Basic subgraphs and graph spectra

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Abstract

Every symmetric matrix has an invertible principal submatrix whose rank is equal to that of the whole matrix. We explore some of the implications of this result for graph spectra. Using the same symbol for a graph and its adjacency matrix, we say that a graph G is *λ-invertible* if $G - \lambda I$ is invertible, and the *λ-rank* of G is the rank of $G - \lambda I$. A *λ-basic subgraph of G* is an induced λ-invertible subgraph H of the same λ-rank as G. Using λ-basic subgraphs, we prove that if $\lambda \notin \{0, -1\}$ then the set of graphs of a given λ-rank is finite; this can be extended to $\lambda = 0$ and $\lambda = -1$ by excluding graphs with what we call *duplicate* and *coduplicate* vertices respectively. We give an algorithm to construct the graphs of a given λ-rank. We show how 0-basic subgraphs can be used to calculate ranks of graphs, using as an example graphs obtained by adding two vertices to a complete graph. We examine some properties of *maximal reduced graphs*, graphs which occur in characterising the graphs of a given rank, and construct some infinite families of such graphs.

1. Introduction

In this paper we use special subgraphs of a graph, known as basic subgraphs, to determine when the addition of a vertex to a graph will increase the (geometric) multiplicity of a particular eigenvalue λ. We apply this technique to obtain several results for the rank of a graph. As a special case of a more general algorithm, we obtain a method to give a finite characterisation of all graphs of a given rank. We calculate the rank of all graphs obtained by adding two vertices to a complete graph. We give some properties of maximal reduced graphs, graphs which occur in the characterisation mentioned above, and construct some infinite families of these graphs.

The investigation of the rank of a graph is a natural part of the study of graph spectra. The rank, or rather its complementary parameter, the nullity of the

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adjacency matrix, has been studied because of applications to the Hückel theory of non-bonding molecular orbitals in chemistry (see [**3**, Sections 8.1 and 8.2]). Interest in the rank was also generated by an old conjecture of van Nuffelen [**9**], which was rediscovered in a weaker form by S. Fajtlowicz's computer program GRAFFITI [4]. The conjecture essentially stated that for a graph with at least one edge, $\chi(G) \leq r(G)$, where χ denotes chromatic number. It was disproved by Alon and Seymour [**2**], who found a 64-vertex counterexample with rank 29 and chromatic number 32. However, it is still an open question as to whether χ can be bounded by a polynomial in r. This question is related to a question in communication complexity, mentioned in a recent survey by Lovász [**7**]. Some other results concerning rank may be found in [**1**, **5**, **6**, **10**, **11**, **12**, **13**].

We begin with some notation. Let F be an arbitrary field. The rank of any matrix M over F will be denoted $r_F(M)$, or just $r(M)$. Let $A = [A_{uv} : u, v \in V]$ be any square matrix, whose rows and columns are indexed by the same set V. If $U, W \subseteq V$ then U and W define a submatrix $A|U \times W = [A_{uw} : u \in U, w \in W]$. The submatrix $A|W \times W$ will be abbreviated to $A|W$. If A has rank r then $A|W$ is called a *basic submatrix of* A if it is an invertible $r \times r$ principal submatrix of A. All vectors will be column vectors unless otherwise stated, and for any vector $x = [x_v : v \in V]$ we define the subvector $x|U = [x_u : u \in U]$.

In general we are interested in the rank not of A, but of $A - \lambda I$, for some λ in F. Therefore, we make the following definitions. The *λ-rank* of A is $r(A - \lambda I)$; A is *λ-invertible* if $A - \lambda I$ is invertible; and $A|W$ is a *λ-basic submatrix* of A if $(A - \lambda I)|W = (A|W) - \lambda I$ is a basic submatrix of $A - \lambda I$. Note that A is λ-invertible if and only if λ is not an eigenvalue of A.

All graphs are finite, with no loops or multiple edges. Suppose that G is a graph. Let $\nu(G)$ denote the number of vertices in G. The *(open) neighbourhood* of a vertex v, consisting of all vertices adjacent to v in G, will be denoted $G(v)$. The *closed neighbourhood* of v is $G[v] = G(v) \cup \{v\}$. If u and v are two vertices of G, the distance between them in G will be denoted $d_G(u, v)$, or just $d(u, v)$. If H is an induced subgraph of G, we write $H \subseteq_I G$. The complement of a graph G will be denoted $\overline{G}$.

Two operations on a graph which will be useful later are *reduction* and *coreduction*. If G has distinct vertices u and v with $G(u) = G(v)$ then u and v are *duplicates*, and $G - u$ is a *reduction* of G. A graph to which no reduction can be applied is *reduced*; all its vertex neighbourhoods are distinct. From every graph we can obtain a unique (up to isomorphism) reduced graph by successive reductions. If G has distinct vertices u and v with $\overline{G}(u) = \overline{G}(v)$ then u and v are *coduplicates*, and $G - u$ is a *coreduction* of G. A graph to which no coreduction can be applied is

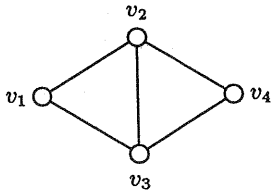


Figure 1.1

coreduced. From every graph we can obtain a unique (up to isomorphism) coreduced graph by successive coreductions. Clearly u and v are coduplicates if and only if they are adjacent and are duplicates in $G - uv$.

We identify some graphical objects with algebraic objects. Thus, we write G to mean either a graph or its adjacency matrix. If G is a graph on vertex set V, and $v \in V$, we use v to denote both the vertex v, and the vector in F^V with $v_v = 1$, and $v_u = 0$ for all u not equal to v. We identify any set $W \subseteq V$ with its characteristic vector in F^V, so that $W_v = 1$ if $v \in W$, and 0 otherwise. Notice that Gv (the product of the matrix G and the vector v) equals $G(v)$ (the characteristic vector of the neighbourhood of v in G). If $W \subseteq V$, $G|W$ means both the subgraph of G induced by W, and the adjacency matrix of this subgraph.

