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Basis Reduction for Cryptogroups and Orthogroups

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Abstract The goal of this note is to provide equivalent bases of identities for subvarieties of completely regular semigroups.

Keywords Equational characterizations · completely regular semigroups · cryptogroups · orthogroups

The search for equivalent definitions of famous classes of algebras has been attracting the attention of mathematicians for the last one hundred years [9–24, 26–28, 30–33]. Especially famous cases are Tarski's single law for abelian groups [30] and Higman and Neumann's single law for groups [10]. Here we carry the same study for semigroups as in [1–6, 8, 29].

We study equivalent definitions for completely regular semigroups, cryptogroups, and orthogroups. Unlike what happens with groups, it is a folklore result in equational algebra that no class of completely regular semigroups in this note is single based [2, 4, 5]. It is worth observing that the difficulty of the results, rather than from proofs, comes more from the task of finding elegant and simpler versions of the original bases of identities.

We will concentrate in finding equivalent set of identities which define classes of completely regular semigroups, cryptogroups and orthogroups, giving an alternative characterization of them, with less number of identities. A principal reference for the definitions and properties of these classes of semigroups is the book [25].

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Throughout the paper $(S,\cdot,')$ is a semigroup with a unary operation, $E(S)$ its set of idempotents, $x^\circ := xx'$ and ${}^\circ x := x'x$, and associativity will be assumed except for completely regular semigroups and left regular semigroups, where the associativity axiom is replaced by another one. The classes induced by the corresponding right notions, that is, right regular (normal) cryptogroup or orthogroup, are treated similarly (see [25, Corollary IV.2.12]).

Table 1 summarizes new characterizations for different subvarieties of completely regular semigroups, orthogroups and cryptogroups. The second column has the definitions, the third the characterization provided by [25] and the last one has ours.

Table 1

Variety	Definition	[25]	Alternative
Completely Regular Semigroup	$\forall a \in S \exists a' \in S:$ $a = aa'a \wedge$ $\wedge a' = a'aa' \wedge$ $\wedge aa' = a'a$	$x(yz) = (xy)z$ (1) $x = x^\circ x$ (2) $x^\circ = {}^\circ x$ (3) $x'' = x$ (4)	$x(yz'') = (xy)z$ (5) $x = x^\circ x$ (2) $x^\circ = {}^\circ x$ (3)
Cryptogroup	Completely Regular Semigroup and Green's relation $\mathscr{H}$ is a congruence	$x(yz) = (xy)z$ (1) $x = x^\circ x$ (2) $x^\circ = {}^\circ x$ (3) $x'' = x$ (4) $(x^\circ y^\circ)^\circ = (xy)^\circ$ (6)	$x(yz) = (xy)z$ (1) $x = x^\circ x$ (2) $x'' = x$ (4) $({}^\circ x y^\circ)^\circ = (xy)^\circ$ (7)
Regular Cryptogroup	Cryptogroup and $S/\mathscr{H}$ is a regular band	$x(yz) = (xy)z$ (1) $x = x^\circ x$ (2) $x^\circ = {}^\circ x$ (3) $x'' = x$ (4) $(xyzx)^\circ = (xyxzx)^\circ$ (8)	$x(yz) = (xy)z$ (1) $x = x^\circ x$ (2) $x'' = x$ (4) ${}^\circ(xyzx) = (xyxzx)^\circ$ (9)
Left Regular Cryptogroup	Cryptogroup and $S/\mathscr{H}$ is a left regular band	$x(yz) = (xy)z$ (1) $x = x^\circ x$ (2) $x^\circ = {}^\circ x$ (3) $x'' = x$ (4) $(xy)^\circ = x^\circ y^\circ x^\circ$ (10)	$x(yz'') = (xy)z$ (5) $x = x^\circ x$ (2) $(xy)^\circ = {}^\circ x(y^\circ {}^\circ x)$ (11)
Normal Cryptogroup	Cryptogroup and $S/\mathscr{H}$ is a normal band	$x(yz) = (xy)z$ (1) $x = x^\circ x$ (2) $x^\circ = {}^\circ x$ (3) $x'' = x$ (4) $(xyzx)^\circ = (xzyx)^\circ$ (12)	$x(yz) = (xy)z$ (1) $x = x^\circ x$ (2) $x'' = x$ (4) $(xyzx)^\circ = {}^\circ(xzyx)$ (13)
Left Normal Cryptogroup	Cryptogroup and $S/\mathscr{H}$ is a left normal band	$x(yz) = (xy)z$ (1) $x = x^\circ x$ (2) $x^\circ = {}^\circ x$ (3) $x'' = x$ (4) $xy^\circ z = xzy^\circ$ (14)	$x(yz) = (xy)z$ (1) $x = x^\circ x$ (2) $(x')^\circ y^\circ z = x^\circ zy^\circ$ (15)
Orthogroup	Completely Regular Semigroup and $E(S)$ is a semigroup	$x(yz) = (xy)z$ (1) $x = x^\circ x$ (2) $x^\circ = {}^\circ x$ (3) $x'' = x$ (4) $(x^\circ y^\circ)^\circ = x^\circ y^\circ$ (16)	$x(yz) = (xy)z$ (1) $x = x^\circ x$ (2) $x^\circ = {}^\circ x$ (3) $(x^\circ y^\circ)^\circ = (x')^\circ y^\circ$ (17)

