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BASIC THEORY OF MAGNETS

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Abstract
The description of two dimensional magnetic fields of magnets used in accelerators is discussed in terms of a harmonic expansion. The expansion is derived for cylindrical components and extended to Cartesian components. The Cartesian components are also described in terms of a complex field. The rules for transformation of the expansion coefficients under various types of coordinate transformation are given. The relationship between a given current distribution and the resulting field harmonics is explored in terms of the vector and complex potentials. Explicit results are presented for some simple geometries. Finally, the harmonics allowed under various symmetries in the magnet current are discussed.

1. MULTIPOLE EXPANSION OF A TWO-DIMENSIONAL FIELD

For most practical purposes related to magnetic measurements in an accelerator magnet, one is interested in the magnetic field in the aperture of the magnet, which is in vacuum and carries no current. Also, most accelerator magnets tend to be long compared to their aperture. Thus, a two dimensional description is valid for most of the magnet, except at the ends. We shall at first confine ourselves to a description of a purely two dimensional field in a current free region. The relationship between the field and the currents will be treated later in Sec. 5 and onwards.

In free space, with no true currents, the curl of the magnetic field, **H**, is zero. Also, the magnetic induction, **B**, is given by $m_0\mathbf{H}$, where $m_0 = 4p\times10^{-7}$ Henry/m is the permeability of free space. Consequently, the magnetic induction, **B**, can be expressed as the gradient of a magnetic scalar potential, Φ_m:

$$\nabla\times\mathbf{H} = 0 \Rightarrow \nabla\times\mathbf{B} = 0 \Rightarrow \mathbf{B} = -\nabla\Phi_m \qquad (1)$$

The magnetic induction, $\mathbf{B} = m_0\mathbf{H}$ also has zero divergence. Combining this fact with Eq. (1), we get Laplace's equation for the scalar potential:

$$\nabla^2\Phi_m = 0 \qquad (2)$$

We choose a cylindrical coordinate system with the Z-axis along the length of the magnet and the origin located at the center of the magnet aperture. For a two dimensional field having no axial component, the scalar potential has no z-dependence. Laplace's equation can be solved by separation of variables and imposing the boundary conditions that Φ_m be periodic in the angular coordinate, q, and be finite at $r = 0$. The resulting general solution is a series expansion of the scalar potential. The radial and the azimuthal components of **B** are then obtained by taking the gradient of the scalar potential. The components of **B** in cylindrical coordinates can be written in the form :

$$B_r(r,q) = -\left(\frac{\P\,\Phi_m}{\P\, r}\right) = \sum_{n=1}^{\infty} C(n)\left(\frac{r}{R_{ref}}\right)^{n-1} \sin[n(q - a_n)] \qquad (3)$$

$$B_q(r,q) = -\left(\frac{1}{r}\right)\left(\frac{\P\,\Phi_m}{\P\, q}\right) = \sum_{n=1}^{\infty} C(n)\left(\frac{r}{R_{ref}}\right)^{n-1} \cos[n(q - a_n)] \qquad (4)$$

where $C(n)$ and a_n are constants and R_{ref} is an arbitrary *reference radius*, typically chosen to be 50-70% of the magnet aperture. For a given r, the n-th term in $B_r(r,q)$ has n maxima and n minima as a function of the azimuthal angle q. These angular positions may be regarded as the locations of magnetic poles. Thus, the n-th terms in Eqs. (3) and (4) correspond to a $2n$-pole field. Accordingly, $C(n)$ is said to be the *amplitude* of the $2n$-pole component of the total field. The locations of the south and the north poles in the $2n$-pole field are:

$$q = \frac{p}{2n} + a_n;\ \frac{5p}{2n} + a_n;\ \frac{9p}{2n} + a_n;\ \ldots. \text{SOUTH POLES} \qquad (5)$$

$$q = \frac{3p}{2n} + a_n;\ \frac{7p}{2n} + a_n;\ \frac{11p}{2n} + a_n;\ \ldots. \text{NORTH POLES} \qquad (6)$$

The parameter a_n defines the orientation of the $2n$-pole component of the field with respect to the chosen X-axis and is called the *phase angle*. Eqs. (3) and (4) represent the multipole expansion of the components of a two dimensional field in a current free region.

1.1 The Cartesian Components

Accelerator physicists often prefer to work with the Cartesian components of the field. The relationship between the Cartesian and the cylindrical components is shown in Fig. 1.

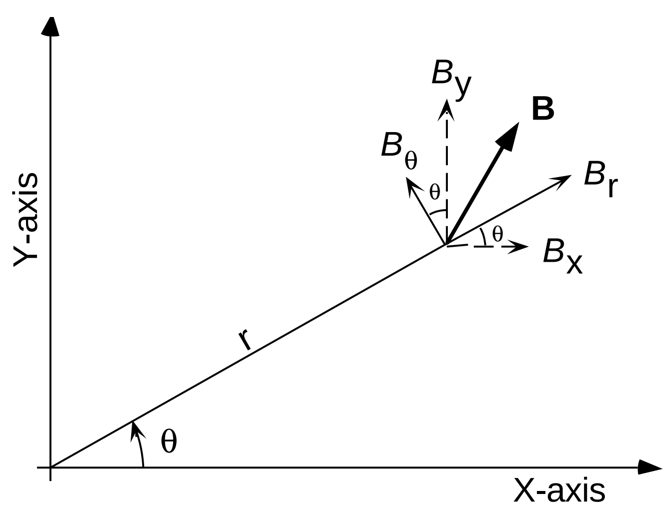

Fig. 1 Cylindrical and Cartesian components of the magnetic induction vector, **B**

Using the expansions for B_r and B_θ from Eqs. (3) and (4), we get the Cartesian components:

$$B_x(r,\theta) = B_r \cos\theta - B_\theta \sin\theta = \sum_{n=1}^{\infty} C(n)\left(\frac{r}{R_{ref}}\right)^{n-1} \sin[(n-1)\theta - n\alpha_n] \tag{7}$$

$$B_y(r,\theta) = B_r \sin\theta + B_\theta \cos\theta = \sum_{n=1}^{\infty} C(n)\left(\frac{r}{R_{ref}}\right)^{n-1} \cos[(n-1)\theta - n\alpha_n] \tag{8}$$

1.2 The Complex Field

The components of a two dimensional field can be very conveniently described in terms of a complex field, $\boldsymbol{B}(\boldsymbol{z})$, defined as a function of the complex variable $\boldsymbol{z} = x + iy = r \cdot \exp(i\theta)$ [a bold and italic font is used for complex quantities in this paper]. Looking at the expansions in Eqs. (7) and (8), one could define the complex field as:

$$\boldsymbol{B}(\boldsymbol{z}) = B_y(x,y) + iB_x(x,y) = \sum_{n=1}^{\infty} \left[C(n)\exp(-in\alpha_n)\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1} \tag{9}$$

The divergence and the curl (in a source free region) of the vector **B** are zero. In two dimensions, we get:

$$\nabla.\mathbf{B} = 0 \Rightarrow \left(\frac{\partial B_x}{\partial x}\right) + \left(\frac{\partial B_y}{\partial y}\right) = 0, \quad \text{or} \quad \left(\frac{\partial B_y}{\partial y}\right) = -\left(\frac{\partial B_x}{\partial x}\right) \tag{10}$$

$$(\nabla \times \mathbf{B})_z = 0 \Rightarrow \left(\frac{\partial B_y}{\partial x}\right) - \left(\frac{\partial B_x}{\partial y}\right) = 0, \quad \text{or} \quad \left(\frac{\partial B_y}{\partial x}\right) = \left(\frac{\partial B_x}{\partial y}\right) \tag{11}$$

Eqs. (10) and (11) are nothing but the well known Cauchy-Riemann conditions for the complex field $\boldsymbol{B}(\boldsymbol{z})$ to be an analytic function of the complex variable $\boldsymbol{z}$. The complex field is sometimes defined as $B_x(x,y) - iB_y(x,y)$, which is also an analytic function of $\boldsymbol{z}$. It should be noted that the quantity $B_x + iB_y$ is not an analytic function of $\boldsymbol{z}$. The analytic property of the complex field has been exploited in solving many two dimensional problems [1-6].

2. TWO-DIMENSIONAL BEHAVIOUR OF THE INTEGRAL FIELD

In magnets of a finite length, the two dimensional representation of the field is valid only in the body of the magnet, sufficiently away from the ends. In regions near the ends of the magnet, the field is three dimensional and the usual multipole expansion is no longer valid. Some examples of a three dimensional treatment may be found in Refs. [7]-[9]. However, for most practical purposes, one is interested in the *integral* of the field (or of its derivatives) over the length of the magnet. This is because most magnets are short compared to the wavelength of the betatron oscillations in the machine and the details of axial variation are of little consequence. Also, a typical rotating coil of a finite length only measures the integral of the field over its length. It can be shown that the integral field behaves as a two dimensional field provided the integration is carried out over an appropriate region [10].

In general, for three dimensional fields in a current free region, the scalar potential satisfies the Laplace's equation:

$$\nabla^2\Phi_m(x,y,z) = \left[\frac{\partial^2\Phi_m}{\partial x^2} + \frac{\partial^2\Phi_m}{\partial y^2} + \frac{\partial^2\Phi_m}{\partial z^2}\right] = 0 \qquad (12)$$

Integrating along the Z-axis from Z_1 to Z_2, we get:

$$\int_{Z_1}^{Z_2}\left[\frac{\partial^2\Phi_m}{\partial x^2} + \frac{\partial^2\Phi_m}{\partial y^2} + \frac{\partial^2\Phi_m}{\partial z^2}\right]dz = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\int_{Z_1}^{Z_2}\Phi_m(x,y,z)\,dz + \int_{Z_1}^{Z_2}\frac{\partial^2\Phi_m}{\partial z^2}\,dz = 0 \qquad (13)$$

We define the z-integrated scalar potential as $\overline{\Phi}_m(x,y) = \int_{Z_1}^{Z_2}\Phi_m(x,y,z)\,dz$. From Eq. (13):

$$\left[\frac{\partial^2\overline{\Phi}_m}{\partial x^2} + \frac{\partial^2\overline{\Phi}_m}{\partial y^2}\right] = -\left[\frac{\partial\Phi_m}{\partial z}\right]_{Z_1}^{Z_2} = B_z(x,y,Z_2) - B_z(x,y,Z_1) \qquad (14)$$

If the region of integration is so chosen that the Z-component of the field is zero at the boundaries of this region, then the right hand side vanishes and the z-integrated scalar potential satisfies the two dimensional Laplace's equation. For example, the points Z_1 and Z_2 could both be chosen well outside the magnet on opposite ends to include the integral of the field over the entire magnet. This situation is commonly encountered in the measurement of the integral field of relatively short magnets with a long integral coil. Alternatively, one could choose Z_1 well outside the magnet and Z_2 well inside the magnet, where the field is again two dimensional. Such a situation would apply to the measurement of the end region in a long magnet with a short measuring coil. The measuring coil for this purpose must have sufficient length so that the condition in Eq. (14) can be satisfied.