As an example, consider the graph G shown in Figure 1.1. Identifying this graph with its adjacency matrix, and taking the vertices in the natural order v_1, v_2, v_3, v_4, we would write

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The vertex v_1 would be identified with the vector $v_1 = [1\ 0\ 0\ 0]^T$, and the product $Gv_1 = [0\ 1\ 1\ 0]^T$ is identified with the set $G(v_1) = \{v_2, v_3\}$, the set of neighbours of v_1. For the set $W = \{v_1, v_3, v_4\}$, $G|W$ is the matrix obtained by taking the rows and columns of G corresponding to these three vertices, so that

$$G|W = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The following theorem is the foundation of our results. It seems to be a known result from matrix theory, although we cannot provide a reference.

Theorem 1.1. *Let $A = [A_{uv} : u, v \in V]$ be a symmetric $\nu \times \nu$ matrix (over any field) with rank r. Suppose that $W \subseteq V$ and $|W| = r$. Let $B = A|W$ and $N = A|W \times V$.*

(i) *The following are equivalent.*
  (a) *The rows of N form a basis for the row space of A.*
  (b) $r(N) = r$.
  (c) *B is a basic submatrix of A.*
  (d) $r(B) = r$.
  (e) *B is invertible.*

(ii) *If any of the conditions in (i) holds then $A = N^t B^{-1} N$.*

Proof. Clearly (a) $\iff$ (b) and (c) $\iff$ (d) $\iff$ (e); also, since B is a submatrix of N and N a submatrix of A, (d) $\implies$ (b). Therefore, for (i) it suffices to prove that (a) $\implies$ (d).

Without loss of generality we may assume that W consists of the first r elements of V. Then, since A is symmetric,

$$A = \begin{bmatrix} B & J \\ J^t & K \end{bmatrix}$$

where $B = A|W$ is an $r \times r$ symmetric matrix, J is $r \times (\nu - r)$, and K is $(\nu - r) \times (\nu - r)$. Now the rows of $[B \quad J]$ are a basis for the row space of A, and therefore there exists an $(\nu - r) \times r$ matrix L such that $[J^t \quad K] = L[B \quad J] = [LB \quad LJ]$. Thus, $J = (J^t)^t = (LB)^t = B^t L^t = BL^t$ since B is symmetric, and so $K = LJ = LBL^t$. But since $J = BL^t$, the r columns of B span the column space of the rank r matrix $N = [B \quad J]$, and so B has rank r, proving (i).

Now we know that

$$A = \begin{bmatrix} B & BL^t \\ LB & LBL^t \end{bmatrix} = \begin{bmatrix} I \\ L \end{bmatrix} B [I \quad L^t] = Q^t B Q, \tag{1.1}$$

where $Q = [I \quad L^t]$. Then, $BQ = B[I \quad L^t] = [B \quad BL^t] = A|W \times V = N$. Thus, $Q = B^{-1}N$, and substituting this into (1.1), remembering that B (and hence B^{-1}) is symmetric, we get (ii). ∎

We apply this to graphs as follows. An induced subgraph $H = G|W$ of a graph G will be called a *λ-basic subgraph of G* if the matrix H is a λ-basic submatrix of the matrix G. Theorem 1.1 (i) guarantees that every graph G has a λ-basic subgraph, by taking $A = G - \lambda I$. In fact, since we can construct a basis of the row space of $A = G - \lambda I$ by starting with any nonzero row, G has a λ-basic subgraph H containing any given vertex v, unless $\lambda = 0$ and v is isolated. Moreover, Stephen

Penrice [personal communication] has shown that if G is connected, then we can choose H to be connected. To do this, construct a basis $\{Aw : w \in W\}$ for the row space of $A = G - \lambda I$ by examining the vectors Aw, w any vertex of G, in ascending order of distance of w from v. For any u not equal to v, W must contain a vertex w with $d(v,w) = d(v,u) - 1$ and w adjacent to u. Thus, we can trace a path in $H = G|W$ from any vertex back to v.

If the field F contains the real numbers $\mathbf{R}$, we can say a little more about λ-basic subgraphs. Since G is symmetric, all eigenvalues of G will be real. Let $n(G,\lambda-)$, $n(G,\lambda+)$ and $n(G,\lambda)$ denote the number of eigenvalues of G (counting multiplicities) which are respectively less than, greater than and equal to λ. Let H be a λ-basic subgraph of G. Then it follows from Cauchy's interlacing inequalities [3, Theorem 0.10] that for any subgraph J of G with $H \subseteq_I J \subseteq_I G$,

$$\begin{aligned} n(J,\lambda-) &= n(G,\lambda-); \\ n(J,\lambda+) &= n(G,\lambda+); \quad \text{and} \\ n(J,\lambda) &= \nu(J) - \nu(H). \end{aligned}$$

In Section 2 we discuss some general results regarding graphs of given λ-rank. In later sections we specialise to the case where $\lambda = 0$ and $F = \mathbf{R}$, giving the ordinary rank of a graph. In Section 3 we show how basic subgraphs can be used to compute the rank of certain graphs. In Section 4 we discuss the maximal reduced graphs of a given rank.

2. Maximal graphs of fixed λ-rank

In this section we show that for $\lambda \notin \{0,-1\}$ and any fixed r there are only a finite number of graphs of λ-rank r. There are infinite numbers of graphs with rank r and (-1)-rank r; however, there are only finitely many reduced graphs of rank r and coreduced graphs of (-1)-rank r. We give an algorithm to find the (reduced or coreduced, if appropriate) graphs of λ-rank r.

We first apply Theorem 1.1 to determine when we can add vertices to a graph G without increasing its λ-rank. We need some notation. Let $\lambda \in F$. Let G be a graph on vertex set V, $H = G|W$ a λ-invertible induced subgraph, and $N = (G - \lambda I)|W \times V$. Define two bilinear forms $\circ$ and $*$ on F^V by

$$\begin{aligned} x \circ y &= x^t(G - \lambda I)y, \\ x * y &= (Nx)^t(H - \lambda I)^{-1}(Ny). \end{aligned} \tag{2.2}$$

Since $G-\lambda I$ and $H-\lambda I$ are symmetric, $x \circ y = y \circ x$ and $x * y = y * x$. For $u, v \in V$, the value of $u \circ v$ is just $(G-\lambda I)_{uv}$, namely $-\lambda$ if $u = v$, 1 if u and v are adjacent, and 0 otherwise. The relationship between $u \circ v$, $u * v$ and the λ-rank of G is given by the following corollary of Theorem 1.1 (ii).

