq 1 \quad .$$

Alternatively, one has the formal relation

$$\|S_t^{(c)}\|_{1 \to 1} = \int_{\mathbf{R}} dx \, |K_t^{(c)}(x)| = \|K^{(c)}\|_1 \quad .$$

But if $\operatorname{Re} \omega_c = 0$ then $K_t^{(c)}$ is not necessarily integrable and $S_t^{(c)}$ is not bounded on L_1.

1.13 Let U be a one-parameter group defined by

$$(\widehat{U_t f})(p) = e^{-ip^2 t} \hat{f}(p) \quad .$$

Thus $f_t = U_t f$ is a solution of the quantum-mechanical Schrödinger equation

$$\frac{\partial f_t}{\partial t}(x) - i \frac{\partial^2 f_t}{\partial x^2}(x) = 0 \quad .$$

Then

$$\|U_t f\|_2 = \|(\widehat{U_t f})\|_2 = \|\hat{f}\|_2 = \|f\|_2$$

and U is a unitary group on $L_2(\mathbf{R})$. If $H = -d^2/dx^2$ then $U_t = e^{-itH}$ and so U formally corresponds to the heat semigroup of Paragraphs 1.5–1.10 but for imaginary time. Then

$$U_t f = L_t * f \quad ,$$

for f which decrease sufficiently fast, where

$$L_t(x) = (2\pi)^{-1} \int_{\mathbf{R}} dp \, e^{ipx} e^{-itp^2} = (4\pi it)^{-1/2} e^{ix^2/4t} \quad .$$

The integral, the Fresnel integral, is evaluated by contour integration. But then it follows that

$$(U_t f)(x) = (4\pi it)^{-1/2} e^{ix^2/4t} \int_{\mathbf{R}} dy \, e^{-ixy/2t} e^{iy^2/4t} \quad .$$

Hence

$$|(U_t f)(x)| = (2|t|)^{-1/2} |\hat{f}_t(x/2t)|$$

where $f_t(x) = e^{ix^2/4t} f(x)$. Therefore

$$\|U_t f\|_p = (2|t|)^{-(1/2-1/p)} \|\hat{f}_t\|_p \quad .$$

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But if $f(x) = e^{-ax^2/4t}$ with $a > 0$ then one can explicitly calculate $\|f\|_p$ and $\|\hat{f}_t\|_p$ to find

$$\|U_t f\|_p / \|f\|_p = (1 + a^2)^{-(1/2 - 1/p)} \quad .$$

Thus if $p < 2$ the operator norm of U_t and be made arbitrarily large. A similar conclusion is reached for $p > 2$ by duality.

1.14 Fourier theory or, more generally, spectral theory can be used to define bounded functions of the operator of differentiation or more general operators. If $F \in L_\infty$ then $F(-d^2/dx^2)$ corresponds to multiplication of the Fourier transform by $F(p^2)$ and hence

$$\|F(-d^2/dx^2)\|_{2 \to 2} \leq \|F\|_\infty \quad .$$

The example of the previous section shows, however, that one can have

$$\|F(-d^2/dx^2)\|_{p \to p} = \infty$$

for all $p \neq 2$. It is of interest to characterize the functions for which

$$\|F(-d^2/dx^2)\|_{p \to p} \leq c_p \|F\|_\infty$$

for all $p \in \langle 1, \infty \rangle$.

2 Contraction semigroups

2.1 In the first lecture we showed how various operators H could be exponentiated by spectral techniques to give a semigroup $S_t = e^{-tH}$ of bounded operators. Fourier transformation reduces differential operators with constant coefficients to multiplication operators whose exponential is defined by the usual numerical algorithm. This technique works well for this special class of operators on the Hilbert space of square-integrable functions but is less well adapted to estimates on other function spaces. In a more general context one needs other techniques and other algorithms for the exponential. For example one has the binomial approximation

$$e^{-x} = \lim_{n\to\infty} (1 - x/n)^n \quad . \tag{16}$$

But a more useful approximation is the inverse binomial form

$$e^{-x} = \lim_{n\to\infty} (1 + x/n)^{-n} \quad . \tag{17}$$

This approximation can be used to construct an interesting family of semigroups, the contraction semigroups.

2.2 If H_c is the differential operator of Paragraph 1.11 and $\omega_c > 0$ then the inverses $(I + H_c/n)^{-1}$ are defined by

$$((I + tH_c/n)^{-1}f)(x) = (2\pi)^{-1/2} \int_{\mathbf{R}} dp\, e^{-ipx} (1 + t\omega_c(p)/n)^{-1} \hat{f}(p) \tag{18}$$

and they are bounded operators,

$$\|(I + tH_c/n)^{-1}f\|_2 \le \|\hat{f}\|_2 = \|f\|_2 \quad .$$

It is then an exercise with (17), (18) and the Plancherel formula to establish that

$$\lim_{n\to\infty} \|S_t^{(c)}f - (I + tH_c/n)^{-n}f\|_2 = 0$$

for all $f \in L_2(\mathbf{R})$ where $S_t^{(c)} = e^{-tH_c}$. Thus the semigroup is constructed as a limit over a sequence of bounded operators. This indicates the general approach to the construction of more general semigroups.

2.3 As a preliminary to the more detailed examination of the exponentiation problem we discuss the inverse problem, differentiation of a continuous semigroup. Let $S = \{S_t\}_{t\ge 0}$ be a family of bounded operators S_t acting on a Banach space $\mathcal{X}$ with the properties

1. $\qquad\qquad S_0 = I$

2. $\qquad\qquad S_s S_t = S_{s+t}$

3. $\lim_{t\to 0} \|S_t x - x\| = 0$ for all $x \in \mathcal{X}$.

Then S is called a (strongly) continuous semigroup.

Next let $D(H)$ denote the subspace spanned by the $x \in \mathcal{X}$ for which $t \mapsto S_t x$ is (strongly) differentiable, i.e., for which $t \mapsto t^{-1}(I - S_t)x$ converges in norm as $t \to 0$. Then for $x \in D(H)$ introduce the linear operator H as the strong derivative of S,

$$Hx = \lim_{t\to 0} t^{-1}(I - S_t)x \quad . \tag{19}$$

The operator H is called the generator of S and the subspace $D(H)$ is the domain of H.

2.4 Since $S_t(I - S_s) = (I - S_s)S_t$ it follows that $S_t D(H) \subseteq D(H)$ and

$$HS_t x = S_t H x$$

for all $x \in D(H)$. Moreover, the fundamental theorem of calculus gives

$$(I - S_t)x = -\int_0^t ds \, HS_s x \tag{20}$$

for all $x \in D(H)$ where the integral is interpreted as a norm-convergent Riemann integral. This last relation is often referred to as the Duhamel equation.

2.5 Duality properties are often important in the theory of semigroups. The dual $\mathcal{X}^*$ of $\mathcal{X}$ induces a weak, or $\sigma(\mathcal{X}^*, \mathcal{X})$-, topology on $\mathcal{X}$. Then $S = \{S_t\}_{t\geq 0}$ is defined to be a weakly, or $\sigma(\mathcal{X}^*, \mathcal{X})$-, continuous semigroup on $\mathcal{X}$ if it satisfies Properties 1. and 2. of Paragraph 2.3 but Property 3. is replaced by

3a. $\lim_{t\to 0}(f, S_t x) = (f, x)$ for all $x \in \mathcal{X}$, $f \in \mathcal{X}^*$.

Somewhat surprisingly the semigroup property allows one to deduce that Properties 3. and 3a. are equivalent, i.e., weak and strong continuity are equivalent for semigroups. Similarly the weak derivative of S coincides with the strong derivative.

2.6 The basic properties of the semigroup S and its generator H are summarized as follows:

1. there exist $M \geq 1$ and $\omega \geq 0$ such that

$$\|S_t\| \leq Me^{\omega t} \quad \text{for all} \quad x \in \mathcal{X} \quad .$$

2. the generator H is densely defined, i.e., $D(H)$ is norm-dense in $\mathcal{X}$.

3. the generator H is (norm-)closed, i.e., if $x_n \in D(H)$, $\|x_n - x\| \to 0$ and $\|Hx_n - y\| \to 0$ as $n \to \infty$ then $x \in D(H)$ and $y \in Hx$.

4. if $\lambda \in \mathbf{C}$ and $\mathrm{Re}\,\lambda > \omega$ then $(\lambda I + H)^{-1}$ is defined by the Laplace transform

$$(\lambda I + H)^{-1}x = \int_0^\infty dt\, e^{-\lambda t} S_t x$$

and one has the bounds

$$\|(\lambda I + H)^{-1}\| \le M (\operatorname{Re}\lambda - \omega)^{-1} \quad . \tag{21}$$

Each of these properties is very important in the abstract analysis of continuous semigroups.

The first property follows because continuity of S implies that S_t is uniformly bounded for small t but the semigroup property means that it can grow at most exponentially for large t.

The second and third property are somewhat more difficult to establish. The third can be expressed in terms of the graph of H, i.e., the pairs $\{x, Hx\}$, $x \in D(H)$. These form a subspace of $\mathcal{X} \times \mathcal{X}$ which can be equipped with the graph norm $x \mapsto \|x\| + \|Hx\|$. Then H is closed if and only if the graph is complete with respect to the graph norm.