NORMAL AND SKEW COMPONENTS

The multipole expansion of the complex field is given by Eq. (9). The *normal* and *skew* components are defined as the real and the imaginary parts of the expansion coefficients:

$$C(n)\exp(-in\alpha_n) = (2n\text{-pole NORMAL Term}) + i(2n\text{-pole SKEW Term}) \qquad (15)$$

Unfortunately, the index n in the expansion coefficient is not the same as the corresponding power $(n-1)$ of $\mathbf{z}$ in Eq. (9). This has led to two different conventions being followed in denoting the normal and the skew terms. The *"US Convention"* denotes the $2n$-pole normal and skew terms with an index of $(n-1)$, to match the corresponding power of $\mathbf{z}$ in Eq. (9):

$$\begin{aligned} 2n\text{-pole Normal Term} &= C(n)\cos(n\alpha_n) \equiv B_{n-1} \\ 2n\text{-pole Skew Term} &= -C(n)\sin(n\alpha_n) \equiv A_{n-1} \end{aligned} \quad \text{"U.S. CONVENTION"} \qquad (16)$$

On the other hand, the *"European Convention"* denotes the 2*n*-pole normal and skew components with an index of *n* to retain the simple relationship between the index and the number of poles:

$$\begin{aligned} 2n\text{-pole Normal Term} &= C(n)\cos(na_n) \equiv B_n \\ 2n\text{-pole Skew Term} &= -C(n)\sin(na_n) \equiv A_n \end{aligned} \qquad \text{"EUROPEAN CONVENTION"} \qquad (17)$$

Even though the two conventions are commonly referred to as the "US" and the "European" notations, their use is not necessarily restricted to the respective geographic regions. This calls for exercising caution in interpreting the notations. One possible indicator of the notation is the presence or the absence of B_0 and A_0 terms.

In terms of the normal and the skew components, the expansion of $\boldsymbol{B}(\boldsymbol{z})$ is:

$$\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x = \sum_{n=0}^{\infty}\left[B_n + iA_n\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^n \qquad \text{"U.S. CONVENTION"} \qquad (18)$$

$$\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x = \sum_{n=1}^{\infty}\left[B_n + iA_n\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1} \qquad \text{"EUROPEAN CONVENTION"} \qquad (19)$$

The corresponding equations for the radial and the azimuthal components of the field are:

$$B_r(r,q) = \sum_{n=0}^{\infty}\left(\frac{r}{R_{ref}}\right)^n \left[B_n \sin\{(n+1)q\} + A_n \cos\{(n+1)q\}\right] \qquad \text{"U.S. Convention"} \qquad (20)$$

$$B_q(r,q) = \sum_{n=0}^{\infty}\left(\frac{r}{R_{ref}}\right)^n \left[B_n \cos\{(n+1)q\} - A_n \sin\{(n+1)q\}\right] \qquad \text{"U.S. Convention"} \qquad (21)$$

$$B_r(r,q) = \sum_{n=1}^{\infty}\left(\frac{r}{R_{ref}}\right)^{n-1} \left[B_n \sin(nq) + A_n \cos(nq)\right] \qquad \text{"European Convention"} \qquad (22)$$

$$B_q(r,q) = \sum_{n=1}^{\infty}\left(\frac{r}{R_{ref}}\right)^{n-1} \left[B_n \cos(nq) - A_n \sin(nq)\right] \qquad \text{"European Convention"} \qquad (23)$$

It should be noted that sometimes the skew component is defined with a sign opposite to that used here [11]. In view of the different notations in existence, it is important to recognize the convention being used in any particular work.

It is possible to assign a physical significance to the normal and the skew components in terms of the derivatives of the field. Using Eq. (18) or Eq. (19), it is easy to show that

$$\left.\begin{aligned} B_n(\text{US}) = B_{n+1}(\text{European}) &= \frac{R_{ref}^n}{n!}\left(\frac{\partial^n B_y}{\partial x^n}\right)\Bigg|_{x=0;\,y=0} \\ A_n(\text{US}) = A_{n+1}(\text{European}) &= \frac{R_{ref}^n}{n!}\left(\frac{\partial^n B_x}{\partial x^n}\right)\Bigg|_{x=0;\,y=0} \end{aligned}\right\} \quad n \geq 0 \qquad (24)$$

A magnet with the $2m$-pole term as the most dominant term is called a *normal* magnet if the $2m$-pole skew term is zero. Similarly, such a magnet is called a *skew* magnet if the normal term is zero. As per Eqs. (16) and (17), the possible values of the phase angle for the $2m$-pole term are $\alpha_m = 0$ or π/m for a normal magnet and $\alpha_m = \pi/(2m)$ or $3\pi/(2m)$ for a skew magnet. The poles in a $2m$-pole magnet [see Eqs. (5) and (6)] are located at $\theta = \pi/(2m)$, $3\pi/(2m)$, $5\pi/(2m)$, etc. for a normal magnet and at $\theta = 0$, π/m, $2\pi/m$, $3\pi/m$, etc. for a skew magnet. A $2m$-pole skew magnet is obtained from a $2m$-pole normal magnet by a clockwise rotation of the magnet by an angle $\pi/(2m)$.

3.1 Fractional Field Coefficients, or "Multipoles"

The expansion coefficients B_n and A_n in Eqs. (18)-(23) are related to the actual field strength in the magnet and are dependent on the excitation level. In describing the field quality of a magnet, one is often interested in the *shape* of the field, rather than its absolute magnitude. This is done by expressing the various harmonic terms in the expansion as a fraction of a *reference field*, B_{ref}. For example, the complex field may be written as:

$$\boldsymbol{B}(\boldsymbol{z}) = B_y(x,y) + iB_x(x,y) = B_{ref}\sum_{n=1}^{\infty}\left[\frac{C(n)\exp(-in\alpha_n)}{B_{ref}}\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1} \qquad (25)$$

Generally, this reference field is chosen as the strength of the most dominant term in the expansion. For a $2m$-pole magnet, it is expected that the term for $n = m$ will be the most dominant one. Hence, B_{ref} may be chosen to be equal to $C(m)$. Sometimes the reference field is chosen as the strength of the dipole field used for bending the beam in a circular accelerator.

The Normal and Skew $2n$-pole *fractional field coefficients*, or *"multipoles"* are defined as:

$$b_{n-1} = \text{Re}\left[\frac{C(n)\exp(-in\alpha_n)}{B_{ref}}\right] = \frac{B_{n-1}}{B_{ref}}, \quad a_{n-1} = \text{Im}\left[\frac{C(n)\exp(-in\alpha_n)}{B_{ref}}\right] = \frac{A_{n-1}}{B_{ref}} \quad \text{(US)} \qquad (26)$$

$$b_n = \text{Re}\left[\frac{C(n)\exp(-in\alpha_n)}{B_{ref}}\right] = \frac{B_n}{B_{ref}}, \quad a_n = \text{Im}\left[\frac{C(n)\exp(-in\alpha_n)}{B_{ref}}\right] = \frac{A_n}{B_{ref}} \quad \text{(European)} \qquad (27)$$

In a typical accelerator magnet, the multipoles are of the order of 10^{-4} when calculated at a reference radius comparable to the size of the region occupied by the beam (~ 50-70% of the coil radius). The coefficients are therefore often quoted after multiplying by a factor of 10^4. With this multiplicative factor, the values of the multipoles are said to be in *"units"*.

3. TRANSFORMATION OF FIELD PARAMETERS

The expansion parameters [$C(n)$, α_n] or [B_n, A_n] depend on the choice of the reference frame. For example, the parameter α_n controls the angular locations of the poles for the 2n-pole component of the field according to Eqs. (5) and (6). If the coordinate axes (or equivalently, the magnet) are rotated, the angular positions of the poles would change, thus necessitating a corresponding change in the parameter α_n. In general, the quantities B_n and A_n should be regarded as expansion coefficients according to Eq. (18) or (19), which clearly depend on the choice of coordinate axes, since the variable $\boldsymbol{z}$ depends on the coordinate axes used. In this section, we shall derive the equations that govern the transformation of these parameters under commonly encountered coordinate transformations.

4.1 Displacement of Axes

Let us consider a frame X'-Y' which is displaced from a frame X-Y by $\boldsymbol{z}_0 = x_0 + iy_0$ as shown in Fig. 2. The axes in the two frames are assumed to be parallel. In other words, there is no rotation of the axes. The field parameters are denoted by [$C(n)$, α_n] or [B_n, A_n] in the X-Y frame and by $\left[C'(n), \alpha'_n\right]$ or $[B'_n, A'_n]$ in the X'-Y' frame.

We consider a point, P, in space located at $\boldsymbol{z} = x + iy$ in the X-Y frame. The same point is located at $\boldsymbol{z}' = x' + iy' = \boldsymbol{z} - \boldsymbol{z}_0$ in the X'-Y' frame, as shown in Fig. 2. Since the coordinate axes in the two frames are parallel to each other, the Cartesian components of the field are the same in the two frames. In other words, the complex field is the same in the two frames. This fact can be used to derive the expansion of the field in the X'-Y' frame as follows:

$$
\begin{aligned}
\boldsymbol{B}(\boldsymbol{z}') &= B_{y'} + iB_{x'} = B_y + iB_x = \boldsymbol{B}(\boldsymbol{z}) \\
&= \sum_{k=0}^{\infty}(B_k + iA_k)\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^k = \sum_{k=0}^{\infty}(B_k + iA_k)\left(\frac{\boldsymbol{z}' + \boldsymbol{z}_0}{R_{ref}}\right)^k \\
&= \sum_{k=0}^{\infty}(B_k + iA_k)\sum_{n=0}^{k}\frac{k!}{n!(k-n)!}\left(\frac{\boldsymbol{z}'}{R_{ref}}\right)^n\left(\frac{\boldsymbol{z}_0}{R_{ref}}\right)^{k-n}
\end{aligned}
\tag{28}
$$

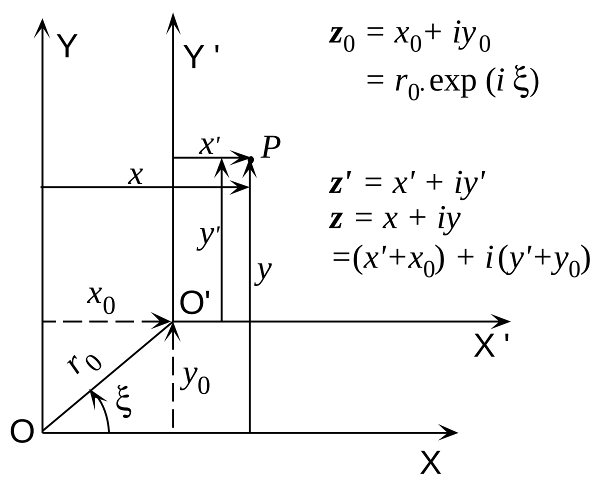

Fig. 2 Displacement of coordinate axes

In writing the above equations, we have used the US convention as per Eq. (18). With a rearrangement of terms in the double summation, it can be shown that

$$\sum_{k=0}^{\infty} \sum_{n=0}^{k} t_{kn} = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} t_{kn} \tag{29}$$

