Baxter's Q-operators and operatorial Bäcklund flow

for quantum (super)-spin chains

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ABSTRACT: We propose the operatorial form of Baxter's TQ-relations in a general form of the operatorial Bäcklund flow describing the nesting process for the inhomogeneous rational $gl(K|M)$ quantum (super)spin chains with twisted periodic boundary conditions. The full set of Q-operators and T-operators on all levels of nesting is explicitly defined. The results are based on a generalization of the identities among the group characters and their group co-derivatives with respect to the twist matrix, found by one of the authors and P.Vieira [1]. Our formalism allows a systematic and rather straightforward derivation of the whole set of nested Bethe ansatz equations for the spectrum of quantum integrable spin chains, starting from the R-matrix.

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Contents

1. Introduction

It has been noticed long ago that the mathematical structures behind the quantum integrable spin chains have many similarities with the theory of classical integrable systems, such as KP or KdV hierarchies. It goes of course not only about the obvious correspondence between the quantum integrable systems and their classical limits when, for example, the quantum transfer matrix of a quantum 1+1-dimensional system becomes the classical monodromy matrix of the corresponding classical Lax connection, but rather about more striking "classical" features of the quantum integrability: the quantum transfer matrix represents a natural (spectral parameter dependant) generalization of the Schur character of a classical algebra [2] given by the so called Bazhanov-Reshetikhin (BR) determinant formula and, as such, it satisfies a certain Hirota bilinear finite difference equation, which appears in the quantum context as a certain fusion relation among the composite quantum states appearing in quantum spin chains as certain bound states ("strings") of Bethe roots [3, 4, 5, 6, 7]. Similar, though a more complicated realization of Hirota discrete "classical" integrable dynamics has been observed in the context of the quantum (1+1)-dimensional QFT's, or sigma-models [6, 8, 9], an observation which appeared to be at the heart of an important advance in the study of the spectrum of the AdS_5/CFT_4 correspondence [10, 11].

Hirota equation immediately brings us to an idea that quantum integrability, at least for certain quantities, such as transfer-matrix eigenvalues, can be viewed as a specific case of classical integrability and of the theory of classical tau-functions. Indeed, the character of a classical group, say $gl(K)$, is nothing but the tau-function of the KdV hierarchy. It was proposed in [1] to view the quantum transfer matrix of a rational quantum Heisenberg-type $gl(K|M)$ (super)-spin chain with twisted boundary conditions as a quantum, operatorial generalization of the character and to construct the transfer matrix (T-operator) by acting on the character in a given irrep, as a function of the group element (twist), by special group derivatives, called the co-derivatives. The formalism of co-derivatives has led to a direct proof of the BR formula (see also [12]), and the basic underlying identity for the characters found in [1] seems to be just a new form of the KdV Hirota identity (the fact yet to be understood).

In the present paper, we want to move even further in this classical interpretation of the quantum integrability and to generalize the basic identity of [1] to include the Baxter's TQ-relations into our formalism. This implies a natural definition of all Baxter's Q-operators, rather different from the one known in the literature [13, 6, 14, 15, 16, 17, 18, 19], and more generally, of the T-operators on all levels of the nesting procedure. This nesting takes a form of a Bäcklund flow, directly for the T- and Q-operators. Due to the fact that all of them belong to a commuting family of operators, all these relations can be immediately transformed into the well known functional form, for their eigenvalues [20, 21, 22, 23].

Our main identity given in the next section offers an interesting alternative and a concise approach to the quantum integrability uncovering the whole structure of the nested Bethe ansatz, from the R-matrix and the Yang-Baxter relations and all the way to the nested Bethe ansatz equations, in the general operatorial form for all the intermediate quantities.

2. Transfer-matrix, co-derivative and TQ-relations

Here we recall that the main object of our study is the transfer matrix of an inhomogeneous quantum spin chain

$$T^{\{\lambda\}}(u) = \mathrm{tr}_\lambda \left(R_N^{\{\lambda\}}(u - \theta_N) \otimes \cdots \otimes R_1^{\{\lambda\}}(u - \theta_1)\, \pi_\lambda(g) \right) \tag{2.1}$$

where $\pi_\lambda(g)$ is a matrix element of a the twist matrix $g \in GL(K)$ in an irrep λ and

$$R_i^{\{\lambda\}}(u) \equiv u + 2\sum_{\alpha\beta} \pi_i(e_{\beta\alpha}) \otimes \pi_\lambda(e_{\alpha\beta}) \equiv u + 2\,\mathcal{P}_{i,\lambda} \tag{2.2}$$

is the R-matrix in irrep λ in auxiliary space (and in fundamental irrep in the quantum space). The $gl(K)$ generator[1] $\pi_i(e_{\alpha\beta})$ corresponds to the ith quantum space (which is in the fundamental representation) and $\pi_\lambda(e_{\alpha\beta})$ - to the auxiliary space. When λ is also fundamental then $\mathcal{P}_{i,\lambda}$ becomes a usual permutation operator $\mathcal{P}_i = \sum_{\alpha\beta} e_{\beta\alpha}^{(i)} \otimes e_{\alpha\beta}$. $\mathcal{P}_i$ permuting the indices of the auxiliary space and the quantum subspace.

The main goal is to find all the eigenvalues of this transfer matrix for the $gl(K)$ quantum spins. For that we work out an operatorial Bäcklund transformation, which can be also called the nesting procedure, whose main goal is to derive, in a deductive way, without any assumptions, the nested system of Bethe ansatz equations defining these eigenvalues. On the way, we will encounter a collection of the intermediate T-operators, and the Baxter's Q-operators as their particular case, at each level of nesting. The operatorial TQ-relations, representing the Bäcklund transformation reducing the problem for a $gl(k)$ subgroup to a similar problem for the $gl(k-1)$ subgroup, in the nesting procedure corresponding to the chain of embeddings $gl(K) \supset gl(K-1) \supset \cdots \supset gl(1)$ were given in their functional form in [21] (for the super-spin chains in [20]). The general TQ-relations, which will be derived in sec.4 in the operatorial form, are given by

$$T_I^s(u)Q_{I,j}(u) = T_{I,j}^s(u)Q_I(u) - x_j T_{I,j}^{s-1}(u+2)Q_I(u-2) \tag{2.3}$$

where the twist $g = \mathrm{diag}(x_1, x_2, \cdots, x_N)$ in the diagonal basis, the superscript s in T-operator denotes the symmetric $\lambda = (s)$ irrep in the auxiliary space, by $I \subset \mathbb{F} = \{1, 2, \cdots, K\}$ we denote a subset of the full set of indices (labeling the eigenvalues) and by $I, j \equiv I \cup \{j\} \subset \mathbb{F}$ we denote a subset with one more index $j \notin I$ [2]. This TQ-relation relates the T-operator T_I^s and the T-operator $T_{I,j}^s$ of the previous level of nesting (which has one more index). A chain of these relations allows to relate the original transfer matrix $T_{\mathbb{F}}^s(u)$ on the highest level of nesting to the u-independent[3] operator $T_\emptyset^s(u)$ on equal to the T-operators taken at an empty Young diagram representation:

$$Q_I(u) = T_I^0(u). \tag{2.4}$$

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[1] The generalization of our construction to the case of $gl(K|M)$ super-spins will be given in sec.5.

[2] In this paper we use an enumeration of all the 2^{K+M} Q-functions for $gl(K|M)$ by index sets and the Hasse diagram discussed in [23].

[3] The exact expression for $T_\emptyset^s(u)$ is given by (4.30)

In the papers [21, 20], all the T- and Q-operators at intermediate steps were assumed, by self-consistency and without a proof, to be polynomials in u. This analyticity assumption immediately leads to the nested Bethe ansatz equations defined by the nesting path. In this paper, we complete the missing link of the chain and find the explicit operatorial form of the Bäcklund flow (2.3).

In what follows, we will extensively use the definitions and the identities of [1]. In particular, the co-derivative $\hat{D}$ defined there and used through the whole current paper is a very simple object defined by its action on any function of g as follows

$$\hat{D}f(g) = \frac{\partial}{\partial \phi} \otimes f(e^{\phi \cdot e}g)\bigg|_{\phi=0} \tag{2.5}$$

where ϕ is a matrix in the fundamental representation and $\phi \cdot e \equiv \sum_{\alpha\beta} e_{\alpha\beta}\phi^{\alpha}_{\beta}$. Its main property, which also could serve as its definition, manifests in its action on the group element in fundamental irrep:

$$\hat{D} \otimes g = \mathcal{P}\left(1 \otimes g\right).$$

where $\mathcal{P}$ is the operator of permutation between the 1st and the 2nd spaces.

Many other useful properties of this co-derivative, mostly following from the application of the standard Leibnitz rule can be found in [1] and some of them are summarized in the appendix A.1.

Using the co-derivative we can for example rewrite the T-operator (2.1) in the following way:

$$T^{\{\lambda\}}(u) = (u_1 + 2\hat{D}) \otimes (u_2 + 2\hat{D}) \otimes \cdots \otimes (u_N + 2\hat{D}) \ \chi_{\{\lambda\}}(g) \tag{2.6}$$

where $u_j = u - \theta_j$ and $\chi_{\{\lambda\}}(g) = \mathrm{tr}\ \pi_\lambda(g)$ is the character of the twist g in the irrep λ. The action of each of N brackets adds a new spin to the system, with its fundamental quantum space.

3. The master identity

A few new properties of the co-derivative given in the appendices A and B lead to the main identity of this paper, which we will call the master identity, the basis of our approach to the quantum integrability. In its simplest form, it reads as follows

$$(t - z)\left[(\mathbb{I} + \hat{D})^{\otimes N} \ w(z)w(t)\Pi\right] \cdot \left[\hat{D}^{\otimes N} \ \Pi\right]$$
$$= t\left[\hat{D}^{\otimes N} \ w(z)\Pi\right] \cdot \left[(\mathbb{I} + \hat{D})^{\otimes N} \ w(t)\Pi\right] - z\left[(\mathbb{I} + \hat{D})^{\otimes N} \ w(z)\Pi\right] \cdot \left[\hat{D}^{\otimes N} \ w(t)\Pi\right]$$
$$\tag{3.1}$$

where

$$w(z) = \det \frac{1}{1 - zg} = \frac{1}{\prod_{j=1}^{K}(1 - zx_j)} = \sum_{s=0}^{\infty} z^s \ \chi_s(g) = \frac{1}{\sum_{a=0}^{\infty}(-1)^a z^a \ \chi^{(a)}(g)}, \tag{3.2}$$

is the generating function of characters in symmetric ($\lambda = (s)$) or antisymmetric ($\lambda = (1^a)$) irreps, x_j are the eigenvalues[4] of the twist matrix g and $\Pi(g)$ is an arbitrary class function

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[4] Throughout this paper, we shall assume that $\forall i \neq j$, $x_i \neq x_j$.

of g, i.e. a symmetric function of its eigenvalues. $\mathbb{I}$ is the identity matrix (in any of N fundamental quantum sub-spaces) and the co-derivatives act on all operators inside the square brackets.

The proof of our main identity (3.1) is given in AppendixB.3, but it can be easily proved directly, for a few small N's, on Mathematica. The identity represents a natural generalization of the identity (20) from[1].