Table 1 continued

Variety	Definition	[25]	Alternative
Regular Orthogroup	Orthogroup and $E(S)$ is a regular band	$x(yz)=(xy)z$ (1) $x=x^\circ x$ (2) $x^\circ={}^\circ x$ (3) $x''=x$ (4) $xyzx=xyx^\circ zx$ (18)	$x(yz)=(xy)z$ (1) $x=x^\circ x$ (2) $x^\circ={}^\circ x$ (3) $x''yzx''=xyx^\circ zx$ (19)
Left Regular Orthogroup	Orthogroup and $E(S)$ is a left regular band	$x(yz)=(xy)z$ (1) $x=x^\circ x$ (2) $x^\circ={}^\circ x$ (3) $x''=x$ (4) $xy=xyx^\circ$ (20)	$x(yz)=(xy)z$ (1) $x=x^\circ x$ (2) $x'y=x^\circ x'y''x^\circ$ (21)
Normal Orthogroup	Orthogroup and $E(S)$ is a normal band	$x(yz)=(xy)z$ (1) $x=x^\circ x$ (2) $x^\circ={}^\circ x$ (3) $x''=x$ (4) $xyz^\circ x=xz^\circ yx$ (22)	$x(yz)=(xy)z$ (1) $x=x^\circ x$ (2) $x^\circ={}^\circ x$ (3) $xy(z')^\circ x=xz^\circ yx$ (23)

The new results described in the fourth column of Table 1 are proved in the following paragraphs.

*Completely regular semigroups.*That identities (1)–(4) imply identity (5) is trivial. Conversely, we only need to prove that identities (2), (3) and (5) imply identity (4). Note that $x^\circ=(x')^\circ$ (*), indeed,

$$x^\circ \overset{(3)}{=} {}^\circ x \overset{(2)}{=} ((x')^\circ x')x \overset{(5)}{=} (x'(x'')^\circ)x \overset{(5)}{=} x'((x'')^\circ x'') \overset{(2)}{=} (x')^\circ.$$

Now,

$$x \overset{(2)}{=} x^\circ x \overset{(*)}{=} (x')^\circ x \overset{(3)}{=} {}^\circ(x')x \overset{(5)}{=} x''(x')^\circ \overset{(*)}{=} x''(x'')^\circ \overset{(*)}{=} x''(x''')^\circ \overset{(5)}{=} (x'')^\circ x'' \overset{(2)}{=} x''.$$

*Cryptogroups.*That identities (2)–(4) and (6) imply identity (7) is trivial. Conversely, we only need to prove that identities (2), (4) and (7) imply identities (3) and (6). Note that $(x^\circ)'=x^\circ$ (*), indeed,

$$\begin{aligned} {}^\circ x x {}^\circ x &\overset{(2)}{=} {}^\circ x(x{}^\circ x)^\circ x{}^\circ x \overset{(7)}{=} {}^\circ x({}^\circ x({}^\circ x)^\circ)^\circ x{}^\circ x \overset{(2),(4)}{=} {}^\circ x({}^\circ x)^\circ({}^\circ x({}^\circ x)^\circ)' x{}^\circ x \\ &\overset{(7)}{=} (x{}^\circ x)^\circ x{}^\circ x \overset{(2)}{=} x{}^\circ x \overset{(2)}{=} x. \quad (**) \end{aligned}$$

and thus,

$$\begin{aligned} (x^\circ)' &\overset{(2),(4)}{=} (x^\circ)'x^\circ(x^\circ)' \overset{(**)}{=} (x^\circ)'x^\circ[(x^\circ)'x^\circ x^\circ]' \overset{(2)}{=} (x^\circ)'x^\circ[(x^\circ)'x^\circ]' \\ &\overset{(2)}{=} ({}^\circ(x^\circ)x^\circ)^\circ \overset{(7)}{=} (x^\circ x)^\circ \overset{(2)}{=} x^\circ. \end{aligned}$$