Applying this identity to Eq. (28), we may write:

$$\boldsymbol{B}(\boldsymbol{z}') = \sum_{n=0}^{\infty} \left[\sum_{k=n}^{\infty} (B_k + iA_k) \frac{k!}{n!(k-n)!} \left(\frac{\boldsymbol{z}_0}{R_{ref}} \right)^{k-n} \right] \left(\frac{\boldsymbol{z}'}{R_{ref}} \right)^{n} = \sum_{n=0}^{\infty} (B'_n + iA'_n) \left(\frac{\boldsymbol{z}'}{R_{ref}} \right)^{n} \tag{30}$$

The expansion coefficients in the displaced frame are therefore given by:

$$(B'_n + iA'_n) = \sum_{k=n}^{\infty} (B_k + iA_k) \left[\frac{k!}{n!(k-n)!} \right] \left(\frac{x_0 + iy_0}{R_{ref}} \right)^{k-n} \quad ; \quad n \geq 0 \qquad \text{(US Notation)} \tag{31}$$

The corresponding equation in the European notation is:

$$(B'_n + iA'_n) = \sum_{k=n}^{\infty} (B_k + iA_k) \left[\frac{(k-1)!}{(n-1)!(k-n)!} \right] \left(\frac{x_0 + iy_0}{R_{ref}} \right)^{k-n} \quad ; \; n \geq 1 \;\; \text{(European Notation)} \tag{32}$$

The transformation for the amplitude and phase of the 2n-pole term is given by:

$$C'(n) \exp(-ina'_n) = \sum_{k=n}^{\infty} [C(k) \exp(-ika_k)] \left[\frac{(k-1)!}{(n-1)!(k-n)!} \right] \left(\frac{x_0 + iy_0}{R_{ref}} \right)^{k-n} \quad ; \; n \geq 1 \tag{33}$$

It is seen from Eqs. (31)-(33) that coefficients of any particular order in the displaced frame are given by a combination of all the terms of equal or higher order in the undisplaced frame. Conversely, a given harmonic term in the undisplaced frame contributes to all the harmonics of equal or lower order in the displaced frame. For example, a displacement in a pure quadrupole magnet will result in a dipole term in addition to the quadrupole term, a displacement in a pure sextupole field will produce quadrupole and dipole fields and so on. This effect is referred to as the *feed down* of harmonics. Eq. (33) is useful in estimating the effect of magnet misalignment in an accelerator, or in calculating the location of measuring coil in the magnet during measurements.

4.2 Rotation of Axes

Let us consider a frame X'-Y' which is rotated with respect to a frame X-Y by an angle f as shown in Fig. 3. The origins of the two frames are the same. The field parameters are denoted by [$C(n)$, a_n] or [B_n, A_n] in the X-Y frame and by $[C'(n), a'_n]$ or $[B'_n, A'_n]$ in the X'-Y' frame. The relationship between the two sets of expansion coefficients can be obtained by using the relationship between the complex coordinates $\boldsymbol{z}$ and $\boldsymbol{z}'$ and the relationship between the Cartesian components in the two frames, as was done for the case of a displacement of axes in Sec. 4.1. For the case of a rotation by an angle f, we have

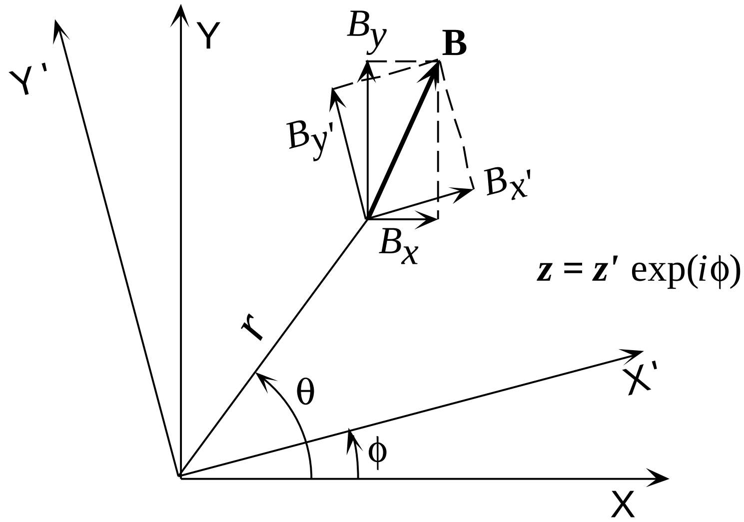

Fig. 3 Rotation of axes

$$B_{x'} = B_x \cos f + B_y \sin f \;\; ; \quad B_{y'} = -B_x \sin f + B_y \cos f \tag{34}$$

$$\boldsymbol{z} = \boldsymbol{z}' \exp(if) \tag{35}$$

Following the US convention, we may write

$$\left.\begin{aligned} \boldsymbol{B}(\boldsymbol{z}') = B_{y'} + iB_{x'} = (B_y + iB_x)\exp(if) = \sum_{n=0}^{\infty}[B_n + iA_n]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^n \exp(if) \\ = \sum_{n=0}^{\infty}[B_n + iA_n]\left(\frac{\boldsymbol{z}'}{R_{ref}}\right)^n \exp[i(n+1)f] = \sum_{n=0}^{\infty}[B_n' + iA_n']\left(\frac{\boldsymbol{z}'}{R_{ref}}\right)^n \end{aligned}\right\} \tag{36}$$

The transformation of coefficients follows immediately from the above equation:

$$(B_n' + iA_n') = (B_n + iA_n)\exp[i(n+1)f]\,; \;\; n \geq 0 \quad \text{(US Notation)} \tag{37}$$

$$(B_n' + iA_n') = (B_n + iA_n)\exp(inf)\,; \quad n \geq 1 \;\; \text{(European Notation)} \tag{38}$$

The transformation for the amplitude and phase of the $2n$-pole term is given by:

$$C'(n)\exp(-ina'_n) = C(n)\exp(-ina_n)\exp(inf) \Rightarrow C'(n) = C(n);\; a'_n = a_n - f;\;\; n \geq 1 \tag{39}$$

As seen from Eqs. (37)-(39), a rotation of axes does not produce any feed down of harmonics, but causes mixing of the normal and the skew components of a given harmonic.

4.3 Reflection of X-Axis: Magnet Viewed from the Opposite End

If a magnet is viewed from the end which is opposite to the end from which the field parameters are measured, then appropriate transformation must be applied to the field parameters. Such a situation may be encountered in practice when all magnets in an

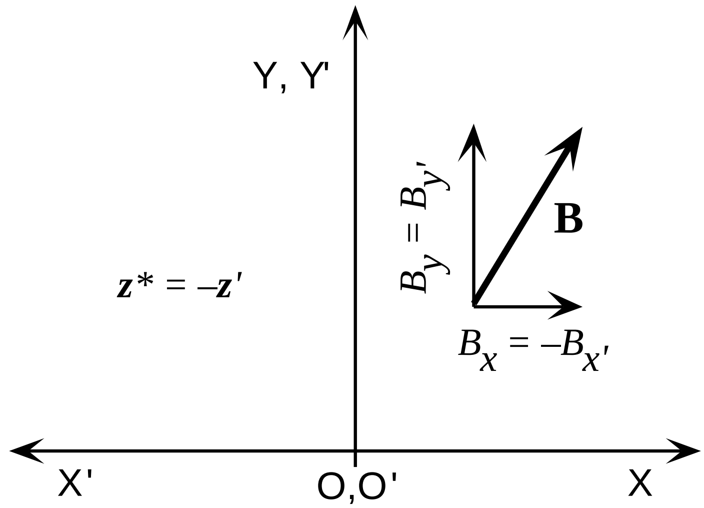

Fig. 4 Reflection of X-Axis: magnet viewed from the opposite end

accelerator are measured from the same end, but all magnets are not installed with the leads facing the same way. The reference frames X-Y and X'-Y' are shown in Fig. 4. The Y'-axis coincides with the Y-axis but the X'-axis points away from the X-axis.

To obtain the transformation, we note that

$$\mathbf{z}^* = x - iy = -x' - iy' = -\mathbf{z}' \tag{40}$$

$$\left.\begin{aligned} B_{y'} + iB_{x'} = B_y - iB_x &= (B_y + iB_x)^* = \sum_{n=0}^{\infty}(B_n + iA_n)^* \left(\frac{\mathbf{z}^*}{R_{ref}}\right)^n \\ &= \sum_{n=0}^{\infty}(B_n - iA_n)(-1)^n \left(\frac{\mathbf{z}'}{R_{ref}}\right)^n = \sum_{n=0}^{\infty}(B'_n + iA'_n)\left(\frac{\mathbf{z}'}{R_{ref}}\right)^n \end{aligned}\right\} \tag{41}$$

From Eq. (41), it immediately follows that

$$B'_n = (-1)^n B_n \; ; \quad A'_n = (-1)^{n+1} A_n \; ; \quad n \geq 0 \qquad \text{(US Notation)} \tag{42}$$

$$B'_n = (-1)^{n+1} B_n \; ; \quad A'_n = (-1)^n A_n \; ; \quad n \geq 1 \quad \text{(European Notation)} \tag{43}$$

$$C'(k) = C(k); \quad a'_{2k-1} = -a_{2k-1}; \quad a'_{2k} = \left(\frac{p}{2k}\right) - a_{2k}; \quad k \geq 1 \tag{44}$$

It should be noted that the transformations given in Eqs. (42) and (43) are for the *unnormalized* normal and skew coefficients. In practice, one often deals with the *normalized* fractional field coefficients, or *multipoles*, defined in Sec.3.1. The transformation of the multipoles can be a source of considerable confusion depending on how the reference field is chosen. For example, in a normal quadrupole magnet, the main field component (the normal quadrupole term) will change sign when viewed from the opposite end according to Eq. (42)

or Eq. (43). However, one may like to normalize the multipoles in such a way that the main multipole, b_1 (or b_2 in the European notation) is always positive. With such a normalization, the normal quadrupole multipole in this case will not change sign under a reflection of axes. Therefore, while deriving the transformation rules for the normalized multipoles, one should also pay attention to the normalization rules. The principal use of the measured multipoles is to provide a series expansion of the field for use in accelerator physics studies. The transformation rules for the multipoles must be derived in such a way that the form of the expansion used in such studies remains valid.

4. RELATIONSHIP BETWEEN CURRENT AND MAGNETIC FIELD

The harmonic expansion of the components of the magnetic induction, **B**, derived in Sec.1 made no reference to the current distribution that produced the field. In practice, the distribution of current in an electromagnet determines the shape of the field, and hence the actual values of the normal and skew components, or the amplitudes and phases of various harmonics. From the point of view of magnet design as well as magnetic measurements, it is essential to have an understanding of the relationship between current distribution in a magnet and the field harmonics.

