BAXTER'S RELATIONS AND SPECTRA OF QUANTUM INTEGRABLE MODELS

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To Victor Kac on his birthday

Abstract. Generalized Baxter's relations on the transfer-matrices (also known as Baxter's TQ relations) are constructed and proved for an arbitrary untwisted quantum affine algebra. Moreover, we interpret them as relations in the Grothendieck ring of the category $\mathcal{O}$ introduced by Jimbo and the second author in [HJ] involving infinite-dimensional representations constructed in [HJ], which we call here "prefundamental". We define the transfer-matrices associated to the prefundamental representations and prove that their eigenvalues on any finite-dimensional representation are polynomials up to a universal factor. These polynomials are the analogues of the celebrated Baxter polynomials. Combining these two results, we express the spectra of the transfer-matrices in the general quantum integrable systems associated to an arbitrary untwisted quantum affine algebra in terms of our generalized Baxter polynomials. This proves a conjecture of Reshetikhin and the first author formulated in 1998 [FR1]. We also obtain generalized Bethe Ansatz equations for all untwisted quantum affine algebras.

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1. Introduction

The partition function Z of a quantum model on an $M \times N$ lattice may be written in terms of the eigenvalues of the row-to-row transfer matrix T:

$$Z = \operatorname{Tr} T^M = \sum_i \lambda_i^M.$$

Therefore, to find Z, one needs to find the spectrum of T.

In his seminal 1971 paper [Ba], R. Baxter tackled this question for the so-called eight-vertex model, in which T acts on the vector space $(\mathbb{C}^2)^{\otimes N}$. In the special case that the parameters satisfy the "ice condition" (then it is called the six-vertex model) the spectrum of the model was previously found by E. Lieb [L1, L2, L3] (see also [Sut]) using an explicit construction of eigenvectors now referred to as Bethe Ansatz. Analyzing this result, Baxter observed that the eigenvalues of T on these eigenvectors always have the form

$$A(z)\frac{Q(zq^2)}{Q(z)}+D(z)\frac{Q(zq^{-2})}{Q(z)}, \tag{1.1}$$

where $Q(z)$ is a polynomial, z, q are parameters of the model, and the functions $A(z), D(z)$ are the same for all eigenvalues. Furthermore, Baxter realized that the condition that the seeming poles of the above expression, occurring at the roots of $Q(z)$, cancel each other is equivalent to the Bethe Ansatz equations guaranteeing that the vectors constructed by the Bethe Ansatz are indeed eigenvectors. Thus, apart from the factors $A(z)$ and $D(z)$ which are universal, the spectrum of T is essentially determined by the polynomials $Q(z)$ satisfying this condition (provided that the Bethe Ansatz gives us all eigenvectors).[1]

The polynomial $Q(z)$ is now called *Baxter's polynomial*, and relation (1.1) is called *Baxter's relation* (or Baxter's TQ relation). It looks rather mysterious. Why should such a relation hold?

To gain insights into this question, we present a modern interpretation of Baxter's result in a broader context of quantum groups. Consider the quantum affine algebra $U_q(\mathfrak{g})$ associated to an untwisted affine Kac–Moody algebra. The completed tensor square of this algebra contains the universal R-matrix $\mathcal{R}$ satisfying the Yang–Baxter relation and other properties. Given a finite-dimensional representation V of $U_q(\mathfrak{g})$, we construct the transfer-matrix

$$t_V(z) = \mathrm{Tr}_V(\pi_{V(z)} \otimes \mathrm{id})(\mathcal{R}),$$

where $V(z)$ is a twist of V by a "spectral parameter" z. It turns out that

$$[t_V(z), t_{V'}(z')] = 0$$

for all V, V' and z, z'. Therefore these transfer-matrices give rise to a family of commuting operators on any finite-dimensional representation W of $U_q(\mathfrak{g})$.

In the special case that $\mathfrak{g} = \widehat{sl}_2$, V a simple two-dimensional representation of $U_q(\widehat{sl}_2)$, and W the tensor product of N two-dimensional representations, the operator $t_V(z)$ acting on W becomes Baxter's transfer-matrix. This makes it clear that an analogue of Baxter's problem may be formulated for an arbitrary quantum affine algebra $U_q(\mathfrak{g})$. Namely, it is the problem of describing the eigenvalues of the transfer-matrices $t_V(z)$ on finite-dimensional representations W of $U_q(\mathfrak{g})$. It is known that these eigenvalues appear as the spectra of quantum integrable systems generalizing the six-vertex model (more precisely, generalizing the XXZ model, whose spectrum is the same as that of the six-vertex model). Hence a solution of this problem has immediate applications in statistical mechanics.

In [FR1], N. Reshetikhin and the first author found a novel and general way to describe the eigenvalues of the transfer-matrices for an arbitrary (untwisted) quantum affine algebra,

[1] In his paper, Baxter went on to generalize this relation to the general eight-vertex model, for which Bethe Ansatz was not available, but this is beyond the scope of the present paper.

generalizing Baxter's formula. The idea was to use the q-characters of finite-dimensional representations of quantum affine algebras introduced in [FR1] (note that a similar notion for representations of the Yangians was introduced earlier by H. Knight). The q-character is a homomorphism of rings

$$\chi_q : \text{Rep } U_q(\mathfrak{g}) \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times},$$

where I is the set of vertices of the Dynkin diagram of the finite-dimensional simple Lie algebra underlying $\mathfrak{g}$. For example, if $\mathfrak{g} = \widehat{sl}_2$ and V is a two-dimensional representation, then there is $a \in \mathbb{C}^*$ such that

$$\chi_q(V) = Y_{1,a} + Y_{1,aq^2}^{-1}.$$

Roughly speaking, the point is that the above Baxter relation (after renaming the variables $z \mapsto aq$) may be obtained from this formula if we substitute

$$Y_{1,a} \mapsto \frac{Q(aq^{-1})}{Q(aq)}$$

(to simplify matters, we are dropping the factors $A(z)$ and $D(z)$ for now; but they can be easily taken into account). This gives us a way to generalize Baxter's formula.

Namely, the following conjecture was proposed in [FR1]: Given a finite-dimensional representation V of $U_q(\mathfrak{g})$, all of the eigenvalues of $t_V(a)$ on any irreducible finite-dimensional representation W may always be written in the following form: we take the q-character $\chi_q(V)$ and substitute in it

$$Y_{i,a} \mapsto \frac{Q_{i,aq_i^{-1}}}{Q_{i,aq_i}}, \qquad i \in I,$$

where $Q_{i,a}$ is the product of two factors: one of them is the same for all eigenvalues (it depends only on W) and the other is a polynomial – these are the analogues of Baxter's polynomial. (Here $q_i = q^{d_i}$, see Section 2.1; the precise statement is in Theorem 5.11.)

We remark that in various special cases, a similar conjectural description of the eigenvalues of the transfer-matrices was proposed by N. Reshetikhin [R1, R2, R3]; V. Bazhanov and N. Reshetikhin [BR]; and A. Kuniba and J. Suzuki [KS].

In this paper we prove the general conjecture of [FR1] about the eigenvalues of the transfer-matrices in a deformed setting (this means that the trace used in the above formula for the transfer-matrix is replaced by the twisted trace that depends on additional parameters $u_i, i \in I$; see Definition 5.1). Among other things, our proof gives a conceptual explanation of Baxter's relation, and its generalizations, in terms of representation theory of quantum affine algebras.

Our proof is based on two results which are of independent interest.

First, we show that the above $Q_{i,a}$ is itself an eigenvalue of a transfer-matrix (a generalization of Baxter's Q-operator) – the one associated to what we call here the ith *prefundamental* representation $L_{i,a}^+$. This is an infinite-dimensional representation of the Borel subalgebra (in the Kac–Moody sense) $U_q(\mathfrak{b})$ of $U_q(\mathfrak{g})$ that was introduced by M. Jimbo and the second author in [HJ].

In order to explain what it is, recall the classification of irreducible finite-dimensional representations of $U_q(\mathfrak{g})$ due to Drinfeld [Dr] and Chari–Pressley [CP]. The algebra $U_q(\mathfrak{g})$

has loop generators $x_{i,n}^{\pm}, i \in I, n \in \mathbb{Z}$; $h_{i,n}, i \in I, n \neq 0$; and $k_i^{\pm 1}, i \in I$. Each irreducible finite-dimensional representation is generated by a "highest weight vector", that is, a vector annihilated by $x_{i,n}^{+}, i \in I, n \in \mathbb{Z}$, which is an eigenvector of the loops to Cartan generators $h_{i,n}, i \in I, n \neq 0$ and $k_i^{\pm 1}$. Furthermore, the eigenvalue of their generating function

$$\phi_i^{\pm}(z) = k_i^{\pm 1} \exp\left(\pm(q_i - q_i^{-1}) \sum_{n>0} h_{i,\pm n} z^{\pm n}\right)$$

is the expansion in $z^{\pm 1}$ of the rational function $q_i^{\deg P_i} P_i(zq_i^{-1})/P_i(zq_i)$, where $P_i(z)$ is a polynomial with constant term 1. These polynomials, called Drinfeld polynomials, record the "highest ℓ-weight" of the representation.

In [HJ], M. Jimbo and the second author extended the category of finite-dimensional representations of $U_q(\mathfrak{g})$ to a category denoted by $\mathcal{O}$. This is a category of (possibly infinite-dimensional) representations of $U_q(\mathfrak{b})$ which have weight decomposition with respect to the finite-dimensional Cartan subalgebra generated by $k_i^{\pm 1}$ and such that all weight components are finite-dimensional. It was shown in [HJ] that irreducible representations of this category are also generated by highest weight vectors (in the above sense), but the corresponding highest ℓ-weights (the eigenvalues of $\phi_j^{\pm}(z)$) are given by arbitrary rational functions which are regular and non-zero at the origin (but may have a zero or a pole at infinity).

The ith prefundamental representation $L_{i,a}^{+}$ is then by definition the representation for which the eigenvalue of $\phi_j^{\pm}(z)$ on the highest weight vector is equal to 1 if $j \neq i$ and to $1 - za$ if $j = i$.

As far as we know, such representations were first constructed in the case of $\mathfrak{g} = \widehat{sl}_2$ by V. Bazhanov, S. Lukyanov, and A. Zamolodchikov [BLZ1, BLZ2]. Their construction was subsequently generalized to $\mathfrak{g} = \widehat{sl}_3$ in [BHK], and to $\mathfrak{g} = \widehat{sl}_{n+1}$ with $i = 1$ in [Ko]. For general $\mathfrak{g}$, the prefundamental representations were constructed in [HJ].

A marvelous insight of [BLZ1, BLZ2] was the identification of the transfer-matrix of this representation in the case of $\mathfrak{g} = \widehat{sl}_2$ with the Baxter operator. From the point of view discussed above, this enables one to interpret Baxter's TQ relation as a relation in the Grothendieck ring of the category $\mathcal{O}$. Here we generalize this result to all untwisted quantum affine algebras.

Namely, we establish the following relation in the Grothendieck ring of $\mathcal{O}$, generalizing the Baxter relation: for any finite-dimensional representation V of $U_q(\mathfrak{g})$, take its q-character and replace each $Y_{i,a}$ by the ratio of the classes of prefundamental representations $[L_{i,aq_i^{-1}}^{+}]/[L_{i,aq_i}^{+}]$ times the class of the one-dimensional representation $[\omega_i]$ of $U_q(\mathfrak{b})$ on which the finite-dimensional Cartan subalgebra acts according to the ith fundamental weight. Then this expression is equal to the class of V in the Grothendieck ring of $\mathcal{O}$ (viewed as a representation of $U_q(\mathfrak{b})$ obtained by restriction from $U_q(\mathfrak{g})$). This is our first main result.

For example, if $\mathfrak{g} = \widehat{sl}_2$ and V is the two-dimensional representation, then we have

$$[V] = [\omega_1] \frac{[L_{1,aq^{-1}}^{+}]}{[L_{1,aq}^{+}]} + [-\omega_1] \frac{[L_{1,aq^3}^{+}]}{[L_{1,aq}^{+}]}, \tag{1.2}$$

or equivalently,

$$[V][L^+_{1,aq}] = [\omega_1][L^+_{1,aq^{-1}}] + [-\omega_1][L^+_{1,aq^3}],$$

which follows from the fact that $V \otimes L^+_{1,aq}$ is an extension of two representations: $[\omega_1]\otimes L^+_{1,aq^{-1}}$ and $[-\omega_1] \otimes L^+_{1,aq^3}$ (see [JMS, Section 2]).