Now,

$$x^\circ \overset{(2)}{=} x^\circ x^\circ \overset{(*)}{=} x^\circ(x^\circ)' = (xx')^\circ \overset{(7),(4)}{=} ({}^\circ x{}^\circ x)^\circ \overset{(2),(4)}{=} ((x')^\circ)^\circ \overset{(*)}{=} (x')^\circ(x')^\circ \overset{(2),(4)}{=} {}^\circ x.$$

Regular Cryptogroups. That identities (2)–(4) and (8) imply identity (9) is trivial. Conversely, we only need to prove that identities (2), (4) and (9) imply identity (3). Note that $x^\circ\, {}^\circ x = {}^\circ x$ (*), indeed,

$$x^\circ\, {}^\circ x \overset{(2)}{=} x^\circ\, {}^\circ(x^\circ x) \overset{(2)}{=} x^\circ\, {}^\circ((x^\circ)^\circ x^\circ x) \overset{(2)}{=} x^\circ\, {}^\circ((x^\circ)^\circ x) \overset{(9)}{=} x^\circ\,(xx'x(xx')'x)^\circ$$
$$\overset{(2)}{=} (xx'x(xx')'x)^\circ \overset{(9)}{=} {}^\circ(xx'(xx')'x) \overset{(2)}{=} {}^\circ((x^\circ)^\circ x^\circ x) \overset{(2)}{=} {}^\circ(x^\circ x) \overset{(2)}{=} {}^\circ x.$$

Now,

$${}^\circ x \overset{(2)}{=} {}^\circ x\, {}^\circ({}^\circ x) \overset{(*)}{=} {}^\circ x\, {}^\circ(xx'x'x) \overset{(9)}{=} {}^\circ x(xx'xx'x)^\circ \overset{(2)}{=} {}^\circ xx^\circ \overset{(4)}{=} (x')^\circ\, {}^\circ(x') \overset{(*)}{=} {}^\circ(x') \overset{(4)}{=} x^\circ.$$

Left Regular Cryptogroups. That identities (1)–(4) and (10) imply axioms (5) and (11) is trivial. To prove the converse, we only need to prove that identities (5), (2) and (11) imply (3) and (4). Note that ${}^\circ x = (x')^\circ$ (*), indeed,

$${}^\circ x \overset{(2)}{=} [(x')^\circ x']x \overset{(5)}{=} [x'(x'')^\circ]x \overset{(5)}{=} x'[(x'')^\circ x''] \overset{(2)}{=} (x')^\circ.$$

Now,

$$x^\circ \overset{(2)}{=} (x^\circ x)^\circ \overset{(5)}{=} (x(x')^\circ)^\circ \overset{(*)}{=} (x\, {}^\circ x)^\circ \overset{(11)}{=} {}^\circ x(({}^\circ x)^\circ\, {}^\circ x) \overset{(2)}{=} {}^\circ x\, {}^\circ x \overset{(*)}{=} (x')^\circ\, (x')^\circ \overset{(5)}{=} (x')^\circ x'x \overset{(2)}{=} {}^\circ x$$

and

$$x \overset{(2)}{=} x^\circ x \overset{(3),(*)}{=} {}^\circ(x')x \overset{(5)}{=} x''(x')^\circ \overset{(*),(3)}{=} x''(x''')^\circ \overset{(5)}{=} (x'')^\circ x'' \overset{(2)}{=} x''.$$