In order to study the relationship between current and the field shape, it is convenient to describe the magnetic induction **B** in terms of the vector potential, **A**. From Maxwell's equations, the divergence of **B** is zero everywhere. So, we may write

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{(ALWAYS HOLDS)} \tag{45}$$

If we restrict ourselves to regions where $\mathbf{B} = m_0\mathbf{H}$, then we also have

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla.\mathbf{A}) - \nabla^2 \mathbf{A} = m_0(\nabla \times \mathbf{H}) = m_0 \mathbf{J} \tag{46}$$

where **J** is the current density vector. Since only the curl of **A** is important, we may add an arbitrary term $\nabla\psi$ to **A** without affecting the results (Gauge Transformation). We choose this term $\nabla\psi$ in such a way that the divergence of **A** becomes zero. With this choice of gauge, we get the Poisson's equation for the vector potential,

$$\nabla^2 \mathbf{A} = -m_0 \mathbf{J} \tag{47}$$

The solution of the Poisson's equation is given by [12]

$$\mathbf{A}(\mathbf{r}) = \left(\frac{m_0}{4p}\right) \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}\mathbf{r}' \tag{48}$$

If the distribution of current is known, one can obtain the vector potential, and hence the magnetic induction, **B**. In the presence of magnetic materials, the current density in Eq. (48) must also include the current loops that effectively produce the magnetization. This complicates the application of Eq. (48) in practice. We shall derive expressions for the magnetic induction for several simple geometries of interest in Sec. 7.

6. THE COMPLEX POTENTIAL

The scalar and the vector potentials can be combined into a single *complex potential*. For a two dimensional field confined to the *X*-*Y* plane, the current and the vector potential have only a *Z*-component. Let us define a function, $\boldsymbol{W}(\boldsymbol{z})$, as a function of the complex variable $\boldsymbol{z}$ by the relation

$$\boldsymbol{W}(\boldsymbol{z}) = A_z(x, y) + i\Phi_m(x, y) \tag{49}$$

where A_z is *Z*-component of the vector potential and Φ_m is the magnetic scalar potential. Then,

$$-\left(\frac{d\boldsymbol{W}(\boldsymbol{z})}{d\boldsymbol{z}}\right) \equiv -\left(\frac{\partial \boldsymbol{W}(\boldsymbol{z})}{\partial x}\right) = -\left(\frac{\partial A_z}{\partial x}\right) - i\left(\frac{\partial \Phi_m}{\partial x}\right) = B_y(x, y) + iB_x(x, y) = \boldsymbol{B}(\boldsymbol{z}) \tag{50}$$

The derivative of $\boldsymbol{W}(\boldsymbol{z})$ in the complex plane is the complex field $\boldsymbol{B}(\boldsymbol{z})$ defined in Eq. (9). We may think of $\boldsymbol{W}(\boldsymbol{z})$ as a *complex potential*. It should be noted that slightly different versions of the complex potential exist in the literature depending on the definition of the complex field. The complex potential for simple current geometries can be easily calculated, leading to several useful results, as we shall see in the next section. The complex potential involves both the vector and the scalar potential, and, therefore, contains more information than is necessary to describe the field. However, it has the advantage of being an analytic function of the complex variable, $\boldsymbol{z}$.

7. FIELD DUE TO SOME SIMPLE CURRENT DISTRIBUTIONS

In this section we shall derive the expressions for the field from some very simple types of current distributions. Since we are dealing with only two dimensional fields, all the current distributions will be assumed to have an infinite extent in the *Z*-direction.

7.1 An Infinitesimally Thin Current Filament

The simplest current configuration is an infinitesimally thin current filament carrying a current, *I*, located at the origin and extending to infinity in the positive and the negative directions along the *Z*-axis, as shown in Fig. 5. Although the vector potential for this configuration can be calculated by explicitly integrating Eq. (48), it is much more convenient in this case to simply utilize Ampère's law to obtain the field, and use Eq. (50) to obtain the complex potential. At any point *P* located at (r, θ), the magnetic field has the same magnitude along a circle of radius *r* and is directed along the azimuthal direction, as shown in Fig. 5. From Ampère's law:

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi r}\hat{\theta} \tag{51}$$

Using Eqs. (7) and (8) for the Cartesian components in terms of the radial and azimuthal components, it is easy to show that the complex field is given by

$$\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x = (B_\theta + iB_r)\exp(-i\theta) = \frac{\mu_0 I}{2\pi r.\exp(i\theta)} = \frac{\mu_0 I}{2\pi \boldsymbol{z}} \tag{52}$$

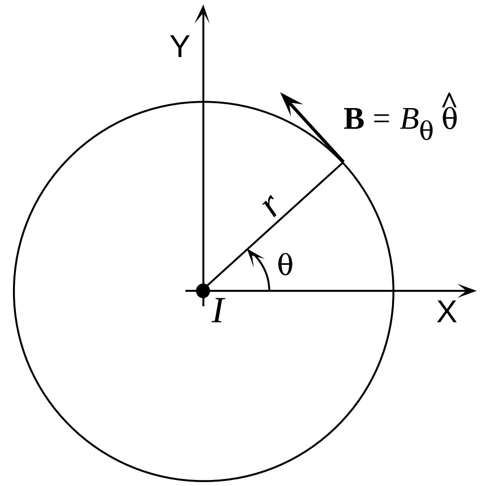

Fig. 5 Field due to an infinitely long, thin current filament, *I*, located at the origin

The complex potential is obtained by integrating the complex field:

$$\boldsymbol{W}(\boldsymbol{z}) = -\int B(\boldsymbol{z})d\boldsymbol{z} = -\left(\frac{m_0 I}{2p}\right)\ln(\boldsymbol{z}) + \text{constant} = -\left(\frac{m_0 I}{2p}\right)[\ln(r) + iq] + \text{constant} \qquad (53)$$

The real and the imaginary parts of $\boldsymbol{W}(\boldsymbol{z})$ give the vector potential, A_z, and the scalar potential, Φ_m, respectively [see Eq. (49)].

A more useful result is for a current filament located at an arbitrary point given by $\boldsymbol{a} = a \cdot \exp(if) = a_x + ia_y$ in the complex plane. In a frame X'-Y' with axes parallel to X-Y and having origin at the current filament (see Fig. 6), the complex field is given by Eq. (52). The complex field and the complex potential in the X-Y frame can be calculated as follows:

$$\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x = B_{y'} + iB_{x'} = \boldsymbol{B}(\boldsymbol{z}') = \left(\frac{m_o I}{2p\,\boldsymbol{z}'}\right) = \frac{m_o I}{2p(\boldsymbol{z} - \boldsymbol{a})} \qquad (54)$$

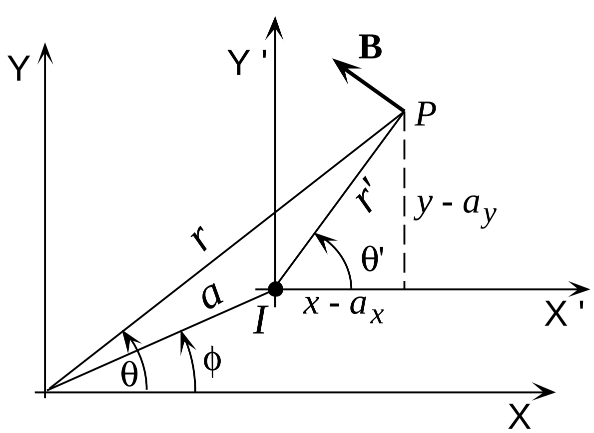

Fig. 6 Calculation of field at *P* due to a current filament at an arbitrary location, ***a***.

$$\boldsymbol{W}(\boldsymbol{z}) = -\int \boldsymbol{B}(\boldsymbol{z})d\boldsymbol{z} = -\left(\frac{m_0 I}{2p}\right)\ln(\boldsymbol{z}-\boldsymbol{a}) + \text{const.} = -\left(\frac{m_0 I}{2p}\right)[\ln(r') + iq'] + \text{const.} \qquad (55)$$

Using Eq. (54), we can determine, in principle, the field due to any arbitrary distribution of current by treating it as a superposition of current filaments. We shall see some simple examples in the rest of this section.

7.2 Cylindrical Current Sheet of Uniform Density

Let us calculate the complex field due to an infinitely long, infinitesimally thin cylindrical conductor of radius a carrying a current I, as shown in Fig. 7. The current density is assumed to be uniform. Also, all currents are assumed to flow parallel to the length of the cylinder, which is along the Z-axis. The complex field at $\boldsymbol{z}$ from a small element of angular width df, located at angle f and carrying a current dI, can be calculated using Eq. (54). The total field can then be obtained by integrating over the entire surface of the cylinder:

$$\boldsymbol{B}(\boldsymbol{z}) = \left(\frac{m_0}{2p}\right)\int \frac{dI}{\boldsymbol{z} - a\exp(if)} = \left(\frac{m_0}{2p}\right)\left(\frac{I}{2p}\right)\int_0^{2p} \frac{df}{\boldsymbol{z} - a\exp(if)} = \left(\frac{m_0 I}{2p}\right)\left(\frac{1}{2p\, i}\right)\oint \frac{d\boldsymbol{z}'}{\boldsymbol{z}'(\boldsymbol{z}-\boldsymbol{z}')} \qquad (56)$$

where we have used the transformation $\boldsymbol{z}' = a.\exp(if)$ and the contour integral is along the surface of the cylinder. The integrand in Eq. (56) has two simple poles, at $\boldsymbol{z}' = 0$ and $\boldsymbol{z}' = \boldsymbol{z}$. If the field point $\boldsymbol{z}$ is outside the cylinder, then only the pole at $\boldsymbol{z}' = 0$ contributes to the integral. Similarly, if the field point $\boldsymbol{z}$ is inside the cylinder, both the poles contribute to the integral. Using the method of residues, it is easy to show that the total complex field in the regions outside and inside the cylinder is given by

$$\boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{m_0 I}{2p\,\boldsymbol{z}}\right); \qquad \boldsymbol{B}_{in}(\boldsymbol{z}) = 0 \qquad (57)$$

Thus, the field inside the cylinder is zero, while for any point outside, the cylinder behaves as an infinitesimally thin current filament located at the center.

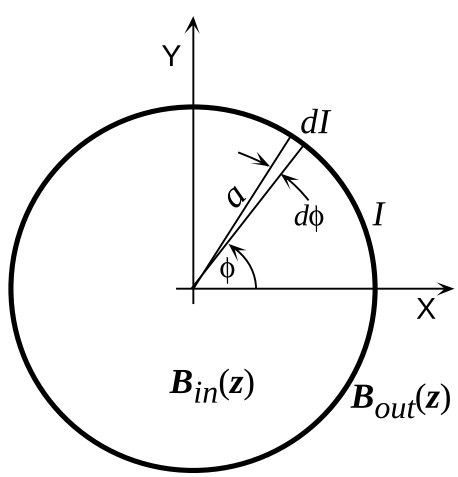

Fig. 7 A section of a thin, cylindrical sheet carrying current I along the Z-axis.

7.3 Solid Cylindrical Conductor with Uniform Current Density

We now consider an infinitely long *solid* cylinder of radius *a* carrying a total current *I* along the *Z*-axis, as shown in Fig. 8. The field at any point can be obtained by dividing the solid cylinder into thin cylindrical shells and using the results of Sec.7.2. For any point outside the cylindrical conductor, all such shells contribute. For any point *P* at a radius $r < a$, as shown in Fig. 8, only shells with radii $x < r$ contribute.