Finally, since S is continuous the Laplace transform in the fourth property is again interpreted as a norm-convergent Riemann integral which defines an element $x_\lambda \in \mathcal{X}$. The identity follows by computing that $(\lambda I + H)x_\lambda = x$. One then has

$$\|(\lambda I + H)^{-1}\| \le M \int_0^\infty dt\, e^{-(\operatorname{Re}\lambda - \omega)t} = M(\operatorname{Re}\lambda - \omega)^{-1} \quad .$$

In particular, the resolvents $(\lambda I + H)^{-1}$ are defined for all λ in a half-plane and if S is uniformly bounded the resolvents are defined for all $\lambda > 0$, i.e., all λ in the open right half-plane. Finally if S is a semigroup of contractions the operators $(\lambda I + H)^{-1}$ are contractions for all $\lambda > 0$.

2.7 The dual, or adjoint, of a densely defined operator X on $\mathcal{X}$ is defined as an operator X^* on $\mathcal{X}^*$ by first introducing $D(X^*)$ as the subspace of $f \in \mathcal{X}^*$ for which $x \in D(X) \mapsto (f, Xx)$ is continuous, i.e., one has an estimate

$$|(f, Xx)| \le c_f \|x\|$$

for all $x \in X$. The adjoint is automatically closed in the weak*, or $\sigma(\mathcal{X}^*, \mathcal{X})$-, topology on $\mathcal{X}^*$. Moreover, $D(X^*)$ is weakly* dense in $\mathcal{X}^*$ if and only if X is (norm-)closed. Finally if X is bounded then the adjoint X^* is also bounded and $\|X\| = \|X^*\|$.

The adjoints $S^* = \{S_t^*\}_{t \ge 0}$ of the strongly continuous semigroup $S = \{S_t\}_{t \ge 0}$ form a weakly* continuous semigroup on $\mathcal{X}^*$. If H is the generator of S the adjoint H^* is the weak* generator of S^*, i.e., its weak* derivative.

The space $\mathcal{X}$ is reflexive if $\mathcal{X}^{**} = \mathcal{X}$ and then the weak* topology and the weak topology coincide. Hence the adjoint S^* is weakly continuous and, by Paragraph 2.4, strongly continuous.

2.8 The properties of Paragraph 2.4 are necessary to ensure that an operator H generates a semigroup. But in the case of contractions the properties of Paragraph 2.4 are also sufficient.

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Theorem 2.1 (Hille–Yosida) *Let H be an operator on a Banach space $\mathcal{X}$. Assume that*

1. *H is densely defined and closed,*

2. *a.* $\quad R(\lambda I + H) = \mathcal{X} \quad$ *for all* $\quad \lambda > 0$,

   b. $\quad \|(\lambda I + H)x\| \geq \|x\| \quad$ *for all* $\quad \lambda > 0$.

Then H is the generator of a contraction semigroup S and

$$\lim_{n \to \infty} \|S_t x - (I + tH/n)^{-n}x\| = 0$$

for all $x \in \mathcal{X}$.

The key to the proof is the limiting operation. If $P_n(t) = (I + tH/n)^{-n}$ then

$$\frac{d}{ds}P_n(s)x = -P_{n+1}(s)Hx$$

for all $x \in D(H)$. Therefore

$$
\begin{aligned}
(P_n(t) - P_m(t))x &= \int_0^t ds \, \frac{d}{ds}P_n(t-s)P_m(s)x \\
&= \int_0^t ds \, P_{n+1}(t-s)P_{m+1}(s)(s/m + (t-s)/n)H^2x
\end{aligned}
$$

for $x \in D(H^2)$ and since the P_n are contractions, by assumption,

$$
\begin{aligned}
\|(P_n(t) - P_m(t))x\| &\leq \int_0^t ds \, (s/m + (t-s)/n)\|H^2x\| \\
&\leq t^2\|H^2x\|(1/2m + 1/2n) \quad .
\end{aligned}
$$

Thus $n \mapsto P_n(t)x$ is a Cauchy sequence. This is the beginning of the proof. Then one must argue that $D(H^2)$ is dense in $\mathcal{X}$ and that the limit defines a semigroup $S_t = e^{-tH}$ as expected from the algorithm (17).

2.9 The utility of the Hille–Yosida theorem is that the conditions which characterize the generator can often be verified without too much trouble. For example it is usually not difficult to show that a differential operator is densely defined. But a densely defined operator is closed if and only if it has a densely defined adjoint and this criterion can be used to verify the closedness. Probably the most difficult condition to verify is the range condition, Condition 2a. This can, however, be replaced by a condition on the adjoint operator.

Theorem 2.2 *Let H be a densely defined, closed operator on a Banach space $\mathcal{X}$.*

Then H is the generator of a contraction semigroup S if and only if

$$\|(\lambda I + H)x\| \geq \|x\|$$

for all $x \in D(H)$, $\lambda > 0$ and

$$\|(\lambda I + H^*)f\| \geq \|f\|$$

for all $f \in D(H^*)$, $\lambda > 0$.

The essential point in this rephrasing of the Hille–Yosida theorem is the following. The range $R(\lambda I + H)$ is closed by the estimate on H. Thus if $R(\lambda I + H) \neq \mathcal{X}$ there is a non-zero $f \in \mathcal{X}^*$ orthogonal to the range of $(\lambda I + H)$. Then

$$|(f, Hx)| = \lambda|(f, x)| \leq \lambda \|f\| \|x\|$$

for all $x \in D(H)$ and this implies $f \in D(H^*)$ and $(\lambda I + H^*)f = 0$. But then $f = 0$ by the estimate on H^*. Thus $R(\lambda I + H) = \mathcal{X}$.

Of course the price paid for this dual reformulation is that one needs to know the domain of $D(H^*)$ and this is not necessarily easier to control than the range of $(\lambda I + H)$.

2.10 Condition 2b. in the Hille–Yosida theorem is often referred to as dissipativity. It has a different more geometric characterization. First recall that an element $f \in \mathcal{X}^*$ is called a tangent functional at $x \in \mathcal{X}$ if $(f, x) = \|f\| \|x\|$.

Proposition 2.3 *Let H be an operator on a Banach space $\mathcal{X}$. The following conditions are equivalent:*

$$1. \qquad \|(I + \alpha H)x\| \geq \|x\|$$

for all $x \in D(H)$ and $\alpha > 0$,

$$2. \qquad \mathrm{Re}\,(f, Hx) \geq 0$$

for one non-zero tangent functional f at each $x \in D(H)$.

Moreover, if H is densely defined these conditions are equivalent to

$$3. \qquad \mathrm{Re}\,(f, Hx) \geq 0$$

for all tangent functionals f at each $x \in D(H)$.

The second form of dissipativity is readily verified for the generator H of a contraction semigroup S. One then has

$$\mathrm{Re}\,(f, S_t x) \leq \|f\| \|x\| = (f, x) = \mathrm{Re}\,(f, x)$$

for every tangent functional f at x. Hence

$$\lim_{t \to 0} \mathrm{Re}\, t^{-1}(f, (I - S_t)x) = \mathrm{Re}\,(f, Hx) \geq 0 \quad .$$

2.11 A continuous (one-parameter) group of isometries is a family $U = \{U_t\}_{t \in \mathbf{R}}$ of isometries U_t satisfying 1. $U_0 = I$, 2. $U_s U_t = U_{s+t}$ for all $s, t \in \mathbf{R}$, and 3.

$$\lim_{t \to 0} \|U_t x - x\| = 0$$

for all $x \in \mathcal{X}$. The group has a generator H defined as before but now $\pm H$ both generate contraction semigroups the semigroups $U^\pm = \{U_{\pm t}\}_{t \geq 0}$. Conversely, if $\pm H$ both generate

contraction semigroups $U^{\pm}$ then one can define a group of isometries with generator H by setting $U_t = U_t^+$, $t \geq 0$, and $U_t = U_t^-$, $t \leq 0$. For example,

$$\frac{d}{dt}U_t U_{-t}x = 0$$

for all $x \in D(H)$. Hence $U_t U_{-t} = I$ and the group property follows easily. But the contraction property gives $1 = \|I\| = \|U_t U_{-t}\| \leq \|U_t\|\,\|U_{-t}\| \leq 1$. Hence the U_t must be isometries.

2.12 Dissipativity takes a particularly easy form on a Hilbert space. Then

$$\|(I + \alpha H)x\|^2 = \|x\|^2 + 2\alpha\,\mathrm{Re}\,(x, Hx) + \alpha^2 \|Hx\|^2 \|x\|^2$$

for all $\alpha > 0$ if and only if

$$\mathrm{Re}\,(x, Hx) \geq 0 \quad .$$

This is also verifiable from the alternative characterization of dissipativity because on a Hilbert space the unique normalized tangent functional at x is given by $x/\|x\|$.

Thus the theorem of Paragraph 2.9 says that a densely defined, closed operator on a Hilbert space generates a continuous contraction semigroup if and only if

$$\mathrm{Re}\,(x, Hx) \geq 0 \quad \text{and} \quad \mathrm{Re}\,(y, H^*y) \geq 0$$

for all $x \in D(H)$ and $y \in D(H^*)$ respectively. Moreover, it follows from the discussion of Paragraph 2.11 that H generates a continous group of isometries if and only if

$$\mathrm{Re}\,(x, Hx) = 0 \quad \text{and} \quad \mathrm{Re}\,(y, H^*y) = 0$$

for all $x \in D(H)$ and $y \in D(H^*)$ respectively. The first of these conditions states that H^* extends $-H$ and the second states that H^{**} extends $-H^*$. Since H automatically extends H^{**} this means that $H = -H^* = H^{**}$, i.e., H is skew-adjoint.