Corollary 2.1. *Let G be a graph on vertex set V with a λ-invertible induced subgraph $H = G|W$, $|W| = r$.*

(i) *If either $u \in W$ or $v \in W$ then $u \circ v = u * v$.*

(ii) *For any X, $W \subseteq X \subseteq V$, the following are equivalent.*

(a) *$G|X$ has λ-rank r.*

(b) *H is a λ-basic subgraph of $G|X$.*

(c) *$u \circ v = u * v$ for all $u, v \in X \backslash W$.*

Now suppose that we are given a λ-invertible graph H (on vertex set W) and we wish to determine the ways of adding vertices to H to obtain a graph G (on vertex set V) with the same λ-rank. From Corollary 2.1 (ii) we can reconstruct any such graph G from H and the $\{0,1\}$-vectors Nv for $v \in V \backslash W$. So, assume we are given H and a collection of $\{0,1\}$-vectors $\mathcal{N} = \{Nv : v \in V \backslash W\}$. Using (2.2), we can construct a symmetric matrix $A = [A_{uv} = u * v : u, v \in V]$. If H and $\mathcal{N}$ determine a graph G, then A should be $G - \lambda I$. However, the elements of $A|V \backslash W$ may be 'ungraphical': there may be diagonal elements not equal to $-\lambda$ (giving diagonal elements not equal to 0 in G), or off-diagonal elements not equal to 0 or 1. Therefore, we have the following *compatibility conditions*, which are necessary and sufficient for H and $\mathcal{N}$ to determine a graph G:

(i) v is compatible with itself, $v * v = -\lambda$, for all $v \in V \backslash W$; and

(ii) u is compatible with v, $u * v \in \{0,1\}$, for all $u, v \in V \backslash W$.

In checking condition (ii), we determine as a byproduct all the adjacencies in G not immediately obvious from H and the Nv's. By Corollary 2.1 (i), we automatically have condition (i) for $v \in W$ and condition (ii) if either $u \in W$ or $v \in W$.

Condition (i) can be viewed as telling us whether the vertex v, adjacent to the vertices of W specified by Nv, increases the λ-rank when added individually to H. In the case where $v * v \neq -\lambda$, the λ-rank does increase, and if $F = \mathbf{R}$, the relationship of $v * v$ to $-\lambda$ gives us some information. Let H' denote H with v added. Using the Courant-Weyl inequalities (see [3, page 51]), it can be shown that if $v * v < -\lambda$ then

$$n(H', \lambda-) = n(H, \lambda-) \qquad \text{and} \qquad n(H', \lambda+) = n(H, \lambda+) + 1,$$

while if $v * v > -\lambda$ then

$$n(H', \lambda-) = n(H, \lambda-) + 1 \qquad \text{and} \qquad n(H', \lambda+) = n(H, \lambda+).$$

Suppose that H and $\mathcal{N}$ satisfy the compatibility conditions and determine a graph G. If $\lambda \notin \{0,-1\}$ we cannot have $Nu = Nv$ for two distinct elements u and v of V, for then $-\lambda = u*u = u*v \in \{0,1\}$. If $\lambda = 0$ and $Nu = Nv$ we have $u*v = 0$ and $u*w = v*w$ for all $w \notin \{u,v\}$, so that u is a duplicate of v. And if $\lambda = -1$ and $Nu = Nv$ then $u*v = 1$ and $u*w = v*w$ for all $w \notin \{u,v\}$, so that u is a coduplicate of v. Thus, we obtain the following.

Lemma 2.2. *Suppose $\lambda \in F$ and G is a graph of λ-rank r.*

(i) *If $\lambda \notin \{0,-1\}$ then $\nu(G) \le 2^r + r - 1$.*

(ii) *If $\lambda = 0$ and G is reduced, $\nu(G) \le 2^r$.*

(iii) *If $\lambda = -1$ and G is coreduced, $\nu(G) \le 2^r - 1$.*

Proof. Take a λ-basic subgraph $H = G|W$ of G, and let $N = (G - \lambda I)|W \times V$. The vectors Nv have length r, are all distinct, are nonzero if $\lambda \neq 0$, and have all entries in $\{0,1\}$, except one entry equal to $-\lambda$ if $v \in W$. The result follows. ∎

Corollary 2.3. *Let r be given.*

(i) *If $\lambda \notin \{0,-1\}$ there are only finitely many graphs of λ-rank r.*

(ii) *There are only finitely many reduced graphs of rank r.*

(iii) *There are only finitely many coreduced graphs of (-1)-rank r.*

Lemma 2.2 (ii) in the case of connected graphs with $F = \mathbf{R}$ was previously known by Torgašev [15], who in fact proved that for fixed q, there are only finitely many reduced graphs having exactly q negative eigenvalues. C. D. Godsil [personal communication] has pointed out that the exponential bound of Lemma 2.2 (ii) cannot be improved to a polynomial bound. For even r, take a set S of cardinality $r/2$, and construct a bipartite graph whose vertices are the elements of S and all subsets of S, and whose edges join $x \in S$ to every $X \subseteq S$ which contains x. This is a reduced graph of rank r (over any field) and order $r/2 + 2^{r/2}$.

The compatibility conditions give us not only the finiteness of the set of (reduced or coreduced, if appropriate) graphs of λ-rank r, but also an algorithm to construct this set, namely Algorithm 2.4.