Normal Cryptogroups. That identities (2)–(4) and (12) imply identity (13) is trivial. To prove the converse, we only need to prove that identities (2), (4) and (13) imply identity (3). Note that $x^\circ\, {}^\circ x = {}^\circ x = xxx'x'$ (*), indeed,

$$x^\circ\, {}^\circ x \overset{(2)}{=} x^\circ\, {}^\circ(x^\circ x) \overset{(2)}{=} x^\circ\, {}^\circ((x^\circ)^\circ x^\circ x) \overset{(2)}{=} x^\circ\, {}^\circ((x^\circ)^\circ x) \overset{(13)}{=} x^\circ\,(x(xx')'x'x)^\circ \overset{(2)}{=} (x(xx')'x'x)^\circ$$
$$\overset{(13)}{=} {}^\circ(xx'(xx')'x) \overset{(2)}{=} {}^\circ((x^\circ)^\circ x^\circ x) \overset{(2)}{=} {}^\circ x$$

and

$$xxx'x' \overset{(2),(4)}{=} xxx'x'xx' \overset{(2)}{=} xxx'x'x(xx'x)' \overset{(*)}{=} (xxx'x'x)^\circ \overset{(13)}{=} {}^\circ(xx'xx'x) \overset{(2)}{=} {}^\circ x.$$

Now,

$${}^\circ x \overset{(*)}{=} xxx'x' \overset{(2),(4)}{=} xx^\circ\, {}^\circ xx' \overset{(*)}{=} x\, {}^\circ xx' \overset{(4)}{=} xx'x''x' \overset{(2)}{=} x^\circ.$$

Left Normal Cryptogroups. That identities (2)–(4) and (14) imply identity (15) is trivial. To prove the converse, we only need to prove that identities (2) and (15) imply identities (4) and (14). Now,

$$xy^\circ z \overset{(2)}{=} x^\circ xy^\circ z \overset{(15)}{=} (x')^\circ y^\circ xz \overset{(15)}{=} x^\circ xzy^\circ \overset{(2)}{=} xzy^\circ$$

Note that $x = xx^\circ$ (*) and $(x')^\circ = x^\circ$ (**), indeed,

$$x \overset{(2)}{=} x^\circ x^\circ x \overset{(14)}{=} x^\circ xx^\circ \overset{(2)}{=} xx^\circ \quad \text{and}$$

$$(x')^\circ \overset{(2)}{=} (x')^\circ (x')^\circ \overset{(2)}{=} (x')^\circ ((x')^\circ)^\circ (x')^\circ \overset{(15)}{=} x^\circ (x')^\circ ((x')^\circ)^\circ \overset{(*)}{=} x^\circ (x')^\circ \overset{(*)}{=} x^\circ$$

thus,

$$x'' \overset{(2)}{=} (x'')^\circ x'' \overset{(**)}{=} (x')^\circ x'' \overset{(**)}{=} x^\circ x'' = x(x')^\circ \overset{(**)}{=} xx^\circ \overset{(*)}{=} x.$$

*Orthogroups.*That identities (2)–(4) and (16) imply identity (17) is trivial. To prove the converse, we only need to prove that identities (2), (3) and (17) imply identity (4). Note that, $(x')^\circ = x^\circ$ (*), indeed,

$$(x')^\circ \overset{(2)}{=} (x')^\circ (x')^\circ \overset{(17)}{=} (x^\circ (x')^\circ)^\circ \overset{(2),(3)}{=} (x^\circ)^\circ \overset{(2)}{=} (x^\circ x^\circ)^\circ \overset{(17)}{=} (x')^\circ x^\circ \overset{(2),(3)}{=} x^\circ$$

Now,

$$x \overset{(2)}{=} x^\circ x \overset{(*)}{=} (x')^\circ x \overset{(3)}{=} x''\,{}^\circ x \overset{(3)}{=} x'' x^\circ \overset{(*)}{=} x''(x')^\circ \overset{(*)}{=} x''(x'')^\circ \overset{(3)}{=} (x'')^\circ x'' \overset{(2)}{=} x''.$$

*Regular Orthogroups.*That identities (2)–(4) and (18) imply identity (19) is trivial. To prove the converse, we only need to prove that identities (2), (3) and (19) imply identity (4). Note that $(x')^\circ = x^\circ$ (*), indeed,

$$(x')^\circ \overset{(3)}{=} {}^\circ(x') \overset{(2),(3)}{=} {}^\circ(x')\,(x')^\circ \overset{(19)}{=} x^\circ x^\circ\,{}^\circ x \overset{(2),(3)}{=} x^\circ x^\circ \overset{(2)}{=} x^\circ$$