This conclusion, that a densely-defined operator on a Hilbert space generates a continuous group of isometries if and only if it is skew-adjoint, was the earliest result in the theory of one-parameter groups and semigroups.

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3 Holomorphic semigroups

3.1 The exponential function has a Cauchy representation

$$e^{-x} = (2\pi i)^{-1} \int_C dz \, \frac{e^{-z}}{x - z}$$

where C is a contour encircling $x > 0$ in a clockwise direction. A standard deformation argument then gives

$$e^{-x} = (2\pi i)^{-1} \int_\gamma dz \, \frac{e^{-z}}{x - z} \tag{22}$$

for all $x > 0$ where γ is the contour defined by the function

$$\gamma(t) \;=\; \begin{cases} te^{i\varphi} & \text{if} \quad t \in \langle 0, \infty \rangle \quad, \\[2mm] -te^{-i\varphi} & \text{if} \quad t \in \langle -\infty, 0 \rangle \quad. \end{cases}$$

This representation can be used to exponentiate a different class of operators to those considered previously.

3.2 If H_c is the differential operator of Paragraph 1.11 and $\omega_c \geq 0$ then one can define $S_z = e^{-zH_c}$, for $z \in \mathbf{C}$ with $\operatorname{Re} z \geq 0$, as a contraction on $L_2(\mathbf{R})$ by the Fourier multiplier method;

$$(S_z f)(x) = (2\pi)^{-1/2} \int_{\mathbf{R}} dp \, e^{-ipx - z\omega_c} \hat{f}(p) \quad.$$

The contractivity follows from the Plancherel formula because $|\exp(-z\omega_c)| \leq 1$. Moreover, $z \mapsto S_z$ is a holomorphic (analytic) function in the L_2-sense in the open right half-plane $\operatorname{Re} z > 0$. Explicitly,

$$\|S_z f - S_{z_0} f\|_2 = \left(\int_{\mathbf{R}} dp \, |e^{-z\omega_c(p)} - e^{-z_0 \omega_c(p)}|^2 |\hat{f}(p)|^2 \right)^{1/2}$$

$$\leq \sup_{p \in \mathbf{R}} |e^{-z\omega_c(p)} - e^{-z_0 \omega_c(p)}| \, \|f\|_2$$

and a simple estimate gives bounds

$$\|S_z f - S_{z_0} f\|_2 / |z - z_0| \leq c(z, z_0) \|f\|_2$$

where $c(z, z_0)$ is bounded for $\operatorname{Re} z, \operatorname{Re} z_0 \geq \varepsilon > 0$.

Alternatively, if the coefficients of H_c are complex-valued the situation is more interesting but more complicated. Suppose one has $\inf_{p \in \mathbf{R}} \omega_c(p) \geq \lambda > 0$ and $\sup_{p \in \mathbf{R}} |\omega_c(p)| \leq \mu < \infty$. Then

$$|e^{-z\omega_c}| \leq e^{-\lambda \operatorname{Re} z + \mu |\operatorname{Im} z|} \leq 1$$

for all $z \in \mathbf{C}$ with $|\arg z| \leq \tan^{-1}(\lambda/\mu)$. Therefore S_z is again defined as a contraction on $L_2(\mathbf{R})$ for all z in the closure of the sector $\Delta(\theta) = \{ z \in \mathbf{C} : |\arg z| < \theta \}$ with

$\theta = \tan^{-1}(\lambda/\mu)$. Moreover, $z \in \Delta(\theta) \mapsto S_z$ is holomorphic in the L_2-sense in the interior of $\Delta(\theta)$.

These examples motivate the following definition.

3.3 A strongly continuous semigroup S on a Banach space $\mathcal{X}$ is defined to be a holomorphic semigroup if there is a $\theta \in \langle 0, \pi/2]$ such that $t \geq 0 \mapsto S_t$ is the restriction to the positive real axis of a holomorphic family of bounded operators $z \in \Delta(\theta) \mapsto S_z$, where $\Delta(\theta) = \{ z \in \mathbf{C} : |\arg z| < \theta \}$, and the following properties are satisfied:

1. $S_{z_1} S_{z_2} = S_{z_1 + z_2}$ for all $z_1, z_2 \in \Delta(\theta)$.

2. $\lim_{z \to 0} \| S_z x - x \| = 0$ for all $x \in \mathcal{X}$.

Moreover, the semigroup is said to be a bounded holomorphic semigroup if in addition the following property is valid:

3. $z \in \Delta(\varphi) \mapsto S_z$ is uniformly bounded for all $\varphi \in [0, \theta\rangle$.

The angle θ is called the holomorphy angle of the semigroup.

The analysis of general holomorphic semigroups can be substantially reduced to the analysis of the bounded semigroups. If S is holomorphic and $\varphi \in \langle -\theta, \theta \rangle$ then $t \geq 0 \mapsto S_{te^{i\varphi}}$ is a strongly continuous semigroup and it follows from the observations of Paragraph 2.6 that one has bounds $\| S_{te^{i\varphi}} \| \leq M_\varphi e^{\omega_\varphi t}$ with $M_\varphi \geq 1$ and $\omega_\varphi \geq 0$. The values of M_φ and ω_φ can diverge as $\varphi \to \pm\theta$ but they will remain bounded in any subsector $\Delta(\varphi_1)$, $\varphi_1 \in \langle 0, \theta \rangle$. Let M_1 and ω_1 denote their upper bounds in the subsector. Then setting $T_z = S_z \exp(-\omega_1 z / \cos\varphi_1)$ one has $\| T_z \| \leq M_1$. Thus T is a bounded holomorphic semigroup and all the properties of S with the exception of the detailed behaviour near the boundary of the holomorphy sector $\Delta(\theta)$ can be inferred from those of T. Therefore we restrict attention to the bounded holomorphic semigroups.

3.4 Although the foregoing definition is in terms of complex analysis there is an equivalent real analytic description. For this one needs to identify conditions on the semigroup and its derivatives which ensure that it can be extended analytically from the real half-axis to a sector $\Delta(\theta)$. As an illustration recall that if f is a complex-valued C^∞-function over $\mathbf{R}$ and $|f^{(n)}(t)| \leq c^n n!$ then f can be extended analytically to the interior of a circle centered at t with radius $1/c$ by setting

$$f(t + z) = f(t) + \sum_{n \geq 1} \frac{z^n}{n!} f^{(n)}(t) \quad .$$

Now a similar construction can be used on operator-valued functions but in the case of a semigroup there are substantial simplifications arising from the semigroup property. It suffices to have a suitable bound on the first derivative.

Theorem 3.1 *Let S be a strongly continuous semigroup on the Banach space $\mathcal{X}$ with generator H.*

Then S is a bounded holomorphic semigroup if and only if there is a $c > 0$ such that

$$\|HS_t\| \leq ct^{-1}$$

for all $t > 0$. In which case the holomorphy angle $\theta \geq \tan^{-1}(1/ce)$.

The estimate in the theorem indicates that $S_t\mathcal{X} \subseteq D(H)$ for $t > 0$ and HS_t is a bounded operator with $\|HS_t\| \leq ct^{-1}$. Since H is the strong derivative of S the estimate states that

$$\|\frac{dS_t}{dt}\| \leq ct^{-1} \quad .$$

But using the semigroup property one has $HS_t\mathcal{X} = S_{t/2}HS_{t/2}\mathcal{X} \subseteq D(H)$ and

$$H^2 S_t = (HS_{t/2})^2 \quad .$$

Therefore

$$\|\frac{d^2 S_t}{dt^2}\| \leq c^2(t/2)^{-2} \quad .$$

Repetition of this reasoning gives

$$\|\frac{d^n S_t}{dt^n}\| = \|(HS_{t/n})^n\|$$

$$\leq c^n n^n t^{-n} \leq (ce)^n n! t^{-n} \quad .$$

Therefore one can analytically extend S as in the case of complex functions by setting

$$S_{t(1+z)} = S_t + \sum_{n \geq 1} \frac{z^n t^n}{n!} \frac{d^n S_t}{dt^n} \quad .$$

The foregoing bounds establish that the series is norm convergent for $|z| < 1/ce$. Hence S extends to the required sector.

It does remain to show that the analytically continued operator satisfies the semigroup property etc. But this follows by calculation with norm convergent power series.

The converse statement follows by using an operator version of Cauchy's theorem,

$$S_t = (2\pi i)^{-1} \int_C dz \, \frac{S_z}{(z-t)}$$

where C is a circle of radius rt centred at t. Then

$$HS_t = (2\pi i)^{-1} \int_C dz \, \frac{S_z}{(z-t)^2} \quad .$$

But since $\|S_z\| \leq M$ one obtains

$$\|HS_t\| \leq ct^{-1}$$

with $c = M/r$.