Algorithm 2.4. *To construct the set $\mathcal{G}$ of all graphs of λ-rank r (reduced if $\lambda = 0$, coreduced if $\lambda = -1$):*

$\mathcal{G} \leftarrow \emptyset$;
for each graph H on vertex set W with $\nu(H) = r$
  if $r(H - \lambda I) = r$ then
    $C \leftarrow$ empty graph;
    for each $\{0,1\}$-vector $Nv \in F^W$
      store $(H - \lambda I)^{-1}(Nv)$
      if Nv is not a row of $H - \lambda I$ and
        $v * v = (Nv)^t(H - \lambda I)^{-1}(Nv) = -\lambda$ then
        add vertex v to C
        for each vertex u of $C \backslash v$
          if $u * v = (Nu)^t(H - \lambda I)^{-1}(Nv) \in \{0,1\}$ then
            add edge uv, coloured with $u * v$, to C
    for each complete subgraph M in C
      construct graph G with H as λ-basic subgraph, using vectors
        Nv for $v \in M$, and colouring of edges of C
      $G' \leftarrow$ canonical form of G
      if $G' \notin \mathcal{G}$ then
        $\mathcal{G} \leftarrow \mathcal{G} \cup \{G'\}$

This algorithm is similar in spirit to the algorithms used by Torgašev [**16**, **17**] to generate the reduced graphs with two and three negative eigenvalues. It is not particularly efficient, because for some graphs many isomorphic copies may be generated. The algorithm does contain minor points aimed at increasing efficiency: for each v we store $(H - \lambda I)^{-1}(Nv)$ for use in calculating $u * v$, and we store each $u * v$, as a colour on the edge uv of the 'compatibility graph' C, for use in constructing G. Algorithm 2.4 requires several things. First, a list of all graphs on r vertices; second, an algorithm to find all complete subgraphs in a graph; and third, an algorithm to put a graph into a canonical form so that it can be checked for isomorphism with other graphs.

When $F = \mathbf{R}$, the remarks at the end of Section 1 show that Algorithm 2.4 can be used to generate not only the graphs of a given λ-rank, but also the graphs G with $n(G, \lambda-) = q$ and $n(G, \lambda+) = p$ for a fixed pair (q, p).

Let $\mathcal{G}(\lambda, r)$ denote the finite set of (reduced or coreduced, if appropriate) graphs of λ-rank r. Instead of listing all elements of $\mathcal{G}(\lambda, r)$, we can characterise it in terms of its $\subseteq_I$-minimal and $\subseteq_I$-maximal elements. The $\subseteq_I$-minimal elements are precisely the λ-invertible graphs of order r, which can be found by examining all graphs of

order r. The $\subseteq_I$-maximal elements can be found by modifying Algorithm 2.4 so that it considers only maximal complete subgraphs of the compatibility graph C, instead of all complete subgraphs.

In the case where $F = \mathbf{R}$ and $\lambda = 0$, the $\subseteq_I$-maximal graphs of a given rank all contain exactly one isolated vertex. Thus, only the nonisolated vertices of such a graph are interesting. We therefore define a *maximal reduced graph* to be a graph with no isolated vertices, which with the addition of an isolated vertex becomes a $\subseteq_I$-maximal reduced graph of real rank r, for some r. A modified version of Algorithm 2.4 for finding such graphs has been implemented both by the author, who found all maximal reduced graphs of rank 6 or less, and, independently, by G. F. Royle, who found all such graphs of rank 7 or less. The author employed the maximal complete subgraph algorithm from [**14**, Section 8.4], and a canonical form algorithm adapted from the one given in [**8**]. The results may be summarised as follows:

rank	2	3	4	5	6	7
number of graphs	1	1	3	8	27	183

A list of the graphs with rank at most 5 appears in Section 4.

Maximal reduced graphs are useful for examining the relationship between rank and graph parameters p which satisfy (i) $p(H) \leq p(G)$ whenever $H \subseteq_I G$, and (ii) $p(H) = p(G)$ if H is a reduction of G. For such a parameter p, the maximum value of p on the infinite set of all graphs of a fixed rank r is equal to the maximum value of p on the finite set of maximal reduced graphs of rank r. The most obvious example of such a graph parameter is chromatic number.

3. Adding vertices to complete graphs

In this and later sections we take our field to be the real numbers $\mathbf{R}$. Here we show that the concept of a basic subgraph can sometimes be used to determine the rank of a family of graphs. In particular, we examine graphs obtained by adding two vertices to a complete graph. The results of this section will be used in Section 4 to prove the existence of two infinite families of the maximal reduced graphs defined in Section 2.

A *nullvector* of a graph G with vertex set V is a vector $x \in \mathbf{R}^V$ such that $Gx = 0$. Thus, x is a nullvector of G if and only if

$$0 = v^t Gx = x^t Gv = v \cdot Gx = x \cdot Gv \qquad \forall\, v \in V.$$

The number of linearly independent nullvectors is just $n(G, 0) = \nu(G) - r(G)$.

It is well known (see [3, Section 2.6]) that $r(K_n) = n$. If we add a single vertex u of degree α, $0 \leq \alpha \leq n$, to K_n, the resulting graph will be denoted $K_n(\alpha)$. We show in (3.3) below that $K_n(\alpha)$ is invertible, and hence has rank $n+1$, except when $\alpha = 0$ (u is isolated) or $\alpha = n-1$ (u is a duplicate). Thus, we see that K_n is a maximal reduced graph of rank n, for all $n \geq 2$.

Now we consider the effect of adding two vertices, u and v, to K_n. The resulting graph G is determined up to isomorphism by five numbers. Let U denote the vertex set of K_n. The five numbers are:

$n = |U|$;
$\alpha = |G(u) \cap U|$;
$\beta = |G(v) \cap U|$;
$\gamma = |G(u) \cap G(v) \cap U|$; and
$\epsilon = 1$ if u and v are adjacent, 0 otherwise.

We shall therefore denote G by $K_n(\alpha, \beta, \gamma; \epsilon)$. Without loss of generality we may assume that $\alpha \geq \beta$. Thus, $n \geq \alpha \geq \beta \geq \gamma \geq 0$, and also $\gamma \geq \alpha + \beta - n$.

Before we proceed to our main classification result, there are some special cases to be dealt with. In Lemma 3.1, we assume that $n \geq 2$, and that the situation of the previous paragraph holds.

