FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Basis of Vector Space

Wojciech A. Trybulec[1]
Warsaw University

Summary. We prove the existence of a basis of a vector space, i.e., a set of vectors that generates the vector space and is linearly independent. We also introduce the notion of a subspace generated by a set of vectors and linear independence of set of vectors.

MML Identifier: VECTSP_7.

The terminology and notation used in this paper are introduced in the following papers: [5], [2], [9], [4], [3], [6], [1], [10], [8], and [7]. For simplicity we follow the rules: x will be arbitrary, G_1 will denote a field, a, b will denote elements of G_1, V will denote a vector space over G_1, W will denote a subspace of V, v, v_1, v_2 will denote vectors of V, A, B will denote subsets of V, and l will denote a linear combination of A. We now define two new predicates. Let us consider G_1, V, A. We say that A is linearly independent if and only if:

(Def.1) for every l such that $\sum l = \Theta_V$ holds $\mathrm{support}\, l = \emptyset$.

We say that A is linearly dependent if A is not linearly independent.

One can prove the following propositions:

(1) A is linearly independent if and only if for every l such that $\sum l = \Theta_V$ holds $\mathrm{support}\, l = \emptyset$.

(2) If $A \subseteq B$ and B is linearly independent, then A is linearly independent.

(3) If A is linearly independent, then $\Theta_V \notin A$.

(4) $\emptyset_{\text{the carrier of the carrier of } V}$ is linearly independent.

(5) $\{v\}$ is linearly independent if and only if $v \neq \Theta_V$.

(6) If $\{v_1, v_2\}$ is linearly independent, then $v_1 \neq \Theta_V$ and $v_2 \neq \Theta_V$.

(7) $\{v, \Theta_V\}$ is linearly dependent and $\{\Theta_V, v\}$ is linearly dependent.

(8) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if $v_2 \neq \Theta_V$ and for every a holds $v_1 \neq a \cdot v_2$.

[1]Supported by RPBP.III-24.C1

ISSN 0777–4028

(9) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if for all a, b such that $a \cdot v_1 + b \cdot v_2 = \Theta_V$ holds $a = 0_{G_1}$ and $b = 0_{G_1}$.

Let us consider G_1, V, A. The functor $\mathrm{Lin}(A)$ yields a subspace of V and is defined by:

(Def.2) the carrier of the carrier of $\mathrm{Lin}(A) = \{\sum l\}$.

The following propositions are true:

(10) If the carrier of the carrier of $W = \{\sum l\}$, then $W = \mathrm{Lin}(A)$.

(11) The carrier of the carrier of $\mathrm{Lin}(A) = \{\sum l\}$.

(12) $x \in \mathrm{Lin}(A)$ if and only if there exists l such that $x = \sum l$.

(13) If $x \in A$, then $x \in \mathrm{Lin}(A)$.

The following propositions are true:

(14) $\mathrm{Lin}(\emptyset_{\text{the carrier of the carrier of } V}) = \mathbf{0}_V$.

(15) If $\mathrm{Lin}(A) = \mathbf{0}_V$, then $A = \emptyset$ or $A = \{\Theta_V\}$.

(16) If $A =$ the carrier of the carrier of W, then $\mathrm{Lin}(A) = W$.

(17) If $A =$ the carrier of the carrier of V, then $\mathrm{Lin}(A) = V$.

(18) If $A \subseteq B$, then $\mathrm{Lin}(A)$ is a subspace of $\mathrm{Lin}(B)$.

(19) If $\mathrm{Lin}(A) = V$ and $A \subseteq B$, then $\mathrm{Lin}(B) = V$.

(20) $\mathrm{Lin}(A \cup B) = \mathrm{Lin}(A) + \mathrm{Lin}(B)$.

(21) $\mathrm{Lin}(A \cap B)$ is a subspace of $\mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

(22) If A is linearly independent, then there exists B such that $A \subseteq B$ and B is linearly independent and $\mathrm{Lin}(B) = V$.

(23) If $\mathrm{Lin}(A) = V$, then there exists B such that $B \subseteq A$ and B is linearly independent and $\mathrm{Lin}(B) = V$.

Let us consider G_1, V. A subset of V is called a basis of V if:

(Def.3) it is linearly independent and $\mathrm{Lin}(\mathrm{it}) = V$.

We now state the proposition

(24) If A is linearly independent and $\mathrm{Lin}(A) = V$, then A is a basis of V.

In the sequel I will denote a basis of V. We now state four propositions:

(25) I is linearly independent.

(26) $\mathrm{Lin}(I) = V$.

(27) If A is linearly independent, then there exists I such that $A \subseteq I$.

(28) If $\mathrm{Lin}(A) = V$, then there exists I such that $I \subseteq A$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[4] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[6] Wojciech A. Trybulec. Finite sums of vectors in vector space. *Formalized Mathematics*, 1(**5**):851–854, 1990.

[7] Wojciech A. Trybulec. Linear combinations in vector space. *Formalized Mathematics*, 1(**5**):877–882, 1990.

[8] Wojciech A. Trybulec. Operations on subspaces in vector space. *Formalized Mathematics*, 1(**5**):871–876, 1990.

[9] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.

[10] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Formalized Mathematics*, 1(**5**):865–870, 1990.

Received July 27, 1990