3.5 If H is the generator of a bounded semigroup S, with $\|S_t\| \leq M$, then the resolvent $(\lambda I + H)^{-1}$ can be defined for $\text{Re}\,\lambda > 0$ by the Laplace transform of Paragraph 2.6,

$$(\lambda I + H)^{-1}x = \int_0^\infty dt\, e^{-\lambda t} S_t x \quad,$$

and then one has the bounds

$$\|(\lambda I + H)^{-1}\| \leq M(\text{Re}\,\lambda)^{-1} \tag{23}$$

which are just a special case of (21). But if in addition S is bounded holomorphic and $|\varphi| < \theta$, the angle of holomorphy, then $e^{i\varphi}H$ is the generator of the bounded semigroup $t \mapsto S_{te^{i\varphi}}$ and therefore $(\lambda I + e^{i\varphi}H)^{-1}$ is also defined as a bounded operator. Since

$$e^{i\varphi}(\lambda I + e^{i\varphi}H)^{-1} = (e^{-i\varphi}\lambda I + H)^{-1} \quad.$$

it follows that $(\mu I + H)^{-1}$ is defined for all $\mu \in \{\lambda e^{i\varphi} : \text{Re}\,\lambda > 0, |\varphi| < \theta\}$. Moreover,

$$(e^{-i\varphi}\lambda I + H)^{-1}x = e^{i\varphi} \int_0^\infty dt\, e^{-\lambda t} S_{te^{i\varphi}} x$$

and hence

$$\|(e^{-i\varphi}\lambda I + H)^{-1}\| \leq M\,(\text{Re}\,\lambda)^{-1}$$

for all $\lambda \in \mathbf{C}$ with $\text{Re}\,\lambda > 0$. Thus one has bounds on $\|(\mu I + H)^{-1}\|$, with $\mu = \lambda e^{-i\varphi}$, which can be optimized by the choice of λ and φ. Consider the case $\arg \mu \geq 0$. There are two possibilities, $\arg \mu < \theta$ and $\arg \mu \in [\theta, \pi/2 + \theta\rangle$. In the first case one can optimize the bounds with λ real and obtain

$$\|(\mu I + H)^{-1}\| \leq M\,|\mu|^{-1} \quad. \tag{24}$$

In the second case, since $\arg \lambda = \arg \mu + \varphi$, one must have $\arg \lambda > 0$. Setting $\arg \mu = \pi/2 + \theta - \varepsilon$ one can choose φ close to $-\theta$ and the optimal bound becomes

$$\|(\mu I + H)^{-1}\| \leq M(\cos(\pi/2 - \varepsilon))^{-1}\,|\mu|^{-1} = M(\sin \varepsilon)^{-1}\,|\mu|^{-1} \quad. \tag{25}$$

The bounds (24) and (25) can be reexpressed in a unified manner. Setting $\Delta^c(\chi) = \mathbf{C}\backslash\Delta(\chi)$ one has

$$\|(\mu I + H)^{-1}\| \leq M\,d(\mu, \Delta^c(\pi/2 + \theta))^{-1} \tag{26}$$

for all $\mu \in \Delta(\pi/2 + \theta)$ where $d(\mu, \Delta^c)$ denotes the distance of μ from the closure of the set Δ^c.

The bounds (26) give precise information on the behaviour of $\|(\mu I + H)^{-1}\|$ as μ approaches the boundary of the resolvent set $\Delta(\pi/2 + \theta)$. Often such precise information is not required and it suffices to note that for each $\varepsilon \in \langle 0, \pi/2 + \theta]$ there is an M_ε such that

$$\|(\mu I + H)^{-1}\| \leq M_\varepsilon\,|\mu|^{-1} \tag{27}$$

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for all $\mu \in \Delta(\pi/2 + \theta - \varepsilon)$

Alternatively, since we intend to use Cauchy integral techniques, it is somewhat more natural to consider the resolvents $(\mu I - H)^{-1}$, i.e., to effectively replace μ by $-\mu$. Then it follows that these resolvents are defined for all $\mu \in \Delta^c(\pi/2 - \theta)$ and for each $\varepsilon \in \langle 0, \pi/2 + \theta]$ there is an M_ε such that

$$\|(\mu I - H)^{-1}\| \leq M_\varepsilon |\mu|^{-1} \tag{28}$$

for all $\mu \in \Delta^c(\pi/2 - \theta + \varepsilon)$

3.6 The properties of Paragraph 3.5 are again not only necessary but sufficient for an operator to generate a bounded holomorphic semigroup. There is a direct analogue of the Hille–Yosida theorem.

Theorem 3.2 *Let H be an operator on a Banach space $\mathcal{X}$ and $\theta \in \langle 0, \pi/2]$. Assume that*

> *1. H is densely defined and closed,*
>
> *2. a.* $\qquad\qquad R(\lambda I - H) = \mathcal{X}$
>
> *for all $\lambda \in \Delta^c(\pi/2 - \theta + \varepsilon)$ and all $\varepsilon \in \langle 0, \pi/2 + \theta]$,*
>
> *b. for each $\varepsilon \in \langle 0, \pi/2 + \theta]$ there is a $c_\varepsilon > 0$ such that*
>
> $$\|(\lambda I - H)x\| \geq c_\varepsilon |\lambda| \|x\|$$
>
> *for all $\lambda \in \Delta^c(\pi/2 - \theta + \varepsilon)$ and all $x \in D(H)$.*

Then H is the generator of a bounded holomorphic semigroup S, with holomorphy angle greater or equal to θ, and

$$S_z x = -(2\pi i)^{-1} \int_\gamma d\lambda \, e^{-\lambda z} (\lambda I - H)^{-1} x \tag{29}$$

for all $z \in \Delta(\theta)$ and $x \in \mathcal{X}$ where $\gamma \in \Delta^c(\pi/2 - \theta)$ is a curve with asymptotes $\pm(\pi/2 - \theta + \delta)$ and $\delta \in \langle 0, \theta \rangle$.

The key to the proof is again the algorithm for constructing S, the Cauchy integral (29).

First, one chooses a γ with the specified asymptotes suited to estimation of convergence of the integral and the proof that $\|S_z\|$ is uniformly bounded. These estimates also establish that $z \in \Delta(\theta) \mapsto S_z$ is an operator-valued holomorphic function and the integral is independent of the choice of γ.

Secondly, one proves the S_z have the semigroup property by using the resolvent equation

$$(\lambda_1 I + H)^{-1}(\lambda_2 I + H)^{-1} = (\lambda_1 - \lambda_2)^{-1}((\lambda_1 I + H)^{-1} - (\lambda_2 I + H)^{-1})$$

to rearrange the double integral representing the product $S_{z_1} S_{z_2}$. The integral can then be partially evaluated to give the representation for $S_{z_1 + z_2}$.

Thirdly, one argues that S is continuous by use of the dominated convergence theorem and Cauchy's theorem. Similar reasoning also establishes that H is the generator of S.

We will only give an indication of the proof of the first step.

Fix $z \in \Delta(\theta - \delta)$. Let $\gamma = \gamma_0 \cup \gamma_+ \cup \gamma_-$ where

$$\gamma_\pm = \{ \lambda \,:\, \lambda = re^{\pm i(\pi/2 - \theta + \delta)} \,,\, r \geq 1 \}$$
$$\gamma_0 = \{ \lambda \,:\, \lambda = e^{i\chi} \,,\, \chi \in [\pi/2 - \theta + \delta \,,\, 3\pi/2 + \theta - \delta] \} \quad .$$

The contour integral I for S_z then splits into a corresponding sum $I_0 \cup I_+ \cup I_-$. The integral for I_0 is obviously convergent. But if $\lambda \in \gamma_\pm$ one has $\mathrm{Re}(\lambda z) = r|z| \sin(\theta \pm \varphi - \delta) > 0$ and hence the integrals $I_\pm$ converge. Thus S_z is well-defined and its definition will be independent of the particular choice of contour.

The proof that $\|S_z\|$ is uniformly bounded relies on a change of variable $\mu = |z|\lambda$ in the contour integral. Then

$$S_z = -(2\pi i |z|)^{-1} \int_{\gamma_z} d\mu \, e^{-\mu z/|z|}((z/|z|)I - H)^{-1}$$
$$= -(2\pi i |z|)^{-1} \int_{\gamma} d\mu \, e^{-\mu z/|z|}((z/|z|)I - H)^{-1}$$

where γ_z is the scaled version of γ and the second identification follows because the integral is independent of the choice of contour. Estimating as before one finds that $\|S_z\|$ is uniformly bounded for all $z \in \Delta(\theta - \varepsilon)$ where ε is small and positive.

3.7 The Cauchy techniques give a different approach to the definition of functions of an unbounded operator H. If H generates a holomorphic semigroup and f is holomorphic in a suitable region one can tentatively define

$$f(H) = -(2\pi i)^{-1} \int_{\gamma} d\lambda \, f(\lambda)(\lambda I - H)^{-1}$$

with γ a contour of the type used above. There are a number of restrictions which have to be placed upon f to ensure that the integral is convergent in some reasonable sense and then it is of interest to analyze the corresponding classes of functions and the properties of the operators $f(H)$.

4 Forms and kernels

4.1 In the previous two lectures we described some of the basic properties of semigroups and their generators on general Banach spaces. In this lecture we examine related questions on particular Banach spaces of functions. There are two key topics in the sequel, 1. quadratic forms and in particular Dirichlet forms, and 2. semigroup kernels. These concepts are especially suited to the description of the evolution semigroups generated by secon-order elliptic partial differential operators but at this stage we will not exploit the detailed differential structure.

In the theory of partial differential operators there is a wide variety of function spaces which plays a useful role. In particular problems it is often necessary to choose a space appropriate to the solution. But then it can be essential to use other spaces to fully analyze properties of the solution. In particular the L_p-spaces and the spaces C_0, and C_b, of continuous functions play a vital auxiliary role. The material in the latter half of this lecture illustrates why the L_p-spaces are so fundamental. But first we examine how one can obtain information about the action of an operator on the L_p-spaces from knowledge of its action on the L_2-space.