As it is proven in appendix B.2.1, the master identity (3.1) can can also be written in the following slightly more general form

$$(t - z)\Big[\otimes_j(2 + u_j + 2\hat{D})w(z)w(t)\Pi\Big] \cdot \Big[\otimes_j(u_j + 2\hat{D})\ \Pi\Big]$$
$$= t\Big[\otimes_j(u_j + 2\hat{D})\ w(z)\Pi\Big] \cdot \Big[\otimes_j(2 + u_j + 2\hat{D})\ w(t)\Pi\Big]$$
$$-z\Big[\otimes_j(2 + u_j + 2\hat{D})\ w(z)\Pi\Big] \cdot \Big[\otimes_j(u_j + 2\hat{D})\ w(t)\Pi\Big] \quad (3.3)$$

where, to avoid the bulky notations, we assume all the terms like u_j and $2 + u_j$ to be multiplied by $\mathbb{I}$. We will mostly use this last form of the master identity. It is important to note that in each of three terms of this master identity both terms in the square brackets commute in the quantum space and can be written in any order. The appendix B shows how this master equation can be proven starting from a particular case, $\Pi = 1$ and $u_i = 0, \forall i$, equivalent to the formula (20) proven in [1],

4. Baxter relations for T- and Q-operators

In this section we will derive from our main identity the operatorial Bäcklund flow in the form of the TQ-relations described above, and even more generally, of TT-relations at every step of the nesting procedure, as well as the QQ-relations [24, 16, 25, 17, 20, 22, 26, 18] (see also earlier papers [27] and a recent presentation in [23]) which give an immediate access to the full set of nested Bethe ansatz equations (also written in an operatorial form in quantum space in the subsection 4.5). At the same time, it will give a natural operatorial definition of these quantities on every step of the nesting, and in particular of the Q-operators. Since all these T- and Q-operators belong to the same family of mutually commuting operators we can transform these relations, at any stage of the nesting procedure, to the operatorial ones, for T- and Q-operators.

4.1 First level of nesting

Now we will obtain from the master identity (3.3) the operatorial Baxter's TQ-relations. We will start from the first level of nesting. In what follows we will frequently use the notation $\overline{I} = \mathbb{F}\backslash I$ for the complimentary set of I.

Definition of Q-operators
From (2.4), the Q-operator on the zero level of nesting is

$$Q_{12...K}(u) \equiv Q_{\overline{\emptyset}}(u) = \left(\prod_i u_i\right) \mathbb{I}^{\otimes N} \quad (4.1)$$

which is a simple function of u_j's, times the identity operator in the full quantum space. In particular, the last factor in the LHS of (3.3) becomes $Q_{\overline{\emptyset}}(u)$ when $\Pi = 1$.

We will see in what follows that the Q-operators of the first level of nesting $Q_{\overline{j}}(u)$ can be defined through the residues at the poles in the expression:

$$(1 - gt)^{\otimes N} \cdot \left(\bigotimes_i (u_i + 2\hat{D} + 2)w(t) \right) = \sum_j \frac{Q_{\overline{j}}(u)}{1 - x_j t} + \text{polynomial in } t \qquad (4.2)$$

where we will see that the normalization factor $(1 - gt)^{\otimes N}$ is necessary in order to have only simple poles. The equivalent definition of the Q-operators is

$$Q_{\overline{j}}(u) = \lim_{t \to \frac{1}{x_j}} (1 - x_j t) \cdot (1 - gt)^{\otimes N} \cdot \left(\bigotimes_i (2 + u_i + 2\hat{D})w(t) \right) \qquad (4.3)$$

Let us note here that this Q-operator acts on the same quantum space $V^{\otimes N}$ (consisting of N copies of the fundamental representations) as the T-operator (2.6) and it is also a polynomial of degree N in u. However, when acting on particular vectors of $V^{\otimes N}$, it looses the dependence on some u_j's and its overall power in u becomes smaller than N. For instance, let us act by $Q_{\overline{j}}(u)$ on a vector $|e_{k_1, k_2, \cdots k_N}\rangle = |e_{k_1}\rangle \otimes |e_{k_2}\rangle \otimes \cdots \otimes |e_{k_N}\rangle$ of the canonical basis of $V^{\otimes N}$. If in the nth factor $2 + u_n + 2\hat{D}$ in (4.3) (this factor has its tensorial indices in the nth quantum space) $\hat{D}$ acts directly on $w(t)$ then it leads to a factor $A = \frac{gt}{1 - gt}$. But in $Q_{\overline{j}}(u)|e_{k_1, k_2, \cdots k_N}\rangle$, this factor acts on the vector $|e_{k_n}\rangle$ such that in the diagonal basis $g|e_{k_n}\rangle = x_{k_n}|e_{k_n}\rangle$, so that the factor A becomes $\frac{x_{k_n} t}{1 - x_{k_n} t}$. This quantity has a pole at $t \to \frac{1}{x_j}$ if and only if $k_n = j$. As a consequence, there is one pole of type $\frac{x_{k_n} t}{1 - x_{k_n} t}$ for each n such that $k_n = j$. On the other hand the factor $(1 - gt)^{\otimes N}$ in (4.3) has a zero of a type $1 - x_{k_n} t$ for each n such that $k_n = j$, and that cancels exactly the poles mentioned above, so that only the poles of order 1 remain[5] in the LHS of (4.2).

An outcome of this pole-regularisation is that when $k_n = j$, then $2 + u_n + 2\hat{D} = \langle \text{finite terms} \rangle + \frac{\langle \text{leading terms} \rangle}{1 - x_j t}$ gets multiplied by $1 - x_j t$, which cancels all finite terms and keeps only the leading terms. In particular, all the dependence in u_n is entirely contained in the finite term, and disappears.

The consequence is that $Q_{\overline{j}}(u)|e_{k_1, k_2, \cdots k_N}\rangle$ is independent of all u_n such that $k_n = j$. For the physical eigenstates used to construct the Bethe ansatz equations, for example by the algebraic Bethe ansatz procedure, we use only the eigenstates where the power in u strictly decreases at each next level of nesting: For example in the standard $su(2)$ Heisenberg spin chain, we can restrict to the subspace of states having at least $N/2$ spins pointing in the direction j, so that the power J of the polynomial Q-function is such that $J \le N/2$.

The same explanation will be true for the operator $T_{\overline{j}}^s(u)$ defined below, and for the further levels of nesting, and it explains that at each step of nesting, the degree in u of some eigenvalues (T-functions) decreases.

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[5]These remaining poles come from the poles in $w(t)$

T-operators and TQ-relations

Now we will transform the master identity (3.3) into a set of TQ-relations (2.3) on the first level of nesting[6]. For that we simply put $\Pi = 1$. Multiplying (3.3) by the matrix $(1 - gt)^{\otimes N}$ which commutes with all the factors of both the LHS and the RHS[7], and picking the poles at $t = 1/x_j$ we come to the equation

$$\lim_{t \to \frac{1}{x_j}} (1 - x_j t) \left[(1 - z/t)(1 - gt)^{\otimes N} \cdot \otimes_j (2 + u_j + 2\hat{D}) w(z) w(t) \right] \cdot Q_{\overline{\emptyset}}(u) = \qquad (4.4)$$

$$= \left[\otimes_j (u_j + 2\hat{D}) \, w(z) \right] \cdot Q_{\overline{j}}(u) - x_j \left[\otimes_j (2 + u_j + 2\hat{D}) \, zw(z) \right] \cdot Q_{\overline{j}}(u - 2) \quad (4.5)$$

It is useful to note that the first factor $(1 - z/t) \sim (1 - zx_j)$ in the LHS can be carried over to the right of the operator containing the co-derivatives $\hat{D}$ allowing to use the relation $(1 - zx_j) \det \frac{1}{1 - zg} = \det \frac{1}{1 - zg_{\overline{j}}}$, where $g_{\overline{j}} = \mathrm{diag}(x_1, x_2, \cdots, x_{j-1}, x_{j+1}, \cdots, x_N)$[8] in the diagonal basis. This is true only because we have the factor $(1 - gt)^{\otimes N}$, introduced to avoid poles of higher orders in (4.2). Indeed, for example in the simplest, one spin case $N = 1$, we can easily check that $(x_j \mathbb{I} - g) \cdot \hat{D} x_j = (x_j \mathbb{I} - g) P_j \cdot x_j = x_j^2 P_j - g P_j x_j = 0$, where P_j is the projector on the jth eigenspace of g. The generalization to any N is rather trivial and is discussed in the Appendix B.2.4.

Now we introduce the symmetric characters of the first level of nesting $\chi_s(g_{\overline{j}}) = \chi_{\{\lambda = 1^s\}}(g_{\overline{j}})$ corresponding to the sub-algebra $gl(K-1) \subset gl(K)$, by the generating function

$$\det \frac{1}{1 - zg_{\overline{j}}} = \sum_{s=1}^{\infty} z^s \, \chi_s(g_{\overline{j}}) \qquad (4.6)$$

and define the T-operators of the first level of nesting, in the symmetric irreps $\lambda = (s)$:

$$T_{\overline{j}}^s(u) = \lim_{t \to \frac{1}{x_j}} (1 - x_j t)(1 - gt)^{\otimes N} \cdot \left(\bigotimes_i (u_i + 2 + \hat{D}) \, \chi_s(g_{\overline{j}}) w(t) \right). \qquad (4.7)$$

The last formula also gives an alternative to (4.2) definition of the Q-operators as the T-operators for an empty Young diagram $s = 0$:

$$Q_{\overline{j}}(u) \equiv T_{\overline{j}}^{s=0}(u) = \lim_{t \to \frac{1}{x_j}} (1 - x_j t)(1 - gt)^{\otimes N} \cdot \left(\bigotimes_i (u_i + 2 + 2\hat{D}) \, w(t) \right) \qquad (4.8)$$

This consecutive "removal" of the poles corresponding to the eigenvalues is at the heart of our nesting procedure, and its repetition defines a certain Bäcklund flow. With

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[6]The "first level of nesting" means that we will relate the original T- and Q-operators labeled by the full set $\overline{\emptyset} = \mathbb{F}$ with some T and Q-operators labeled by $\overline{j}$, which has one less index.

[7]which is clear in the basis of a diagonal g since $(1 - gt)^{\otimes N}$ obviously commutes with permutations, and with tensorial product of diagonal matrices, hence with any operator of the form $\otimes_j (a_j + \hat{D}) \, w(b)$, due to its diagrammatic expansion given in appendix A.1.

[8]We will only be interested in the characters of $g_{\overline{j}}$, which are insensitive to the diagonalization basis. But a more invariant definition of $g_{\overline{j}}$ can be $g_{\overline{j}} = g - x_j P_j$, where P_j is the projector on the jth eigenspace of g.

this definition, we obtain from (4.4) the simplest Baxter's TQ-relation, the first of the chain of Bäcklund transformations

$$T_{\overline{j}}^s(u)Q_{\overline{\emptyset}}(u) = T_{\overline{\emptyset}}^s(u)Q_{\overline{j}}(u) - x_j T_{\overline{\emptyset}}^{s-1}(u+2)Q_{\overline{j}}(u-2) \tag{4.9}$$

among the commuting T- and Q-operators of the zero[th] and first level of nesting known in the operatorial form for a long time [21]. Here $T_{\overline{\emptyset}}^s \equiv T^s$ is the T-operator of the zero level of nesting, the original transfer matrix (2.1), or (2.6), in the symmetric irrep. The T- and Q-operators labeled by $\overline{j} = \mathbb{F} \setminus \{j\}$ have $K - 1$ indices, and are considered the first level of nesting[9]

Let us also note that the T-operators can be also defined as the residues at the poles:

$$\sum_j \frac{T_{\overline{j}}^s(u)}{1 - x_j t} = (1 - gt)^{\otimes N} \cdot \left(\bigotimes_i (u_i + 2 + 2\hat{D}) \left(\chi_{\{s\}}(g) - t\, \chi_{\{s-1\}}(g) \right) w(t) \right)$$
$$+ \text{ polynomial in } t \tag{4.10}$$

It is clear from these definitions and from (B.2) that all these T- (and hence also the Q-) operators, independently of the level of nesting, belong to the same family of commuting operators $[T_I^s(u), T_{I'}^{s'}(u')] = 0$.