Lemma 3.1. *If $\alpha \geq n-1$ or $\beta = 0$ then $r(G) = n+2$, except when at least one of u or v is an isolated or duplicate vertex.*

Proof. If $\alpha = n$ then $G \cong K_{n+1}(\beta + \epsilon)$ and the result follows.

If $\beta = 0$, then when $\epsilon = 0$ the vertex v is isolated. If $\epsilon = 1$, then any nullvector x of G has $0 = x \cdot Gv = x \cdot u = x_u$, and so $x|U$ is a nullvector of K_n. But then $x = 0$ and $r(G) = n+2$.

If $\alpha = n-1$, then in $G|U \cup \{u\}$ the vertex u has a duplicate u'. If $\gamma + \epsilon = \beta$, then u and u' are duplicates in G. If $\gamma + \epsilon \neq \beta$, then v is adjacent to exactly one of u and u', say to u. Let x be a nullvector of G. Then $0 = 0 - 0 = x \cdot Gu - x \cdot Gu' = x \cdot (Gu - Gu') = x \cdot v = x_v$. Therefore, $x|U \cup \{u\}$ is a nullvector of $G|U \cup \{u\} = K_n(n-1)$. Now $n(K_n(n-1), 0) = 1$, so any nullvector of this $K_n(n-1)$ must be a multiple of the nullvector $u - u'|U \cup \{u\}$. Thus, $x_u = -x_{u'}$ and $x_w = 0$ for all $w \in U \backslash \{u'\}$. But then $0 = x \cdot Gv = x_u$. Hence $x = 0$ and $r(G) = n+2$. ∎

We may now assume that $1 \leq \beta \leq \alpha \leq n-2$. Let $V = U \cup \{u, v\}$ and $W = U \cup \{u\}$. Suppose that we construct $G = K_n(\alpha, \beta, \gamma; \epsilon)$ by first adding the vertex u, then adding the vertex v. When we add u, we get $H = G|W = K_n(\alpha)$ of rank $n+1$. Adding v, the rank can either remain at $n+1$, or increase to $n+2$. Let $Nv \in \mathbf{R}^W$ denote the characteristic vector of the set of neighbours of v in W, which in this case is the set of all neighbours of v. By Corollary 2.1 (ii), the rank

will remain $n+1$ if and only if $v \circ v = v * v$, or $0 = (Nv)^t H^{-1}(Nv)$. To determine when this is true, we first must find $H^{-1} = K_n(\alpha)^{-1}$.

If we order the vertices of $W = U \cup \{u\}$ with the α elements of $U \cap G(u)$ first, then the $n-\alpha$ elements of $U \backslash G(u)$, and finally u, we obtain

$$H = \left[\begin{array}{ccc|ccc|c} & & & & & & 1 \\ & K_\alpha & & & 1 & & \vdots \\ & & & & & & 1 \\ \hline & & & & & & 0 \\ & 1 & & & K_{n-\alpha} & & \vdots \\ & & & & & & 0 \\ \hline 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \end{array}\right],$$

It is not difficult to verify that this is invertible, with inverse

$$H^{-1} = \left[\begin{array}{ccc|ccc|c} \frac{1}{\alpha}-1 & & \frac{1}{\alpha} & & & & \frac{1}{\alpha} \\ & \ddots & & & 0 & & \vdots \\ \frac{1}{\alpha} & & \frac{1}{\alpha}-1 & & & & \frac{1}{\alpha} \\ \hline & & & \frac{1}{\omega}-1 & & \frac{1}{\omega} & -\frac{1}{\omega} \\ & 0 & & & \ddots & & \vdots \\ & & & \frac{1}{\omega} & & \frac{1}{\omega}-1 & -\frac{1}{\omega} \\ \hline \frac{1}{\alpha} & \cdots & \frac{1}{\alpha} & -\frac{1}{\omega} & \cdots & -\frac{1}{\omega} & \frac{1}{\alpha}+\frac{1}{\omega} \end{array}\right], \tag{3.3}$$

where $\omega = n - \alpha - 1$.

Now $r(G) = n+1$ if and only if 0 is equal to $(Nv)^t H^{-1}(Nv)$, which is the sum of all entries in the principal submatrix $H^{-1}|Nv$ of H^{-1}. The vertex v has β neighbours in U, of which γ are in $U \cap G(u)$ and $\beta - \gamma$ in $U \backslash G(u)$; also, if $\epsilon = 1$ then v is adjacent to u. Thus, the necessary and sufficient conditions for $r(G)$ to be $n+1$ are: if $\epsilon = 0$,

$$\begin{aligned} 0 = v * v &= \gamma^2(\frac{1}{\alpha}) - \gamma + (\beta-\gamma)^2(\frac{1}{\omega}) - (\beta-\gamma) \\ &= \frac{\gamma(\gamma-\alpha)}{\alpha} + \frac{(\beta-\gamma)(\beta-\gamma-\omega)}{\omega}, \end{aligned} \tag{3.4}$$

and if $\epsilon = 1$,

$$\begin{aligned} 0 &= v * v \\ &= \gamma^2(\frac{1}{\alpha}) - \gamma + 2\gamma(\frac{1}{\alpha}) + (\beta-\gamma)^2(\frac{1}{\omega}) - (\beta-\gamma) - 2(\beta-\gamma)(\frac{1}{\omega}) + (\frac{1}{\alpha}+\frac{1}{\omega}) \\ &= \frac{(\gamma+1)(\gamma+1-\alpha)}{\alpha} + \frac{(\beta-\gamma-1)(\beta-\gamma-1-\omega)}{\omega}, \end{aligned} \tag{3.5}$$

where $0 \le \gamma \le \alpha$ and $0 \le \beta - \gamma \le \omega + 1$ in both cases. Equations (3.4) and (3.5) are very similar in form, and it is not too difficult to find all integer values of α, β, γ and $\omega = n - \alpha - 1$, subject to the restrictions just mentioned, which satisfy them. We omit the details. We are led to the following classification theorem.