4.2 Let H be a linear operator on a complex Hilbert space $\mathcal{H}$. Then

$$\varphi, \psi \in D(H) \mapsto h(\varphi, \psi) = (\varphi, H\psi) \in \mathbb{C}$$

is a densely-defined, sesquilinear form, i.e.,

$$h(\lambda_1 \varphi_1 + \lambda_2 \varphi_2, \psi) = \overline{\lambda}_1 h(\varphi_1, \psi) + \overline{\lambda}_2 h(\varphi_2, \psi)$$

$$h(\varphi, \mu_1 \psi_1 + \mu_2 \psi_2) = \mu_1 h(\varphi, \psi_1) + \mu_2 h(\varphi, \psi_2) \quad .$$

One may associate a quadratic form $\varphi \in D(H) \mapsto h(\varphi) \in \mathbb{C}$ to the sesquilinear form by setting

$$h(\varphi) = h(\varphi, \varphi) = (\varphi, H\varphi) \quad .$$

Conversely, one can recuperate the sesquilinear form from the quadratic form by use of the polarization identity

$$h(\varphi, \psi) = \frac{1}{4} \sum_{n=0}^{3} i^{-n} h(\varphi + i^n \psi) \quad .$$

The quadratic form automatically satisfies the parallelogram law

$$h(\varphi + \psi) + h(\varphi - \psi) = 2h(\varphi) + 2h(\psi)$$

for all $\varphi, \psi \in D(H)$.

Note that if H is self-adjoint then the sesquilinear form is symmetric, i.e., it satisfies the reality condition

$$\overline{h(\varphi, \psi)} = h(\psi, \varphi)$$

for all $\varphi, \psi \in D(H)$, and the associated quadratic form is real. Moreover, if the spectrum of H lies in the interval $[\lambda, \mu]$ then

$$\lambda \|\varphi\|^2 \le h(\varphi) \le \mu \|\varphi\|^2$$

for all $\varphi \in D(H)$.

Finally, if H is a positive self-adjoint operator then it has a positive self-adjoint square root $H^{1/2}$ with $D(H) \subseteq D(H^{1/2})$ and

$$h(\varphi) = (\varphi, H\varphi) = (H^{1/2}\varphi, H^{1/2}\varphi)$$

for $\varphi \in D(H)$. Therefore h can be extended to $D(H^{1/2})$ by use of this last relation. Alternatively, this extension can be constructed by closure. If $\varphi_n \in D(H)$ is strongly convergent to φ and if $h(\varphi_n - \varphi_m)$ is a Cauchy sequence then it follows that $\varphi \in D(H^{1/2})$ and $h(\varphi_n) \to \|H^{1/2}\varphi\|^2$ as $n \to \infty$.

4.3 The construction of the previous paragraph can often be reversed: one can pass from a quadratic form to an operator if the form satisfies suitable requirements. There are two types of requirement. First, the form should be lower semibounded, i.e., there should be a $\lambda \in \mathbf{R}$ such that

$$h(\varphi) \ge \lambda \|\varphi\|^2$$

for all $\varphi \in D(h)$. This requirement can be somewhat relaxed but there nevertheless has to be some form of boundedness. Secondly, the form should be closed, i.e., if $\varphi_n \in D(h)$, $\|\varphi_n - \varphi\| \to 0$ as $n \to \infty$, and $h(\varphi_n - \varphi_m)$ is a Cauchy sequence then $\varphi \in D(h)$ and $h(\varphi_n - \varphi) \to 0$ as $n \to \infty$. This latter property is essential for the reconstruction of an operator from a form as the next result indicates.

Proposition 4.1 *Let h be a positive quadratic form with domain $D(h)$ a dense subspace of the Hilbert space $\mathcal{H}$. The following conditions are equivalent;*

1. *h is closed,*

2. *h is the form of a positive self-adjoint operator H, i.e., $D(h) = D(H^{1/2})$ and*

$$h(\varphi) = \|H^{1/2}\varphi\|^2$$

for all $\varphi \in D(h)$.

Moreover, if these conditions are satisfied then $\psi \in D(H)$ if and only if $\varphi \in D(h) \mapsto h(\varphi, \psi)$ is continuous and then $h(\varphi, \psi) = (\varphi, H\psi)$ for all $\varphi \in D(h)$.

4.4 Let S be a densely-defined, closed, operator on $\mathcal{H}$ and define h by $D(h) = D(S)$ and

$$h(\varphi, \psi) = (S\varphi, S\psi)$$

for $\varphi, \psi \in D(S)$. It follows that the associated quadratic form h is positive and closed. Therefore there is a self-adjoint operator H with $D(H) \subseteq D(S)$ such that

$$h(\varphi, \psi) = (\varphi, H\psi) = (S\varphi, S\psi)$$

for all $\varphi \in D(S)$ and $\psi \in D(H)$. But $\varphi \in D(S) \mapsto (S\varphi, S\psi)$ is continuous if and only if $S\psi \in D(S^*)$ and hence $D(H) = \{\psi \in D(S) : S\psi \in D(S^*)\}$ and $H = S^*S$.

4.5 Let (X, ρ) be a σ-finite measure space, H a positive self-adjoint operator on the Hilbert space $L_2(X; \rho)$ and h the corresponding closed quadratic form. Then H generates a self-adjoint contraction semigroup $S_t = e^{-tH}$ and it is of interest to find conditions which ensure that S extends to a continuous contraction semigroup on the L_p-spaces $L_p(X; \rho)$. There is a simple abstract criterion in terms of the form h if S is positive in the sense that it maps positive functions into positive functions. Moreove, the positivity can also be simply characterized by properties of the form.

Theorem 4.2 (Beurling–Deny) *Let h be a positive, densely-defined, closed, quadratic form over $L_2(X; \rho)$, H the corresponding positive, self-adjoint, operator and $S_t = e^{-tH}$ the associated self-adjoint, contraction, semigroup.*

The following conditions are equivalent:

1. *S is positive,*

2. *if $\varphi \in D(h)$ then $|\varphi| \in D(h)$ and*

$$h(|\varphi|) \leq h(\varphi) \quad .$$

Moreover, if S is positive the following conditions are equivalent;

$1'$. *S extends to a continuous contraction semigroup on each of the spaces $L_p(X; \rho)$, $p \in [1, \infty]$,*

$2'$. *if $\varphi \in D(h)$ and $\varphi \geq 0$ then $\varphi \wedge 1 \in D(h)$ and*

$$h(\varphi \wedge 1) \leq h(\varphi) \quad .$$

Positive, closed, quadratic forms satisfying Conditions 2. and $2'$. of the theorem are called Dirichlet forms. Semigroups S satisfying Conditions 1. and $1'$. of the theorem are called Markov semigroups. Thus there is a one-to-one correspondence between Dirichlet forms and the generators of Markov semigroups.

The striking feature of the theorem is that L_2-estimates on the form h lead to L_p-estimates on the semigroup S. In practise estimates on the Hilbert space of L_2-functions are relatively easy to obtain and it is remarkable that these can yield estimates on the Banach spaces of L_p-functions. In the theory of partial differential operators it is a standard stratagem to try to derive 'hard' Banach space estimates from 'easy' Hilbert space estimates but usually this is only possible in the simplest cases such as second-order operators with real coefficients.

4.6 Let $S = d/dx$ acting on $L_2(\mathbb{R})$. Thus S corresponds to multiplication of the Fourier transform $\hat{\varphi}$ of $\varphi \in L_2$ by ip (see Paragraph 1.7) and S is closed on the domain of $\varphi \in L_2$ such that $p \mapsto p\hat{\varphi}(p)$ is square-integrable. Then

$$h(\varphi, \psi) = (S\varphi, S\psi)$$

is a positive, closed, quadratic form and the associated positive self-adjoint operator is $H = S^*S = -d^2/dx^2$. But h is also a Dirichlet form. This can be established by verifying Conditions 2. and 2'. of the Beurling–Deny theorem but it follows more easily by observing that the semigroup $S_t = e^{-tH}$ acts by convolution with the Gaussian kernel.

4.7 Let $c \in L_\infty(\mathbf{R})$ and consider the 'operator'

$$H_c = -\frac{d}{dx}c(x)\frac{d}{dx} \quad .$$

If c is not differentiable then the interpretation of this operator is not straightforward. But it can be viewed as a form on $L_2(\mathbf{R})$,

$$h_c(\varphi, \psi) = (S\varphi, cS\psi) \quad ,$$

where S again denotes the closed operator of differentiation. Now suppose $c \geq 0$. Then h_c is a positive, closed, quadratic form and there is a corresponding positive, self-adjoint, operator H_c with $D(H_c) \subseteq D(S)$ and

$$(S\varphi, cS\psi) = (\varphi, H_c\psi)$$

for all $\varphi \in D(S)$ and $\psi \in D(H_c)$.

It is not quite so easy to verify that h_c is a Dirichlet form but this can be achieved by an approximation argument.

First, since h_c is the form of a positive self-adjoint operator it is a lower semi-continuous function. Therefore if $\varphi_n \in D(h_c)$, $\|\varphi_n - \varphi\| \to 0$ and $n \to h(\varphi_n)$ is uniformly bounded then $\varphi \in D(h_c)$ and

$$h_c(\varphi) \leq \lim_{n \to \infty} h_c(\varphi_n) \quad .$$

Secondly, let F be a smooth, positive, function over $\mathbf{R}$ with $F(0) = 0$, $|F'(x)| \leq 1$ and $F(x) = |x| - 1$ for $|x| \geq 2$. Given $\varphi \in D(S)$ real set

$$\varphi_\varepsilon = \varepsilon F(\varepsilon^{-1}\varphi)$$

for all $\varepsilon > 0$. Then

$$\lim_{\varepsilon \to 0} \|\varphi_\varepsilon - \varphi\| = 0 \quad .$$

Moreover,

$$h_c(\varphi_\varepsilon) = (S\varphi, F'(\varepsilon^{-1}\varphi)cF'(\varepsilon^{-1}\varphi)S\varphi) \leq h_c(\varphi) \quad .$$

It then follows by the first observation that $|\varphi| \in D(h_c)$ and

$$h_c(|\varphi|) \leq \lim_{\varepsilon \to 0} h_c(\varphi_\varepsilon) \leq h_c(\varphi) \quad .$$

Thus the first criterion of the Beurling–Deny theorem is verified for real φ and it then follows easily for complex functions. The second criterion is verified by a similar approximation argument.