4.2 Next levels of nesting

Now we will generalize this procedure, and the corresponding TQ-relations, to all nesting levels. Suppose we want to consecutively "remove" from the T- and Q-operators the eigenvalues $x_{j_1}, x_{j_2}, \ldots, x_{j_k}$, where $\overline{I} = \{j_1, j_2, \ldots, j_k\}$ is a subset of the full set of indices: $\overline{I} \subset \mathbb{F}$ (their order is not important but they are all different). At such arbitrary level of nesting, we define a normalization operator

$$\mathcal{B}_{\overline{I}} = \prod_{j \in \overline{I}} (1 - x_j\, t_j) \cdot (1 - g\, t_j)^{\otimes N} \tag{4.11}$$

and take as a class function $\Pi(g)$ the following product of generating functions of characters

$$\Pi_{\overline{I}} = \prod_{j \in \overline{I}} w(t_j) \tag{4.12}$$

The definition of the Q-operator labeled by a nesting path $I = \mathbb{F} \setminus \overline{I}$ becomes

$$Q_I(u) = \lim_{\substack{t_j \to \frac{1}{x_j} \\ j \in \overline{I}}} \mathcal{B}_{\overline{I}} \cdot \left(\bigotimes_i (2|\overline{I}| + u_i + 2\hat{D}) \right) \Pi_{\overline{I}} \tag{4.13}$$

and once again, it is an operator on the quantum space $V^{\otimes N}$, which is polynomial (of degree N if $I \neq \emptyset$) in the spectral parameter u. Its eigenvalues have degree $\leq N$, and it

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[9]In the same spirit, the kth level of nesting will involve the quantities with $K - k$ indices.

can be shown, as in section 4.1, that $Q_I(u)|e_{k_1,k_2,\cdots k_N}\rangle$ is independent of all u_n such that $k_n \in \overline{I}$.

We will show how to write a TQ relation between the T- and Q-operators labeled by I and the operators of the previous level of nesting, labeled by $I \cup j_k$:

Let us first generalize (4.6) into

$$w_I(z) \equiv \det \frac{1}{1-zg_I} \equiv \prod_{j\in I} \frac{1}{1-zx_j} \equiv \sum_{s=1}^{\infty} z^s \, \chi_s(g_I) = \frac{w(z)}{w_{\overline{I}}(z)}. \tag{4.14}$$

If we chose in the master identity (3.3), $\Pi = \Pi_{\overline{I\cup j_k}} \equiv \Pi_{\overline{I}\backslash j_k}$, $t = t_{j_k}$, and $u \to u+2|\overline{I \cup j_k}| = u + 2|\overline{I}| - 2$, then after multiplying[10] it by $\frac{1}{w_{\overline{I}\backslash j_k}(z)}\mathcal{B}_{\overline{I}} \cdot \mathcal{B}_{\overline{I\cup j_k}}$ and taking the limit $t_j \to \frac{1}{x_j}$, we get

$$\lim_{\substack{t_j \to \frac{1}{x_j} \\ j\in\overline{I}\cup j_k}} \left((1 - z/t_{j_k})\mathcal{B}_{\overline{I}} \left(\otimes_j (2|\overline{I}| + u_j + 2\hat{D}) \frac{w(z)}{w_{\overline{I\cup j_k}}(z)} w(t_{j_k}) \Pi_{\overline{I\cup j_k}} \right) \right) \cdot Q_{I\cup j_k}(u) =$$

$$= \lim_{\substack{t_j \to \frac{1}{x_j} \\ j\in\overline{I}}} \left[\left(\mathcal{B}_{\overline{I\cup j_k}} \left(\otimes_j (2|\overline{I}| - 2 + u_j + 2\hat{D}) \frac{w(z)}{w_{\overline{I\cup j_k}}(z)} \Pi_{\overline{I\cup j_k}} \right) \right) \cdot Q_I(u) \right.$$

$$\left. - x_{j_k} \left(\mathcal{B}_{\overline{I\cup j_k}} \left(\otimes_j (2|\overline{I}| + u_j + 2\hat{D}) z \frac{w(z)}{w_{\overline{I\cup j_k}}(z)} \Pi_{\overline{I\cup j_k}} \right) \right) \cdot Q_I(u - 2) \right] \tag{4.15}$$

where $\overline{I \cup j_k} = \{j_1, j_2, \ldots, j_{k-1}\}$. These expressions are obtained by rewriting the factors where $\hat{D}$ acts only on $w(t_j)$'s using the formula (4.13). For instance, the last factor of the last term obtained from (3.3) is $\mathcal{B}_{\overline{I\cup j_k}} \otimes_j (2|\overline{I}| - 2 + u_j + 2\hat{D})w(t_{j_k})\Pi_{\overline{I}\backslash j_k}$ which becomes $Q_I(u - 2)$ when the limit $t_j \to \frac{1}{x_j}$ is taken.

If we define the T-operators for symmetric representations as follows

$$T_I^s(u) = \lim_{\substack{t_j \to \frac{1}{x_j} \\ j\in\overline{I}}} \mathcal{B}_{\overline{I}} \left(\bigotimes_{i=1}^{N} (u_i + 2\hat{D} + 2|\overline{I}|) \ \chi_s(g_I)\Pi_{\overline{I}} \right) \tag{4.16}$$

In the definition, $\chi_s(g_I)$ is defined by (4.14), and it is a priori non trivial how co-derivatives act on $\chi_s(g_I)$. The recipe to avoid this problem and compute T-operators is given in

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[10]Once again, the normalization factors $\mathcal{B}_{\overline{I}}$ and $\mathcal{B}_{\overline{I\cup j_k}}$ commute with all the other factors, because they commute with all permutations, and with all operators $g^{i_1} \otimes g^{i_2} \otimes \cdots g^{i_N}$, which are the building blocks of all other factors.

On the other hand, we will see a posteriori in appendix B.2.4 that the factor $w_{\overline{I}}(z)$ can be freely move across the $\hat{D}$'s.

appendix (B.2.4), and it states that

$$\sum_{s\geq 0} z^s T_I^s(u) = \lim_{\substack{t_j \to \frac{1}{x_j} \\ j \in \overline{I}}} \mathcal{B}_{\overline{I}} \left(\bigotimes_{i=1}^{N} (u_i + 2\hat{D} + 2|\overline{I}|) \ \frac{w(z)}{w_{\overline{I}}(z)} \Pi_{\overline{I}} \right) \tag{4.17}$$

$$= \frac{1}{w_{\overline{I}}(z)} \lim_{\substack{t_j \to \frac{1}{x_j} \\ j \in \overline{I}}} \mathcal{B}_{\overline{I}} \left(\bigotimes_{i=1}^{N} (u_i + 2\hat{D} + 2|\overline{I}|) \ w(z) \Pi_{\overline{I}} \right) \tag{4.18}$$

And if we use the results of this appendix B.2.4 to move $(1 - z/t_{j_k}) \sim (1 - zx_{j_k})$, then the LHS of (4.15) contains $(1 - zx_{j_k})\frac{w(z)}{w_{\overline{I \cup j_k}}(z)} = w_I(z)$, so that (4.15) becomes the following operatorial TQ-relation

$$T_I^s(u)Q_{I \cup j_k}(u) = T_{I \cup j_k}^s(u)Q_I(u) - x_{j_k}T_{I \cup j_k}^{s-1}(u+2)Q_I(u-2) \tag{4.19}$$

It generalizes the similar relation[11] among the characters of symmetric irreps:

$$\chi_s(g_I) = \chi_s(g_{I,j}) - x_j\chi_{s-1}(g_{I,j}) \tag{4.20}$$

Notice that again the Q-operator on any level of nesting is equal to the T-operator, with the same nesting path $\overline{I}$, for the singlet representation $s = 0$, or in general, for $\lambda = \emptyset$

$$Q_I(u) = T_I^{\{\emptyset\}}(u). \tag{4.21}$$

4.3 Generalization to any representations

There is a natural way to generalize the T-operators to any irreps λ in the auxiliary space:

$$T_I^{\{\lambda\}}(u) = \lim_{\substack{t_j \to \frac{1}{x_j} \\ j \in \overline{I}}} \mathcal{B}_{\overline{I}} \left(\bigotimes_{i=1}^{N} (u_i + 2\hat{D} + 2|\overline{I}|) \ \chi_\lambda(g_I)\Pi_{\overline{I}} \right) \tag{4.22}$$

where the $gl(K - |\overline{I}|)$ characters of the irreps λ are given through the symmetric characters $\chi_s(g_I)$ by the Jacobi-Trudi determinant formula

$$\chi_{\{\lambda\}}(g_I) = \det_{1 \leq i,j \leq a} \chi_{\lambda_j+i-j}(g_I), \tag{4.23}$$

where a is the number of rows in the Young diagram λ. It is noteworthy that, due to the definition (4.22), $T_I^{\{\lambda\}} = 0$ if λ has more than $|I|$ rows, because $\chi_{\{\lambda\}}(g_I) = 0$.

We believe (although we did not prove it yet) that the Bazhanov-Reshetikhin formula proven in [1] at the zero[th] level of nesting is also satisfied by these nested T-operators. At least it is clear that the Bazhanov-Reshetikhin determinant holds for rectangular representations because the Hirota relation proven in [1] at the zero[th] level of nesting is easily

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[11]obvious in terms of the generating function

generalized, by using the properties[12] of appendix B.2, to the same expression for nested T-operators:

$$T_I^{(a,s)}(u+1)T_I^{(a,s)}(u-1) = T_I^{(a+1,s)}(u+1)T_I^{(a-1,s)}(u-1) + T_I^{(a,s+1)}(u-1)T_I^{(a,s-1)}(u+1) \tag{4.24}$$

generalizing the Hirota[13] fusion relations for the main transfer matrix $T^\lambda \equiv T_{\overline{\emptyset}}^\lambda$ (2.1) in the irreps a^s, with rectangular Young tableaux. In particular, this Hirota relation, together with the TQ-relation (4.19) proven above, implies the following bilinear relations on the nested T-operators [20]:

$$T_{I,j}^{(a+1,s)}(u)T_I^{(a,s)}(u) - T_{I,j}^{(a,s)}(u)T_I^{(a+1,s)}(u) = x_j T_{I,j}^{(a+1,s-1)}(u+2)T_I^{(a,s+1)}(u-2) \tag{4.25}$$

$$T_{I,j}^{(a,s+1)}(u)T_I^{(a,s)}(u) - T_{I,j}^{(a,s)}(u)T_I^{(a,s+1)}(u) = x_j T_{I,j}^{(a+1,s)}(u+2)T_I^{(a-1,s+1)}(u-2) \tag{4.26}$$

The TQ-relation (4.19) is a particular case of equation (4.25) when $a = 0$, and the two equations (4.25,4.26) coincide with the definition of Bäcklund flow given in $[22]^{14}$, so that the definitions for nested T- and Q-operators explicitly give the solution of this linear system.