Theorem 3.2. *Suppose that $n \ge 2$ and we add two vertices u and v to K_n as described above to obtain a graph $G = K_n(\alpha, \beta, \gamma; \epsilon)$ (with $\alpha \ge \beta$). Suppose that neither u nor v has a duplicate in G and that neither u nor v is an isolated vertex. Then $r(G) = n + 2$ unless G is one of the following graphs with $r(G) = n + 1$:*

(i) $K_n(\alpha, n-\alpha-1, 0; 0)$ *for* $1 \le \alpha \le n-2$;

(ii) $K_n(\alpha, \alpha, \alpha-1; 1)$ *for* $1 \le \alpha \le n-2$; *or*

(iii) *sixteen graphs with 11, 12 or 13 vertices –*

$K_9(6,6,3;0)$, $K_9(6,2,2;1)$, $K_{10}(6,6,2;0)$,$K_{10}(8,6,4;0)$,
$K_{10}(6,1,1;1)$,$K_{10}(6,3,3;1)$,$K_{10}(8,3,3;1)$,$K_{11}(8,5,2;0)$,
$K_{11}(9,5,3;0)$,$K_{11}(9,8,6;0)$,$K_{11}(5,1,1;1)$,$K_{11}(5,2,2;1)$,
$K_{11}(8,1,1;1)$,$K_{11}(8,5,5;1)$,$K_{11}(9,2,2;1)$,$K_{11}(9,5,5;1)$.

This theorem gives us a complete description of the rank of graphs obtained by adding two vertices to a complete graph.

The following result concerning the sixteen exceptional graphs of Theorem 3.2 will be useful in Section 4.

Observation 3.3. *For the sixteen exceptional graphs $G = K_n(\alpha, \beta, \gamma; \epsilon)$ of Theorem 3.2 (iii),*

(i) *if $\epsilon = 0$ then $U \subseteq G(u) \cup G(v)$, and*

(ii) *if $\epsilon = 1$ then $U \not\subseteq G(u) \cup G(v)$ and $G(v) \cap U \subseteq G(u) \cap U$.*

4. Maximal reduced graphs

In this section we prove some simple properties of the maximal reduced graphs defined in Section 2, describe three infinite families of maximal reduced graphs, and list the maximal reduced graphs of rank at most 5.

Some elementary properties of maximal reduced graphs can be deduced from the following lemma.

Lemma 4.1. *Suppose that G is a maximal reduced graph, and that u and v are vertices with $G[u] \cap G[v] = \emptyset$. Then there exists a vertex w with $G(w) = G(u) \cup G(v)$.*

Proof. If no vertex w with $G(w) = G(u) \cup G(v)$ exists in G, then we can add such a w without creating a duplicate, creating an isolated vertex (since u and v are not isolated), or increasing the rank (since the adjacency matrix of the new graph is obtained from the adjacency matrix of G by adjoining first the sum of two rows and then the sum of two columns). Thus, G is not maximal, a contradiction. ∎

It follows from this that every maximal reduced graph must be connected (otherwise there is no w for u and v in different components) and, except for $K_2 = K_{1,1}$, not bipartite (since there would be no w for nonadjacent u and v in different parts of the bipartition). We can also bound the diameter of maximal reduced graphs.

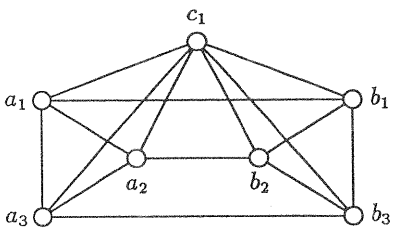
Corollary 4.2. *Every maximal reduced graph has diameter at most 3.*

Proof. Let u and v be any two vertices in a maximal reduced graph G. We must show that $d(u,v)$, the distance from u to v in G, is at most 3. Since u and v are not isolated we can choose $x \in G(u)$ and $y \in G(v)$. If $G[x] \cap G[v] \neq \emptyset$, then $d(x,v) \leq 2$ and hence $d(u,v) \leq 3$. Otherwise, by Lemma 4.1 there exists a vertex w with $G(w) = G(x) \cup G(v)$; then $u \in G(x) \subseteq G(w)$ and $y \in G(v) \subseteq G(w)$, so that G contains the walk $uwyv$ of length 3, and $d(u,v) \leq 3$. ∎

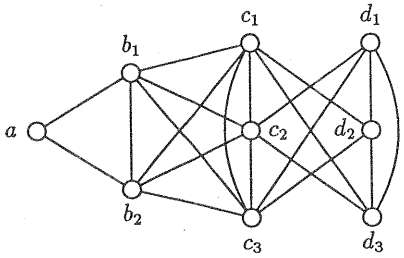
Corollary 4.2 is best possible, as will be demonstrated by the family $M(m,n)$ constructed below: the graphs in this family of maximal reduced graphs all have diameter 3.

We now describe three infinite families of maximal reduced graphs. We have already mentioned, in Section 3, that complete graphs form a family of maximal reduced graphs. Our first new family consists of graphs which, like the complete graphs, are both maximal reduced and invertible. Let $F = F(n)$ denote a graph with

$$\begin{aligned} V(F) &= \{a, b_1, \ldots, b_n, c_1, \ldots, c_n\}, \\ E(F) &= \{ab_i, ac_i, b_ic_i : 1 \leq i \leq n\}; \end{aligned}$$



(a) $L(3,1)$



(b) $M(2,3)$

Figure 4.1

$F(n)$ is sometimes called a *friendship graph*. The following lemma describes the effect on the rank of adding a vertex to $F(n)$.

Lemma 4.3. *Let G be a graph obtained from $F = F(n)$ by adding a non-isolated non-duplicate vertex v. Let ϵ be 1 if v is adjacent to a and 0 otherwise. Let α be the number of indices i such that v is adjacent to exactly one of b_i and c_i, and let β be the number of indices i for which v is adjacent to both b_i and c_i. Then $r(G) = r(F)$ if and only if*

$$4n\beta = (\alpha + 2\beta - \epsilon)^2. \tag{4.6}$$

The proof of this lemma involves finding $F(n)^{-1}$ and calculating $v * v$, as we did for $K_n(\alpha)$ in Section 3. Equation (4.6) can be satisfied for $\beta < n$ if n has a factor k^2, $k \geq 2$, and not otherwise; therefore, $F(n)$ is maximal reduced if and only if n is square-free, $n \geq 1$.