4.8 The examples given in the last two paragraphs extend naturally to higher dimensions and to partial differential operators. For example, if $c_{ij} \in L_\infty(\mathbf{R}^d)$ for $i,j = 1,\dots,d$ are real-valued functions and $C = (c_{ij})$ as a matrix is symmetric and satisfies $0 < \lambda I \le C \le \mu I$ uniformly on $\mathbf{R}^d$ then the quadratic form corresponding to

$$h_c(\varphi,\psi) = \sum_{i,j=1}^d (\frac{\partial\varphi}{\partial x_i}, c_{ij}\frac{\partial\psi}{\partial x_j})$$

is a Dirichlet form. The positive self-adjoint operator associated with h_c can be thought of as

$$H_c = -\sum_{i,j=1}^d \frac{\partial}{\partial x_i} c_{ij} \frac{\partial}{\partial x_j}$$

even if the c_{ij} are not differentiable.

4.9 The heat semigroup S described in Paragraph 1.10 is given by convolution with the Gaussian kernel and hence S automatically acts on each of the L_p-spaces. It is natural to ask under what conditions a semigroup which acts on the L_p-spaces is determined by a similar kernel. The answer to this question is governed by the properties of the semigroup acting between the L_p-spaces. To describe this result we need to first recall the notion of a cross-norm.

Let S be an operator on the spaces $L_p(X\,;\rho)$, i.e., S is defined on $L_p \cap L_r$ and $SL_p \cap L_r \subseteq L_p \cap L_r$ for each pair $p,r \in [1,\infty]$. Then $\|S\|_{p\to r}$, the cross-norm of S from L_p to L_r, is defined by

$$\|S\|_{p\to r} = \sup\{\|S\psi\|_r : \|\psi\|_p \le 1\} \quad .$$

Note that if $1/p + 1/q = 1$ and $1/r + 1/s = 1$ then

$$\|S\|_{p\to r} = \sup\{|(\varphi, S\psi)| : \|\varphi\|_s \le 1,\ \|\psi\|_p \le 1\}$$

$$= \sup\{|(S^*\varphi, \psi)| : \|\varphi\|_s \le 1,\ \|\psi\|_{\check{p}} \le 1\}$$

$$= \sup\{\|S^*\varphi\|_q : \|\varphi\|_s \le 1\} = \|S^*\|_{s\to q} \quad .$$

Moreover, if $p \le t \le r$ then

$$\|S_1 S_2 \varphi\|_r \le \|S_1\|_{t\to r}\|S_2\varphi\|_t$$

for all $\varphi \in L_p$ and hence

$$\|S_1 S_2\|_{p\to r} \le \|S_2\|_{p\to t}\|S_1\|_{t\to r} \quad .$$

4.10 Let S be an operator on the spaces $L_p(X\,;\rho)$, with (X,ρ) separable, which is bounded on each of the spaces. We are interested in the question of the existence of a kernel s as a function over $X \times X$ such that

$$(S\varphi)(x) = \int_X d\rho(y)\, s(x\,;y)\varphi(y)$$

for all $\varphi \in L_p(x\,;\rho)$, $p \in [1,\infty]$. If s exists then the boundedness properties of S imply integrability properties of s and it is useful to introduce the norms

$$|||s|||_p = \sup_{x\in X} \left(\int_X d\rho(y)\, |s(x\,;y)|^p \right)^{1/p} \quad .$$

We use $\mathcal{L}_p$ to denote the space of functions for which this norm is finite.

Theorem 4.3 (Dunford–Pettis) *The following conditions are equivalent;*

1. *S is bounded from L_p to L_∞,*
2. *S has a kernel $s \in \mathcal{L}_1 \cap \mathcal{L}_q$ where $1/p + 1/q = 1$.*

Moreover, if these conditions are satisfied then

$$\|S\|_{p\to\infty} = |||s|||_q \quad , \quad \|S\|_{\infty\to\infty} = |||s|||_1 \quad .$$

The theorem is particularly interesting for $p = 1$ in which case $s \in \mathcal{L}_1 \cap \mathcal{L}_\infty$. Consequently, by interpolation theory (see Paragraph 4.12), $s \in \mathcal{L}_r$ for all $r \in [1,\infty]$.

4.11 If $t \geq 0 \mapsto S_t$ is a continuous semigroup of bounded operators on the spaces $L_p(X\,;\rho)$ and each S_t, $t \geq 0$, is bounded from L_1 to L_∞ then S is determined by a family of kernels $K_t \in \mathcal{L}_1 \cap \mathcal{L}_\infty$,

$$(S_t\varphi)(x) = \int_X d\rho(y)\, K_t(x\,;y)\varphi(y) \quad .$$

Then the semigroup property $S_{s+t} = S_s S_t$ implies that the K_t satisfy the convolution semigroup property

$$K_{s+t}(x\,;y) = \int_X d\rho(z)\, K_s(x\,;z)K_t(z\,;y) \quad .$$

Moreover, the continuity implies that $K_t(x\,;y) \to \delta(x-y)$ weakly as $t \to 0$.

4.12 We conclude this lecture by remarking that interpolation theory is one of the most useful tools in estimating operators on L_p-spaces, or more general scales of spaces. There is a wide variety of interpolation techniques and their description goes well beyond the scope of these lectures but to give the flavour of the theory we quote a classic result, the Riesz–Thorin theorem. *Assume S is a bounded operator from L_{p_1} to L_{r_1} and from L_{p_2} to L_{r_2}. Further for $\alpha \in \langle 0,1\rangle$ set*

$$p^{-1} = \alpha p_1^{-1} + (1-\alpha)p_2^{-1} \quad , \quad r^{-1} = \alpha r_1^{-1} + (1-\alpha)r_2^{-1} \quad .$$

It follows that S is bounded from L_p to L_r and

$$\|S\|_{p\to r} \leq (\|S\|_{p_1\to r_1})^\alpha (\|S\|_{p_2\to r_2})^{1-\alpha} \quad .$$

In particular if S is bounded on L_{p_1} and L_{p_2} then it is bounded on L_p for all $p \in [p_1, p_2]$. Thus the boundedness property interpolates to the interior values from the extremal values.

5 Heat kernel estimates

5.1 In this last lecture we describe some results specific to second-order elliptic operators and the associated semigroups. These results are of recent provenance and illustrate one type of problem which remains of great interest in current research.

In Paragraph 4.8 we argued that the operator

$$H_c = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} c_{ij} \frac{\partial}{\partial x_j} \quad ,$$

with the symmetric matrix of real-valued L_∞-coefficients $C = (c_{ij})$ satisfying $0 < \lambda I \le C \le \mu I$ uniformly on $\mathbf{R}^d$, can be defined in terms of a Dirichlet form. Then H_c generates a positive, continuous, contraction semigroup $S_t = e^{-tH_c}$ on $L_2(\mathbf{R}^d)$ which extends to a continuous contraction semigroup on each of the L_p-spaces. Our aim is to establish by use of the Dunford–Pettis theorem that the action of S is determined by a convolution semigroup kernel K. Then we will prove that K satisfies 'Gaussian' bounds.

5.2 To establish the existence of the kernel K it is necessary to prove that the cross-norms $\|S_t\|_{1\to\infty}$ are bounded. But

$$\|S_t\|_{1\to\infty} = \|S_{t/2} S_{t/2}\|_{1\to\infty} \le \|S_{t/2}\|_{1\to2} \|S_{t/2}\|_{2\to\infty}$$

and since S is self-adjoint

$$\|S_{t/2}\|_{1\to2} = \|S_{t/2}\|_{2\to\infty} \quad .$$

Therefore

$$\|S_t\|_{1\to\infty} \le (\|S_{t/2}\|_{1\to2})^2$$

and it suffices to prove that $\|S_t\|_{1\to2}$ is bounded. This will be achieved by use of a Nash inequality which we next derive.

5.3 Let B_r denote the ball of radius r in $\mathbf{R}^d$ centred at the origin and $|B_r|$ the volume of the ball. If $\varphi \in L_2(\mathbf{R}^d)$ has square-integrable derivatives then the Plancherel formula gives

$$\|\varphi\|_2^2 = \|\hat{\varphi}\|_2^2 \le r^{-2} \int_{\mathbf{R}^d} dp\, p^2 |\hat{\varphi}(p)|^2 + \int_{B_r} dp\, |\hat{\varphi}(p)|^2$$

$$\le r^{-2} \|\nabla\varphi\|_2^2 + |B_r|\, \|\hat{\varphi}\|_\infty^2$$

$$\le r^{-2} \|\nabla\varphi\|_2^2 + r^d |B_1|\, \|\varphi\|_1^2$$

for all $r > 0$. This is the first form of the required Nash inequality. One can obtain a second version of the inequality by minimizing the right hand side over r. One finds

$$\|\varphi\|_2 / \|\varphi\|_1 \le k \big(\|\nabla\varphi\|_2 / \|\varphi\|_1 \big)^{d/(2+d)}$$

where the value of the constant k depends only on the dimension d.