4.4 TT- and QQ-relations

In the previous subsections, the TQ-relations where proven using the formula (3.3) which contains the explicit factors $w(z)$ and $w(t)$, by incorporating $w(t)$ into $\Pi_{\overline{I}}$ and using $w(z) = \sum z^s \chi_s$ to relate it to the T-operators labeled by symmetric irrep (s).

It is also possible to incorporate both $w(t)$ and $w(z)$ into $\Pi_{\overline{I}}$, giving rise to different nesting paths. In this case, the character present in the definition of $T^{\{\lambda\}}$ can be incorporated into the quantity Π of (3.3). Then we immediately get the following TT relation:

$$(x_i - x_j)\, T_I^{\{\lambda\}}(u-2)T_{I,i,j}^{\{\lambda\}}(u) = x_i T_{I,j}^{\{\lambda\}}(u-2)T_{I,i}^{\{\lambda\}}(u) - x_j T_{I,j}^{\{\lambda\}}(u)T_{I,i}^{\{\lambda\}}(u-2) \tag{4.27}$$

As a particular case, if we choose $\{\lambda\} = \{\emptyset\}$, we get the well known QQ-relations, this time for Q-operators :

$$(x_i - x_j)\, Q_I(u-2)Q_{I,i,j}(u) = x_i Q_{I,j}(u-2)Q_{I,i}(u) - x_j Q_{I,j}(u)Q_{I,i}(u-2) \tag{4.28}$$

We can immediately solve these QQ-relations for all the Q-operators to get

$$Q_{I,J}(u) = \frac{\det\left(Q_{I,j}(u-2k)x_j^{|J|-1-k}\right)_{\substack{j\in J \\ 0\le k\le|J|-1}}}{\left(\prod_{k=1}^{|J|-1}Q_I(u-2k)\right)\det\left(x_j^{|J|-1-k}\right)_{\substack{j\in J \\ 0\le k\le|J|-1}}} \tag{4.29}$$

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[12]The properties proven in appendix B.2 allow to generalize bilinear equations of a quite general form to any level nesting. That is why it applies directly to the Hirota equation, but not to the Bazhanov-Reshetikhin formula which is not bilinear. On the other hand, the proof of the Bazhanov-Reshetikhin determinant given in [1] relies mainly on the vanishing of a 2×2 minors of the determinant, which is a bilinear relation. That is why we can expect this proof to be generalizable to the nested T-operators.

[13]Hirota relation can also be written in terms of the T-functions defined as follows: $\mathbf{T}^{(a,s)}(u) = T^{(a,s)}(u+a-s)$. In term of these $\mathbf{T}$-functions, (4.24) takes the usual form $\mathbf{T}^{(a,s)}(u+1)\mathbf{T}^{(a,s)}(u-1) = \mathbf{T}^{(a+1,s)}(u)\mathbf{T}^{(a-1,s)}(u) + \mathbf{T}^{(a,s+1)}(u)\mathbf{T}^{(a,s-1)}(u)$.

[14]In the literature these equations are usually written in an equivalent form in terms of $\mathbf{T}$ functions.

In particular, choosing $I = \emptyset$ gives the expression of any Q-operator in terms of $N + 1$ Q-operators, namely, the N single indexed Q_i operators describing the last level of nesting, and the u-independent operator $Q_\emptyset(u)$. More explicitly, $Q_\emptyset$ can be defined by its action on the canonical basis of the quantum space by

$$Q_\emptyset(u)|e\rangle \equiv T_\emptyset(u)|e\rangle = \sum_{\sigma \in \mathcal{S}_N} \delta_{i_1, i_{\sigma(1)}} \delta_{i_2, i_{\sigma(2)}} \cdots \delta_{i_N, i_{\sigma(N)}} |e\rangle \qquad (4.30)$$

where $|e\rangle = |e_{i_1}\rangle \otimes |e_{i_2}\rangle \otimes \cdots \otimes |e_{i_N}\rangle \otimes |e_{i_N}\rangle$.

4.5 Operatorial Bethe equations

In this subsection we derive the set of nested Bethe ansatz equations.

From the QQ-relations (4.28) one immediately sees that since $Q_{I,j}(u)$ should be, by its definition, a polynomial then

$$Q_{I,i}(u) \mid (x_i - x_j)Q_I(u-2)Q_{I,i,j}(u) + x_j Q_{I,j}(u)Q_{I,i}(u-2) \qquad (4.31)$$

$$Q_{I,i}(u) \mid (x_i - x_j)Q_I(u)Q_{I,i,j}(u+2) - x_i Q_{I,j}(u)Q_{I,i}(u+2) \qquad (4.32)$$

where $P|P'$ denotes the fact that the polynomial P' contains the polynomial P as a factor. By adding $x_i Q_{I,i}(u+2)$ times the first line to $x_j Q_{I,i}(u-2)$ times the second line, one gets

$$\begin{aligned} Q_{I,i}(u) \mid &x_i Q_I(u-2)Q_{I,i,j}(u)Q_{I,i}(u+2) \\ &+ x_j Q_I(u)Q_{I,i,j}(u+2)Q_{I,i}(u-2) \end{aligned} \qquad (4.33)$$

This is written for the Q-operators, but for their eigenvalues these equations are equivalent to the usual nested Bethe ansatz equations (68) of [22]. Indeed, now we know, by construction, that the Q-operators are polynomial[15] and therefore their eigenvalues are also polynomials of a degree $K_I \leq N$ in u:

$$Q_I(u) = c_I \prod_{k=1}^{K_I} (u - u_k^{(I)}) \qquad (4.34)$$

When acting on particular eigenstates, the operatorial Bethe equations (4.33) become the usual polynomial Bethe equations on the Bethe roots $u_k^{(I)}$ along a chosen nesting path. In case of the Q-operators they are formal polynomials $Q_I(u) = \sum_{k=1}^{N} u^k \, \hat{c}_k^{(I)}$ where the coefficients $\{\hat{c}_k^{(I)}\}$ are also operators in the quantum space. On particular quantum states, or eigenvectors of the spin chain, a part of them becomes zero, which explains the fact that the power of Q-functions - the eigenvalues of the Q-operators - can diminish on each step of the Bäcklund procedure.

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[15]This fact was missing in the analytic Bethe ansatz construction of [22] and appeared there only as a hypothetic ansatz for the solution of Hirota equation by the Bäcklund procedure

5. Generalization to the supersymmetric case

In the case of the $gl(K|M)$ super-spin chain, the co-derivative can be defined by

$$\hat{D} \otimes f(g) = e_{ij} \frac{\partial}{\partial \phi_i^j} \otimes f\left(e^{\phi_l^k e_{kl}} g\right)_{\phi=0} \quad , \quad \frac{\partial}{\partial \phi_{i_1}^{j_1}} \phi_{j_2}^{i_2} \equiv \delta_{j_1}^{i_2} \delta_{j_2}^{i_1} (-1)^{p_{j_1}} \tag{5.1}$$

where ϕ and g are matrices in the fundamental representation of $GL(K)$, and p_j is the grading :

$$p_b = 0, \qquad 1 \le b \le K; \qquad\qquad p_f = 1, \qquad K+1 \le f \le M \tag{5.2}$$

As explained in [1], the properties of co-derivatives are exactly the same as in the bosonic case, including the expression of T-operators in terms of co-derivatives (at the zero[th] nesting level). The diagrammatics of the co-derivative is also the same as in the bosonic case, except the signs to be introduced into every permutation operator, to get $\mathcal{P} = \sum_{\alpha\beta}(-)^{p_\alpha} e_{\beta\alpha} \otimes e_{\alpha\beta}$. In particular, the formula (3.3) still holds in the super-case but (3.2) has to be substituted by

$$w(z) = \text{sdet} \frac{1}{1-zg} = \frac{\prod_{j=1}^M (1-zy_j)}{\prod_{j=1}^K (1-zx_j)} = \sum_{s=0}^\infty z^s \; \chi_s(g) = \frac{1}{\sum_{s=0}^\infty (-1)^a z^a \; \chi^{(a)}(g)} \tag{5.3}$$

With slight generalizations of the definitions w.r.t. the bosonic case, all the supersymmetric TQ- and QQ-relations follow from (3.3) if we define T- and Q-operators in the following way:

$$\Pi_{\bar{I}} = \prod_{j \in \bar{I}} w(t_j)^{((-1)^{p_j})} \qquad\qquad \mathcal{B}_{\bar{I}} = \prod_{j \in \bar{I}} (1 - \xi_j \; t_j) \cdot (1 - g \; t_j)^{\otimes N} \tag{5.4}$$

$$T_I^{\{\lambda\}}(u) = \lim_{\substack{t_j \to \frac{1}{x_j} \\ j \in \bar{I}}} \mathcal{B}_{\bar{I}} \left(\bigotimes_{i=1}^N (u_i + 2\hat{D} + 2n_b - 2n_f) \; \chi_\lambda(g_I)\Pi_{\bar{I}} \right) \tag{5.5}$$

$$Q_I^{\{\lambda\}}(u) = \lim_{\substack{t_j \to \frac{1}{x_j} \\ j \in \bar{I}}} \mathcal{B}_{\bar{I}} \left(\bigotimes_{i=1}^N (u_i + 2\hat{D} + 2n_b - 2n_f) \; \Pi_{\bar{I}} \right) \tag{5.6}$$

where $(\xi_1, \cdots, \xi_{K+M}) = (x_1, \cdots, x_K, y_1, \cdots, y_M)$ are the eigenvalues of g, $n_b = |I \cap \{1, 2, \cdots, K\}|$, and[16] $n_f = |I \cap \{K+1, \cdots, M\}|$,

Then the TQ relation (4.19) becomes

$$T_I^s(u)Q_{I,j}(u) = T_{I,j}^s(u)Q_I(u) - x_j T_{I,j}^{s-1}(u+2)Q_I(u-2) \qquad \text{if } j \le K \tag{5.7}$$

$$T_{I,j}^s(u)Q_I(u) = T_I^s(u)Q_{I,j}(u) - y_k T_I^{s-1}(u+2)Q_{I,j}(u-2) \quad \text{if } j = K+k \ge K \tag{5.8}$$

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[16]As before, $|I|$ denotes card(I)

And the QQ-relations also become grading-dependent:

$$(x_i - x_j) Q_I(u-2)Q_{I,i,j}(u) = x_i Q_{I,j}(u-2)Q_{I,i}(u) - x_j Q_{I,j}(u)Q_{I,i}(u-2) \qquad (5.9)$$

$$(x_i - \xi_l) Q_{I,l}(u-2)Q_{I,i}(u) = x_i Q_I(u-2)Q_{I,i,l}(u) - \xi_l Q_I(u)Q_{I,i,l}(u-2) \qquad (5.10)$$

$$(\xi_l - \xi_m) Q_{I,l,m}(u-2)Q_I(u) = \xi_l Q_{I,l}(u-2)Q_{I,m}(u) - \xi_m Q_{I,l}(u)Q_{I,m}(u-2) \qquad (5.11)$$

$$\text{where } i,j \leq K \qquad \text{and } l,m > K \qquad (5.12)$$

As explained in [28], these QQ-relations are all obtained from the bosonic one by a "bosonization trick" which allows, in particular, to write very simple Wronskian determinants for Q-operators, even in the super-case. We can also write the Wronskian expressions through certain Q-operators, for all the T-operators at all levels of nesting. They do not differ in the form from the operatorial relations found in [23].