The current density *J* is given by $I/(p\,a^2)$. The current carried by a thin shell of radius *x* and thickness *dx* is given by $dI = J.2px.dx$. For any point inside the cylinder, the total complex field is:

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = \int\left(\frac{m_o dI}{2p\boldsymbol{z}}\right) = \left(\frac{m_o J}{\boldsymbol{z}}\right).\int_0^r x.dx = \left(\frac{m_o J r^2}{2\boldsymbol{z}}\right) = \left(\frac{m_o J}{2}\right)\boldsymbol{z}^* \qquad (58)$$

where we have used the fact that $r^2 = \boldsymbol{zz^*}$. For any point outside the solid cylinder, the total complex field is:

$$\boldsymbol{B}_{out}(\boldsymbol{z}) = \int\left(\frac{m_o dI}{2p\boldsymbol{z}}\right) = \left(\frac{m_o J}{\boldsymbol{z}}\right).\int_0^a x.dx = \left(\frac{m_o J a^2}{2\boldsymbol{z}}\right) = \left(\frac{m_o J}{2}\right)\left(\frac{a^2}{\boldsymbol{z}}\right) \qquad (59)$$

It should be noted that $\boldsymbol{B}_{in}(\boldsymbol{z})$ is a function of $\boldsymbol{z^*}$, and hence is *not* an analytic function of $\boldsymbol{z}$. This result is not unexpected, because the points inside the cylinder are not in a source free region. The analyticity of the complex field defined as $B_y + iB_x$ was shown in Sec.2 to follow from Maxwell's equations in a current free region only [see Eq. (11)].

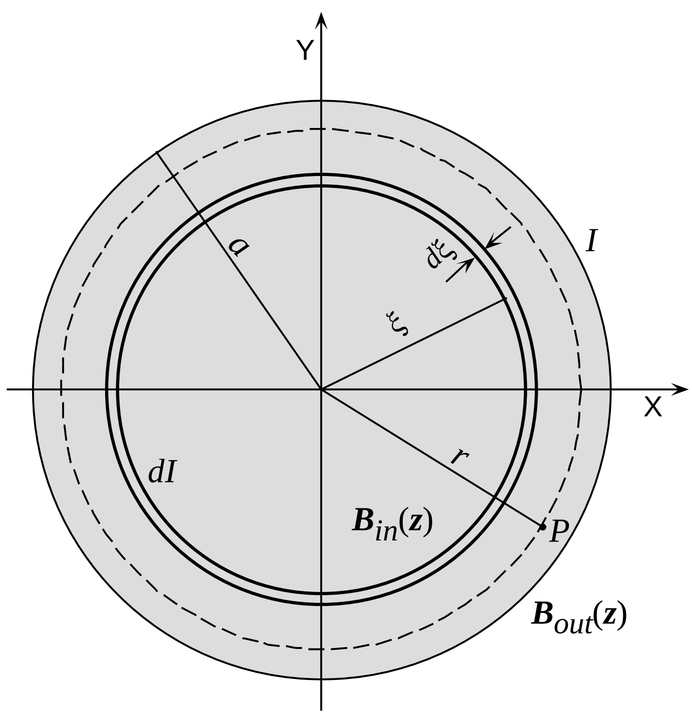

Fig. 8 Calculation of field due to a solid cylindrical conductor.

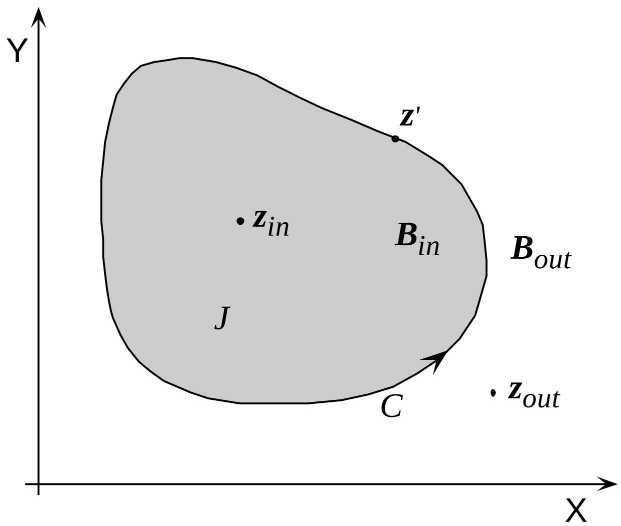

Fig. 9 Section of an infinitely long solid conductor of arbitrary cross section.

7.4 Solid Conductor of Arbitrary Cross Section

Let us now consider a solid conductor of arbitrary cross section defined by the contour C in the complex plane, as shown in Fig. 9. In principle, one could calculate the field from such a conductor by integrating the result for a thin current filament [see Eq. (54)] over the cross section of the conductor. This approach requires a two dimensional integral to be evaluated and is not always convenient. A much simplified result has been obtained by Beth[2] by defining a complex function which is analytic everywhere, including the region of the conductor, and then evaluating the field as a contour integral around the path C. The result of Beth, referred to as the *integral formula*, is briefly summarized here.

Inside the region of the conductor, Maxwell's equations give:

$$(\nabla \times \mathbf{B})_z = \left(\frac{\partial B_y}{\partial x}\right) - \left(\frac{\partial B_x}{\partial y}\right) = \mu_0 J_z(x, y); \quad \nabla \cdot \mathbf{B} = \left(\frac{\partial B_x}{\partial x}\right) + \left(\frac{\partial B_y}{\partial y}\right) = 0 \qquad (60)$$

where $J_z(x, y)$ is the current density at the point (x, y). For a constant current density over the cross section of the conductor, $J_z(x,y) = J$, it can be easily shown that the function:

$$\boldsymbol{F}(\boldsymbol{z}) = B_y(x, y) + iB_x(x, y) - \left(\frac{\mu_0 J}{2}\right)\boldsymbol{z}^* = \boldsymbol{B}(\boldsymbol{z}) - \left(\frac{\mu_0 J}{2}\right)\boldsymbol{z}^* \qquad (61)$$

satisfies Cauchy-Riemann conditions and is, therefore, an analytic function of the complex variable, $\boldsymbol{z}$. For points outside the conductor, $J = 0$, and $\boldsymbol{F}(\boldsymbol{z})$ becomes identical to $\boldsymbol{B}(\boldsymbol{z})$. It has been shown by Beth [2] that this function, $\boldsymbol{F}(\boldsymbol{z})$, can be evaluated as a contour integral over the boundary of the conductor. The complex field is then calculated using Eq. (61). The final result is given by:

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = \boldsymbol{F}(\boldsymbol{z}) + \left(\frac{\mu_0 J}{2}\right)\boldsymbol{z}^* = i\left(\frac{\mu_0 J}{4\pi}\right)\oint_C \frac{\boldsymbol{z}'^*}{\boldsymbol{z}' - \boldsymbol{z}}\, d\boldsymbol{z}' + \left(\frac{\mu_0 J}{2}\right)\boldsymbol{z}^*; \text{ for } \boldsymbol{z} = \boldsymbol{z}_{in} \qquad (62)$$

$$\boldsymbol{B}_{out}(\boldsymbol{z}) = \boldsymbol{F}(\boldsymbol{z}) = i\left(\frac{m_0 J}{4p}\right)\oint_C \frac{\boldsymbol{z}'^*}{\boldsymbol{z}' - \boldsymbol{z}}\, d\boldsymbol{z}' \text{ , for } \boldsymbol{z} = \boldsymbol{z}_{out} \tag{63}$$

The expressions for the field are thus reduced to an integral along the *perimeter* of the conductor cross section, rather than a two dimensional integral over the *area* of the conductor cross section. It should be noted that the integrand in Eqs. (62) and (63) contains $\boldsymbol{z}'^*$, and is not an analytic function of the complex variable. The method of residues is not applicable in general and the integral must be explicitly evaluated.

A quick verification of the integral formula can be made for the case of a solid circular cylinder, for which the expression for field was derived in Sec.7.3. For this special case, $\boldsymbol{z}'^* = a.\exp(-if) = a^2/\,\boldsymbol{z}'$, which is an analytical function with a simple pole at $\boldsymbol{z}' = 0$. The integral in Eqs. (62) and (63) becomes identical to the one in Eq. (56) evaluated earlier in Sec.7.2. Using the results of Sec.7.2, it is easy to verify that Eqs. (62) and (63) give the same results for a solid cylindrical conductor as Eqs. (58) and (59).

7.5 Application of the Integral Formula: A Solid Elliptical Conductor

A good illustration of the application of the integral formula is to calculate the expressions for the field from a solid conductor of an elliptical cross section with semi-axes a and b along the X and Y directions, as shown in Fig. 10. The detailed procedure to carry out the integrations in Eqs. (62) and (63) is described in Ref. [2] for this special case. The final result obtained is:

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = \frac{m_o J}{(a+b)}[bx - iay]; \quad \boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{m_o J}{2}\right)\left[\frac{2ab}{\boldsymbol{z} + \sqrt{\boldsymbol{z}^2 - (a^2 - b^2)}}\right] \tag{64}$$

For the special case of $b = a$, Eq. (64) reduces to the results in Eqs. (58) and (59) for a solid cylindrical conductor.

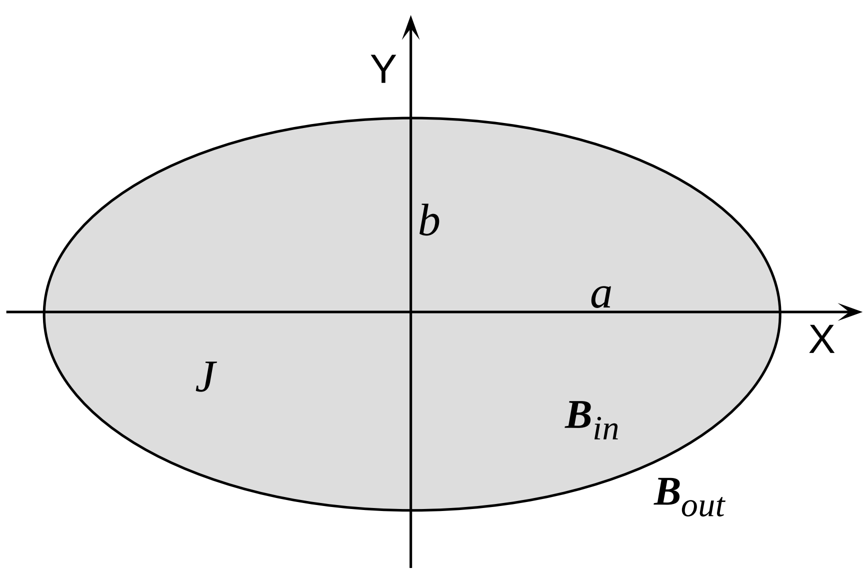

Fig. 10 Cross section of an infinitely long elliptical conductor.

8. GENERATING PURE DIPOLE AND QUADRUPOLE FIELDS USING OVERLAPPING CYLINDERS

Magnets that generate a pure dipole or a pure quadrupole field are of utmost importance in accelerators due to their use as bending and focussing elements. Using the result in Eq. (64) for a solid elliptical conductor, it can be shown that, at least in principle, a perfect dipole or a quadrupole field can be produced by overlapping two elliptical conductors carrying equal and opposite current densities, as shown in Figs. 11 and 12.