Now,

$$x \overset{(2)}{=} x^\circ x \overset{(*)}{=} (x')^\circ x \overset{(3)}{=} x''\,{}^\circ x \overset{(3)}{=} x'' x^\circ \overset{(*)}{=} x''(x')^\circ \overset{(*)}{=} x''(x'')^\circ \overset{(3)}{=} (x'')^\circ x'' \overset{(2)}{=} x''.$$

*Left Regular Orthogroups.*That identities (2)–(4) and (20) imply identity (21) is achieved as follows (note that $(x')^\circ = x^\circ$ (•)):

$$x'y \overset{(20)}{=} x'y(x')^\circ \overset{(2)}{=} (x')^\circ x'y(x')^\circ \overset{(\bullet)}{=} x^\circ x'yx^\circ \overset{(4)}{=} x^\circ x'y''x^\circ.$$

To prove the converse, we only need to prove that identities (2) and (21) imply identities (3), (4) and (20). Note that $x^\circ x' = x'$ (*), $x = x''x^\circ$ (**) and $x'' = (x')^\circ x$ (***), indeed,

$$x^\circ x' \overset{(2)}{=} x^\circ (x')^\circ x' \overset{(21)}{=} x^\circ x^\circ x'x''''x^\circ x' \overset{(2)}{=} x^\circ x'x''''x^\circ x' \overset{(21)}{=} (x')^\circ x' \overset{(2)}{=} x'.$$

The second one is given by:

$$x \overset{(2)}{=} x^\circ x \overset{(21)}{=} xx^\circ (x')^\circ x^\circ \overset{(*)}{=} x^\circ x''x^\circ \overset{(*)}{=} x^\circ (x')^\circ x''x^\circ \overset{(*)}{=} (x')^\circ x''x^\circ \overset{(*)}{=} x''x^\circ,$$

and finally the third auxiliary identity is achieved in the following way:

$$\begin{aligned} x'' &\overset{(2)}{=} (x'')^\circ x'' \overset{(*)}{=} x''(x'')^\circ\,{}^\circ(x'') \overset{(*)}{=} x''x'x''x''x'''x'''x'' \overset{(*)}{=} x''x'x''x'''x'' \\ &\overset{(2)}{=} {}^\circ(x')x'' \overset{(*)}{=} x'x''x''x'x'' \overset{(*)}{=} x'(x')^\circ x''x''(x')^\circ \overset{(21)}{=} (x')^\circ x. \end{aligned}$$

Now,

$$x'' \overset{(***)}{=} (x')^\circ x \overset{(**)}{=} (x')^\circ x''x^\circ \overset{(*)}{=} x''x^\circ \overset{(**)}{=} x$$

and

$$x^\circ \overset{(**)}{=} xx'''(x')^\circ \overset{(4)}{=} x^\circ x'x \overset{(*)}{=} {}^\circ x.$$

Finally,

$$xy \overset{(4)}{=} x''y \overset{(21)}{=} (x')^\circ x''y''(x')^\circ \overset{(4)}{=} (x')^\circ x''y(x')^\circ \overset{(*)}{=} x''y(x')^\circ \overset{(4)}{=} xy^\circ x \overset{(3)}{=} xyx^\circ.$$

*Normal Orthogroups.*That identities (2)–(4) and 22 imply identity (23) is trivial. To prove the converse, we only need to prove that identities (2), (3) and (23) imply identity (4). Note that $xyx = xy''x$ (*), indeed,

$$xyx \overset{(2)}{=} xy^\circ yx \overset{(23)}{=} xy(y')^\circ x = xy^\circ y''x \overset{(23)}{=} xy''(y')^\circ x \overset{(3)}{=} x(y')^\circ y''x \overset{(23)}{=} xy''(y'')^\circ x \overset{(3),(2)}{=} xy''x.$$

Now,

$$x'' \overset{(2)}{=} (x'')^\circ x'' \overset{(*),(3)}{=} (x')^\circ x'' \overset{(2)}{=} (x')^\circ (x')^\circ x'' \overset{(*)}{=} {}^\circ x(x')^\circ x'' \overset{(3)}{=} xx'x''x'x'' \overset{(*)}{=} xx'xx'x''$$
$$\overset{(3)}{=} xxx'x''x' \overset{(*)}{=} xxx'xx' \overset{(3),(2)}{=} x.$$

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