Our second main result is that the (twisted) transfer-matrix associated to $L^+_{i,a}$ is well-defined (despite the fact that $L^+_{i,a}$ is infinite-dimensional), and further, all of its eigenvalues on any irreducible finite-dimensional representation W of $U_q(\mathfrak{g})$ are *polynomials* up to one and the same factor that depends only on W (more precisely, we prove this for the prefundamental representations in the category dual to $\mathcal{O}$). Denoting these eigenvalues by $Q_{i,a}$, and combining our two results, we obtain the proof of the conjecture of Reshetikhin and the first author.

For example, if $\mathfrak{g} = \widehat{sl}_2$ and V is the two-dimensional representation, then (1.2) implies the Baxter equation (1.1) for the eigenvalues of the transfer-matrix of V (after renaming the variables $a \mapsto zq^{-1}$).

As explained in [FR1, Section 6.3] and in Section 5.6 below, the formula for the eigenvalues of the transfer-matrices in terms of the polynomials $Q_{i,a}$ leads to a natural system of equations on the roots of these polynomials. These equations ensure the cancellation of the apparent poles in the eigenvalues of the transfer-matrices due to the appearance of the polynomials $Q_{i,a}$ in the denominator. These are the generalized Bethe Ansatz equations, which are suitably modified equations (6.6) of [FR1] (they are modified because we use the twisted trace in the definition of the transfer-matrices, which depends on additional parameters). We conjecture that the solutions of these generalized Bethe Ansatz equations are in one-to-one correspondence with the eigenvalues of the (twisted) transfer-matrices.

If $\mathfrak{g} = \widehat{sl}_2$ and V is the two-dimensional representation, these equations are precisely the Bethe Ansatz equations of the six-vertex model. This observation was used by N. Reshetikhin as a guiding principle for writing conjectural Bethe Ansatz equations and the eigenvalues of the transfer-matrices for some $\mathfrak{g}$ [R1, R2, R3] – a procedure he dubbed "analytic Bethe Ansatz" (see also [BR, KS]). The results of [FR1] and the present paper give us a conceptual explanation of this procedure.

We close this Introduction with the following three remarks.

(1) Our results show that the prefundamental representations have an important role to play in representation theory of quantum affine algebra. They are infinite-dimensional, but in many ways they have a simpler structure and behavior than finite-dimensional representations. And they can be used effectively to prove results about finite-dimensional representations that were previously out of reach, such as the conjecture on the spectra of transfer-matrices that we prove in this paper. As another application, we use our results on the polynomiality of the transfer matrix of $L^+_{i,a}$ to show that a certain generating function of the Drinfeld's Cartan elements $h_{i,n}$ is, up to a universal factor, a polynomial on any finite-dimensional irreducible representation of $U_q(\mathfrak{g})$.

(2) As we learned from Nikita Nekrasov and Andrei Okounkov, Baxter's polynomials $Q_{i,a}$ should have a geometric interpretation. Finite-dimensional representations of $U_q(\mathfrak{g})$ may be realized in equivariant K-theory of quiver varieties as shown by H. Nakajima [N1],

and similarly, finite-dimensional representations of the Yangian are realized in equivariant cohomology of these varieties [V] (see also [MO]). In the Yangian case, the Baxter polynomial is expected to be equal to the operator of (quantum) multiplication by the Chern polynomial of a certain tautological vector bundle on the quiver variety (see [NS], p. 15), and there is a similar conjecture in the case of $U_q(\mathfrak{g})$. It would be interesting to connect our results with the geometry of quiver varieties. In particular, it is an interesting question to find a geometric realization of the prefundamental representations analogous to that of finite-dimensional representations of $U_q(\mathfrak{g})$.

We remark that the category of finite-dimensional representations of $U_q(\mathfrak{g})$ is equivalent to that of the Yangian $Y_h(\mathfrak{g})$ (where $q = e^{\pi i h}$ and h is not a rational number), as shown by S. Gautam and V. Toledano Laredo [GTL].

(3) After this paper was finished, we learned from N. Nekrasov about his joint work with V. Pestun and S. Shatashvili (subsequently published as [NPS]), in which the q-characters are used to describe quantum geometry of the Ω-deformations of 5D supersymmetric quiver gauge theories (and similarly for 4D theories, with quantum affine algebras replaced by Yangians). Note however that in [NPS] the analytic properties of the functions $Q_{i,a}$ and $t_V(a)$ are quite different from ours. Thus, it seems that the q-characters represent a rather general algebraic structure that can be used (by imposing various analytic conditions on $Q_{i,a}$ and $t_V(a)$) to describe not only the models of statistical mechanics discussed in the present paper, but also other models of quantum physics.

As explained in [FR1], the q-character map is a q-deformation of the Miura transformation. And so the q-characters play a role similar to that of the Miura transformation in the Gaudin model and its generalizations: describing the spectra of the Hamiltonians of the model [FFR, F]. Moreover, the Miura transformation (resp., the q-character map) arises naturally from the center of a completed enveloping algebra at the critical level of an affine Kac–Moody algebra, as explained in [FFR, F] (resp., quantum affine algebra, as explained in [FR1]). Thus, it is this center that is the fundamental algebraic object governing a large class of quantum integrable models.

The paper is organized as follows. In Section 2 we recall the definitions and the main properties of quantum affine (or loop) algebras and the corresponding Borel subalgebras. In Section 3 we recall important results about their representations; in particular, finite-dimensional representation as well as those from the category $\mathcal{O}$ (such as the prefundamental representations), introduced in [HJ]. In Section 4 we prove a uniform explicit q-character formula for positive prefundamental representations (Theorem 4.1). We prove that it implies our first main result: the realization of generalized Baxter's relations in the Grothendieck ring of $\mathcal{O}$ (Theorem 4.8). We also prove that an arbitrary tensor product of positive prefundamental representations is simple (Theorem 4.11). In Section 5 we state our second main result: polynomiality of the twisted transfer-matrices associated to prefundamental representations (Theorem 5.9). Our main application is the proof of a deformed version of the conjecture of Reshetikhin and the first author (Theorem 5.11). We use this result to write down the system of Bethe Ansatz equations explicitly in Section 5.6. We also derive the polynomiality of Drinfeld's Cartan elements on finite-dimensional representations (Theorem 5.17) and prove commutativity of the twisted transfer-matrices associated to representations in the category $\mathcal{O}$ (Theorem 5.3). In Section 6, we establish the existence of a certain grading on positive prefundamental representations (Theorem 6.1). This result is used in Section 7 to conclude the proof of Theorem 5.9.

Acknowledgments. We are grateful to Nikita Nekrasov and Andrei Okounkov for valuable discussions. We also thank Ilaria Damiani for communications on root vectors.

This paper was completed while the first author was visiting Université Paris-Diderot Paris 7, which he thanks for hospitality. He also acknowledges support of the NSF.

The results of this paper were presented at the conference "Symmetries in Mathematics and Physics II" in honor of Victor Kac's 70th birthday at IMPA (Rio de Janeiro) in June 2013. It is a pleasure to dedicate this paper to him.

2. Quantum loop algebra and Borel algebras

2.1. **Quantum loop algebra.** Let $C = (C_{i,j})_{0\leq i,j\leq n}$ be an indecomposable Cartan matrix of untwisted affine type. We denote by $\mathfrak{g}$ the Kac–Moody Lie algebra associated with C. Set $I = \{1, \ldots, n\}$, and denote by $\dot{\mathfrak{g}}$ the finite-dimensional simple Lie algebra associated with the Cartan matrix $(C_{i,j})_{i,j\in I}$. Let $\{\alpha_i\}_{i\in I}$, $\{\alpha_i^\vee\}_{i\in I}$, $\{\omega_i\}_{i\in I}$, $\{\omega_i^\vee\}_{i\in I}$, $\dot{\mathfrak{h}}$ be the simple roots, the simple coroots, the fundamental weights, the fundamental coweights and the Cartan subalgebra of $\dot{\mathfrak{g}}$, respectively. We set $Q = \oplus_{i\in I}\mathbb{Z}\alpha_i$, $Q^+ = \oplus_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$, $P = \oplus_{i\in I}\mathbb{Z}\omega_i$. Let $D = \mathrm{diag}(d_0 \ldots, d_n)$ be the unique diagonal matrix such that $B = DC$ is symmetric and d_i's are relatively prime positive integers. We denote by $(\ ,\ ) : Q \times Q \to \mathbb{Z}$ the invariant symmetric bilinear form such that $(\alpha_i, \alpha_i) = 2d_i$. We use the numbering of the Dynkin diagram as in [Ka]. Let $a_0, \cdots, a_n$ stand for the Kac label ([Ka], pp.55-56). We have $a_0 = 1$ and we set $\alpha_0 = -(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$.

Throughout this paper, we fix a non-zero complex number q which is not a root of unity. We set $q_i = q^{d_i}$. We also set $h \in \mathbb{C}$ such that $q = e^h$, so that q^r is well-defined for any $r \in \mathbb{Q}$. We will use the standard symbols for q-integers

$$[m]_z = \frac{z^m - z^{-m}}{z - z^{-1}}, \quad [m]_z! = \prod_{j=1}^m [j]_z, \quad \begin{bmatrix} s \\ r \end{bmatrix}_z = \frac{[s]_z!}{[r]_z![s-r]_z!}.$$

We will use the quantum Cartan matrix $C(q) = (C_{i,j}(q))_{i,j\in I}$ defined by $C_{i,j}(q) = [C_{i,j}]_q$ if $i \neq j$ in I and $C_{i,i}(q) = [2]_{q_i}$ for $i \in I$. The symmetrized quantum Cartan matrix $B(q) = (B_{i,j}(q))_{i,j\in I}$ is defined by $B_{i,j}(q) = [d_i]_q C_{i,j}(q)$ for $i, j \in I$. We denote by $\tilde{B}(q)$ (resp. $\tilde{C}(q)$) the inverse of $B(q)$ (resp. $C(q)$).

The quantum loop algebra $U_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra defined by generators e_i, f_i, $k_i^{\pm 1}$ $(0 \leq i \leq n)$ and the following relations for $0 \leq i, j \leq n$.

$$k_i k_j = k_j k_i, \quad k_0^{a_0} k_1^{a_1} \cdots k_n^{a_n} = 1, \qquad k_i e_j k_i^{-1} = q_i^{C_{i,j}} e_j, \quad k_i f_j k_i^{-1} = q_i^{-C_{i,j}} f_j,$$

$$[e_i, f_j] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-C_{i.j}} (-1)^r e_i^{(1-C_{i,j}-r)} e_j e_i^{(r)} = 0 \quad (i \neq j), \qquad \sum_{r=0}^{1-C_{i.j}} (-1)^r f_i^{(1-C_{i,j}-r)} f_j f_i^{(r)} = 0 \quad (i \neq j)\,.$$

Here we have set $x_i^{(r)} = x_i^r/[r]_{q_i}!$ $(x_i = e_i, f_i)$. The algebra $U_q(\mathfrak{g})$ has a Hopf algebra structure given by

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \quad \Delta(k_i) = k_i \otimes k_i\,,$$
$$S(e_i) = -k_i^{-1}e_i, \quad S(f_i) = -f_i k_i, \quad S(k_i) = k_i^{-1}\,,$$

where $i = 0, \cdots, n$.

The algebra $U_q(\mathfrak{g})$ can also be presented in terms of the Drinfeld generators [Dr, Be]

$$x_{i,r}^{\pm}\ (i \in I, r \in \mathbb{Z}), \quad \phi_{i,\pm m}^{\pm}\ (i \in I, m \geq 0), \quad k_i^{\pm 1}\ (i \in I).$$

Example 2.1. *In the case* $\dot{\mathfrak{g}} = sl_2$*, we have* $e_1 = x_{1,0}^+$*,* $e_0 = k_1^{-1}x_{1,1}^-$*,* $f_1 = x_{1,0}^-$ *and* $f_0 = x_{1,-1}^+k_1$*.*

We shall use the generating series $(i \in I)$:

$$\phi_i^{\pm}(z) = \sum_{m\geq 0} \phi_{i,\pm m}^{\pm} z^{\pm m} = k_i^{\pm 1}\mathrm{exp}\left(\pm(q_i - q_i^{-1})\sum_{m>0} h_{i,\pm m} z^{\pm m}\right).$$

We also set $\phi_{i,\pm m}^{\pm} = 0$ for $m < 0$, $i \in I$.