The Bethe ansatz equation are obtained in the same manner as in (4.33). For example, the "fermionic-fermionic" relation is obtained by isolating $Q_{I,l}$ in (5.11) which gives

$$Q_{I,l}(u)|\xi_m Q_{I,l,m}(u-2)Q_I(u)Q_{I,l}(u+2) + \xi_l Q_{I,l,m}(u)Q_I(u+2)Q_{I,l}(u-2) \qquad (5.13)$$

which is, at the level of eigenvalues, exactly the equation (69) of [22].

On the other hand, the bosonic-fermionic relation are trivially obtained from (5.10) which immediately gives

$$Q_{I,i}(u) \mid x_i Q_I(u-2)Q_{I,i,l}(u) - \xi_l Q_I(u)Q_{I,i,l}(u-2) \qquad (5.14)$$

$$Q_{I,l}(u) \mid x_i Q_I(u)Q_{I,i,l}(u+2) - \xi_l Q_I(u+2)Q_{I,i,l}(u) \qquad (5.15)$$

In terms of eigenvalues, the equation (5.14) [resp (5.15)] exactly becomes the equation (71) [resp (70)] of [22].

6. Conclusions

The co-derivative formalism and the master identity (3.1),(3.3), together with the definitions (4.13),(4.22) of nested T- and Q-operators proposed in this paper can serve as an alternative approach to the quantum integrability, rather different from the popular algebraic Bethe ansatz (see for example [29] and the references therein). It allows to complete the whole procedure of diagonalization of transfer-matrix of the inhomogeneous twisted $su(K|M)$ (super)spin chain, all the way from its construction from R-matrices obeying the Yang-Baxter relations and till the nested system of Bethe ansatz equations, directly in terms of the operators acting in the quantum space. The master identity (3.1) presented at the beginning of the paper and generalizing a similar identity from [1] is the basis of this approach, encoding all possible operatorial TT-, QQ- and Baxter's TQ-relations at every step of nesting, or of the operatorial Bäcklund flow, generalizing the operatorial Bäcklund transformations of [21, 20]. Remarkably, the master identity takes a bilinear form with respect to the $gl(K|M)$ characters, or their generating functions. Since the characters can be viewed as the tau-functions of KdV hierarchy one can speculate that this identity is simply a particular case of the general Hirota identity for the KdV tau-functions, with

$\tau_n = \frac{1}{n}\text{tr } g^n$ playing the role of the KdV "times". It would be an interesting relation between the quantum and classical integrability, showing that, paradoxically, the former is a particular case of the latter.

This could provide us also with a simpler and more natural proof of our master identity than the proof we give in appendix B.

It would be interesting to generalize our approach to the case of non-compact representations of $gl(K|M)$ in the auxiliary space, following the observations made in [30, 28, 31] for the characters and Q-operators for $U(2,2|4)$. This might teach us how to deal with one of the most interesting integrable physical systems, $\mathcal{N} = 4$ SYM theory and its AdS dual - the Metsaev-Tseytlin sigma-model having the $PSU(2,2|4)$ global symmetry. In general, the Y-systems for sigma-models and their Wronskian solutions [8, 9] might be also an interesting subject for their operatorial generalization in the quantum (physical) space and might give us an interesting tool for the study of the spectrum of excited states and shed some light on the formulas for the energy of an excited state conjectured in the literature for relativistic sigma models [32, 33, 9] and for the AdS/CFT [34].

Acknowledgments

Note added

When finishing this paper we learned about the results of [19] which deal with the same problem, the operatorial formulation of the Q-operators and of the TQ-relations. The objects studied in that forthcoming paper are the same, but the formalism is radically different from our.

A. Diagrammatics of co-derivatives

A.1 Co-Derivatives and characters

As explained in [1], the action of the co-derivative on characters and their generating function: For instance we can write

$$\hat{D} \otimes \hat{D} w(x) = \left(\big| \big| + \big\rangle\!\big\langle \right) w(x) \tag{A.1}$$

where $|$ (resp $\vdots$) stands for $\frac{gx}{1-gx}$ (resp $\frac{1}{1-gx}$), so that $||=\frac{gx}{1-gx}\otimes\frac{gx}{1-gx}$ and $\diagdown\!\!\!\!\diagup=\mathcal{P}_{1,2}(\frac{1}{1-gx}\otimes\frac{gx}{1-gx})$, where $\mathcal{P}_{1,2}$ denotes the usual permutation between the 1st and 2nd quantum space[17]. The general expression for $\hat{D}^{\otimes N}w(x)$ is given by the formula (30) of [1], and it represents the sum of diagrams corresponding to all possible permutations, with dashing all the lines going from lower to upper nodes and directed to the right.

In terms of characters of the symmetric irreps $\chi_s(g)$ the equation (A.1) reads $\hat{D}\otimes\hat{D}\,\chi_t=\left(||+\diagdown\!\!\!\!\diagup\right)\chi_t$ where $|$ (resp $\vdots$) stands for $\frac{ge^{-\partial_t}}{1-ge^{-\partial_t}}$ (resp $\frac{1}{1-ge^{-\partial_t}}$). If we identify $x=e^{-\partial_t}$ there is no ambiguity between these two definitions of the diagrammatics.

This diagrammatics can first be extended for the inclusion of the parameters u_j, i.e.

$$(u_1+\hat{D})\otimes(u_2+\hat{D})w(x)=\left(||+||+||+||+\diagdown\!\!\!\!\diagup\right)w(x) \tag{A.2}$$

where $|$ stands for u_1 or u_2, according to its position (for instance, $||=u_1\mathbb{I}\otimes\frac{gx}{1-gx}$, and $||=\frac{gx}{1-gx}\otimes\mathbb{I}u_2$).

Then a simple computation shows that

$$(u_1+\hat{D})\otimes(u_2+\hat{D})w(x)^\alpha=\left(||+\alpha||+\alpha||+\alpha\diagdown\!\!\!\!\diagup+\alpha^2||\right)w(x)^\alpha \tag{A.3}$$

In full generality, one can show that the power of α in a given diagram is the number of "x"-cycles of the overall permutation in the quantum space (ie cycles which have lines of type $|$ or $\vdots$).

A.2 Co-derivatives of products

In this paper, we often use the co-derivatives acting on products, defining the quantities like $\hat{D}^{\otimes N}\,\Pi\cdot w(x)$ for an arbitrary class function Π. Then each co-derivative can act either on Π or on $w(x)$, and at two spins, for instance, the Leibnitz rule gives

$$\hat{D}\otimes\hat{D}\quad\Pi\cdot w(x)=\hat{D}\otimes\left(\left(\hat{D}\Pi\right)w(x)+\Pi\left(\hat{D}w(x)\right)\right) \tag{A.4}$$

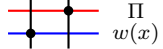$$=\left(\hat{D}\otimes\hat{D}\,\,\Pi\right)w(x)+\left(\mathbb{I}\otimes\hat{D}\,\,\Pi\right)\left(\hat{D}\otimes\mathbb{I}\,\,w(x)\right)$$

$$+\left(\hat{D}\otimes\mathbb{I}\,\,\Pi\right)\left(\mathbb{I}\otimes\hat{D}\,\,w(x)\right)+\Pi\left(\hat{D}\otimes\hat{D}\,\,w(x)\right) \tag{A.5}$$
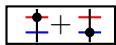
$$=\quad\begin{array}{c}\text{(diagram)}\end{array}\Pi\atop w(x)+\begin{array}{c}\text{(diagram)}\end{array}\Pi\atop w(x)$$

$$+\begin{array}{c}\text{(diagram)}\end{array}\Pi\atop w(x)+\begin{array}{c}\text{(diagram)}\end{array}\Pi\atop w(x) \tag{A.6}$$

$$=\quad\begin{array}{c}\text{(boxed diagram)}\end{array}\Pi(g)\atop w(x) \tag{A.7}$$

The equality between (A.4) and (A.5) is just the Leibnitz rule, while (A.6) defines the graphical representation of each term of (A.5). Each black dot stands for a co-derivative, acting on what lies on its right on the same horizontal line (horizontal lines actually hide auxiliary spaces whose characters contribute to $\Pi(g)$ [resp $w(x)$]). The vertical lines

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correspond to the quantum space on which the whole operator acts, and the crossings without dots stand for $\mathbb{I}$. When operators are multiplied in the quantum spaces, they are represented one above another. For instance $\left(\mathbb{I} \otimes \hat{D} \ \Pi \right) \left(\hat{D} \otimes \mathbb{I} \ w(x) \right) = $.

The generalization to an arbitrary number of spins N is straightforward - one has N blocks (instead of two) of the form in this diagrammatic representation.

One can also see that due to the relation $\hat{D} \det(g) = \mathbb{I} \det(g)$ and to the Leibnitz rule, we have very generally $\bigotimes_i (u_1 + 2\hat{D})\Pi \det(g)^a = \det(g)^a \bigotimes_i (u_1 + 2a + 2\hat{D})\Pi$.

A.3 Number of cycles

As noticed in equation (A.3) there exists a natural notion of the "number of cycles" corresponding to the expansion in powers of α (if $w(x)^\alpha$ is substituted for every $w(x)$).

We therefore introduce the operator $\mathcal{C}_n^{(x)}$, extensively used in the section B.2.2, which keeps only the diagrams with exactly n cycles where the lines are of the type $\vert = \frac{gx}{1-gx}$ or $\vdots = \frac{1}{1-gx}$. For instance,

$$\mathcal{C}_1^{(x)} \left((u_1 + \hat{D}) \otimes (u_2 + \hat{D}) w(x)^\alpha \right) = \left(\vert\vert + \vert\vert + \rangle\langle \right) w(x)^\alpha$$

In particular, for an A being x-independent, $\mathcal{C}_n^{(x)}(A) = \delta_{n,0} \ A$.

If $\mathcal{C}_n^{(x)}$ acts on a product of operators, we take the definition $\mathcal{C}_n^{(x)} (A \cdot B) = \sum_{k \leq n} \left(\mathcal{C}_k^{(x)} A \right) \left(\mathcal{C}_{n-k}^{(x)} B \right)$, in agreement with the interpretation of $\mathcal{C}_n^{(x)}$ as the sum of terms of degree n in α.

B. Proof of the formulae

B.1 Yang-Baxter Relation

All over this file, the quantities of interest are of the form $\left(\bigotimes_{i=1}^N (u_i + \hat{D}) \ \Pi \right)$, where Π is an arbitrary, class-invariant function.

In particular it can be written as a sum of characters, and lots of properties where already proven in [1] for $T^{\{\lambda\}} = \left(\bigotimes_{i=1}^N (u_i + \hat{D}) \ \chi_{\{\lambda\}} \right)$, which correspond to the case when Π is a character.

One of these properties is the Yang-Baxter relation, between the R matrices, which, in particular gives the relation

$$(v - u + \mathcal{P}_{1,2}) \left((u + \hat{D}_1)(v + \hat{D}_2) \ A \right) = \left((v + \hat{D}_2)(u + \hat{D}_1) \ A \right) (v - u + \mathcal{P}_{1,2}) \quad \text{(B.1)}$$

where $A = \bigotimes_{i=1}^N (u_i + \hat{D})\Pi$ and $\hat{D}_j$ is a co-derivative whose tensorial indices live in the jth quantum space. In other words, $\hat{D}_1 \equiv \mathbb{I} \otimes D$, $\hat{D}_2 \equiv \hat{D} \otimes \mathbb{I}$, $\hat{D}_2\hat{D}_1 \equiv P_{1,2} \left(\hat{D} \otimes \hat{D} \right) P_{1,2}$. It is easily proven for an arbitrary class function Π by writing it as a sum of characters, and writing the action of $u_i + \hat{D}$ on a character as an R-matrix.