8.1 Generating a Pure Dipole Field

To generate a pure normal-dipole field, the two elliptical cylinders are displaced with respect to each other by a distance x_0 along the X-axis, as shown in Fig. 11. To produce a skew-dipole field, the arrangement should be rotated by 90^o. Since the cylinders carry equal and opposite current densities, the region of overlap effectively carries no current, and may be removed, thus producing the “aperture” of the magnet. Any point, $\boldsymbol{z}_{in}$, inside this aperture is also inside both the ellipses. The complex field due to cylinder 1 (see Fig. 11) is obtained by substituting $\boldsymbol{z} = \boldsymbol{z}_1 = \boldsymbol{z}_{in} + x_0/2$ in Eq. (64). Similarly, the complex field due to cylinder 2 is obtained by substituting $\boldsymbol{z} = \boldsymbol{z}_2 = \boldsymbol{z}_{in} - x_0/2$. The overall field is given by:

$$\boldsymbol{B}_{in}(\boldsymbol{z}_{in}) = \frac{m_o J}{(a+b)}\left[b\left(x + \frac{x_0}{2}\right) - iay - b\left(x - \frac{x_0}{2}\right) + iay \right] = \frac{m_o J b x_0}{(a+b)} = B_y(\boldsymbol{z}_{in}) + iB_x(\boldsymbol{z}_{in}) \quad (65)$$

It is seen from Eq. (65) that the y-component of the field is the same everywhere in the aperture. Also, the x-component is zero. This is characteristic of a pure, normal-dipole field [see Eq. (18) or (19)]. Similarly, a pure skew-dipole field can be generated by displacing the two ellipses along the y-axis instead of the x-axis. As the separation, x_0, between the two ellipses increases, the aperture size reduces, the total current carried by the conductors increases, and the dipole field increases. It should be noted that a pure dipole field can also be obtained by two overlapping *circles* of radius a, instead of ellipses. The magnitude of the field in this case is given by Eq. (65), with $b = a$, to be simply $(m_0 J/2)\, x_0$. For a fixed separation and current density, larger circles use more total current and provide a larger aperture than smaller circles, but circles of all sizes give the same dipole field strength.

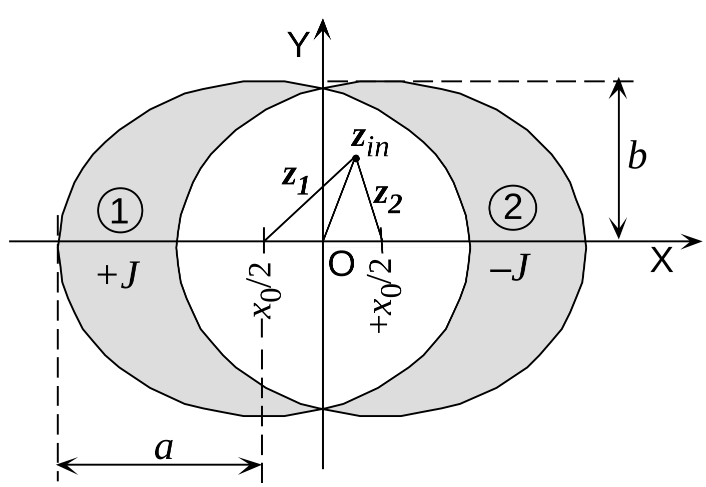

Fig. 11 Using two overlapping elliptical cylinders to generate a pure normal-dipole field.

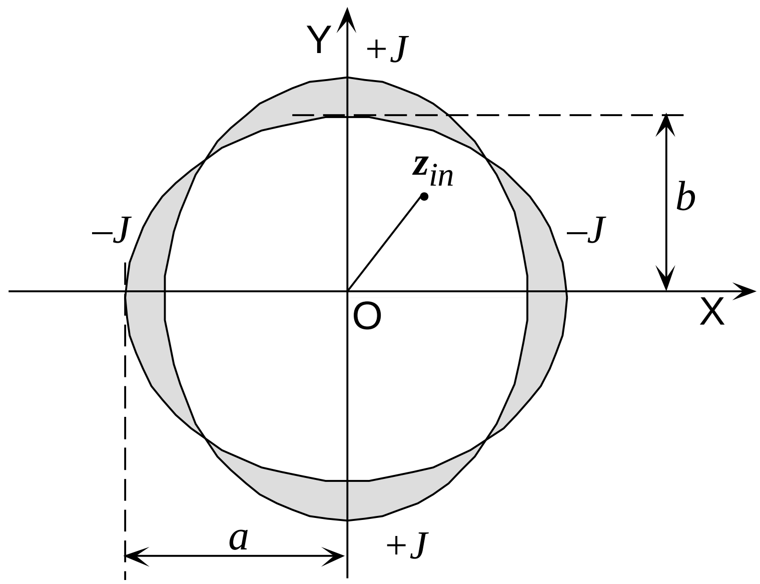

Fig. 12 Generation of a pure normal-quadrupole field using two overlapping cylinders.

8.2 Generating a Pure Quadrupole Field

A scheme for generating a pure normal-quadrupole field is shown in Fig. 12. Two cylinders of the same elliptical cross section carrying equal and opposite current densities are made to intersect at right angles to each other. The region of overlap carries no current, and can be treated as the aperture of the magnet. Any point inside this "aperture" is also inside both the cylinders. The total complex field at any point $\boldsymbol{z_{in}}$ is given by:

$$\boldsymbol{B}_{in}(\boldsymbol{z_{in}}) = \frac{\mu_o J}{(a+b)}\left[-bx + iay + ax - iby\right] = \frac{\mu_o J(a-b)}{(a+b)}(x+iy) = \frac{\mu_o J(a-b)}{(a+b)}\boldsymbol{z_{in}} \qquad (66)$$

This represents a pure normal-quadrupole field [see Eq.(18) or (19)]. The gradient is given by:

$$G = \left(\frac{\partial B_y}{\partial x}\right) = \left(\frac{\partial B_x}{\partial y}\right) = \frac{\mu_o J(a-b)}{(a+b)} = \text{ constant.} \qquad (67)$$

To obtain a skew quadrupole field, the arrangement of Fig. 12 should be rotated by 45 degrees.The gradient is proportional to the difference in the semi-major and semi-minor axes of the ellipse, as seen from Eq. (67). Physically, more oblong ellipses produce smaller apertures and use more total current, thus giving a larger gradient. Clearly, circular cylinders ($b = a$) can not be used to produce a quadrupole field in this fashion.

9. MULTIPOLE EXPANSION OF FIELD DUE TO A CURRENT FILAMENT

In Sec.7-8, we discussed the field produced by several simple current distributions. The expressions derived so far were for the total complex field. In practice, one is also interested in a harmonic description of the field. Once the total field is known for a given current distribution, one could obtain the various harmonic components simply by a series expansion

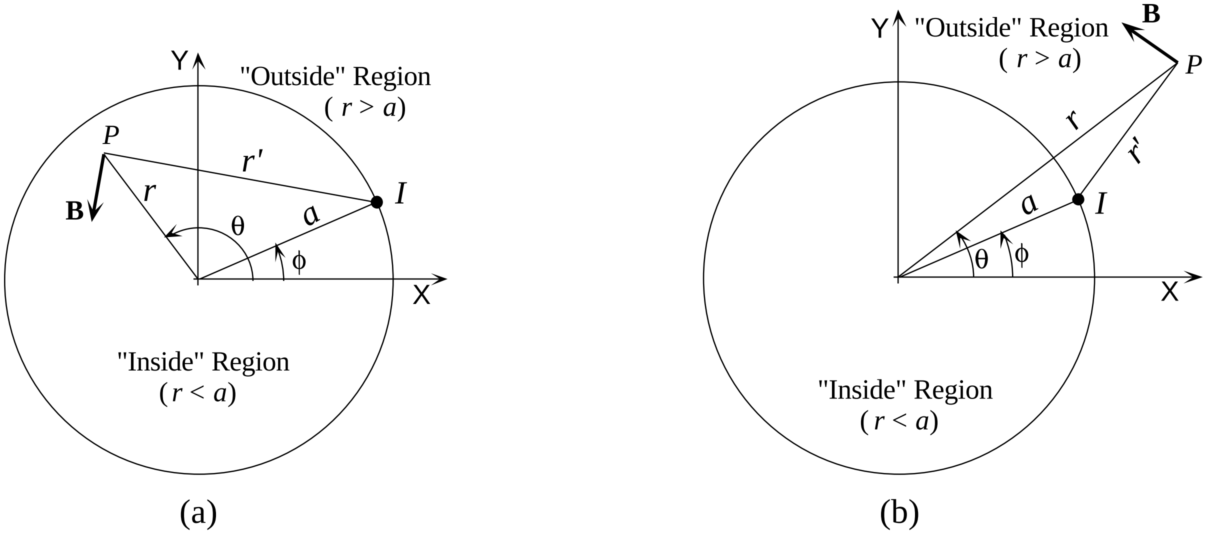

Fig. 13 Calculation of multipole expansion of the field due to a current filament. A field point, *P*, is shown in (a) the region $r < a$, and (b) the region $r > a$.

in the form of Eqs. (18)-(19), or use the differential formulae in Eq. (24). A more convenient approach is to start again with the simplest current distribution given by an infinitely long, infinitesimally thin current filament located at an arbitrary location and express the field produced by this filament [see Eq. (54)] in terms of a multipole expansion. The multipole expansion for any arbitrary current distribution can then be obtained by integrating each harmonic term over the appropriate regions.

The complex field at any point, *P*, due to a current filament located at $\boldsymbol{z} = \boldsymbol{a} = a.\exp(if)$, as shown in Fig. 13, is given by Eq. (54). This expression is valid for all values of $\boldsymbol{z}$ and has a singularity at $\boldsymbol{z} = \boldsymbol{a}$. Various multipole terms in this field can be obtained by a power series expansion. Because of the singularity, such an expansion has a radius of convergence given by $|\boldsymbol{z}| = a$. In any case, a single power series expansion with non-zero coefficients would tend to diverge as $|\boldsymbol{z}| \to \infty$. As a result, we must separate the entire complex plane into two regions – an “inside” region extending to a circle of radius *a*, and an “outside” region extending beyond the radius *a* to infinity, as shown in Fig. 13.