Remark 2.2. *In* [FR1]*, the notation* $h_{i,m}$ *is used for* $[d_i]_q h_{i,m}$*.*

The algebra $U_q(\mathfrak{g})$ has a $\mathbb{Z}$-grading defined by $\deg(e_i) = \deg(f_i) = \deg(k_i^{\pm 1}) = 0$ for $i \in I$ and $\deg(e_0) = -\deg(f_0) = 1$. It satisfies $\deg(x_{i,m}^{\pm}) = \deg(\phi_{i,m}^{\pm}) = m$ for $i \in I$, $m \in \mathbb{Z}$. For $a \in \mathbb{C}^*$, there is a corresponding automorphism $\tau_a : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ such an element g of degree $m \in \mathbb{Z}$ satisfies $\tau_a(g) = a^m g$.

The algebra $U_q(\mathfrak{g})$ has a Q-grading defined by $\deg(x_{i,m}^{\pm}) = \pm\alpha_i$, $\deg(\phi_{i,m}^{\pm}) = 0$ for $i \in I$ and $m \in \mathbb{Z}$.

By Chari's result [C1, Proposition 1.6], there is an involutive automorphism $\hat{\omega} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by $(i \in I,\ m, r \in \mathbb{Z},\ r \neq 0)$

$$\hat{\omega}(x_{i,m}^{\pm}) = -x_{i,-m}^{\mp}, \quad \hat{\omega}(\phi_{i,\pm m}^{\pm}) = \phi_{i,\mp m}^{\mp}, \quad \hat{\omega}(h_{i,r}) = -h_{i,-r}.$$

Besides, it satisfies (see the proof of [C1, Proposition 1.6]):

$$\hat{\omega}(e_0) \in \mathbb{C}^* f_0 \text{ and } \hat{\omega}(e_i) = -f_i \text{ for } i \in I). \tag{2.3}$$

Let $U_q(\mathfrak{g})^{\pm}$ (resp. $U_q(\mathfrak{g})^0$) be the subalgebra of $U_q(\mathfrak{g})$ generated by the $x_{i,r}^{\pm}$ where $i \in I, r \in \mathbb{Z}$ (resp. by the $\phi_{i,\pm r}^{\pm}$ where $i \in I$, $r \geq 0$). We have a triangular decomposition [Be]

$$U_q(\mathfrak{g}) \simeq U_q(\mathfrak{g})^- \otimes U_q(\mathfrak{g})^0 \otimes U_q(\mathfrak{g})^+. \tag{2.4}$$

2.2. **Borel algebra.**

Definition 2.3. *The Borel algebra* $U_q(\mathfrak{b})$ *is the subalgebra of* $U_q(\mathfrak{g})$ *generated by* e_i *and* $k_i^{\pm 1}$ *with* $0 \leq i \leq n$*.*

This is a Hopf subalgebra of $U_q(\mathfrak{g})$. The algebra $U_q(\mathfrak{b})$ contains the Drinfeld generators $x_{i,m}^+$, $x_{i,r}^-$, $k_i^{\pm 1}$, $\phi_{i,r}^+$ where $i \in I$, $m \geq 0$ and $r > 0$.

Let $U_q(\mathfrak{b})^\pm = U_q(\mathfrak{g})^\pm \cap U_q(\mathfrak{b})$ and $U_q(\mathfrak{b})^0 = U_q(\mathfrak{g})^0 \cap U_q(\mathfrak{b})$. Then we have

$$U_q(\mathfrak{b})^+ = \langle x^+_{i,m}\rangle_{i\in I, m\geq 0}, \quad U_q(\mathfrak{b})^0 = \langle \phi^+_{i,r}, k_i^{\pm 1}\rangle_{i\in I, r>0}.$$

It follows from [Be] that we have a triangular decomposition

$$U_q(\mathfrak{b}) \simeq U_q(\mathfrak{b})^- \otimes U_q(\mathfrak{b})^0 \otimes U_q(\mathfrak{b})^+. \tag{2.5}$$

Denote $\mathfrak{t} \subset U_q(\mathfrak{b})$ the subalgebra generated by $\{k_i^{\pm 1}\}_{i\in I}$.

3. Representations of Borel algebras

In this section we review results on representations of the Borel algebra $U_q(\mathfrak{b})$, in particular on the category $\mathcal{O}$ defined in [HJ] and on finite-dimensional representations of $U_q(\mathfrak{g})$.

3.1. **Highest ℓ-weight modules.** Set $\mathfrak{t}^* = \left(\mathbb{C}^\times\right)^I$, and endow it with a group structure by pointwise multiplication. We define a group morphism $\overline{} : P \longrightarrow \mathfrak{t}^*$ by setting $\overline{\omega_i}(j) = q_i^{\delta_{i,j}}$. We shall use the standard partial ordering on $\mathfrak{t}^*$:

$$\omega \leq \omega' \quad \text{if } \omega\omega'^{-1} \text{ is a product of } \{\overline{\alpha}_i^{-1}\}_{i\in I}. \tag{3.6}$$

For a $U_q(\mathfrak{b})$-module V and $\omega \in \mathfrak{t}^*$, we set

$$V_\omega = \{v \in V \mid k_i\, v = \omega(i) v \ (\forall i \in I)\}\,, \tag{3.7}$$

and call it the weight space of weight ω. For any $i \in I$, $r \in \mathbb{Z}$ we have $\phi^\pm_{i,r}(V_\omega) \subset V_\omega$ and $x^\pm_{i,r}(V_\omega) \subset V_{\omega\overline{\alpha}_i^{\pm 1}}$. We say that V is $\mathfrak{t}$-diagonalizable if $V = \bigoplus_{\omega\in\mathfrak{t}^*} V_\omega$.

Definition 3.1. *A series $\mathbf{\Psi} = (\Psi_{i,m})_{i\in I, m\geq 0}$ of complex numbers such that $\Psi_{i,0} \neq 0$ for all $i \in I$ is called an ℓ-weight.*

We denote by $\mathfrak{t}^*_\ell$ the set of ℓ-weights. Identifying $(\Psi_{i,m})_{m\geq 0}$ with its generating series we shall write

$$\mathbf{\Psi} = (\Psi_i(z))_{i\in I}, \quad \Psi_i(z) = \sum_{m\geq 0} \Psi_{i,m} z^m.$$

Since each $\Psi_i(z)$ is an invertible formal power series, $\mathfrak{t}^*_\ell$ has a natural group structure. We have a surjective morphism of groups $\varpi : \mathfrak{t}^*_\ell \to \mathfrak{t}^*$ given by $\varpi(\mathbf{\Psi})(i) = \Psi_{i,0}$. For a $U_q(\mathfrak{b})$-module V and $\mathbf{\Psi} \in \mathfrak{t}^*_\ell$, the linear subspace

$$V_{\mathbf{\Psi}} = \{v \in V \mid \exists p \geq 0, \forall i \in I, \forall m \geq 0, (\phi^+_{i,m} - \Psi_{i,m})^p v = 0\} \tag{3.8}$$

is called the ℓ-weight space of V of ℓ-weight $\mathbf{\Psi}$.

Definition 3.2. *A $U_q(\mathfrak{b})$-module V is said to be of highest ℓ-weight $\mathbf{\Psi} \in \mathfrak{t}^*_\ell$ if there is $v \in V$ such that $V = U_q(\mathfrak{b})v$ and the following hold:*

$$e_i\, v = 0 \quad (i \in I)\,, \qquad \phi^+_{i,m} v = \Psi_{i,m} v \quad (i \in I,\ m \geq 0)\,.$$

The ℓ-weight $\mathbf{\Psi} \in \mathfrak{t}^*_\ell$ is uniquely determined by V. It is called the highest ℓ-weight of V. The vector v is said to be a highest ℓ-weight vector of V.

Proposition 3.3. [HJ] *For any $\mathbf{\Psi} \in \mathfrak{t}^*_\ell$, there exists a simple highest ℓ-weight module $L(\mathbf{\Psi})$ of highest ℓ-weight $\mathbf{\Psi}$. This module is unique up to isomorphism.*

The submodule of $L(\mathbf{\Psi})\otimes L(\mathbf{\Psi'})$ generated by the tensor product of the highest ℓ-weight vectors is of highest ℓ-weight $\mathbf{\Psi}\mathbf{\Psi'}$. In particular, $L(\mathbf{\Psi}\mathbf{\Psi'})$ is a subquotient of $L(\mathbf{\Psi})\otimes L(\mathbf{\Psi'})$.

Definition 3.4. [HJ] *For $i \in I$ and $a \in \mathbb{C}^\times$, let*

$$L_{i,a}^{\pm} = L(\mathbf{\Psi}_{i,a}) \quad \textit{where} \quad (\mathbf{\Psi}_{i,a})_j(z) = \begin{cases} (1-za)^{\pm 1} & (j = i)\,, \\ 1 & (j \neq i)\,. \end{cases} \tag{3.9}$$

We call $L_{i,a}^+$ (resp. $L_{i,a}^-$) a positive (resp. negative) prefundamental representation in the category $\mathcal{O}$.

Example 3.5. *In the case $\dot{\mathfrak{g}} = sl_2$, $L_{1,a}^+$ carries a basis $L_{1,a}^+ = \oplus_{j\geq 0}\mathbb{C}v_j$ with the explicit action ($r, j \geq 0$, $p > 0$, $v_{-1} = 0$):*

$$x_{1,r}^+ v_j = \delta_{r,0} v_{j-1}\,, \quad x_{1,p}^- v_j = \frac{-aq^{-j}\delta_{p,1}[j+1]_q}{q-q^{-1}} v_{j+1}\,, \quad \phi_1^+(z)v_j = q^{-2j}(1-za)v_j.$$

Definition 3.6. [HJ] *For $\omega \in \mathfrak{t}^*$, let*

$$[\omega] = L(\mathbf{\Psi}_\omega) \quad \textit{where} \quad (\mathbf{\Psi}_\omega)_i(z) = \omega(i) \quad (i \in I).$$

Note that the representation $[\omega]$ is 1-dimensional with a trivial action of $e_0, \cdots, e_n$. It is called a zero prefundamental representation. For $\lambda \in P$, we will simply use the notation $[\lambda]$ for the representation $[\overline{\lambda}]$.

For $a \in \mathbb{C}^\times$, the subalgebra $U_q(\mathfrak{b})$ is stable under τ_a. Denote its restriction to $U_q(\mathfrak{b})$ by the same letter. Then the pullbacks of the $U_q(\mathfrak{b})$-modules $L_{i,b}^{\pm}$ by τ_a is $L_{i,ab}^{\pm}$.

3.2. **Category $\mathcal{O}$.** For $\lambda \in \mathfrak{t}^*$, we set $D(\lambda) = \{\omega \in \mathfrak{t}^* \mid \omega \leq \lambda\}$.

Definition 3.7. [HJ] *A $U_q(\mathfrak{b})$-module V is said to be in category $\mathcal{O}$ if:*

i) V is $\mathfrak{t}$-diagonalizable,

ii) for all $\omega \in \mathfrak{t}^*$ we have $\dim(V_\omega) < \infty$,

iii) there exist a finite number of elements $\lambda_1, \cdots, \lambda_s \in \mathfrak{t}^*$ such that the weights of V are in $\bigcup_{j=1,\cdots,s} D(\lambda_j)$.

The category $\mathcal{O}$ is a monoidal category.

Let $\mathfrak{r}$ be the subgroup of $\mathfrak{t}_\ell^*$ consisting of $\mathbf{\Psi}$ such that $\Psi_i(z)$ is rational for any $i \in I$.

Theorem 3.8. [HJ] *Let $\mathbf{\Psi} \in \mathfrak{t}_\ell^*$. The simple module $L(\mathbf{\Psi})$ is in category $\mathcal{O}$ if and only if $\mathbf{\Psi} \in \mathfrak{r}$. Then it is a subquotient of a tensor product of (positive, negative, zero) prefundamental representations. Moreover, for V in category $\mathcal{O}$, $V_{\mathbf{\Psi}} \neq 0$ implies $\mathbf{\Psi} \in \mathfrak{r}$.*

Let $\mathcal{E}_\ell \subset \mathbb{Z}^{\mathfrak{r}}$ be the ring of maps $c : \mathfrak{r} \to \mathbb{Z}$ satisfying $c(\mathbf{\Psi}) = 0$ for all $\mathbf{\Psi}$ such that $\varpi(\mathbf{\Psi})$ is outside a finite union of sets of the form $D(\mu)$ and such that for each $\omega \in \mathfrak{t}^*$, there are finitely many $\mathbf{\Psi}$ such that $\varpi(\mathbf{\Psi}) = \omega$ and $c(\mathbf{\Psi}) \neq 0$. Similarly, let $\mathcal{E} \subset \mathbb{Z}^{\mathfrak{t}^*}$ be the ring of maps $c : \mathfrak{t}^* \to \mathbb{Z}$ satisfying $c(\omega) = 0$ for all ω outside a finite union of sets of the form $D(\mu)$. The map ϖ is naturally extended to a surjective ring morphism $\varpi : \mathcal{E}_\ell \to \mathcal{E}$.