The main consequence of the relation (B.1) is that for any permutation[18] σ of $\{1, \cdots N\}$, there exists a similarity transformation, independent of Π, which maps all $\left(\bigotimes_{i=1}^{N} (u_i + \hat{D}) \ \Pi \right)$ to $\left(\bigotimes_{i=1}^{N} (u_{\sigma(i)} + \hat{D}) \ \Pi \right)$.

The Yang-Baxter equations can also be used to prove that

$$\left[\left(\bigotimes_{i=1}^{N} (u_i + \hat{D}) \ \Pi \right), \left(\bigotimes_{i=1}^{N} (u_i + v + \hat{D}) \ \Pi' \right) \right] = 0. \tag{B.2}$$

This is proven in [1] when Π, Π' are characters, and can be generalized by writing Π and Π' as sums of characters.

B.2 General results for co-derivatives

B.2.1 Shift of spectral parameter

The first property we will prove is the fact that, if $(A_j(g))_{1 \leq j \leq k}$ and $(B_j(g))_{1 \leq j \leq k}$ are arbitrary class functions of $g \in GL(K)$, such that for all $N \in \mathbb{N}$ and for all g

$$\sum_j \left(\hat{D}^{\otimes N} A_j(g) \right) \cdot \left(\hat{D}^{\otimes N} B_j(g) \right) = 0 \tag{B.3}$$

then for all $N \in \mathbb{N}, (u_1, u_2, \cdots, u_N) \in \mathbb{C}^N$ and for all g

$$\sum_j \left(\bigotimes_{i=1}^{N} (u_i + 2\hat{D}) \ A_j(g) \right) \cdot \left(\bigotimes_{i=1}^{N} (u_i + 2\hat{D}) \ B_j(g) \right) = 0 \tag{B.4}$$

Proof

Let us denote by $\mathfrak{A} = \sum_j \left(\bigotimes_{i=1}^{N_0} (u_i + 2\hat{D}) \ A_j \cdot w(x) \right) \cdot \left(\bigotimes_{i=1}^{N_0} (u_i + 2\hat{D}) \ B_j \cdot w(x) \right)$ the left hand side of (B.4). Then we see that the coefficient of the term of degree 1 in u_1 is $2 \sum_j \left(\left(\hat{D} \otimes E_j \right) \cdot (\mathbb{I} \otimes F_j) + (\mathbb{I} \otimes E_j) \cdot \left(\hat{D} \otimes F_j \right) \right) = 2\hat{D} \otimes \sum_j E_j \cdot F_j$ where $E_j = \left(\bigotimes_{i=2}^{N_0} (u_i + 2\hat{D}) \ A_j \cdot w(x) \right)$ and $F_j = \left(\bigotimes_{i=2}^{N_0} (u_i + 2\hat{D}) \ B_j \cdot w(x) \right)$. We also see that the term of degree 2 in u_1 is $\mathbb{I} \otimes \sum_j E_j \cdot F_j$.

Then by induction over the number N of spins, we see that if (B.4) is true for $N - 1$ spins, then $\sum_j E_j \cdot F_j = 0$ and $\mathfrak{A}$ is u_1-independent. But due to the Yang-Baxter relation (B.1), acting by the permutation $u_i \mapsto u_{\sigma(i)}$ rotates all A_j's and B_j's by the same similarity transformation. As a consequence, if $\mathfrak{A}$ is u_1-independent, then it is u_j-independent for all j. As a consequence, it is equal to $\mathfrak{A}(u_1 = u_2 = \cdots = 0)$, which is zero due to (B.3).

Remark

By rewriting $A_j = A'_j \det(g)^{a_j}$ (resp $B_j = B'_j \det(g)^{a_j}$) we immediately see that, if for all $N \in \mathbb{N}$ and for all g

$$\sum_j \left((a_j + \hat{D})^{\otimes N} A'_j(g) \right) \cdot \left((b_j + \hat{D})^{\otimes N} B'_j(g) \right) = 0 \tag{B.5}$$

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[18] Actually (B.1) proves the result only for transposition between consecutive indices. But these transposition generate all permutations, hence this more general claim.

then for all $N \in \mathbb{N}, (u_1, u_2, \cdots, u_N) \in \mathbb{C}^N$ and for all g

$$\sum_j \left(\bigotimes_{i=1}^N (u_i + 2a_j + 2\hat{D}) \quad A_j'(g) \right) \cdot \left(\bigotimes_{i=1}^N (u_i + 2b_j + 2\hat{D}) \quad B_j'(g) \right) = 0 \qquad \text{(B.6)}$$

This proves, for instance, the generalization of (3.1) into (3.3).

B.2.2 Addition of cycles

A second property which we will prove is the fact that if $(A_j(g))_{1 \leq j \leq k}$ and $(B_j(g))_{1 \leq j \leq k}$ are class functions of $g \in GL(K)$, such that for all $N \in \mathbb{N}$ and for all g

$$\sum_j \left(\hat{D}^{\otimes N} A_j(g) \right) \cdot \left(\hat{D}^{\otimes N} B_j(g) \right) = 0 \qquad \text{(B.7)}$$

then[19] for all $N \in \mathbb{N}, x \in \mathbb{C}$ and for all g

$$\mathcal{C}_1^{(x)} \circ \sum_j \left(\hat{D}^{\otimes N} \quad A_j(g) \cdot w(x)) \right) \cdot \left(\hat{D}^{\otimes N} \quad B_j(g) \cdot w(x) \right) = 0 \qquad \text{(B.8)}$$

Proof

We will first use the commutation relations (stated at the end of section B.1) to prove the intermediate formula (B.9), and then we will use diagrammatic arguments to show that (B.9) proves (B.8) (by showing the cancellation of some "unwanted terms"):

From (B.7),(B.4) and the commutation relation (B.2), we get

$$\mathfrak{B} \equiv \mathcal{D}_1^{(u)} \circ \mathcal{C}_1^{(x)} \left[\sum_j \left(\bigotimes_{i=0}^N (u_i + \hat{D}) \; A_j(g)) \right) \right.$$
$$\left. \left(\bigotimes_{i=0}^N (u_i + \hat{D}) \; w(x)) \right) \cdot \left(\bigotimes_{i=0}^N (u_i + \hat{D}) \; B_j(g) \right) \right] = 0 \qquad \text{(B.9)}$$

where $\mathcal{C}_1^{(x)}$ is the cycle counting operator introduced in section A.3 and $\mathcal{D}_1^{(u)}$ is defined by

$$\mathcal{D}_1^{(u)} [f(u_1, \cdots, u_N)] \equiv \partial_{u_1} \circ \partial_{u_2} \circ \cdots \circ \partial_{u_N} f(u_1, \cdots, u_N)|_{u_i=0}. \qquad \text{(B.10)}$$

This means that we keep only the linear term in all the variables u_i.

The equation (B.9) has to be compared with the equation $\mathfrak{A} = 0$ where

$$\mathfrak{A} \equiv \frac{1}{w(x)} \mathcal{C}_1^{(x)} \sum_j \left(\hat{D}^{\otimes N} \quad A_j \cdot w(x) \right) \cdot \left(\hat{D}^{\otimes N} \quad B_j \cdot w(x) \right) \qquad \text{(B.11)}$$

Due to the definition and properties of $\mathcal{C}_1^{(x)}$ given in section A.3[20], we get $\mathfrak{A} = \sum_j \mathcal{C}_1^{(x)} \left[\left(\hat{D}^{\otimes N} \quad A_j \cdot w(x) \right) \cdot \left(\hat{D}^{\otimes N} \quad B_j \right) + \left(\hat{D}^{\otimes N} \quad A_j \right) \cdot \left(\hat{D}^{\otimes N} \quad B_j \cdot w(x) \right) \right]$, which is represented diagrammatically in the case of 2 spins in the fig.1a, using the diagrammatic rules of section A.2. The same kind of representation can be used for $\mathfrak{B}$,
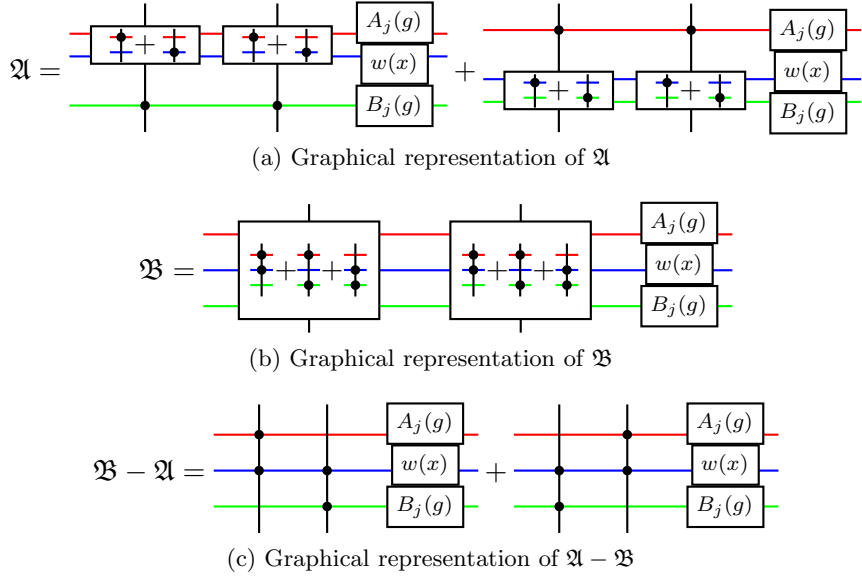
(a) Graphical representation of $\mathfrak{A}$

(b) Graphical representation of $\mathfrak{B}$

(c) Graphical representation of $\mathfrak{A} - \mathfrak{B}$

Figure 1: Comparison of the terms $\mathfrak{A}$ and $\mathfrak{B}$. The subfigure 1c shows what we will call the "unwanted terms". A sum over j, as well as the operation $\mathcal{C}_1^{(x)}$ is implicitly taken on all diagrams.

where the operator $\mathcal{D}_1^{(u)}$ has the effect of leaving exactly one empty crossing (without co-derivative) on every vertical line (quantum space). This is illustrated in figure 1, where $\mathfrak{B} - \mathfrak{A}$ is written in the case of 2 spins.

In the more general case of N spins, it is clear that $\mathfrak{B}$ is made of the full set of all the diagrams where each vertical line contains two dots and an empty crossing (and where the horizontal, middle line has at least one dot, due to the operator $\mathcal{C}_1^{(x)}$), while $\mathfrak{A}$ contains the subset of the diagrams where either the first or the last horizontal line is fully filled with dots.

As we want to prove $\mathfrak{A} = 0$ from $\mathfrak{B} = 0$, we will call "unwanted terms" the diagrams of $\mathfrak{B} - \mathfrak{A}$, i.e. the subset of the diagrams which have at least one empty crossing on the upper line and one empty crossing on the lower line.

To prove that these "unwanted" diagrams cancel, we can start by investigating the basic properties of these terms : the figure 2 gives a very general example of such term. The middle line is of the form $\mathcal{C}_1^{(x)} \left(\hat{D} \otimes \hat{D} \otimes \mathbb{I} \otimes \hat{D} \otimes \cdots w(x) \right)$ and has a diagrammatic representation explained in section A.1, which allows to write it as a sum of terms such as the term drawn in fig 2b. There $\|$ stands for $\mathbb{I}$ (corresponding to the crossings without dot in 2a. In this diagram, two continuous paths are of special importance : they are the paths connecting two $\|$, and they are emphasized in fig 2c. We will call them the u-paths, and extensively use them in our proof.