9.1 The “Inside” region

In a typical accelerator magnet, all the current carrying conductors are external to the beam tube. The “inside” region, defined by $r < a$, therefore, is the region of primary importance for accelerator magnets. A multipole expansion of the field in the form of Eq. (18)-(19) is possible in this region. For the case of an infinitely long, thin current filament, we have, from Eq. (54):

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = \left(\frac{m_0 I}{2p}\right).\frac{1}{\boldsymbol{z}-\boldsymbol{a}} = -\left(\frac{m_0 I}{2p\,a\exp(if)}\right)\left[1-\left(\frac{r}{a}\right)\exp\{i(q-f)\}\right]^{-1} \qquad (68)$$

Using the binomial expansion $(1-x)^{-1} = \sum_{n=1}^{\infty} x^{n-1}$, and introducing a *reference radius*, we get,

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = -\left(\frac{m_0 I}{2p\, a}\right) \sum_{n=1}^{\infty} \exp(-inf) \left(\frac{R_{ref}}{a}\right)^{n-1} \left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1} \tag{69}$$

which is in the desired form of Eq. (9) or Eqs. (18)-(19). Comparing with Eq. (9), the amplitude and phase of the 2*n*-pole term are given by:

$$C(n) = \left|\frac{m_0 I}{2p\, a}\right| . \left(\frac{R_{ref}}{a}\right)^{n-1} \;; \quad a_n = f + \frac{p}{n} \;\text{ for }\; I > 0\,; \quad a_n = f \;\text{ for }\; I < 0 \tag{70}$$

where a positive value of current corresponds to current flowing along the positive *Z*-axis. It should be noted that the amplitude is always defined to be a positive quantity and any change in the field direction is absorbed in the phase angle. The NORMAL and SKEW components of the 2*n*-pole field in the “US” and “European” notations are:

$$\left.\begin{aligned} B_{n-1}(\text{US}) = B_n(\text{European}) &= -\left(\frac{m_0 I}{2p\, a}\right) . \left(\frac{R_{ref}}{a}\right)^{n-1} \cos(nf) \\ A_{n-1}(\text{US}) = A_n(\text{European}) &= \left(\frac{m_0 I}{2p\, a}\right) . \left(\frac{R_{ref}}{a}\right)^{n-1} \sin(nf) \end{aligned}\right\} \tag{71}$$

9.2 The “Outside” region

For the “outside” region, we expand in a power series of $(a/\mathbf{z})$ instead of $(\mathbf{z}/a)$, since $|(a/\mathbf{z})| < 1$ in this case. Starting again with Eq. (54), we write:

$$\boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{m_0 I}{2p}\right) \frac{1}{\boldsymbol{z} - \boldsymbol{a}} = \left(\frac{m_0 I}{2p\, r \exp(iq)}\right) \left[1 - \left(\frac{a}{r}\right) \exp\{i(f - q)\}\right]^{-1} \tag{72}$$

Expressing the binomial expansion in the form $(1 - \xi)^{-1} = 1 + \sum_{n=1}^{\infty} \xi^n$, we can write,

$$\boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{m_0 I}{2p\, \boldsymbol{z}}\right) \left[1 + \sum_{n=1}^{\infty} [\cos(nf) + i \sin(nf)] \left(\frac{a}{\boldsymbol{z}}\right)^n\right] \tag{73}$$

This series is NOT in the usual form of the multipole expansion for the field inside a magnet aperture. However, it can be seen that this expansion gives a field which reduces with distance and converges for $|\mathbf{z}| \to \infty$ to $\boldsymbol{B}_{out}(\mathbf{z}) \to [m_0 I/(2p\, \mathbf{z})]$, as expected.

It may seem puzzling at first as to why one is forced to use two different expansions starting from the same expression for the complex field, which is valid everywhere. It should be pointed out that the division of the complex plane into “inner” and “outer” regions is somewhat artificial, and depends solely on the location of the origin, the point around which the series expansion is based. A point that lies in the “outside” region can be brought into the “inside” region by choosing a different origin. Starting from the series expansion for the “inside” region, one can, in fact, obtain the field everywhere, including the points in the “outside” region, by a multistep process. For example, one could use the normal and skew terms at the origin, given by Eq. (71), to calculate the terms at a new origin near the periphery (but still within it) of the “inside” region, using the coordinate transformations described in Sec.4 [see Eqs. (31)-(32)]. This new series expansion will be valid within a circle of a new radius, centered at the new origin. Such a circle will encompass some regions of the complex plane that were earlier in the “outer” region. This is essentially the technique of *analytic continuation*.

9.3 Effect of an Iron Yoke

In accelerator magnets, an iron yoke is almost invariably used as part of the mechanical and magnetic design. In the design of practical magnets, one must use sophisticated numerical calculations to include the detailed geometry as well as non-linear magnetic properties of the iron. However, analytical results can be obtained under the assumption of a constant relative permeability, m_r, and a circular aperture in an infinitely large yoke, as shown in Fig. 14. This gives reasonable results at low fields where the iron is not saturated.

We consider an infinitely long current filament located at $\boldsymbol{a} = a.\exp(if)$, inside a circular aperture of radius R_{yoke} in the iron. The effect of the iron is obtained by applying the appropriate boundary conditions on the magnetic induction, **B**, and the magnetic field, **H**, at the inner surface of the yoke. For calculating the field inside the aperture of the yoke, this effect can be described by replacing the iron with an *image current* of magnitude I' located at $\boldsymbol{a}' = a'.\exp(if')$ where

$$a' = R_{yoke}^2 / a; \quad I' = \left[(m_r - 1)/(m_r + 1)\right] I; \quad f' = f \tag{74}$$

Using Eq. (70), the amplitude and phase of the 2n-pole term are given by:

$$C(n)\exp(-ina_n) = -\left(\frac{m_0 I}{2pa}\right)\left(\frac{R_{ref}}{a}\right)^{n-1}\left[1 + \left(\frac{m_r - 1}{m_r + 1}\right)\left(\frac{a}{R_{yoke}}\right)^{2n}\right].\exp(-inf) \tag{75}$$

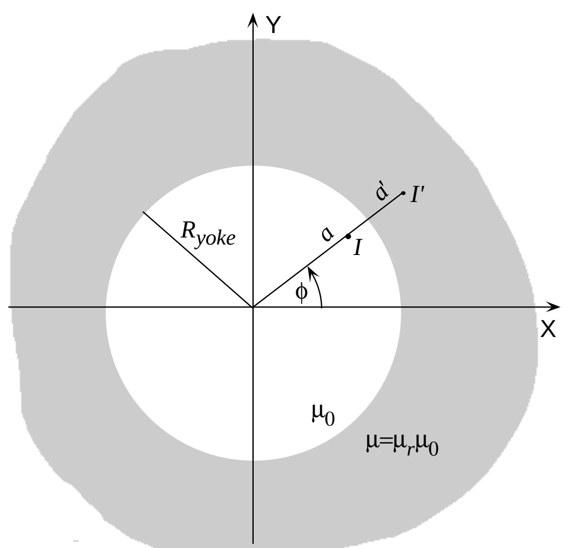

Fig. 14 A current filament inside a circular aperture in an infinite iron yoke.

A comparison of Eqs. (75) and (70) shows that the presence of the iron yoke results in an increase in the field strength. Typically, for a superconducting dipole magnet, the contribution of the iron yoke to the total field is between 10-35%. This can result in substantial savings in the superconductor. A drawback of having a large contribution from the iron is the non-linear behaviour at higher fields, often leading to undesirable higher harmonics due to saturation. Such effects can be minimized to tolerable levels by a careful placement of holes and other features in the iron yoke [13,14].

10. GENERATING A PURE 2*m*-POLE FIELD

We had seen in Sec.8 how simple-looking current distributions obtained by overlapping cylinders could be used to generate a pure dipole or a pure quadrupole field. Another approach towards generating a field of any given multipolarity can be arrived at by looking at the multipole expansion of the field from a thin current filament [see Sec.9].

To generate a pure 2*m*-pole field, the goal is to have all the terms in the expansion vanish, except for $n = m$. Let us assume that such a field is generated by using a thin, cylindrical current shell of radius a, similar to that shown in Fig. 7, except that the current is not uniformly distributed. Instead, the current is a function of the azimuthal angle, ϕ. The complex field in the region inside the current shell is obtained by integrating Eq. (69):

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = -\left(\frac{\mu_0}{2\pi a}\right)\int_0^{2\pi}\sum_{n=1}^{\infty}\left(\frac{\boldsymbol{z}}{a}\right)^{n-1}\exp(-in\phi)I(\phi)d\phi \tag{76}$$

In order to make all the terms except $n = m$ vanish, it is clear that the current distribution $I(\phi)$ must be orthogonal to the function $\exp(-in\phi)$ for all n, except $n = m$. Clearly, such functions are $\sin(n\phi)$ and $\cos(n\phi)$, since

$$\int_0^{2\pi}\cos(n\phi)\cos(m\phi)d\phi = \pi\delta_{mn}; \quad \int_0^{2\pi}\sin(n\phi)\cos(m\phi)d\phi = 0 \tag{77}$$

Therefore, for a "cosine-theta" current distribution given by

$$I(\phi) = I_0\cos(m\phi) \tag{78}$$

the complex field is given by

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = -\left(\frac{\mu_0 I_0}{2a}\right)\left(\frac{\boldsymbol{z}}{a}\right)^{m-1} \tag{79}$$

which represents a pure 2*m*-pole (normal) field. Similarly, for a "sine-theta" current distribution given by $I(\phi) = I_0\sin(m\phi)$, the field is a skew 2*m*-pole field.

Unfortunately, neither the intersecting cylinders, nor the "cosine-theta" distributions can be accurately reproduced in practice. For example, the intersecting cylinders have sharp edges that must be rounded off in practice, leading to unacceptable harmonic distortions. In the design of conductor-dominated magnets (such as superconducting magnets), one has to approximate these distributions with several turns of a finite sized conductor carrying a given current. This has led to the concept of “current blocks” approximation to the “cosine-theta” distribution. In this approach, a *nearly pure* multipole field is generated by employing one or more “current blocks”. Each of the blocks is formed by one or more turns of a conductor with a fixed cross section. The current is usually kept the same in each of the blocks for practical reasons. A description of this approach can be found in reference [15]. As an example, the coils for the arc dipole magnets for the Relativistic Heavy ion Collider (RHIC), being built at the Brookhaven National Laboratory, consist of four current blocks with 9, 11, 8 and 4 turns [16]. Such a design attempts to zero out several of the lower order harmonics and also reduce the remaining higher order terms to negligible levels.

11. CURRENT SYMMETRIES AND ALLOWED HARMONICS

For a general current distribution, all harmonic terms in the multipole expansion of the field would be present. The current distributions in accelerator magnets have various symmetry properties depending on the type of the magnet. These symmetries imply that the normal and/or skew components of certain harmonics must vanish [17]. Such harmonic terms are referred to as the *unallowed* terms. The harmonics that can *possibly* be non-zero are called the *allowed* terms. It should be noted that a particular term may be *allowed*, but *absent* due to a careful design of the magnet. On the other hand, a particular term may be *unallowed*, but *present* in a magnet due to violation of the relevant symmetry as a result of construction errors. In this section, we shall discuss the various symmetries and derive the corresponding allowed terms.

We shall assume that the current density, J, is a function of the azimuthal angle only. This assumption is valid for the current blocks formed with finite sized conductors. Based on Eq. (69), the amplitude and phase of the 2n-pole term are given by:

$$C(n)\exp(-ina_n) \propto \int_0^{2p} J(f) e^{-inf}\, df \qquad (80)$$

The normal and the skew components are proportional to the real and the imaginary parts of the integral in Eq. (80). The fundamental symmetry in a 2m-pole magnet is a 2m-fold rotational antisymmetry, since the north and south poles of such a magnet are interchanged under a rotation by (p/m) radians. In addition, a magnet may have left-right, or top-bottom symmetry (or antisymmetry) if the X-Y axes are properly oriented.