It is not so easy to prove that the graphs in our next two families are in fact maximal reduced graphs. The second family contains graphs $L(m,n) = (K_m \times K_2) + K_n$, with $m \geq 3$ and $n \geq 0$, where $\times$ denotes Cartesian product, and $+$ denotes join. We shall think of a graph $L = L(m,n)$ as having vertex set and edge set

$$V(L) = A \cup B \cup C = \{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_m\} \cup \{c_1, \ldots, c_n\},$$
$$E(L) = K(A) \cup K(B) \cup K(C) \cup P(A,B) \cup K(A,C) \cup K(B,C),$$

where $K(V)$ denotes the edge set of a complete graph on V, $K(U,V)$ denotes the set of edges joining every vertex in U to every vertex in V, and for two sets $U = \{u_1, \ldots, u_k\}$ and $V = \{v_1, \ldots, v_k\}$ the set $P(U,V) = \{u_i v_i : 1 \leq i \leq k\}$ forms a perfect matching between U and V. Figure 4.1 (a) shows $L(3,1)$.

The third family consists of graphs $M(m,n)$, with $m \geq 1$ and $n \geq 2$, where $M = M(m,n)$ has vertex set and edge set

$$V(M) = \{a\} \cup B \cup C \cup D = \{a\} \cup \{b_1, \ldots, b_m\} \cup \{c_1, \ldots, c_n\} \cup \{d_1, \ldots, d_n\},$$
$$E(M) = K(B) \cup K(C) \cup K(D) \cup K(\{a\}, B) \cup K(B, C) \cup \overline{P}(C, D),$$

where for two sets U and V with $|U| = |V|$, $\overline{P}(U,V) = K(U,V) \backslash P(U,V)$. Figure 4.1 (b) depicts $M(2,3)$.

All but a finite number of graphs $L(m,n)$ and $M(m,n)$ are maximal reduced, as the following two theorems state.

Theorem 4.4. *If $m \geq 3$ and $n \geq 0$, then $L(m,n)$ is a maximal reduced graph unless it is one of*

$$L(4,5), \quad L(3,7), \quad L(5,5), \quad L(3,8), \quad L(4,7), \quad L(7,4).$$

Theorem 4.5. *If $m \geq 1$ and $n \geq 2$, then $M(m,n)$ is a maximal reduced graph unless it is one of*

$$M(6,3), \quad M(6,4), \quad M(8,2), \quad M(5,6), \quad M(8,3), \quad M(9,2).$$

Note that the twelve exceptional graphs in the above theorems have $m+n = 9$, 10 or 11: they come from the sixteen exceptional cases of Theorem 3.2. To prove these two theorems, we use Corollary 2.1 to examine the effect on the rank of adding a vertex to L or M. We shall give the complete proof for Theorem 4.4; the proof of Theorem 4.5 is similar.

It is clear that the graphs $L(m,n)$ are reduced. The first step towards proving that these graphs are maximal reduced graphs is to find the rank and a basic subgraph.

Lemma 4.6. *The graph $L = L(m,n)$, $m \geq 3, n \geq 0$, has rank $m+n+1$ and a basic subgraph $H = L|C \cup B \cup \{a_1\}$.*

Proof. For each i, $1 \leq i \leq m$, we have

$$La_i + Lb_i = (A - a_i + b_i + C) + (a_i + B - b_i + C) = A + B + 2C.$$

Therefore $La_i + Lb_i = La_1 + Lb_1$ and we have $m-1$ linearly independent nullvectors $a_i + b_i - a_1 - b_1$, $2 \leq i \leq m$, for L, showing that $r(L) \leq \nu(L) - (m-1) = m+n+1$. Now $H = L|C \cup B \cup \{a_1\}$ is isomorphic to $K_{n+m}(n+1)$, and hence has rank $m+n+1 = \nu(H)$ from Section 3, since $m \geq 3$. This proves the lemma. ∎

Now we use the basic subgraph H to determine the effect on the rank of adding a vertex to L. The reader should notice the way in which conditions for the rank to remain the same are derived from equations such as (4.7) below.

Lemma 4.7. *Suppose that $m \geq 3$, $n \geq 0$, and $L = L(m,n)$. If we add a non-isolated non-duplicating vertex v to L to form G, then $r(G) = r(L)$ if and only if the following both hold:*

(i) *v is adjacent to all vertices in A and no vertices in B, or vice versa; and*

(ii) *letting $\xi = |G(v) \cap C|$, (m,n,ξ) is one of –*

$(4,5,2)$, $(3,7,3)$, $(5,5,1)$, $(5,5,3)$, $(3,8,2)$, $(3,8,5)$,
$(4,7,1)$, $(4,7,5)$, $(7,4,1)$, $(7,4,2)$.

Proof. Let $V = C \cup B \cup A \cup \{v\}$, $W = C \cup B \cup \{a_1\}$, $H = G|W$, $N = G|W \times V$ and $r = r(L) = m + n + 1$. From Lemma 4.6, H is a basic submatrix of L. From Corollary 2.1 (ii), $r(G) = r \iff s \circ t = s * t$ for all $s,t \in V \backslash W$. But since $r(L) = r$, by Corollary 2.1 (ii) $s \circ t = s * t$ whenever neither s nor t is equal to v. Therefore $r(G) = r \iff v \circ t = v * t$ for all $t \in V \backslash W \iff v \circ v = v * v$ and $v \circ a_i = v * a_i$, $2 \leq i \leq m$.