5.3 Next we bound $\|S_t\|_{1\to 2}$ and hence deduce that S_t has an integral kernel K_t. Since H_c is a real operator, it maps real functions into real functions, the semigroup S is also real. Therefore the crossnorms can be estimated on the corresponding spaces of real functions. Hence in the sequel all spaces and functions will be real.

The starting point of the estimations is the differential equation

$$\frac{d}{dt}\|S_t\varphi\|_2^2 = -2h_c(S_t\varphi)$$

where h_c is the Dirichlet form used to define the generator H_c of S. But

$$h_c(\psi) = \sum_{i,j=1}^d (\partial_i\varphi, c_{ij}\partial_j\varphi)$$

$$\geq \lambda \sum_{i=1}^d \|\partial_i\psi\|_2^2 = \lambda \|\nabla\psi\|_2^2$$

where $\partial_i = \partial/\partial x_i$ and λ is the uniform lower bound on the matrix C of coefficients, the ellipticity constant. Therefore

$$\frac{d}{dt}\|S_t\varphi\|_2^2 \leq -2\|\nabla S_t\varphi\|_2^2 \quad .$$

But since S is a contraction semigroup on L_1 the Nash inequalities give

$$\|\nabla S_t\varphi\|_2 \geq a\|S_t\varphi\|_2^{1+2/d}\|S_t\varphi\|_1^{-2/d} \geq a\|S_t\varphi\|_2^{1+2/d}\|\varphi\|_1^{-2/d}$$

for some $a > 0$. Hence one has

$$\frac{d}{dt}\|S_t\varphi\|_2^2 \leq -2a^2\lambda\|S_t\varphi\|_2^{2(1+2/d)}\|\varphi\|_1^{-4/d}$$

or, equivalently,

$$\frac{d}{dt}\|S_t\varphi\|_2^{-4/d} \geq (8a^2\lambda/d)\|\varphi\|_1^{-4/d} \quad .$$

Thus by integration one obtains

$$\|S_t\varphi\|_2^{-4/d} \geq \|S_t\varphi\|_2^{-4/d} - \|\varphi\|_2^{-4/d} \geq (8a^2\lambda/d)t\|\varphi\|^{-4/d} \quad .$$

Therefore

$$\|S_t\varphi\|_2 \leq a't^{-d/4}\|\varphi\|_1 \quad .$$

Hence we conclude that

$$\|S_t\|_{1\to\infty} \leq \left(\|S_t\|_{1\to 2}\right)^2 \leq a't^{-d/2}$$

and S_t has a kernel K_t satisfying

$$\||K_t\||_\infty \leq a''t^{-d/2} \quad .$$

Note that the various positive constants a, a', a'' depend only on the dimension d and the ellipticity constant λ. Further remark that since S is positive $(\varphi, S_t \psi) \geq 0$ for all positive $\varphi, \psi \in L_2$ and hence $K_t \geq 0$. Thus one has bounds

$$0 \leq K_t(x\,; y) \leq a\,t^{-d/2} \tag{30}$$

uniformly for $x, y \in \mathbf{R}^d$.

5.4 The foregoing estimates provide the basis for more elaborate arguments which lead to refined 'Gaussian' upper bounds

$$0 \leq K_t(x\,; y) \leq at^{-d/2}e^{-d_c(x;y)^2/4t}$$

with d_c an appropriate distance function. Therefore it is relevant to emphasize that there are two pieces of data used in the estimates. First, the lower bound

$$h_c(\varphi) \geq \lambda \, \|\nabla\varphi\|_2^2$$

on the L_2-form of the generator. Secondly, the L_1-contractivity

$$\|S_t\|_{1\to 1} \leq 1$$

of the semigroup. The first estimate is a direct consequence of the bound $C \geq \lambda I$ on the matrix of coefficients. The second is a more subtle indirect implication of the Beurling–Deny theorem.

There is an alternative, more elaborate, derivation of the uniform upper bounds (30) which avoids use of the L_1-estimates and hence is more easily generalized. Therefore at the risk of repetition we give a second derivation of the uniform upper bounds which depends only on L_2-estimates.

5.5 The strategy of this second argument is to bound $\|S_t\|_{1\to\infty}$ by first bounding $\|S_t\|_{2\to\infty}$. The $2 \to \infty$ cross-norm is bounded by using L_2-estimates to successively bound $\|S_t\|_{2\to 4}$, $\|S_t\|_{2\to 8}$ .... The new idea is that $\varphi \in L_{2p}$ is equivalent to $\varphi^p \in L_2$. Therefore L_{2p}-estimates on φ can be converted into L_2-estimates on φ^p.

The starting point now is the differential equation

$$\frac{d}{dt}\|S_t\varphi\|_{2p}^{2p} = -2p\,h_c((S_t\varphi)^{2p-1}, S_t\varphi)$$

for real-valued $\varphi \in L_{2p}$. But

$$h_c((S_t\varphi)^{2p-1}, S_t\varphi) = (2p-1)\sum_{i,j=1}^{d}((S_t\varphi)^{p-1}\partial_i S_t\varphi), c_{ij}(S_t\varphi)^{p-1}\partial_j S_t\varphi))$$

$$= ((2p-1)/p^2)\,h_c((S_t\varphi)^p)$$

where $\partial_i = \partial/\partial x_i$. Hence

$$\frac{d}{dt}\|S_t\varphi\|_{2p}^{2p} = -2((2p-1)/p)\,h_c((S_t\varphi)^p) \leq -2((2p-1)/p)\lambda\,\|\nabla(S_t\varphi)^p\|_2^2 \quad . \tag{31}$$

Now we solve this differential inequality with the aid of the Nash inequalities as before. But we successively consider $p = 2, 4, \ldots$.

First, if $p = 2$ then the Nash inequality gives

$$\|\nabla(S_t\varphi)^2\|_2 \geq a \, \|(S_t\varphi)^2\|_2^{1+2/d} \|(S_t\varphi)^2\|_1^{-2/d}$$
$$= a \, \|S_t\varphi\|_4^{2(1+2/d)} \|S_t\varphi\|_2^{-4/d}$$
$$\geq a \, \|S_t\varphi\|_4^{2(1+2/d)} \|\varphi\|_2^{-4/d}$$

where the last estimate uses the L_2-contractivity of S. Consequently, the differential inequality (31) yields

$$\frac{d}{dt}\|S_t\varphi\|_4^4 \leq -3a^2\lambda \, \|S_t\varphi\|_4^{4(1+2/d)} \|\varphi\|_2^{-8/d}$$

or, equivalently,

$$\frac{d}{dt}\|S_t\varphi\|_4^{-8/d} \geq a' \|\varphi\|_2^{-8/d} \, .$$

Then by integration one obtains bounds

$$\|S_t\|_{2\to4} \leq a_2 \, t^{-d/8} \quad . \tag{32}$$

Secondly, if $p = 4$ the Nash inequality gives

$$\|\nabla(S_t\varphi)^4\|_2 \geq a \, \|(S_t\varphi)^4\|_2^{1+2/d} \|(S_t\varphi)^4\|_1^{-2/d}$$
$$= a \, \|S_t\varphi\|_8^{4(1+2/d)} \|S_t\varphi\|_4^{-8/d}$$
$$\geq a't \, \|S_t\varphi\|_8^{4(1+2/d)} \|\varphi\|_2^{-8/d}$$

where the last bound uses the $\|S_t\|_{2\to4}$ estimate (32). Therefore the differential inequality (31) yields

$$\frac{d}{dt}\|S_t\varphi\|_8^8 \leq -at^2\|S_t\varphi\|_8^{8(1+2/d)} \|\varphi\|_2^{-16/d}$$

which can be rewritten as

$$\frac{d}{dt}\|S_t\varphi\|_8^{-16/d} \geq a't^2\|\varphi\|_2^{-16/d} \quad .$$

Then by integration one obtains bounds

$$\|S_t\|_{2\to8} \leq a_3 \, t^{-3d/16} \quad . \tag{33}$$

Now this can be used as initial data in an estimate of $\|S_t\|_{2\to16}$ etc.

This iterative argument gives bounds

$$\|S_t\|_{2\to2^n} \leq a_n \, t^{-d(1/2-1/2^n)/2} \quad .$$

A little care with the calculation shows, however, that the a_n are uniformly bounded in n. Hence one can take a limit $n \to \infty$ to obtain the bounds

$$\|S_t\|_{2\to\infty} \leq a_\infty \, t^{-d/4} \quad .$$

One then has, as before,

$$\|S_t\|_{1\to\infty} \le \left(\|S_{t/2}\|_{2\to\infty}\right)^2 \le a\,t^{-d/2}$$

but the important point is that the proof only uses L_2-estimates. There are again two basic estimates involved, the L_2-contractivity of S which follows from the positivity $h_c(\varphi) \ge 0$ and the crossnorm bounds which use the more stringent bound $h_c(\varphi) \ge \lambda\|\varphi\|_2^2$.