For $\mathbf{\Psi} \in \mathfrak{r}$ (resp. $\omega \in \mathfrak{t}^*$), we define $[\mathbf{\Psi}] = \delta_{\mathbf{\Psi},.} \in \mathcal{E}_\ell$ (resp. $[\omega] = \delta_{\omega,.} \in \mathcal{E}$).

Let V be a $U_q(\mathfrak{b})$-module in category $\mathcal{O}$. We define [FR1, HJ] the q-character of V

$$\chi_q(V) = \sum_{\mathbf{\Psi}\in\mathfrak{r}} \dim(V_{\mathbf{\Psi}})[\mathbf{\Psi}] \in \mathcal{E}_\ell\,. \tag{3.10}$$

Example 3.9. *For $\omega \in \mathfrak{t}^*$, the q-character of the 1-dimensional representation $[\omega]$ is just its ℓ-highest weight $\chi_q([\omega]) = [\omega]$. That is why the use of the same notation $[\omega]$ will not lead to confusion.*

Similarly we define the ordinary character of V to be an element of $\mathcal{E}$

$$\chi(V) = \varpi(\chi_q(V)) = \sum_{\omega\in\mathfrak{t}^*} \dim(V_\omega)[\omega]\,. \tag{3.11}$$

For V in category $\mathcal{O}$ which has a unique ℓ-weight $\mathbf{\Psi}$ whose weight is maximal, we also consider its normalized q-character $\tilde{\chi}_q(V)$ and normalized character $\tilde{\chi}(V)$ by

$$\tilde{\chi}_q(V) = [\mathbf{\Psi}^{-1}]\cdot\chi_q(V)\,, \quad \tilde{\chi}(V) = \varpi(\tilde{\chi}_q(V))\,.$$

Let $\mathrm{Rep}(U_q(\mathfrak{b}))$ be the Grothendieck ring of the category $\mathcal{O}$. By [FR1, Theorem 3] and [HJ, Proposition 3.12], we have the following.

Proposition 3.10. *The q-character morphism*

$$\chi_q : Rep(U_q(\mathfrak{b})) \to \mathcal{E}_\ell, \quad [V] \mapsto \chi_q(V),$$

is an injective ring morphism.

3.3. **Finite-dimensional representations.** Let $\mathcal{C}$ be the category of (type 1) finite-dimensional representations of $U_q(\mathfrak{g})$.

For $i \in I$, let $P_i(z) \in \mathbb{C}[z]$ be a polynomial with constant term 1. Set

$$\mathbf{\Psi} = (\Psi_i(z))_{i\in I}, \quad \Psi_i(z) = q_i^{\deg(P_i)} \frac{P_i(zq_i^{-1})}{P_i(zq_i)}.$$

Then $L(\mathbf{\Psi})$ is finite-dimensional. Moreover the action of $U_q(\mathfrak{b})$ can be uniquely extended to an action of the full quantum affine algebra $U_q(\mathfrak{g})$, and any simple object in the category $\mathcal{C}$ is of this form.

Remark 3.11. *Let $L(\mathbf{\Psi}')$ be any finite-dimensional module in the category $\mathcal{O}$. We claim that there is $\mathbf{\Psi}$ as above and $\omega \in \mathfrak{t}^*$ such that $L(\mathbf{\Psi}') \simeq L(\mathbf{\Psi}) \otimes [\omega]$. Since $[\omega]$ is just a one-dimensional representation, this means that, up to a slight twisting of the action of the Cartan elements, $L(\mathbf{\Psi}')$ is a representation of $U_q(\mathfrak{g})$. This statement is known in the case $\dot{\mathfrak{g}} = sl_2$* [BT]. *To prove it in general, it suffices to prove that $\mathbf{\Psi}' = \mathbf{\Psi}\mathbf{\Psi}_\omega$. This is clear by sl_2-reduction as, for each $i \in I$, the subalgebra of $U_q(\mathfrak{b})$ generated by the $k_i^{\pm 1}$, $x_{i,m}^+$, $x_{i,m+1}^-$, $\phi_{i,m}^+$, $m \geq 0$ is isomorphic to the Borel algebra of $U_{q_i}(\widehat{sl}_2)$.*

Following [FR1], consider the ring of Laurent polynomials $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$ in the indeterminates $\{Y_{i,a}\}_{i\in I, a\in\mathbb{C}^*}$. Let $\mathcal{M}$ be the group of monomials of $\mathcal{Y}$. For example, for $i \in I, a \in \mathbb{C}^*$, define $A_{i,a} \in \mathcal{M}$ to be

$$Y_{i,aq_i^{-1}} Y_{i,aq_i} \Big(\prod_{\{j\in I | C_{j,i}=-1\}} Y_{j,a} \prod_{\{j\in I | C_{j,i}=-2\}} Y_{j,aq^{-1}} Y_{j,aq} \prod_{\{j\in I | C_{j,i}=-3\}} Y_{j,aq^{-2}} Y_{j,a} Y_{j,aq^2} \Big)^{-1}.$$

For a monomial $m = \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}}$, we consider its 'evaluation on $\phi^+(z)$'. By definition it is an element $m(\phi(z)) \in \mathfrak{r}$ given by

$$m\big(\phi(z)\big) = \prod_{i\in I, a\in\mathbb{C}^*} (Y_{i,a}(\phi(z)))^{u_{i,a}} \text{ where } \Big(Y_{i,a}\big(\phi(z)\big)\Big)_j = \begin{cases} q_i \dfrac{1-aq_i^{-1}z}{1-aq_iz} & (j=i),\\ 1 & (j\neq i).\end{cases}$$

This defines an injective group morphism $\mathcal{M}\to\mathfrak{r}$. We identify a monomial $m\in\mathcal{M}$ with its image in $\mathfrak{r}$. Note that $\varpi(Y_{i,a}) = \overline{\omega_i}$.

It is proved in [FR1] that a finite-dimensional $U_q(\mathfrak{g})$-module V satisfies $V = \bigoplus_{m\in\mathcal{M}} V_{m(\phi(z))}$. In particular, $\chi_q(V)$ can be viewed as an element of $\mathcal{Y}$.

A monomial $M\in\mathcal{M}$ is said to be dominant if $M\in\mathbb{Z}[Y_{i,a}]_{i\in I, a\in\mathbb{C}^*}$. For $L(\mathbf{\Psi})$ a finite-dimensional simple $U_q(\mathfrak{g})$-module, $\mathbf{\Psi} = M\big(\phi(z)\big)$ holds for some dominant monomial $M\in\mathcal{M}$. This representation will be denoted by $L(M)$.

For example, for $i\in I$, $a\in\mathbb{C}^*$ and $k\geq 0$, we have the Kirillov-Reshetikhin (KR) module

$$W_{k,a}^{(i)} = L(Y_{i,a}Y_{i,aq_i^2}\cdots Y_{i,aq_i^{2(k-1)}})\,. \tag{3.12}$$

The representations $W_{1,a}^{(i)} = L(Y_{i,a})$ are called fundamental representations.

Example 3.12. *In the case $\dot{\mathfrak{g}} = sl_2$, we have ($k\geq 0$, $a\in\mathbb{C}^*$)* [FR1]*:*

$$\chi_q(W_{k,aq^{1-2k}}^{(1)}) = Y_{aq^{-1}}Y_{aq^{-3}}\cdots Y_{aq^{-2k+1}}(1 + A_{1,a}^{-1} + A_{1,a}^{-1}A_{1,aq^{-2}}^{-1} + \cdots + A_{1,a}^{-1}\cdots A_{1,aq^{-2(k-1)}}^{-1})\,,$$

and $W_{k,aq^{1-2k}}^{(1)}$ carries a basis $(w_0,\cdots,w_k)$ with the explicit action ($r\in\mathbb{Z}$, $0\leq j\leq k$, $w_{-1} = w_{k+1} = 0$):

$$x_{1,r}^+ w_j = a^r q^{2r(-j+1)} w_{j-1}\,,\quad x_{1,r}^- w_j = a^r q^{-2rj}[j+1]_q[k-j]_q w_{j+1}\,,$$

$$\phi_1^{\pm}(z) w_j = q^{k-2j}\frac{(1-q^{-2k}za)(1-q^2za)}{(1-q^{-2j+2}za)(1-q^{-2j}za)} w_j\,.$$

For m a dominant monomial, we will denote

$$\tilde{L}(m) = L(m(\varpi(m))^{-1}). \tag{3.13}$$

3.4. **The dual category $\mathcal{O}^*$.** For V a $\mathfrak{t}$-diagonalizable $U_q(\mathfrak{b})$-module, we define a structure of $U_q(\mathfrak{b})$-module on its graded dual $V^* = \oplus_{\beta\in\mathfrak{t}^*} V_\beta^*$ by

$$(x\,u)(v) = u\big(S^{-1}(x)v\big)\quad (u\in V^*,\ v\in V,\ x\in U_q(\mathfrak{b})).$$

Definition 3.13. *Let $\mathcal{O}^*$ be the category of $\mathfrak{t}$-diagonalizable $U_q(\mathfrak{b})$-modules V such that V^* is in category $\mathcal{O}$.*

A $U_q(\mathfrak{b})$-module V is said to be of lowest ℓ-weight $\mathbf{\Psi}\in\mathfrak{t}_\ell^*$ if there is $v\in V$ such that $V = U_q(\mathfrak{b})v$ and the following hold:

$$U_q(\mathfrak{b})^- v = \mathbb{C}v\,,\qquad \phi_{i,m}^+ v = \Psi_{i,m} v\quad (i\in I,\ m\geq 0)\,.$$

For $\mathbf{\Psi}\in\mathfrak{t}_\ell^*$, we have the simple $U_q(\mathfrak{b})$-module $L'(\mathbf{\Psi})$ of lowest ℓ-weight Ψ. We have the notion of characters and q-characters for category $\mathcal{O}^*$ as in Section 3.2.

Proposition 3.14. [HJ] *For $\mathbf{\Psi}\in\mathfrak{t}_\ell^*$ we have $(L'(\mathbf{\Psi}))^*\simeq L(\mathbf{\Psi}^{-1})$.*

We will consider the prefundamental representations $R^\pm_{i,a}$ in $\mathcal{O}^*$ defined by $(R^\pm_{i,a})^* \simeq L^\mp_{i,a}$.

Example 3.15. *In the case $\dot{\mathfrak{g}} = sl_2$, $R^+_{1,a}$ carries a basis $R^+_{1,a} = \oplus_{j\geq 0}\mathbb{C}v_j^*$ with the explicit action ($r, j \geq 0$, $p > 0$, $v^*_{-1} = 0$):*

$$x^+_{1,r}v_j^* = \delta_{r,0}q^{2j}v^*_{j+1}\,, \quad x^-_{1,p}v_j^* = \frac{-aq^{1-j}\delta_{p,1}[j]_q}{q-q^{-1}}v^*_{j-1}\,, \quad \phi_1^+(z)v_j^* = q^{2j}(1-za)v_j^*.$$

3.5. **The opposite Borel and the category $\overline{\mathcal{O}}$.** It will be also convenient to use the opposite Borel $U_q(\mathfrak{b}^-) = \hat{\omega}(U_q(\mathfrak{b}))$. By (2.3), $U_q(\mathfrak{b}^-)$ is the subalgebra of $U_q(\mathfrak{g})$ generated by f_i and $k_i^{\pm 1}$ with $0 \leq i \leq n$. Hence it is a Hopf subalgebra of $U_q(\mathfrak{g})$. Let further $U_q(\mathfrak{b}^-)^\pm = U_q(\mathfrak{g})^\pm \cap U_q(\mathfrak{b}^-)$ and $U_q(\mathfrak{b}^-)^0 = U_q(\mathfrak{g})^0 \cap U_q(\mathfrak{b})$. Then we have

$$U_q(\mathfrak{b}^-)^- = \langle x^-_{i,-m}\rangle_{i\in I, m\geq 0}, \quad U_q(\mathfrak{b}^-)^0 = \langle \phi^-_{i,-r}, k_i^{\pm 1}\rangle_{i\in I, r>0}. \tag{3.14}$$

We have a triangular decomposition

$$U_q(\mathfrak{b}^-) \simeq U_q(\mathfrak{b}^-)^- \otimes U_q(\mathfrak{b}^-)^0 \otimes U_q(\mathfrak{b}^-)^+. \tag{3.15}$$

By mimicking the definition of the category $\mathcal{O}$, we can define the category $\overline{\mathcal{O}}$ of $U_q(\mathfrak{b}^-)$-modules. For V a $U_q(\mathfrak{b})$-module, we have a structure of $U_q(\mathfrak{b}^-)$-module on V denoted by $V^{\hat{\omega}}$ and defined by twisting the action by the automorphism $\hat{\omega}$. In particular, we get the simple objects $\overline{L}(\mathbf{\Psi}) = (L'(\mathbf{\Psi}))^{\hat{\omega}}$ of the category $\overline{\mathcal{O}}$. Hence, we have a parametrization of simple objects, as well as q-character theory, in the category $\overline{\mathcal{O}}$ as for the category $\mathcal{O}$. In particular we have the prefundamental representations $\overline{L}^\pm_{i,a} = (R^\pm_{i,a^{-1}})^{\hat{\omega}}$ in the category $\overline{\mathcal{O}}$.