11.1 Allowed harmonics in a 2*m*-pole magnet

It was shown in Sec.10 that a pure 2m-pole field is produced by a $\cos(mq)$ or a $\sin(mq)$ distribution of current. Such a current distribution is antisymmetric under a rotation by (p/m) radians and is symmetric under a rotation by ($2p/m$) radians. Even when the $\cos(mq)$ distribution is approximated by current blocks, this rotational symmetry is still preserved for a 2m-pole magnet. The current density at any azimuthal angle $f + p/m$ is opposite that at f, and so on, as shown in Fig. 15. Using this fact, the integral in Eq. (80) can be written as a sum of 2m terms:

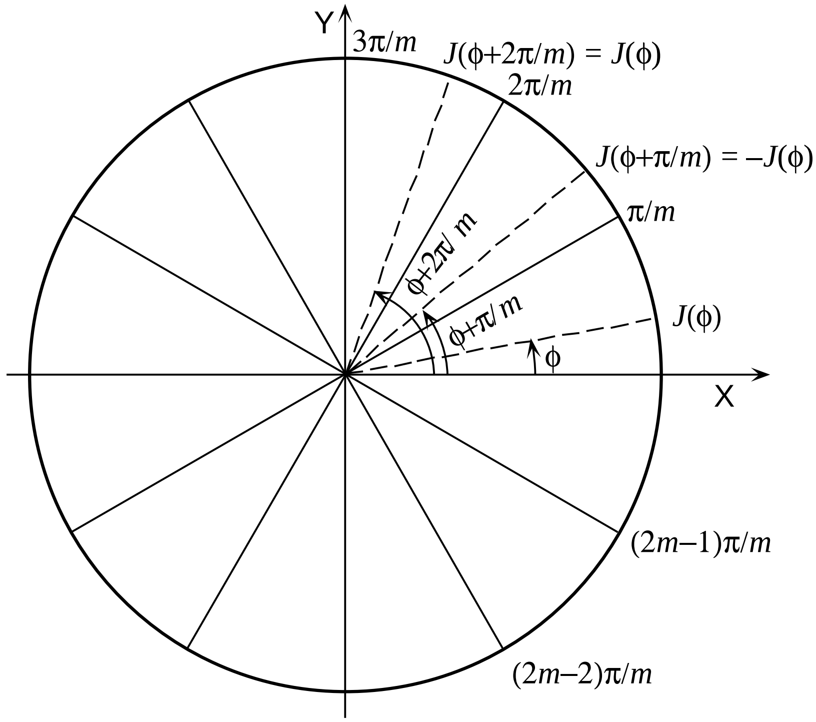

Fig. 15 Rotational symmetry of current density in a 2*m*-pole magnet.

$$C(n)\exp(in\boldsymbol{a}_n) \propto \int_0^{p/m} J(f)e^{inf}[1-e^{inp/m}+e^{2inp/m}+\cdots\ -e^{i(2m-1)p/m}]df \qquad (81)$$

$$\text{or,}\quad C(n)\exp(in\boldsymbol{a}_n) \propto \left[\int_0^{p/m} J(f)e^{inf}\,df\right]\cdot\left[\frac{1-e^{2inp}}{1+e^{inp/m}}\right] \qquad (82)$$

Since n is always an integer, $\exp(2inp) = 1$ and the right hand side in the above integral vanishes, unless the denominator also vanishes. This requires that n must be an odd multiple of m. Thus, for a dipole magnet ($m = 1$), only terms with $n = 1,3,5,...$ are allowed. For a quadrupole magnet (m=2), terms with $n = 2,6,10,...$ are allowed, and so on.

11.2 "Top-Bottom" Symmetry or Anti-symmetry

A "top-bottom" symmetry about the X-axis implies $J(2p-f) = J(f)$, as shown in Fig. 16. In this case, Eq. (80) can be written as:

$$C(n)\exp(in\boldsymbol{a}_n) \propto \int_0^{2p} J(f)e^{inf}\,df = \int_0^{p} J(f)\left[e^{inf}+e^{in(2p-f)}\right]df = \int_0^{p} J(f)\cos(nf)\,df \qquad (83)$$

The result of integration has no imaginary part in this case. This implies that all the skew terms vanish and only the normal terms are allowed as a result of the "top-bottom" symmetry in the current distribution. It is obvious that the current distribution in all normal magnets, such as a normal dipole, a normal quadrupole, etc. must have top-bottom symmetry.

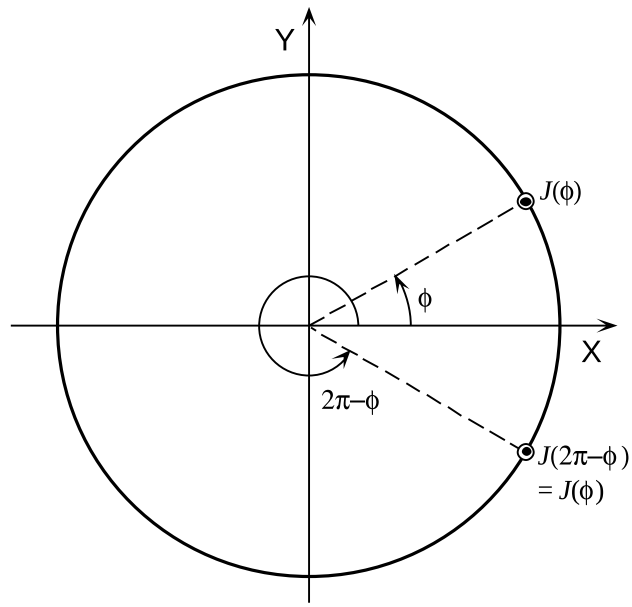

Fig. 16 “Top-bottom” symmetry in the current distribution.

The case of the “top-bottom” antisymmetry is similar, except that the current density satisfies $J(2p - f) = -J(f)$. The integral in Eq. (80) becomes,

$$C(n)\exp(ina_n) \propto \int_0^{2p} J(f)e^{inf}\,df = \int_0^{p} J(f)\left[e^{inf} - e^{in(2p-f)}\right]df = i\int_0^{p} J(f)\sin(nf)\,df \quad (84)$$

The result of integration has no real part in this case. This implies that all the normal terms vanish and only the skew terms are allowed as a result of the “top-bottom” antisymmetry in the current distribution. All skew magnets, such as a skew dipole, a skew quadrupole, etc. have a “top-bottom” antisymmetry.

11.3 “Left-Right” Symmetry or Anti-symmetry

A “left-right” symmetry about the Y-axis implies $J(p - f) = J(f)$, as shown in Fig. 17. In this case, Eq. (80) can be written as:

$$C(n)\exp(ina_n) \propto \int_{-p/2}^{p/2} J(f)\left[e^{inf} + e^{in(p-f)}\right]df = \int_{-p/2}^{p/2} J(f)\left[e^{inf} + (-1)^n e^{-inf}\right]df \quad (85)$$

$$\left.\begin{aligned} C(n)\exp(ina_n) &\propto i\int_{-p/2}^{p/2} J(f)\sin(nf)\,df \quad \text{for ODD } n \\ C(n)\exp(ina_n) &\propto \int_{-p/2}^{p/2} J(f)\cos(nf)\,df \quad \text{for EVEN } n \end{aligned}\right\} \text{Left-Right Symmetry} \quad (86)$$

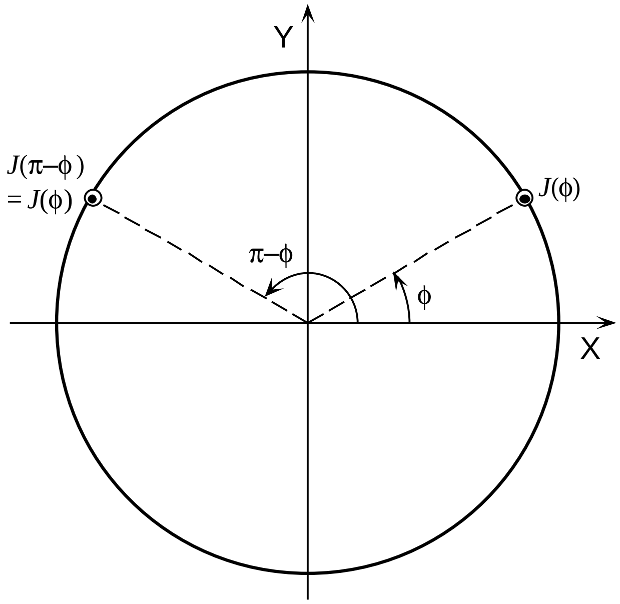

Fig. 17 “Left-Right” symmetry in the current distribution.

The result of integration has no real part for odd multipoles and has no imaginary part for even multipoles. This implies that all the odd normal terms (such as the normal dipole, the normal sextupole, etc.) and all the even skew terms (such as the skew quadrupole, the skew octupole, etc.) vanish as a result of the “left-right” symmetry in the current distribution. All the normal magnets of even order, such as a normal quadrupole, and all the skew magnets of odd order, such as the skew dipole, have this type of current symmetry.

Similarly, a “left-right” antisymmetry implies $J(\mathbf{p} - f) = -J(f)$. In this case, Eq. (80) can be written as:

$$C(n)\exp(ina_n) \propto \int_{-p/2}^{p/2} J(f)\left[e^{inf} - e^{in(p-f)}\right] df = \int_{-p/2}^{p/2} J(f)\left[e^{inf} - (-1)^n e^{-inf}\right] df \qquad (87)$$

$$\left.\begin{aligned} C(n)\exp(ina_n) &\propto i\int_{-p/2}^{p/2} J(f)\sin(nf)\,df \quad \text{for EVEN } n \\ C(n)\exp(ina_n) &\propto \int_{-p/2}^{p/2} J(f)\cos(nf)\,df \quad \text{for ODD } n \end{aligned}\right\} \text{Left-Right Antisymmetry} \qquad (88)$$

The result of integration has no real part for even multipoles and has no imaginary part for odd multipoles. This implies that all the even normal terms (such as the normal quadrupole, the normal octupole, etc.) and all the odd skew terms (such as the skew dipole, the skew sextupole, etc.) vanish as a result of the “left-right” antisymmetry in the current distribution. All the normal magnets of odd order, such as a normal dipole, and all the skew magnets of even order, such as the skew quadrupole, have this type of current symmetry.

The unallowed harmonics under various symmetries are summarized in Table 1. The entries for various normal and skew terms in this table follow the "European notation". To obtain a table in the US notation, the indices should be reduced by one. It should be noted that these symmetries refer to the *current distribution* in the magnet, and not to the symmetry in the shapes of the *pole-tips*. One can obtain similar results for symmetries in the shapes of the pole-tips also [10].

Table 1

Allowed harmonics under various symmetries in the current distribution

Type of Symmetry in the Current Distribution	**Example**	**Normal Terms***	**Skew Terms***
Left-Right Symmetry	Normal Quadrupole	$B_{2k+1} = 0$	$A_{2k} = 0$
Left-Right Anti-symmetry	Normal Dipole	$B_{2k} = 0$	$A_{2k+1} = 0$
Top-Bottom Symmetry	Normal Dipole	—	$A_k = 0$
Top-Bottom Anti-symmetry	Skew Dipole	$B_k = 0$	—

[* *The "European Notation" is used in this table to denote the normal and skew terms.*]

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