5.6 The standard strategy to obtain the Gaussian bounds referred to in Paragraph 5.4 from the uniform bounds is a perturbation argument. Let ψ be a real-valued smooth function over $\mathbf{R}^d$ with compact support and introduce the one-parameter family of bounded multiplication operators $U_\alpha = \exp\{-\alpha\psi\}$. Then consider the continuous semigroups

$$S_t^\alpha = U_\alpha S_t U_{-\alpha} \quad .$$

These semigroups are no longer self-adjoint or contractive but they do have kernels K^α with

$$K_t^\alpha(x\,;y) = e^{-\alpha(\psi(x)-\psi(y))} K_t(x\,;y) \quad .$$

Therefore if one can establish new uniform bounds

$$\||K_t^\alpha|\|_\infty \le a\,t^{-d/2} e^{\alpha^2 \omega t} \tag{34}$$

with a,ω independent of α then one has

$$0 \le K_t(x\,;y) \le a\,t^{-d/2} e^{\alpha^2 \omega t + \alpha(\psi(x)-\psi(y))} \quad .$$

Hence minimizing over α gives

$$0 \le K_t(x\,;y) \le a\,t^{-d/2} e^{-|\psi(x)-\psi(y)|^2/4\omega t} \quad .$$

Finally one can minimize over ψ to obtain optimal bounds but for this one has to control the dependence of a and ω on ψ.

5.7 The bounds (34) can be established by a variation of the calculations of Paragraph 5.6 applied to S^α in place of S.

The generator of S^α is $H_c^\alpha = U_\alpha H_c U_\alpha^{-1}$ and since $U_\alpha \partial_i U_\alpha^{-1} = \partial_i + \alpha(\partial_i\psi)$ one has formally

$$H_c^\alpha = -\sum_{i,j=1}^d \left(\partial_i + \alpha(\partial_i\psi)\right)c_{ij}\left(\partial_j + \alpha(\partial_j\psi)\right)$$

$$= H_c - \alpha\sum_{i,j=1}^d \left(\partial_i c_{ij}(\partial_j\psi) + (\partial_i\psi))c_{ij}\partial_j\right) - \alpha^2 c_\psi$$

where

$$c_\psi = \sum_{i,j=1}^d c_{ij}(\partial_i\psi)(\partial_j\psi)$$

30

Moreover, the quadratic form h_c^α corresponding to H_c^α is given by

$$h_c^\alpha(\varphi) = h_c(\varphi) - \alpha^2(\varphi, c_\psi \varphi)$$

because the terms linear in α vanish as a result of the symmetry of the matrix of coefficients C. Therefore

$$\frac{d}{dt}\|S_t^\alpha \varphi\|_2^2 = -2\,\mathrm{Re}\,h_c^\alpha(S_t^\alpha \varphi) \leq 2\alpha^2(S_t^\alpha \varphi, c_\psi S_t^\alpha \varphi) \quad .$$

Hence for the restricted class of ψ for which $\|c_\psi\|_\infty \leq 1$ one has

$$\frac{d}{dt}\|S_t^\alpha \varphi\|_2^2 \leq 2\alpha^2 \|S_t^\alpha \varphi\|_2^2 \quad .$$

Consequently, by integration,

$$\|S_t^\alpha\|_{2\to 2} \leq e^{\alpha^2 t} \tag{35}$$

which is a bound of the required type.

Next consider the $\|S_t^\alpha\|_{2\to 4}$ bound. Again one begins with the differential equation

$$\frac{d}{dt}\|S_t^\alpha \varphi\|_4^4 = -4\,\mathrm{Re}\,h_c^\alpha((S_t^\alpha \varphi)^3, S_t^\alpha \varphi)$$

with the sesquilinear form h_c^α quadratic in α. The zero-order term gives the same contribution as in the calculations of Paragraph 5.5 but S is replaced by S^α. The second-order term in α is straightforward to estimate and if $\|c_\psi\|_\infty \leq 1$ it is bounded by $4\alpha^2 \|S_t^\alpha \varphi\|_4^4$. It remains to bound the first-order terms. But these can be reexpressed in terms of $(S_t^\alpha \varphi)^2$ and then dominated by the other terms by use of the inequality

$$2\,\mathrm{Re}(\varphi, C\chi) \leq \varepsilon(\varphi, C\varphi) + \varepsilon^{-1}(\chi, C\chi)$$

which is valid for any positive matrix C and all $\varepsilon > 0$. Therefore by choosing ε small one obtains an analogue of (31)

$$\frac{d}{dt}\|S_t^\alpha \varphi\|_4^4 \leq -a\,\|\nabla(S_t^\alpha \varphi)^2\|_2^2 + 4b\alpha^2 \|S_t^\alpha \varphi\|_4^4 \tag{36}$$

with $a, b > 0$. But this can be rewritten as

$$\frac{d}{dt}\left(e^{-b\alpha^2 t}\|S_t^\alpha \varphi\|_4\right)^4 \leq -a\,e^{-4b\alpha^2 t}\|\nabla(S_t^\alpha \varphi)^2\|_2^2 \quad .$$

Thus additional factors $e^{-b\alpha^2 t}$ enter the calculation and lead to less favourable estimates. Nevertheless one can proceed as previously with the Nash inequalities and the estimate (35) instead of L_2-contractivity to obtain estimates, analogous to (32),

$$\|S_t^\alpha\|_{2\to 4} \leq a_2\,e^{\alpha^2 \omega_2 t} t^{-d/8} \quad , \tag{37}$$

with a_2 and ω_2 independent of α. But these can again be used as initial data in an estimate of $\|S_t^\alpha\|_{2\to 8}$ and one finds

$$\|S_t^\alpha\|_{2\to 8} \leq a_3\,e^{\alpha^2 \omega_3 t} t^{-3d/16} \tag{38}$$

in place of (33). The only difference between the new estimates on S^α and the old estimates on S are that (37) and (38) now contain additional factors $e^{\alpha^2 \omega t}$.

Iteration of this procedure and some care with estimation of the various constants leads as before to bounds

$$\|S_t^\alpha\|_{2\to\infty} \le a\, e^{\alpha^2 \omega t} t^{-d/4} \quad . \tag{39}$$

Finally the value of ω in (39) can be decreased at the expense of increasing the value of a by noting that

$$\|S_t^\alpha\|_{2\to\infty} \le \|S_{(1-\delta)t}^\alpha\|_{2\to 2} \|S_{\delta t}^\alpha\|_{2\to\infty}$$

for all $\delta \in \langle 0,1 \rangle$. Thus combining (35) and (39) gives bounds

$$\|S_t^\alpha\|_{2\to\infty} \le a\, e^{\alpha^2(1+\varepsilon)t} (\varepsilon t)^{-d/4} \tag{40}$$

for all $\varepsilon > 0$. Since

$$\|S_t^\alpha\|_{1\to\infty} \le \|S_{t/2}^\alpha\|_{1\to 2} \|S_{t/2}^\alpha\|_{2\to\infty}$$
$$\le \|S_{t/2}^{-\alpha}\|_{2\to\infty} \|S_{t/2}^\alpha\|_{2\to\infty}$$

one then obtains bounds

$$\||K_t^\alpha\||_\infty = \|S_t^\alpha\|_{1\to\infty} \le a\, e^{\alpha^2(1+\varepsilon)t} (\varepsilon t)^{-d/2} \tag{41}$$

for all $\varepsilon > 0$ with a different choice of a.

In conclusion the argument of Paragraph 5.6 gives bounds

$$0 \le K_t(x\,;y) \le a\, (\varepsilon t)^{-d/2} e^{-|\psi(x)-\psi(y)|^2/4(1+\varepsilon)t}$$

uniformly for all ψ satisfying $c_\psi \le 1$. Hence introducing a distance function d_c by

$$d_c(x\,;y) = \sup\{\, |\psi(x)-\psi(y)| \,:\, \psi \in C_c^\infty(\mathbf{R}^d) \,,\, \sum_{i,j=1}^d c_{ij}(\partial_i\psi)(\partial_j\psi) \le 1 \,\}$$

one finds

$$0 \le K_t(x\,;y) \le a\, (\varepsilon t)^{-d/2} e^{-d_c(x\,;y)^2/4(1+\varepsilon)t} \tag{42}$$

for all $\varepsilon > 0$.

5.8 The distance d_c which occurs in the bounds (42) can be reinterpreted as the geodesic distance between x and y corresponding to the Riemmannian metric C^{-1}. In particular if the coefficients c_{ij} are constant then

$$d_c(x\,;y)^2 = ((x-y), C^{-1}(x-y))$$

where the scalar product is on $\mathbf{R}^d$.

In the case of constant coefficients on can calculate K_t explicitly and one finds

$$K_t(x\,;y) = (4\pi t) - d/2(\det C)^{-1/2} e^{-d_c(x\,;y)^2/4t} \quad .$$

Thus the Gaussian factor in (42) has almost the correct form.

The situation for variable coefficients is more complicated. Different factors enter the small t and the large t behaviour of the kernel. It appears reasonable, for small t, that the kernel behaves like a Gaussian governed by the geodesic distance. But if t is large there are reasons to believe that a different larger distance is relevant. For example, if the coefficients are periodic it is natrual that the effect of the periodicity is averaged out over a large time. This indeed appears to be the case, the Riemmanian metric C^{-1} is replaced by its average. This last statement is not quite accurate, unless $d = 1$, but it should be emphasized that the average of C^{-1} is not the same as the inverse of the average of C. The averaging is apparently at the level of the effective metric and not the coefficients.

5.9 Although we have concentrated on obtaining upper bounds on the kernel one can also derive Gaussian lower bounds,

$$K_t(x\,;y) \geq a\,t^{-d/2}e^{-bd_c(x\,;y)^2/4t}\quad .$$

But the observations of Paragraph 5.8 indicate that one cannot generally expect the value of b to be close to one. The best lower bounds currently known involve a distance related to a t-dependent Riemannian metric.