Example 3.16. *In the case $\dot{\mathfrak{g}} = sl_2$, $\overline{L}^+_{1,a}$ carries a basis $\overline{L}^+_{1,a} = \oplus_{j\geq 0}\mathbb{C}v_j^*$ with the explicit action ($r, j \geq 0$, $p > 0$, $v^*_{-1} = 0$):*

$$x^-_{1,-r}v_j^* = -\delta_{r,0}q^{2j}v^*_{j+1}\,, \quad x^+_{1,-p}v_j^* = \frac{a^{-1}q^{1-j}\delta_{p,1}[j]_q}{q-q^{-1}}v^*_{j-1}\,, \quad \phi_1^-(z)v_j^* = q^{2j}(1-(za)^{-1})v_j^*.$$

4. Baxter's relations in category $\mathcal{O}$

In this section we prove a uniform explicit q-character formula for positive prefundamental representations (Theorem 4.1): it is equal to the product of the highest ℓ-weight and the ordinary character (which does not depend on the spectral parameter). We prove that this implies for each finite-dimensional representation V of $U_q(\mathfrak{g})$ the existence of a relation in the Grothendieck ring of $\mathcal{O}$ obtained from the q-character of V (Theorem 4.8). This is our first main result, which is a generalization of Baxter's TQ relations discussed in the Introduction. We also prove that an arbitrary tensor product of positive prefundamental representations is simple (Theorem 4.11).

4.1. q-characters of positive prefundamental representations.

Theorem 4.1. *Let $i \in I$. Then we have for any $a \in \mathbb{C}^*$,*

$$\chi_q(L^+_{i,a}) = \mathbf{\Psi}_{i,a} \times \chi(L^+_{i,a}) \text{ and } \chi_q(R^+_{i,a}) = \mathbf{\Psi}_{i,a} \times \chi(R^+_{i,a}).$$

Remark 4.2. *(i) The characters* $\chi(L_{i,a}^+) = \chi(L_{i,a}^-)$*, are explicitly known and are equal to each other* [HJ, Theorem 6.4]*. Since* $(R_{i,a}^+)^* \simeq L_{i,a}^-$*, the character* $\chi(R_{i,a}^+)$ *is also explicitly known. Besides, the formula is uniform. Hence, as the highest* ℓ*-weight* $\mathbf{\Psi}_{i,a}$ *is known, the statement of Theorem 4.1 is an explicit uniform q-character formula.*

(ii) Theorem 4.1 implies that the normalized q-characters $\tilde{\chi}_q(L_{i,a}^+) = \chi(L_{i,a}^+)$*,* $\tilde{\chi}_q(R_{i,a}^+) = \chi(R_{i,a}^+)$ *do not depend on the spectral parameter a.*

(iii) When the multiplicity N_i *of* α_i *in the maximal root of* $\dot{\mathfrak{g}}$ *is equal to* 1*, this result was established in* [HJ]*. It relies on an asymptotic construction of the representation* $L_{i,a}^+$ *which is only valid if* $N_i = 1$*. Our proof is different and works for all cases.*

Example 4.3. *In the case* $\dot{\mathfrak{g}} = sl_2$*, we have*

$$\chi_q(L_{1,a}^+) = [(1-za)]\left(\sum_{r\geq 0}[\overline{-2r\omega_1}]\right)\ ,\ \chi_q(R_{1,a}^+) = [(1-za)]\left(\sum_{r\geq 0}[\overline{2r\omega_1}]\right).$$

Remark 4.4. *Although positive and negative prefundamental representations have the same character, their q-characters are very different. For instance, in the case* $\dot{\mathfrak{g}} = sl_2$*, we have*

$$\chi_q(L_{1,a}^-) = [(1-za)^{-1}]\left(\sum_{r\geq 0}(A_{1,a}A_{1,aq^{-2}}\cdots A_{1,aq^{-2(r-1)}})^{-1}\right),$$

$$\chi_q(R_{1,a}^-) = [(1-za)^{-1}]\left(\sum_{r\geq 0}(A_{1,a}A_{1,aq^{2}}\cdots A_{1,aq^{2(r-1)}})\right).$$

The reader may also look at the geometric q-character formulas for negative prefundamental representations established in [HL] *(see* [HL, Remark 4.19] *for details).*

4.2. **Proof of Theorem 4.1.** We will use the following technical result.

Lemma 4.5. *Let* $i \in I$*,* $a \in \mathbb{C}^*$*,* $0 \leq K \leq k$*. Let* m *be a monomial occurring in* $\tilde{\chi}_q(W^{(i)}_{k,aq_i^{1-2k}})$ *such that the multiplicity of* $-\alpha_i$ *in* $\varpi(m)$ *is lower than* K*. Then* m *is a monomial of* $\tilde{\chi}_q(W^{(i)}_{K,aq_i^{1-2K}})$*.*

Proof. Let us prove the result by induction on $k - K \geq 0$. It it is trivial if $k = K$. Now, suppose in general that $k > K$. We have [H2, Lemma 5.8]

$$\tilde{\chi}_q(W^{(i)}_{k,aq_i^{1-2k}}) \in \tilde{\chi}_q(W^{(i)}_{k-1,aq_i^{3-2k}}) + (A_{i,a}A_{i,aq_i^{-2}}\cdots A_{i,aq_i^{2-2k}})^{-1}\mathbb{Z}[A_{j,b}^{-1}]_{j\in I, b\in\mathbb{C}^*}. \tag{4.16}$$

Hence m is a monomial of $\tilde{\chi}_q(W^{(i)}_{k-1,aq_i^{3-2k}})$. We can conclude with the induction hypothesis. □

Now we complete the proof of the Theorem.

Proof. Let us explain the proof for the $L_{i,a}^+$. The same proof gives the analogous result for the $\overline{L}_{i,a}^+$. By using $\hat{\omega}$ this implies the result for the $R_{i,a}^+$. For $k \geq 0$, $L^+_{i,aq_i^{-2k}}$ is a subquotient of

$$L^+_{i,a} \otimes \tilde{L}(Y_{i,aq_i^{-1}} Y_{i,aq_i^{-3}} \cdots Y_{i,aq_i^{-2k+1}})$$

where we use the notation of Equation (3.13). Suppose that an ℓ-weight $\mathbf{\Psi} = (\Psi_i(z))_{i\in I}$ occurring in $\tilde{\chi}_q(L^+_{i,a})$ has a pole or a zero $b \in \mathbb{C}^*$. Since $(\mathbf{\Psi})_{q_i^{-2k}} = (\Psi_i(zq_i^{-2k}))_{i\in I}$ occurs in $\tilde{\chi}_q(L^+_{i,aq_i^{-2k}})$, it can be factorized into

$$(\mathbf{\Psi})_{q_i^{-2k}} = \mathbf{\Psi}'_k m_k$$

where $\mathbf{\Psi}'_k$ (resp. m_k) occurs in $\tilde{\chi}_q(L^+_{i,a})$ (resp. in $\tilde{\chi}_q(W_{k,aq_i^{1-2k}})$). Let K be the multiplicity of $-\alpha_i$ in $\varpi(\mathbf{\Psi}) \in -Q^+$. By Lemma 4.5, the monomial m_k occurs in $\tilde{\chi}_q(W_{K,aq_i^{1-2K}})$. We have proved that $(\mathbf{\Psi})_{q_i^{-2k}}$ occurs in

$$\tilde{\chi}_q(L^+_{i,a})\tilde{\chi}_q(\tilde{L}(Y_{i,aq_i^{-1}} Y_{i,aq_i^{-3}} \cdots Y_{i,aq_i^{1-2K}})).$$

In this product, there is only a finite number of terms of weight $\varpi(\mathbf{\Psi})$. But for each $k \geq K$, one of this term has a pole or a zero bq_i^{-2k}. Contradiction. So

$$\tilde{\chi}_q(L^+_{i,a}) = \chi(L^+_{i,a}) = \chi(L^+_{i,1}).$$

□

4.3. **Baxter's relations.** Now we have the following.

Corollary 4.6. *For $i \in I$ and $a \in \mathbb{C}^*$, we have*

$$[\overline{\omega_i}]\frac{\chi_q(L^+_{i,aq_i^{-1}})}{\chi_q(L^+_{i,aq_i})} = [\overline{\omega_i}]\frac{\chi_q(R^+_{i,aq_i^{-1}})}{\chi_q(R^+_{i,aq_i})} = Y_{i,a}.$$

Proof. First, by definition of $Y_{i,a}$, we have the relation for highest ℓ-weights:

$$[\overline{\omega_i}]\frac{\mathbf{\Psi}_{i,aq_i^{-1}}}{\mathbf{\Psi}_{i,aq_i}} = Y_{i,a}.$$

Now the character of prefundamental representations do not depend on the spectral parameter:

$$\chi(L^+_{i,aq_i^{-1}}) = \chi(L^+_{i,aq_i}) \text{ and } \chi(R^+_{i,aq_i^{-1}}) = \chi(R^+_{i,aq_i}).$$

Hence Theorem 4.1 implies the result. □

Remark 4.7. *The formulas we obtain in Corollary 4.6 can also be seen as a change of variables (analogous to those used in* [HL, Section 5.2.2]*).*

We can now prove generalized Baxter's relations in the category $\mathcal{O}$ (and $\mathcal{O}^*$).

Theorem 4.8. *Let V be a finite-dimensional representation of $U_q(\mathfrak{g})$. Replace in $\chi_q(V)$ each variable $Y_{i,a}$ by $[\omega_i]\frac{[L^+_{i,aq_i^{-1}}]}{[L^+_{i,aq_i}]}$ and $\chi_q(V)$ by $[V]$. Then, multiplying by denominators, we get a relation in the Grothendieck ring of $\mathcal{O}$.*

Similarly, replacing $Y_{i,a}$ by $[\omega_i]\frac{[R^+_{i,aq_i^{-1}}]}{[R^+_{i,aq_i}]}$, we get a relation in the Grothendieck ring of $\mathcal{O}^*$.

Proof. Since the q-character morphism is injective, the result follows from Corollary 4.6. We have also used the q-character formula $\chi_q([\omega_i]) = [\overline{\omega_i}]$ for the 1-dimensional representation $[\omega_i]$, as explained in Example 3.9. □

Example 4.9. *(i) Our result generalizes the following known example in the case $\dot{\mathfrak{g}} = sl_2$. We have for $a \in \mathbb{C}^*$ a relation in the Grothendieck ring of $\mathcal{O}$:*

$$[L(Y_{1,a})][L^+_{1,aq}] = [L^+_{1,aq^{-1}}][\omega_1] + [L^+_{1,aq^3}][-\omega_1].$$

Similarly, we have in the Grothendieck ring of the category $\mathcal{O}^*$:

$$[L(Y_{1,a})][R^+_{1,aq}] = [R^+_{1,aq^{-1}}][\omega_1] + [R^+_{1,aq^3}][-\omega_1].$$

(ii) For $\dot{\mathfrak{g}} = sl_3(\mathbb{C})$, we have

$$\chi_q(L(Y_{1,1})) = Y_{1,1} + Y^{-1}_{1,q^2}Y_{2,q} + Y^{-1}_{2,q^3}.$$

Hence we have the following Baxter relation in the Grothendieck ring of $\mathcal{O}$:

$$[L(Y_{1,1})][L^+_{1,q}][L^+_{2,q^2}] = [L^+_{1,q^{-1}}][L^+_{2,q^2}][\omega_1] + [L^+_{1,q^3}][L^+_{2,1}][\omega_2 - \omega_1] + [L^+_{1,q}][L^+_{2,q^4}][-\omega_1].$$

(iii) Let us give another example for $\dot{\mathfrak{g}}$ of type B_2. The q-character of the 4-dimensional fundamental representation $L(Y_{2,1})$ is

$$\chi_q(L(Y_{2,1})) = Y_{2,1} + Y^{-1}_{2,q^2}Y_{1,q} + Y^{-1}_{1,q^5}Y_{2,q^4} + Y^{-1}_{2,q^6}.$$