First one can see that every "unwanted term" has at least one u-path. This is because if such paths did not exist, it would mean that the blue permutation in the middle row

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[19]The operator $\mathcal{C}_1^{(x)}$ is defined in appendix A.1.

[20]We also use the obvious fact that $\mathcal{C}_0^{(x)}(\hat{D}^{\otimes N} A w(x)) = w(x)(\hat{D}^{\otimes N} A)$, when A is x-independent.
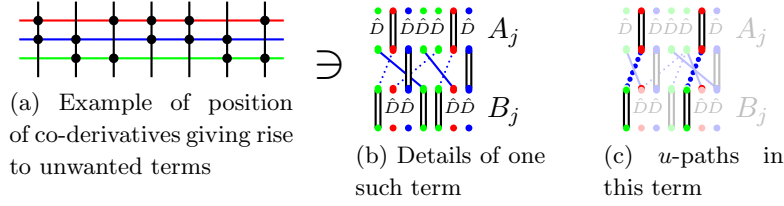
(a) Example of position of co-derivatives giving rise to unwanted terms

(b) Details of one such term

(c) u-paths in this term

Figure 2: Example of unwanted term, and its u-paths.

$$\mathcal{D}_1^{(u)} \circ \mathcal{C}_1^{(x)} \quad \cdots \quad = \quad \mathfrak{P}_2 \cdot \left(\frac{1}{1-ge^{-\partial_t}} \otimes \frac{1}{1-ge^{-\partial t}} \otimes \left[\left(\bigotimes_{i=1}^{N-2}(u_i + \hat{D})A_j \right) \cdot \left(\bigotimes_{i=1}^{N-2}(u_i + \hat{D})w(x) \right) \cdot \left(\bigotimes_{i=1}^{N-2}(u_i + \hat{D})Bj \right) \right] \right) \mathfrak{P}_1$$

Figure 3: **Manipulations on graphs with u-paths corresponding to consecutive indices.** The first equality shows the manipulation on an unwanted term having u-paths with consecutive indices, and the same manipulation gives a general expression for the sum of all diagrams having (at least) these two u-paths (second equality, where $\hat{D}$ stands for $u_i + \hat{D}$ in the LHS to make the notations less bulky.). Here, $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are permutation operations interchanging the indices, and are imposed by the position of the u-paths.

maps the set of the columns where ‖ appears on the first row (where red dots were put, in fig. 2b) to itself, and the set of the columns where ‖ appears on the third row (where green dots were put, in fig.2b) to itself. But that would contradict the fact that this permutation has only one "x"-cycle. In other words, the operator $\mathcal{C}_1^{(x)}$ inserted into the definition of $\mathfrak{B}$ is such that all "unwanted terms" have at least one u-path.

Now we will show the cancellation of all the terms which have the same u-paths, and we will start with the simple case of u-paths corresponding to two consecutive columns[21], like in the fig.2. The fig.3 indicates how to factor out these paths to transform such terms into exactly the diagrams of fixed[22] degree in u_i's of the quantity $\mathfrak{C}_j = \mathcal{C}_1^{(x)}\mathfrak{P}_2 \cdot \left(\left(\frac{1}{1-ge^{-\partial_t}} \right)^{\otimes k} \otimes \left[\left(\bigotimes_{i=1}^{N-k}(u_i + \hat{D})A_j \right) \cdot \left(\bigotimes_{i=1}^{N-k}(u_i + \hat{D})w(x) \right) \cdot \left(\bigotimes_{i=1}^{N-k}(u_i + \hat{D})Bj \right) \right] \right) \mathfrak{P}_1$ where k is the number of u-paths which are factored out.

After a summation on the index j, the identity[23] $\sum_j \mathfrak{C}_j = 0$ shows that all the terms

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[21]More explicitly, that means u-paths made of one ‖ at position i on the third row and ‖ at position $i+1$ on the first row.

[22]The relevant degree in u's, as well as the choice of the permutations $\mathfrak{P}_1$ and $\mathfrak{P}_1$ depend on the position of the paths.

[23]This identity is proven in exactly the same manner as (B.9).

having the same u-paths cancel, it the case of u-paths defined by consecutive indices.

In the case of u-paths corresponding to non-consecutive indices, one can easily convince himself that the multiplication by the appropriate permutations maps, due to the Yang-Baxter relation (B.1), the set of all "unwanted diagrams" having a common u-path into the set of unwanted terms having another common u-path, but which now corresponds to consecutive indices. That gives the cancellation of all "unwanted terms".

Finally, by proving that all "unwanted terms" cancel, we have shown that (B.8) was implied by the relation (B.9)

B.2.3 Fundamental property

As a consequence of the two previous properties, we easily get that if $(A_j(g))_{1 \le j \le k}$ and $(B_j(g))_{1 \le j \le k}$ are class functions of $g \in GL(K)$, such that for all $N \in \mathbb{N}$ and for all g

$$\sum_j \left(\hat{D}^{\otimes N} A_j(g) \right) \cdot \left(\hat{D}^{\otimes N} B_j(g) \right) = 0 \tag{B.12}$$

then for any class function $\Pi(g)$, for all N and g

$$\sum_j \left(\bigotimes_{i=1}^{N} (u_i + \hat{D}) \quad A_j \Pi \right) \cdot \left(\bigotimes_{i=1}^{N} (u_i + \hat{D}) \quad B_j \Pi \right) = 0 \tag{B.13}$$

Proof

The proof can be achieved in the case of $\Pi = \prod_k w(t_k)^{\alpha_k}$ by proving that all terms of a fixed power in each α_k do cancel. This is done by induction over this power, by adding cycles of each type (some cycles of $w(t_1)$, some cycles of $w(t_2)$, etc.), one after another. The induction can be done using the property of section B.2.2.

At the end of the induction, we can use the property of section B.2.1 to add the dependence on $u_1, u_2, \cdots u_N$.

B.2.4 Co-derivative and "removal" of eigenvalues

The co-derivative of $\chi_s(g_I)$ a priori does not have such a simple expression (in terms of diagrams) as the co-derivative of $\chi_s(g)$.

The claim which was already given in (4.17) is that the computation of T-operators is done by commuting a factor $\frac{1}{w_{\overline{I}}(z)}$ to the left of the co-derivatives. At one spin, in (4.7), we saw that this was made possible by the factor $(1 - gt)^{\otimes N}$.

For more spins, we will show that in (4.17), the multiplication by $\mathcal{B}_{\overline{I}}$ defined in (4.11) allows to commute any function of x_j (where $j \in \overline{I}$) across the co-derivatives, so that

$$\sum_s T_I^s z^s = \left(\prod_{j \in \overline{I}} (1 - x_j z) \right) \lim_{\substack{t_j \to \frac{1}{x_j} \\ j \in \overline{I}}} \mathcal{B}_{\overline{I}} \left(\bigotimes_{i=1}^{N} (u_i + 2\hat{D} + 2|\overline{I}|) \quad w(z) \Pi_{\overline{I}} \right) \tag{B.14}$$

which can be easily computed by diagrammatic methods.

In the case of 1 spin, This is checked by computing[24]

$$\lim_{t \to \frac{1}{x_j}} (1 - gt) \left[(u + 2\hat{D}), x_j \right] = \left(1 - \frac{g}{x_j} \right) \cdot \left(2\hat{D}x_j \right) = 2 \left(1 - \frac{g}{x_j} \right) x_j Pj = 0 \qquad \text{(B.15)}$$

We see that the key point in this commutation is the multiplication by $(1-gt) \simeq (1-g/x_j)$, which cancels the terms in $\hat{D} \, x_j = x_j Pj$ due to the property $(1 - g/x_j) \, Pj = 0$.

At $N = m + n$ spins, the analogous relation is

$$C_{m,n} = 0 \qquad \text{where } C_{m,n} \equiv (1 - g/x_j)^{\otimes(m+n)} \, B_{m,n} \qquad \text{(B.16)}$$

$$B_{m,n} \equiv \left(\bigotimes_{i=1}^{m} (u_i + 2\hat{D}) \right) \otimes \left[(u + 2\hat{D}), x_j \right] \otimes \left(\bigotimes_{i=m+2}^{m+n} (u_i + 2\hat{D}) \right)$$

and it can be proven by the recurrence over m : if we assume that $C_{m,n} = 0, \forall g$, then

$$0 = \left((1 - g/x_j) \otimes \mathbb{I}^{\otimes(m+n)} \right) \cdot \left((u + 2\hat{D}) \; C_{m,n} \right) \qquad \text{(B.17)}$$

$$= C_{m+1,n} + 2 \left((1 - g/x_j) \otimes \mathbb{I}^{\otimes(m+n)} \right) \cdot \left[\hat{D}, (1 - g/x_j)^{\otimes(m+n)} \right] B_{m,n} \qquad \text{(B.18)}$$

and the second term in (B.18) can be expanded according to the Leibnitz rule which gives two kinds of terms

- Each term where $\hat{D}$ acts on a $1/x_j$ vanishes[25] because they are left-multiplied by $((1 - g/x_j) \otimes \mathbb{I}^{\otimes m+n})$.

- In the other terms $\hat{D}$ acts on g. But as[26] $\hat{D}g = (g \otimes \mathbb{I}) \cdot \mathcal{P}$, these terms are of the form

$$t_i = 2 \left(\left(\frac{g}{x_j} \left(1 - \frac{g}{x_j} \right) \right) \otimes \mathbb{I}^{\otimes(m+n)} \right) \cdot \mathcal{P}_{0,i} \cdot \mathbb{I} \otimes \left(\bigotimes_{k=1}^{m+n} \left(1 - \frac{g}{x_j} \right)^{1-\delta_{k,i}} B_{m,n} \right)$$

$$= 2 \left(\frac{g}{x_j} \otimes \mathbb{I}^{\otimes(m+n)} \right) \cdot \mathcal{P}_{0,i} \cdot \mathbb{I} \otimes \left(\left(1 - \frac{g}{x_j} \right)^{\otimes(m+n)} B_{m,n} \right) = 0 \qquad \text{(B.19)}$$

This completes the proof of the fact that the second term of the RHS of (B.18) is zero, but the LHS is also zero, so that $C_{m+1,n} = 0$.

As a consequence, we can indeed commute the factors $\frac{1}{w_{\bar{\jmath}}(z)}$ to the left of all co-derivatives in (4.16) in (4.17) and the previous relations.

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[24] Here, $[A, B]$ denotes the commutator $AB - BA$.

[25] By the same argument as in (B.15)

[26] The whole argument here is based of "commutation relations" of the form $\mathcal{P} \cdot (I \otimes g) = (g \otimes \mathbb{I}) \cdot \mathcal{P}$.

B.3 Proof of the master identity

The proof of the main identity relies on the above-mentioned properties. In order to prove (3.1), we only have to chose

$$\begin{cases} A_1 = (t - z)w(z)w(t)\det(g) \\ A_2 = -tw(z) \\ A_3 = +zw(z)\det(g) \end{cases} \qquad \begin{cases} B_1 = 1 \\ B_2 = w(t)\det(g) \\ B_3 = w(z) \end{cases} \qquad (\text{B.20})$$

Then the relation $\sum_{j=1}^{3}(\hat{D}^{\otimes N}A_j)(\hat{D}^{\otimes N}B_j) = 0$ is trivial when $N = 0$ and reduces, when $N \geq 1$, to the equation (20) in [1], because $\hat{D}^{\otimes N}B_1 = 0$.

All that means that (3.1), and more generally (3.3), are consequence of the above-mentioned relation.