Let $H' = G|W \cup \{v\}$. Notice that v is neither isolated nor a duplicate in H' because $Nv \neq 0$ and $Nv \neq Nw$ for $w \in W$. By Corollary 2.1 (ii), $v \circ v = v * v \iff r(H') = r$. Let $k = m + n = r - 1$; then $H' \backslash \{a_1, v\} \cong K_k$ and so H' is isomorphic to some $K_k(\alpha,\beta,\gamma;\epsilon)$ with no isolated or duplicate vertices. Therefore it has rank $r = k + 1$ if and only if it is one of the graphs of Theorem 3.2. Also, for any i, $2 \leq i \leq m$, we have

$$Na_i = Na_1 + Nb_1 - Nb_i \tag{4.7}$$

and so
$$\begin{aligned} v * a_i &= v * a_1 + v * b_1 - v * b_i \\ &= v \circ a_1 + v \circ b_1 - v \circ b_i, \end{aligned}$$

where the last equality follows because, from Corollary 2.1 (i), $v * t = v \circ t$ for all $t \in W$. Therefore $v \circ a_i = v * a_i$ if and only if

$$v \circ a_i = v \circ a_1 + v \circ b_1 - v \circ b_i \tag{4.8}.$$

Thus, $r(G) = r(L)$ if and only if

(iii) H' is a graph from Theorem 3.2, and (4.8) holds for $2 \leq i \leq m$.

We must now show that (iii) is equivalent to [(i) and (ii)].

Suppose that (iii) holds. Assume first that H' is a graph $K_k(\alpha, k - \alpha - 1, 0; 0)$ from Theorem 3.2 (i). Then $v \circ a_1 = \epsilon = 0$. Since $\gamma = 0$, v and a_1 have no common neighbours in $W \backslash \{a_1\}$, and therefore $G(v) \cap C = \emptyset$ and $v \circ b_1 = 0$. Therefore, by (4.8), we have $v \circ a_i = v \circ b_i = 0$, $2 \leq i \leq m$. But then v is isolated, a contradiction.

Next assume that H' is a graph $K_k(\alpha, \alpha, \alpha - 1; 1)$ from Theorem 3.2 (ii). Let $X = G(a_1) \cap (B \cup C) = C \cup \{b_1\}$ and $Y = G(v) \cap (B \cup C)$. Since $\alpha = \beta$ and

$\gamma = \alpha - 1$, we have $Y = X\backslash\{x\} \cup \{b_j\}$ for some $x \in X$ and $j \neq 1$. Since $m \geq 3$, there is at least one i not equal to 1 or j, and for each such i, $v \circ b_i = 0$; hence, from (4.8), $v \circ a_i = 1$ and $v \circ b_1 = 0$, implying that $x = b_1$. Since $v \circ b_j = 1$, (4.8) yields that $v \circ a_j = 0$. So $G(v) = C \cup \{b_j\} \cup A\backslash\{a_j\} = G(a_j)$ and v is a duplicate of a_j, a contradiction.

Now assume that H' is a graph $K_k(\alpha, \beta, \gamma; \epsilon)$ from Theorem 3.2 (iii). We shall analyse the cases $\epsilon = 0$ and $\epsilon = 1$ separately.

Suppose that $0 = \epsilon = v \circ a_1$. Then from Observation 3.3 (i), $B \cup C \subseteq G(a_1) \cup G(v)$, and hence $v \circ b_i = 1$ for all $i \neq 1$. Thus we must have $v \circ b_1 = 1$ and $v \circ a_i = 0$ for $i \neq 1$, from (4.8). Hence v is adjacent to all vertices of B and no vertices of A, proving (i). Now we must have $k = m + n$ and either $\alpha = n + 1$ and $\beta = \xi + m$ or $\beta = n + 1$ and $\alpha = \xi + m$; thus from each $K_k(\alpha, \beta, \gamma; 0)$ of Theorem 3.2 (iii) we obtain two triples: $(m, n, \xi) = (k + 1 - \alpha, \alpha - 1, \alpha + \beta - k - 1)$ and $(k+1-\beta, \beta-1, \alpha+\beta-k-1)$. We obtain twelve triples, two of which are repeated, reducing to the ten triples listed in (ii).

Now suppose that $1 = \epsilon = v \circ a_1$. Then from Observation 3.3 (ii), either $G(a_1) \cap (B \cup C) \subseteq G(v)$ or $G(v) \cap (B \cup C) \subseteq G(a_1)$. Suppose that $G(a_1) \cap (B \cup C) \subseteq G(v)$; then $v \circ b_1 = 1$ and so from (4.8) $v \circ a_i = v \circ b_i = 1$ for all $i \neq 1$. But then $B \cup C \subseteq G(a_1) \cup G(v)$, contradicting Observation 3.3 (ii). Therefore we have $G(v) \cap (B \cup C) \subseteq G(a_1)$; thus, $v \circ b_i = 0$ for $i \neq 1$ and from (4.8) we get $v \circ b_1 = 0$ and $v \circ a_i = 1$ for $i \neq 1$. So, v is adjacent to all of A and none of B, proving (i). Now we must have $k = m + n$ and $\alpha = n + 1 > \beta = \xi$; thus from each of the ten graphs $K_k(\alpha, \beta, \gamma; 1)$ of Theorem 3.2 (iii) we obtain a triple $(m, n, \xi) = (k+1-\alpha, \alpha-1, \beta)$, again giving the ten triples of (ii).

So we have shown that (iii) implies [(i) and (ii)]. It is not difficult to reverse the reasoning above and prove the converse. ∎

Now, Theorem 4.4 follows as an immediate consequence of Lemma 4.7. The information provided by Lemma 4.7 can also be used to construct all maximal reduced graphs which contain the six exceptional graphs of Theorem 4.4. Theorem 4.5 can be deduced in a similar way by analysing the ways to add a vertex to $M(m, n)$ without increasing the rank, using the basic subgraph $H = M|B \cup C \cup \{a\}$.

Below is a list of all the maximal reduced graphs of rank at most 5, generated as described in Section 2.

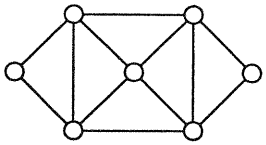


Figure 4.2: P

rank 2	K_2
rank 3	K_3
rank 4	K_4, $L(3,0)$, $M(1,2)$
rank 5	K_5, $L(4,0)$, $L(3,1)$, $M(1,3)$, $M(2,2)$, $F(2)$, C_5, P (see Figure 4.2)

As mentioned earlier, for rank 6 there are 27 maximal reduced graphs, and for rank 7 there are 183 such graphs.

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