Hence we have the following Baxter relation in the Grothendieck ring of $\mathcal{O}$:

$$[L(Y_{2,1})][L^+_{2,q}][L^+_{1,q^3}][L^+_{2,q^5}] = [L^+_{2,q^{-1}}][L^+_{1,q^3}][L^+_{2,q^5}][\omega_2] + [L^+_{2,q^3}][L^+_{1,q^{-1}}][L^+_{2,q^5}][\omega_1 - \omega_2]$$

$$+[L^+_{2,q^3}][L^+_{1,q^7}][L^+_{2,q}][\omega_2 - \omega_1] + [L^+_{2,q^7}][L^+_{2,q}][L^+_{1,q^3}][-\omega_2].$$

Remark 4.10. *(i) By taking the duals, Baxter's relations in Theorem 4.8 may also be written in terms of negative prefundamental representations. For example, in the case $\dot{\mathfrak{g}} = sl_2$, for $a \in \mathbb{C}^*$ we get in the Grothendieck ring of $\mathcal{O}$:*

$$[L(Y_{1,aq^2})][L^-_{1,aq}] = [L^-_{1,aq^{-1}}][-\omega_1] + [L^-_{1,aq^3}][\omega_1],$$

and in the Grothendieck ring of the category $\mathcal{O}^*$:

$$[L(Y_{1,aq^{-2}})][R^-_{1,aq}] = [R^-_{1,aq^{-1}}][-\omega_1] + [R^-_{1,aq^3}][\omega_1].$$

(ii) For quantum affine algebras of classical types, some conjectural relations in the Grothendieck ring of the category $\mathcal{O}$ have been proposed in [Sun]. *It is not clear to us whether there is a connection between them and the generalized Baxter relations that we establish in this paper.*

4.4. **Baxter's relations as tensor product decomposition.** In this subsection we give an additional interpretation of Baxter's relations of Theorem 4.8. This is only included for completeness of the paper as the results of this subsection will not be used in the other sections.

Let us first prove the following additional application of Theorem 4.1.

Theorem 4.11. *An arbitrary tensor product of positive (resp. negative) prefundamental representations in the category $\mathcal{O}$ is simple. The same holds in the category $\mathcal{O}^*$.*

Proof. First let us prove the result for a tensor product T of negative prefundamental representations in the category $\mathcal{O}$. It can be written in form

$$T = \bigotimes_{a\in(\mathbb{C}^*/q^{\mathbb{Z}})} \bigotimes_{i\in I, r\in\mathbb{Z}} (L^-_{i,aq^r})^{\otimes n_{i,aq^r}}$$

where we have chosen a representative $a \in \mathbb{C}^*$ for each class in $\mathbb{C}^*/q^{\mathbb{Z}}$ so that for any $i \in I$, $r \leq d_i$, we have $n_{i,aq^r} = 0$. The ordering in the tensor product is not relevant for this proof as the Grothendieck ring of the category $\mathcal{O}$ is commutative. Let $\mathbf{\Psi}$ be the highest ℓ-weight of T. We prove that $L(\mathbf{\Psi})$ is isomorphic to T. First $L(\mathbf{\Psi})$ is a subquotient of T. So it suffices to prove that the dimensions of weight spaces of $L(\mathbf{\Psi})$ are greater than those of T.

For $R \leq 0$, consider the simple module $L_R = \tilde{L}(M_R)$ where the monomial M_R is defined by

$$M_R = \prod_{a\in(\mathbb{C}^*/q^{\mathbb{Z}})} \prod_{i\in I, r>0} \left(\prod_{\{r'\in r+2d_i\mathbb{Z} | r\geq r' > R\}} Y_{i,aq^{r'}}^{n_{i,aq^{r+d_i}}} \right).$$

Then L_R is isomorphic to a tensor product of (normalized) Kirillov-Reshetikhin modules

$$L_R \simeq \bigotimes_{a\in(\mathbb{C}^*/q^{\mathbb{Z}})} \bigotimes_{i\in I, r\in\mathbb{Z}} \left(\tilde{L} \left(\prod_{\{r'\in r+2d_i\mathbb{Z} | r\geq r' > R\}} Y_{i,aq^{r'}} \right) \right)^{\otimes n_{i,aq^r}}.$$

Indeed it suffices to prove the irreducibility of the tensor product when we replace spectral parameters by their inverse (see [H4, Proposition 4.13]). Then the result becomes clear as the q-character of the tensor product has a unique dominant monomial (see for example [H2, Proposition 5.3]).

Now, by [HJ, Theorem 6.1], the character of T is the limit (as a formal power series in the negative simple roots) of the character of L_R when $R \to -\infty$. So it suffices to prove that the dimension of weight spaces of L_R are lower than those of $L(\mathbf{\Psi})$.

Consider the tensor product

$$T' = L(\mathbf{\Psi}) \otimes \left(\bigotimes_{a\in(\mathbb{C}^*/q^{\mathbb{Z}})} \bigotimes_{i\in I, r\in\mathbb{Z}} (L^+_{i,aq^{R_{a,i,r}}})^{\otimes n_{i,aq^r}} \right)$$

where $R_{a,i,r}$ is the lowest integer r' such that $r' \in r + 2d_i\mathbb{Z}$ and $r' > R - d_i$. Since T' and L_R have the same highest ℓ-weight, L_R is a subquotient of T'.

Recall that the ℓ-weights of L_R are the product of the highest ℓ-weight $M_R(\varpi(M_R))^{-1}$ multiplied by a product of $A^{-1}_{j,b}$, $j \in i$, $b \in \mathbb{C}^*$ [FM, Theorem 4.1]. Hence, by Theorem 4.1, an ℓ-weight of T' is an ℓ-weight of L_R only if it of the form

$$\mathbf{\Psi'}(M_R(\varpi(M_R))^{-1}\mathbf{\Psi}^{-1})$$

where $\mathbf{\Psi'}$ is an ℓ-weight of $L(\mathbf{\Psi})$ and $(M_R(\varpi(M_R))^{-1}\mathbf{\Psi}^{-1})$ is the highest ℓ-weight of the remaining tensor product of positive prefundamental representations. We get the result.

The same proof gives the result for negative prefundamental representations in the category $\mathcal{O}^*$. By duality, we get the result for prefundamental representations in the category $\mathcal{O}$ as well as in the category $\mathcal{O}^*$. □

The tensor product of a simple representation in $\mathcal{O}$ (resp. in $\mathcal{O}^*$) by a 1-dimensional representation $[\omega]$, $\omega \in \mathfrak{t}^*$, is clearly simple. So, multiplying Baxter's relation of Theorem 4.8 by the denominators and using Theorem 4.11, we get the following.

Corollary 4.12. *Baxter's relation of Theorem 4.8 may be interpreted as the decomposition, in the Grothendieck ring of $\mathcal{O}$ (resp. of $\mathcal{O}^*$), of the class of the tensor product of two simple representations into a sum of classes of simple representations.*

5. Transfer-matrices and polynomiality

In this section we state our second main result: polynomiality of the twisted transfer-matrices (and of their eigenvalues) associated to the prefundamental representations (Theorem 5.9). Our main application is the proof of a version of the conjecture of Reshetikhin and the first author (Theorem 5.11). We use this result to write down the system of Bethe Ansatz equations explicitly in Section 5.6. We also derive the polynomiality of Drinfeld's Cartan elements on finite-dimensional representations (Theorem 5.17) and prove commutativity of the twisted transfer-matrices associated to representations in the category $\mathcal{O}$ (Theorem 5.3).

5.1. **Universal R-matrix, L-operators and transfer-matrices.** The universal R-matrix $\mathcal{R}$ of $U_q(\mathfrak{g})$ belongs to the tensor product $U_q(\mathfrak{g})\hat{\otimes}U_q(\mathfrak{g})$ (completed for the $\mathbb{Z}$-grading of $U_q(\mathfrak{g})$). The Cartan subalgebra of $U_q(\mathfrak{g})$ is also slightly completed: elements $e^{h\omega}$, $\omega \in P\otimes_{\mathbb{Z}}\mathbb{C}$, are added so that $k_i = e^{h\alpha_i}$ for $i \in I$, as in [CP] (see also [Da1]). We will use analogous completions of Borel subalgebras. The completed algebras still act on the representations we consider.

The universal R-matrix satisfies the Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{5.17}$$

In fact, it is known that $\mathcal{R}$ belongs to $U_q(\mathfrak{b})\hat{\otimes}U_q(\mathfrak{b}^-)$. Hence for V a $U_q(\mathfrak{b})$-module, we may define the L-operator associated to V

$$L_V(z) = (\pi_{V(z)} \otimes \mathrm{Id})(\mathcal{R}) \in End(V)[[z]]\hat{\otimes}U_q(\mathfrak{b}^-),$$

where $\pi_V(z) : U_q(\mathfrak{b}) \to End(V)[[z]]$ is the representation morphism of V with the $\mathbb{Z}$-grading of $U_q(\mathfrak{b})$.

Let V in the category $\mathcal{O}^*$ whose weights are in $\overline{P} \subset \mathfrak{t}^*$. For $g \in U_q(\mathfrak{b})$ (or in $\mathrm{End}(V)$), the twisted trace of g on V is

$$Tr_{V,u}(g) = \sum_{\lambda \in \mathfrak{t}^*} Tr_{V_\lambda}(\pi_V(g)) \left(\prod_{i\in I} u_i^{\lambda_i}\right) \in \mathbb{C}[[u_i^{\pm 1}]]_{i\in I},$$

where $\lambda_i \in \mathbb{Z}$ is defined by $\lambda(i) = q_i^{\lambda_i}$.

Definition 5.1. *The twisted transfer-matrix associated to V is*

$$t_V(z,u) = (Tr_{V,u} \otimes Id)(L_V(z)) \in U_q(\mathfrak{b}^-)[[z, u_i^{\pm 1}]]_{i\in I}.$$

More precisely we have

$$t_V(z,u) \in U_q(\mathfrak{b}^-)[u_i^{\pm 1}]_{i\in I}[[z, v_i]]_{i\in I}$$

where for $i \in I$

$$v_i = \prod_{j\in I} u_j^{C_{j,i}}$$

corresponds to a simple root. Hence the product of twisted transfer-matrices is well-defined.

Note that for $V = R_{i,a}^+$, we have $t_V(z,u) \in U_q(\mathfrak{b}^-)[[z, v_i]]_{i\in I}$.

Example 5.2. *Let $i \in I$. The zero prefundamental representations$L(\overline{\alpha_i})$, $L(\overline{\omega_i})$ are defined in Section 3.1. By using Section 7.1, we get*

$$t_{L(\overline{\alpha_i})} = v_i k_i^{-1} \text{ and } t_{L(\overline{\omega_i})}(z,u) = u_i \tilde{k}_i^{-1}$$

where the $\tilde{k}_i = \prod_{j\in I} k_j^{(C^{-1})_{j,i}}$ are characterized by the relations

$$k_i = \prod_{j\in I} \tilde{k}_j^{C_{j,i}}.$$

For V in $\mathcal{C}$, $t_V(z,u)$ is a finite sum and the variables u_i can be specialized to any non zero complex values. For example, consider the specialization

$$u_i = q_i^{2\sum_{j\in I}(C^{-1})_{j,i}} = q^{2\sum_{j\in I}(DC^{-1})_{i,j}}.$$

This means $v_i = q_i^2$ as DC^{-1} is symmetric (and $u = q$ if $\dot{\mathfrak{g}} = sl_2$). Then we recover the standard non-twisted transfer-matrix

$$t_V(z) = (Tr_V \otimes \mathrm{Id})(((\prod_{1\leq i\leq n} \tilde{k}_i^2) \otimes 1)L_V(z)) \in U_q(\mathfrak{b}^-)[[z]].$$

5.2. **Commutativity of twisted transfer-matrices.** It can be proved as in [FR1, Lemma 2] that for V, V' in the category $\mathcal{O}^*$ whose weights are in $\overline{P} \subset \mathfrak{t}^*$, and for W an extension of V and V' in the category $\mathcal{O}^*$, we have

$$t_W(z,u) = t_V(z,u) + t_{V'}(z,u) \text{ and } t_{V\otimes V'}(z,u) = t_V(z,u)t_{V'}(z,u). \tag{5.18}$$

Let us prove the following stronger result.