References

[1] V. Kazakov and P. Vieira, "From Characters to Quantum (Super)Spin Chains via Fusion," JHEP **0810** (2008) 050 [arXiv:0711.2470 [hep-th]].

[2] V. Bazhanov and N. Reshetikhin, "Restricted Solid On Solid Models Connected With Simply Based Algebras And Conformal Field Theory," J. Phys. A **23**, 1477 (1990).

[3] A. Klümper and P. A. Pearce, *"Conformal weights of RSOS lattice models and their fusion hierarchies"* Physica A **183** (1992) 304.

[4] A. Kuniba, T. Nakanishi and J. Suzuki, *"Functional relations in solvable lattice models. 1: Functional relations and representation theory,"* Int. J. Mod. Phys. A **9** (1994) 5215 [arXiv:hep-th/9309137].

[5] A. Kuniba, S. Nakamura and R. Hirota, J. Phys. A **29**, 1759 (1996) [arXiv:hep-th/9509039].

[6] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *"Integrable Structure of Conformal Field Theory II. Q-operator and DDV equation,"* Commun. Math. Phys. **190** (1997) 247 [arXiv:hep-th/9604044].

[7] Z. Tsuboi, *"Analytic Bethe ansatz and functional equations for Lie superalgebra $sl(r + 1|s + 1),$"* J. Phys. A **30**, 7975 (1997) [arXiv:0911.5386 [math-ph]].

[8] N. Gromov, V. Kazakov and P. Vieira, *"Finite Volume Spectrum of 2D Field Theories from Hirota Dynamics,"* JHEP **0912**, 060 (2009) [arXiv:0812.5091 [hep-th]].

[9] V. Kazakov and S. Leurent, *"Finite Size Spectrum of SU(N) Principal Chiral Field from Discrete Hirota Dynamics,"* arXiv:1007.1770 [hep-th].

[10] N. Gromov, V. Kazakov and P. Vieira, *"Exact Spectrum of Anomalous Dimensions of Planar N=4 Supersymmetric Yang-Mills Theory,"* Phys. Rev. Lett. **103** (2009) 131601 [arXiv:0901.3753 [hep-th]].

[11] D. Bombardelli, D. Fioravanti and R. Tateo, *"Thermodynamic Bethe Ansatz for planar AdS/CFT: a proposal,"* J. Phys. A **42**, 375401 (2009) [arXiv:0902.3930 [hep-th]]. ★
N. Gromov, V. Kazakov, A. Kozak and P. Vieira, *"Integrability for the Full Spectrum of Planar AdS/CFT II,"* Lett. Math. Phys. **91**, 265 (2010) [arXiv:0902.4458 [hep-th]]. ★
G. Arutyunov and S. Frolov, *"Thermodynamic Bethe Ansatz for the $AdS_5 \times S^5$ Mirror Model,"* JHEP **0905**, 068 (2009) [arXiv:0903.0141 [hep-th]].

[12] I. Cherednik, *On special basis of irreducible representations of degenerated affine Hecke algebras*, Funk. Analys. i ego Prilozh. **20:1** (1986) 87-88 (in Russian) ★ I. Cherednik, *Quantum groups as hidden symmetries of classical representation theory* , Proceed. of 17th Int. Conf. on diff. geom. methods in theoretical physics, World Scient. (1989), **47**. ★ I. Cherednik, *On irreducible representations of elliptic quantum R-algebras*, Dokl. Akad. Nauk SSSR 291:1, 49-53 (1986) Translation: M 34-1987, 446-450. ★ I. Cherednik, *An analogue of character formula for Hecke algebras*, Funct. Anal. and Appl. 21:2, 94-95 (1987) (translation: pgs 172-174).

[13] R.J. Baxter, Partition function of the eight-vertex lattice model, Ann. Phys. 70 (1972) 193-228.

[14] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *"Integrable structure of conformal field theory. III: The Yang-Baxter relation,"* Commun. Math. Phys. **200**, 297 (1999) [arXiv:hep-th/9805008].

[15] V. Pasquier, M. Gaudin, The periodic Toda chain and a matrix generalization of the Bessel function recursion relations, J. Phys. A: Math. Gen. 25 (1992) 5243-4252; ★ K. Hikami, Baxter Equation for Quantum Discrete Boussinesq Equation, Nucl. Phys. B604 (2001) 580-602 [arXiv:nlin/0102021]; ★ K. Fabricius, B.M. McCoy, New Developments in the Eight Vertex Model, J.Statist.Phys. 111 (2003) 323-337 [arXiv:cond-mat/0207177]; ★ V.B. Kuznetsov, V.V. Mangazeev, E.K. Sklyanin, Q-operator and factorised separation chain for Jack polynomials, Indag. Math. 14 (2003) 451-482 [arXiv:math/0306242[math.CA]]; ★ P. P. Kulish, A. M. Zeitlin, "Superconformal field theory and SUSY N=1 KDV hierarchy II: The Q-operator," Nucl. Phys. **B709** (2005) 578 [hep-th/0501019]; ★ C. Korff, A Q-Operator Identity for the Correlation Functions of the Infinite XXZ Spin-Chain, J.Phys. A: Math. Gen. 38 (2005) 6641-6658 [arXiv:hep-th/0503130]; ★ A.G. Bytsko, J. Teschner, Quantization of models with non-compact quantum group symmetry. Modular XXZ magnet and lattice sinh-Gordon model, J.Phys.A 39 (2006) 12927-12981[arXiv:hep-th/0602093]; ★ H. Boos, M. Jimbo, T. Miwa, F. Smirnov, Y. Takeyama, Hidden Grassmann structure in the XXZ model, Commun. Math. Phys. 272 (2007) 263-281 [arXiv:hep-th/0606280]; ★ T. Kojima, *"The Baxter's Q-operator for the W-algebra W_N,"* J.Phys.A: Math. Theor. 41 (2008) 355206 [arXiv:0803.3505 [nlin.SI]]. ★ S.E. Derkachov, A.N. Manashov, Factorization of R-matrix and Baxter Q-operators for generic $sl(N)$ spin chains, J.Phys.A: Math. Theor. 42 (2009) 075204; [arXiv:0809.2050 [nlin.SI]]. ★ H. Boos, F. Göhmann, A. Klümper, K.S. Nirov, A.V. Razumov, Exercises with the universal R-matrix J. Phys. A: Math. Theor. 43 (2010) 415208 [arXiv:1004.5342 [math-ph]]. ★ V. V. Bazhanov, T. Lukowski, C. Meneghelli and M. Staudacher, *"A Shortcut to the Q-Operator,"* arXiv:1005.3261 [hep-th]. ★ S.E. Derkachov, A.N. Manashov, *"Noncompact $sl(N)$ spin chains: Alternating sum representation for finite dimensional transfer matrices,"* arXiv:1008.4734 [nlin.SI].

[16] V. V. Bazhanov, A. N. Hibberd and S. M. Khoroshkin, "Integrable structure of W_3 conformal field theory, quantum Boussinesq theory and boundary affine Toda theory," Nucl. Phys. B **622** (2002) 475 [arXiv:hep-th/0105177].

[17] A. V. Belitsky, S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, *"Baxter Q-operator for graded $SL(2|1)$ spin chain,"* J. Stat. Mech. **0701**, P005 (2007) [arXiv:hep-th/0610332].

[18] V. V. Bazhanov and Z. Tsuboi, *"Baxter's Q-operators for supersymmetric spin chains,"* Nucl. Phys. B **805**, 451 (2008) [arXiv:0805.4274 [hep-th]];

[19] V. V. Bazhanov, R. Frassek, T. Lukowski, C. Meneghelli and M. Staudacher, *"Q-Operators and Representations of Yangians,"* (to appear).

[20] V. Kazakov, A. S. Sorin and A. Zabrodin, *"Supersymmetric Bethe ansatz and Baxter equations from discrete Hirota dynamics,"* Nucl. Phys. B **790**, 345 (2008) [arXiv:hep-th/0703147].

[21] I. Krichever, O. Lipan, P. Wiegmann and A. Zabrodin, *"Quantum integrable models and discrete classical Hirota equations,"* Commun. Math. Phys. **188** (1997) 267 [arXiv:hep-th/9604080].

[22] A. Zabrodin, "Backlund transformations for difference Hirota equation and supersymmetric Bethe ansatz," arXiv:0705.4006 [hep-th].

[23] Z. Tsuboi, *"Solutions of the T-system and Baxter equations for supersymmetric spin chains,"* Nucl. Phys. B **826** (2010) 399 [arXiv:0906.2039 [math-ph]].

[24] G.P. Pronko, Yu.G. Stroganov, *"Families of solutions of the nested Bethe Ansatz for the A_2 spin chain,"* J. Phys. A: Math. Gen. 33 (2000) 8267-8273 [arXiv:hep-th/9902085]; ★ P. Dorey, C. Dunning, D. Masoero, J. Suzuki, R. Tateo, *"Pseudo-differential equations, and the Bethe Ansatz for the classical Lie algebras,"* Nucl. Phys. B 772 (2007) 249-289 [arXiv:hep-th/0612298].

[25] F. Göhmann, A. Seel: *"A note on the Bethe Ansatz solution of the supersymmetric t-J model,"* Czech.J.Phys. 53 (2003) 1041 [arXiv:cond-mat/0309138].

[26] N. Gromov and P. Vieira, *"Complete 1-loop test of AdS/CFT,"* JHEP **0804**, 046 (2008) [arXiv:0709.3487 [hep-th]].

[27] F. Woynarovich, *"Low-energy excited states in a Hubbard chain with on-site attraction,"* J.Phys.C: Solid State Phys. 16 (1983) 6593; ★ P.A. Bares, I.M.P. Carmelo, J. Ferrer, P. Horsch, *"Charge-spin recombination in the one-dimensional supersymmetric t-J model,"* Phys. Rev. B46 (1992) 14624. ★ Z. Tsuboi, *"Analytic Bethe Ansatz and functional equations associated with any simple root systems of the Lie superalgebra $sl(r+1|s+1)$"*, Physica A **252**, 565 (1998) [arXiv:0911.5387 [math-ph]].

[28] N. Gromov, V. Kazakov, S. Leurent and Z. Tsuboi, "Wronskian Solution for AdS/CFT Y-system," arXiv:1010.2720 [hep-th].

[29] L. D. Faddeev, arXiv:hep-th/9605187. ★ Andreas Klümper, *Integrability of quantum chains: theory and applications to the spin-1/2 XXZ chain*, arXiv:cond-mat/0502431.

[30] N. Gromov, V. Kazakov and Z. Tsuboi, *"$PSU(2,2|4)$ Character of Quasiclassical AdS/CFT,"* JHEP**1007**(2010)097 [arXiv:1002.3981 [hep-th]].

[31] A. Hegedus, *"Discrete Hirota dynamics for AdS/CFT,"* Nucl. Phys. B **825**, 341 (2010) [arXiv:0906.2546 [hep-th]].

[32] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, "Quantum field theories in finite volume: Excited state energies," Nucl. Phys. B **489** (1997) 487 [arXiv:hep-th/9607099].

[33] P. Dorey and R. Tateo, "Excited states by analytic continuation of TBA equations," Nucl. Phys. B **482**, 639 (1996) [arXiv:hep-th/9607167].

[34] N. Gromov, V. Kazakov and P. Vieira, *"Exact AdS/CFT spectrum: Konishi dimension at any coupling,"* Phys. Rev. Lett. **104**, 211601 (2010) [arXiv:0906.4240 [hep-th]].