Theorem 5.3. *For V, V' in the category $\mathcal{O}^*$ whose weights are in $\overline{P} \subset \mathfrak{t}^*$, we have*

$$t_V(z,u)t_{V'}(w,u) = t_{V'}(w,u)t_V(z,u).$$

Remark 5.4. *(i) The result is well-known when V, V' are finite-dimensional in the category $\mathcal{C}$ (see* [FR1, Lemma 2]*). But in general the action of $U_q(\mathfrak{b})$ on V, V' can not be extended to an action of $U_q(\mathfrak{g})$ and we can not evaluate $\pi_{V(z)} \otimes \pi_{V'(w)}$ on the universal R-matrix $\mathcal{R}$. That is why we can not use directly the standard proof for the general representations in $\mathcal{O}^*$ we consider.*

(ii) The same proof as below gives the result for representations in the category $\mathcal{O}$.

Proof. From (5.18), $t_V(z)$ depends only of the class of V in the Grothendieck ring of $\mathcal{O}^*$. But by [HJ, Remark 3.13], this ring is commutative. Hence

$$t_V(z,u)t_{V'}(z,u) = t_{V'}(z,u)t_V(z,u).$$

Now let $a \in \mathbb{C}^*$ and V_a be the $U_q(\mathfrak{b})$-module obtained from V by twisting by τ_a. Then V_a is in the category $\mathcal{O}^*$ and

$$t_{V_a}(z,u) = t_V(za,u). \tag{5.19}$$

Hence for any $a \in \mathbb{C}^*$ we have

$$t_V(za,u)t_{V'}(z,u) = t_{V'}(z,u)t_V(za,u).$$

This implies the result. □

For V in category $\mathcal{O}^*$ with weights in $\overline{P}$, we will denote

$$t_V(z,u) = \sum_{m\geq 0} t_V[m](u)z^m.$$

5.3. **Deformation of $U_q(\mathfrak{b}^-)^0$.** Let $t(v)$ be the $\mathbb{C}[[v_i]]_{i\in I}$-subalgebra of $U_q(\mathfrak{b}^-)[[v_i]]_{i\in I}$ generated by the $t_{R^+_{i,a}}[m](u)$ ($i \in I$, $a \in \mathbb{C}^*$, $m \geq 0$) and the $t_{L(\overline{\omega})}[m](u)$ ($\omega \in Q^+$). We recall that the zero prefundamental representations $L(\overline{\omega})$ are defined in Section 3.1.

Proposition 5.5. *$t(v)$ is a commutative subalgebra of $U_q(\mathfrak{b}^-)[[v_i]]_{i\in I}$ which is a deformation of $U_q(\mathfrak{b}^-)^0$.*

More precisely, let $i \in I$. Then the limit at $v_j \to 0$ ($j \in I$) of $t_{R^+_{i,a}}(z,u)$ is $T_i(za)$ where

$$T_i(z) = exp\left(\sum_{m>0} z^m \frac{\tilde{h}_{i,-m}}{[d_i]_q[m]_{q_i}}\right)$$

and

$$\tilde{h}_{i,-m} = \sum_{j\in I}[d_j]_q\tilde{C}_{j,i}(q^m)h_{j,-m}.$$

The commutativity is a direct consequence of Theorem 5.3. The rest of the Proposition will be proved in Section 7.2.

Example 5.6. *In the case $\dot{\mathfrak{g}} = sl_2$, we have $T_1(z) = exp\left(\sum_{m>0} z^m \frac{h_{1,-m}}{[m]_q(q^m+q^{-m})}\right)$.*

The next lemma follows from [FM, Lemma 3.1].

Lemma 5.7. *$T_i(z)$ commutes with the $x^{\pm}_{j,r}$ for $j \neq i$ and $r \in \mathbb{Z}$.*

Let W be a tensor product of simple objects in $\mathcal{C}$. It has a highest weight vector w of highest ℓ-weight $m = \prod_{j\in I, b\in\mathbb{C}^*} Y_{j,b}^{u_{j,b}}$. Then w is an eigenvector of $T_i(z)$. Let us denote by $f_i(z)$ the corresponding eigenvalue. A straightforward computation gives

$$f_i(z) = \prod_{j\in I, b\in\mathbb{C}^*} \exp\left(u_{j,b}(m)\sum_{r>0}(zb^{-1})^r \frac{\tilde{C}_{i,j}(q^r)}{r}\right). \tag{5.20}$$

By [FM, Theorem 4.1], the other ℓ-weights of W are of the form

$$M = mA_{i_1,a_1}^{-1}A_{i_2,a_2}^{-1}\cdots A_{i_N,a_N}^{-1}$$

where $i_1,\cdots,i_N\in I$ and $a_1,\cdots,a_N\in\mathbb{C}^*$.

Proposition 5.8. *The eigenvalue of $T_i(z)$ on the ℓ-weight space W_M is equal to*

$$f_i(z)\times\prod_{1\leq k\leq N, i_k=i}(1-za_k^{-1}).$$

Proof. Let us replace in $T_i(z)$ each $\tilde{h}_{j,-m}$ by the associated eigenvalue encoded by the monomial $A_{k,b}^{-1}$ ($k\in I$, $b\in\mathbb{C}^*$). We get

$$\exp\left(-\sum_{j\in I, m>0} z^m\frac{\tilde{B}_{i,j}(q^m)[d_j]_q}{[m]_q}C_{j,k}(q^m)\frac{(q_j^m-q_j^{-m})}{m(q_j-q_j^{-1})}b^{-m}\right)$$

$$= \exp\left(-\sum_{j\in I,m>0}(zb^{-1})^m\frac{\tilde{B}_{i,j}(q^m)B_{j,k}(q^m)}{m}\right) = \exp\left(-\sum_{m>0}\frac{(zb^{-1})^m\delta_{i,k}}{m}\right) = (1-zb^{-1})^{\delta_{i,k}}.$$

Hence the result. □

5.4. **Baxter polynomiality.** Let W be a tensor product of simple objects in the category $\mathcal{C}$ of finite-dimensional representations of $U_q(\mathfrak{g})$. It has a highest weight vector w of weight ω. For $i\in I$, let $f_i(z)\in\mathbb{C}[[z]]$ be the eigenvalue of $T_i(z)$ on w as given in (5.20). Let λ be a weight of W and $ht_i(\omega-\lambda)$ be the multiplicity of α_i in $\omega-\lambda$; that is

$$\omega-\lambda = \sum_{i\in I} ht_i(\omega-\lambda)\cdot\alpha_i.$$

One of the main results of this paper is the following Baxter polynomiality.

Theorem 5.9. *Let $i\in I, a\in\mathbb{C}^*$ and $V=R_{i,a}^+$. Then the operator*

$$(f_i(az))^{-1}t_V(z,u)\in((End((W)_\lambda))[[v_j]]_{j\in I})[z]$$

is a polynomial in z of degree $ht_i(\omega-\lambda)$.

This Theorem 5.9 will be proved in Section 6 and Section 7. By Theorem 5.3 and Proposition 5.5, it implies immediately the following:

Corollary 5.10. *Let $i\in I$, $a\in\mathbb{C}^*$ and $V=R_{i,a}^+$. Then the eigenvalues of $t_V(z,u)$ on $(W)_\lambda$ are of the form*

$$f_i(za)Q_i(za,u),$$

where $Q_i(z,u)$ is a polynomial in z of degree $ht_i(\omega-\lambda)$.

We now give some applications of these results.

5.5. **Proof of the conjecture of Reshetikhin and the first author.** Our main application is the proof of a version of the conjecture of Reshetikhin and the first author from [FR1] about the spectra of transfer-matrices. It is a consequence of Theorem 5.9 and Corollary 4.6. Let us use the notation of the previous section.

Theorem 5.11. *Let W be as above and V be a finite-dimensional representation of $U_q(\mathfrak{g})$. Then every eigenvalue of $t_V(z,u)$ on $(W)_\lambda$ may be expressed as $\chi_q(V)$ in which we replace each variable $Y_{i,a}$ by*

$$q_i^{ht_i(\omega-\lambda)} a_i u_i \frac{f_i(azq_i^{-1})Q_i(azq_i^{-1},u)}{f_i(azq_i)Q_i(azq_i,u)}, \tag{5.21}$$

where the series $f_i(z) \in \mathbb{C}[[z]]$ given by formula (5.20) and the numbers

$$a_i = \prod_{j\in I} q_j^{-(C^{-1})_{j,i}\omega(\alpha_i^\vee)}, \qquad i \in I, \tag{5.22}$$

depend only on W (and not on the eigenvalue), whereas $Q_i(z,u)$ is a polynomial in z of degree $ht_i(\omega-\lambda)$ that depends on the eigenvalue.

The polynomials $Q_i(z,u)$ are the analogues of the Baxter polynomials (see the Introduction for more details).

For $\omega = \lambda$, Theorem 5.11 follows from the definition of $\chi_q(V)$, as explained in [FR1, Section 6.1]. In this case, $Q_i(z,u) = 1$ for all $i \in I$.

Proof. By Theorem 5.3, it suffices to replace in Baxter's formula of Corollary 4.6 the classes of representations $R_{i,a}^+$, $L(\overline{\omega_i})$ by the eigenvalues of their respective twisted transfer-matrices. For $R_{i,a}^+$, we use the formula in Corollary 5.10. For $L(\overline{\omega_i})$, by Example 5.2, we have to compute the eigenvalue of $u_i\tilde{k}_i^{-1}$. Let a_i be the eigenvalue of $\tilde{k}_i^{-1}$ on w. We get the formula $a_iu_iq_i^{ht_i(\omega-\lambda)}$ because for $j\in I$, $r\in\mathbb{Z}$ we have $\tilde{k}_ix_{j,r}^- = q_i^{-\delta_{i,j}}x_{j,r}^-\tilde{k}_i$. □

Remark 5.12. *(i) Suppose that V is irreducible and let m_V be its highest weight dominant monomial. By* [FM, Theorem 4.1]*, any monomial in $\chi_q(V)$ has the form*

$$M_V = m_V \prod_{k=1}^N A_{i_k,a_k}^{-1}.$$

Let us write out M_V as the product of the $Y_{i,a}$, and then replace each $Y_{i,a}$ by the corresponding factor $a_iu_if_i(zaq_i^{-1})/f_i(zaq_i)$ appearing in Theorem 5.11. We obtain a scalar function, which we denote by F_{M_V}. A straightforward computation using formula (5.20) yields that the ratio between F_{M_V} and F_{m_V} is given by the following rational function in z:

$$\frac{F_{M_V}}{F_{m_V}} = \prod_{k=1}^N v_{i_k}^{-1} q_{i_k}^{-\deg(P_{i_k})} \frac{P_{i_k}(z^{-1}a_k^{-1}q_{i_k})}{P_{i_k}(z^{-1}a_k^{-1}q_{i_k}^{-1})},$$

where the $P_j, i\in I$, are the Drinfeld polynomials of the highest weight vector w of W.

Therefore, it follows from Theorem 5.11 that all eigenvalues of $t_V(z,u)$ on W are rational functions in z up to one and the same overall scalar factor, namely $F_{m_V} =$

$$\prod_{i,j\in I, a\in\mathbb{C}^*}\left((v_j q_j^{-\omega(\alpha_i^\vee)})^{(C^{-1})_{j,i}}\exp\left(\sum_{r>0,b\in\mathbb{C}^*}\frac{u_{j,b}(m)(zab^{-1})^r(q_i^{-r}-q_i^r)\tilde{C}_{i,j}(q^r)}{r}\right)\right)^{u_{i,a}(m_V)}$$

where $u_{i,a}(m_V)$ (resp. $u_{i,a}(m)$) is the power of $Y_{i,a}$ in m_V (resp. in the highest monomial m of W).

This is in agreement with the fact that up to a scalar function the operator $\pi_W(t_V(z))$ is rational in z (see [FR2] *and* [EFK, Proposition 9.5.3]*). Our result gives a description of the eigenvalues of the u-deformation of this operator.*

This is also in agreement with the calculations in [FR1, Section 6].

(ii) It follows from the definition that each z^m-coefficient of $Q_i(z,u)$ is a root of a polynomial whose coefficients are in the ring of formal Taylor power series in the v_j, $j\in I$ (hence it belongs to the algebraic closure of the field of fractions of this ring). We expect these series to be expansions of rational functions near the point $v_j=0$, $j\in I$. It would be interesting to prove that this is indeed the case and to describe the poles of these rational functions.

Example 5.13. *(i) For $\dot{\mathfrak{g}}=sl_2(\mathbb{C})$ and $V=L(Y_{1,q^{-1}})$, we get as in (1.1)*

$$D(z)\frac{Q_1(zq^{-2},u)}{Q_1(z,u)}+A(z)\frac{Q_1(zq^2,u)}{Q_1(z,u)},$$

$$\text{with } A(z)=(D(zq^2))^{-1}=a_1^{-1}u_1^{-1}q^{-ht_1(\omega-\lambda)}\frac{f_1(zq^2)}{f_1(z)}.$$

Dividing by $D(z)=F_{Y_{1,q^{-1}}}(z)$ we get a rational fraction in z (see Remark 5.12 (i)):

$$\frac{Q_1(zq^{-2},u)}{Q_1(z,u)}+v_1^{-1}q^{-\deg(P)-2ht_1(\omega-\lambda)}\frac{P(z^{-1}q)}{P(z^{-1}q^{-1})}\frac{Q_1(zq^2,u)}{Q_1(z,u)},$$

where P is the Drinfeld polynomial corresponding to the highest monomial m of W.

(ii) In general there are more than 2 terms. For $\dot{\mathfrak{g}}=sl_3(\mathbb{C})$ and $V=L(Y_{1,q^{-1}})$, we get

$$D_1(z)\frac{Q_1(zq^{-2},u)}{Q_1(z,u)}+(D_1(zq^2))^{-1}D_2(z)\frac{Q_1(zq^2,u)Q_2(zq^{-1},u)}{Q_1(z,u)Q_2(zq,u)}+(D_2(zq^2))^{-1}\frac{Q_2(zq^3,u)}{Q_2(zq,u)},$$

$$\text{with } D_1(z)=a_1u_1q^{ht_1(\omega-\lambda)}\frac{f_1(zq^{-2})}{f_1(z)},\quad \text{and}\quad D_2(z)=a_2u_2q^{ht_2(\omega-\lambda)}\frac{f_2(zq^{-1})}{f_2(zq)}.$$

Dividing by $D_1(z)=F_{Y_{1,q^{-1}}}(z)$ we get a rational fraction in z (see Remark 5.12 (i)):

$$\frac{Q_1(zq^{-2},u)}{Q_1(z,u)}+v_1^{-1}q^{-\deg(P_1)+(ht_2-2ht_1)(\omega-\lambda)}\frac{P_1(z^{-1}q)}{P_1(z^{-1}q^{-1})}\frac{Q_1(zq^2,u)Q_2(zq^{-1},u)}{Q_1(z,u)Q_2(zq,u)}$$
$$+v_1^{-1}v_2^{-1}q^{-\deg(P_1P_2)-(ht_1+ht_2)(\omega-\lambda)}\frac{P_1(z^{-1}q)P_2(z^{-1})}{P_1(z^{-1}q^{-1})P_2(z^{-1}q^{-2})}\frac{Q_2(zq^3,u)}{Q_2(zq,u)},$$

where P_1,P_2 are the Drinfeld polynomials corresponding to the highest monomial m of W.

5.6. **Bethe Ansatz equations.** We now derive the generalized Bethe Ansatz equations from Theorem 5.11 following [FR1, Section 6.3].

According to Theorem 5.11, each eigenvalue of $t_V(z,u)$ on a finite-dimensional representation W is a sum of terms, each having the product of the functions of the form $f_i(azq_i)Q_i(azq_i,u)$ in the denominator.

Suppose that $W = \otimes_{j=1}^N L(\boldsymbol{\Psi}_j)$, where

$$\boldsymbol{\Psi}_j = (\Psi_{j,i}(z))_{i\in I}, \quad \Psi_{j,i}(z) = q_i^{\deg(P_i)} \frac{P_{j,i}(zq_i^{-1})}{P_{j,i}(zq_i)}, \tag{5.23}$$

where $P_{j,i}(z)$ are polynomials (see Section 3.3). The zeros of $f_i(azq_i)$ differ from the roots of $P_{j,i}(z)$ by powers of q, and therefore the corresponding poles in the eigenvalues of $t_V(z,u)$ on W are to be expected. But the roots of $Q_i(azq_i,u)$ give rise to extraneous poles, which we do not expect to have in the eigenvalues of $t_V(z,u)$. Therefore they should cancel each other, and this must happen uniformly for all V.

We expect that, at least for generic values of q, the only possible way for this to happen is for the poles in the monomials of the form M and MA_{i,aq_i}^{-1} in the q-character of V to cancel out.

We also expect that each root of $Q_i(z,u)$ has multiplicity one. Then each cancellation of this type gives rise to an equation, which says that the sum of the residues of the terms in the eigenvalues corresponding to M and MA_{i,aq_i}^{-1} at the poles coming from the roots of $Q_i(zaq_i,u)$ is equal to 0.

To write down these equations explicitly, let us set

$$Q_i(z,u) = \prod_{k=1}^{m_i} (w_k^{(i)} - z), \qquad m_i = ht_i(\omega - \lambda).$$

Recall that $Q_i(z,u)$ is a polynomial in z whose coefficients are in the algebraic closure of the field of fractions of the ring of formal Taylor power series in the $v_i, i \in I$. Hence each root $w_k^{(i)}$ belongs to the same field. Further, since the $Q_i(z,u)$ enter the eigenvalues through the ratios (5.21), we do not lose any generality by normalizing $Q_i(z,u)$ this way.

An explicit calculation along the lines of those in [FR1, Section 6] gives us the following system of equations on the $w_k^{(i)}$:

$$v_i \prod_{j=1}^N q_i^{\deg P_{j,i}} \frac{P_{j,i}(q_i^{-1}/w_k^{(i)})}{P_{j,i}(q_i/w_k^{(i)})} = \prod_{s\neq k} q_i^2 \frac{w_k^{(i)} - w_s^{(i)} q_i^{-2}}{w_k^{(i)} - w_s^{(i)} q_i^2} \prod_{l\neq i} \prod_{s=1}^{m_l} q^{C_{li}} \frac{w_k^{(i)} - w_s^{(l)} q^{-C_{li}}}{w_k^{(i)} - w_s^{(l)} q^{C_{li}}}. \tag{5.24}$$

These are the generalized Bethe Ansatz equations corresponding to a given collection of polynomials $P_{j,i}, j = 1,\ldots,N; i \in I$.

These equations come from "local" pole cancellations, in the sense that they occur between monomials of the form M and MA_{i,aq_i}^{-1}. Therefore the equations are the same for any choice of the representation V.

If $\mathfrak{g} = \widehat{sl}_2$ and $W = \otimes_{j=1}^N W^{(1)}_{R_j, b_j q^{-R_j+1}}$ (we use the notation of Example 3.12), then these equations become

$$v \prod_{j=1}^{N} q^{R_j} \frac{w_k - b_j q^{-R_j}}{w_k - b_j q^{R_j}} = \prod_{s \neq k} q^2 \frac{w_k - w_s q^{-2}}{w_k - w_s q^2}, \qquad k = 1, \ldots, ht(\omega - \lambda). \tag{5.25}$$

Formulas (5.24) and (5.25) specialize to formulas (6.6) and (6.5) of [FR1], respectively, if we set $v_i = q_i^2$ (the factor $-q^{-N}$ on the RHS of formula (6.5) in [FR1] should be replaced by q^{2m-4}, and similarly for formula (6.6) in [FR1]).

Example 5.14. *For $N = R_1 = 1 = k$ and $b_1 = q^{-1}$, we recover the well-known relations*

$$vq \frac{w_1 - q^{-2}}{w_1 - 1} = 1 \text{ and } w_1 = \frac{1 - vq^{-1}}{1 - vq}.$$

We expect that for generic q and generic polynomials $P_{j,i}$, it is possible prove along the lines of the above argument that any eigenvalue of the transfer-matrices $t_V(z, u)$ on $(W)_\lambda$ gives rise to a solution of the Bethe Ansatz equations (5.25). Furthermore, we expect that the converse is true as well. Thus, we arrive at the following conjecture, which is a version of the "completeness of Bethe Ansatz" (at the level of eigenvalues).

Conjecture 5.15. *For generic q and generic polynomials $P_{j,i}$, there is a one-to-one correspondence between the eigenvalues of the transfer-matrices $t_V(z, u)$ on $(W)_\lambda$, where $W = \otimes_{j=1}^N L(\mathbf{\Psi}_j)$, with $\mathbf{\Psi}_j$ given by formula (5.23) and the solutions of the Bethe Ansatz equations (5.24) with $w_k^{(i)}, k = 1, \ldots, ht_i(\omega - \lambda); i \in I$.*

5.7. **Example.** We suppose that $\dot{\mathfrak{g}} = sl_2$, $V = R^+_{1,1}$, and $W = W^{(1)}_{N, q^{1-2N}}$ is a KR-module (we use the notation of Example 3.12). There is an explicit formula for the R-matrix (see Section 7.1 below). We have $(\pi_W(x_0^-))^{N+1} = (\pi_W(k x^+_{-1}))^{N+1} = 0$. So the image $\mathcal{L}_V(z)$ of $L_V(z)$ in $(\mathrm{End}(V) \otimes \mathrm{End}(W))[[z]]$ is a product

$$\mathcal{L}_V(z) = \mathcal{L}_V^+(z)(\mathrm{Id}_V \otimes \pi_W(T(z)))\mathcal{L}_V^-(z)\mathcal{L}^\infty$$

$$\mathcal{L}_V^+(z) = \sum_{0 \leq r \leq N} \frac{((q^{-1} - q)\pi_V(x^+_{1,0}) \otimes \pi_W(x^-_{1,0}))^r}{q^{\frac{r(r-1)}{2}}[r]_q!},$$

$$\mathcal{L}_V^-(z) = \sum_{0 \leq r \leq N} \frac{((q - q^{-1})z\pi_V(k_1^{-1}x^-_{1,1}) \otimes \pi_W(x^+_{1,-1})k_1)^r}{q^{\frac{r(r-1)}{2}}[r]_q!}$$

$\mathcal{L}_V^\infty = (\pi_V \otimes \pi_W)(\mathcal{R}^\infty)$ does not depend on z.

By taking the twisted trace, the image of the twisted transfer-matrix in $(\mathrm{End}(W))[[u, z]]$ is

$$\sum_{0 \leq r \leq N} \frac{(-(q - q^{-1})^2 z)^r}{q^{r(r-1)}([r]_q!)^2} (Tr_{V,u} \otimes \pi_W)((x^+_{1,0} \otimes x^-_{1,0})^r (1 \otimes T_1(z))(k_1^{-1}x^-_{1,1} \otimes x^+_{1,-1}k_1)^r \mathcal{L}_V^\infty)$$

$$= \sum_{0 \leq r \leq N} \frac{((q - q^{-1})z)^r}{[r]_q!} \pi_W((x^-_{1,0})^r T_1(z)(x^+_{1,-1}k_1)^r) \sum_{m \geq r} u^{2m} \begin{bmatrix} m \\ r \end{bmatrix}_q q^{\frac{r(3-r)}{2} - rm} \pi_W(k_1^{-m}).$$

Let $0 \leq j \leq N$. Note that $h_{1,-m}.w_0 = \frac{q^{Nm}[Nm]_q}{m} w_0$ for $m > 0$. Then $(x_{1,-1}^+ k_1)^r k_1^m.w_j = q^{r(N-2)-m(N-2j)} w_{j-r}$ is an eigenvector of $T_1(z)$ with eigenvalue $f_1(z)P_{j-r}(z)$ where

$$f_1(z) = \exp\left(\sum_{m>0} z^m \frac{q^{Nm}[Nm]_q}{m[m]_q(q^m + q^{-m})}\right)$$

does not depend of r, j and

$$P_{j-r}(z) = (1-z)(1-zq^2)\cdots(1-zq^{2(j-r-1)})$$

is a polynomial in z of degree $j-r$ (this can be computed by hand or with Proposition 5.8). Since $(x_{1,0}^-)^r w_{j-r} = \frac{[j]_q![N-j+r]_q!}{[j-r]_q![N-j]_q!} w_j$, this implies that w_j is an eigenvector of $(f_1(z))^{-1} t_V(z,u)$ with eigenvalue $Q_1(z,u)$ equal to

$$\sum_{0\leq r\leq N} \frac{(q-q^{-1})^r [N-j+r]_q! \begin{bmatrix} j \\ r \end{bmatrix}_q z^r (1-z)\cdots(1-zq^{2(j-r-1)})}{q^{r(\frac{r+1}{2}-N)}[N-j]_q!} \sum_{m\geq r} u^{2m} \begin{bmatrix} m \\ r \end{bmatrix}_q q^{-m(r+N-2j)}$$

which is a polynomial in z. It is clear that the degree is at most j. By taking the limit $u=0$, only the term with $m=r=0$ contributes and we see that the degree is exactly j.

Note that in addition each z^m-coefficient of $Q_1(z,u)$ is rational in $v = u^2$.

Example 5.16. *For $N = j = 1$, we get the well-known Baxter polynomial*

$$Q_1(z,u) = \frac{(1-z)(1-u^2q^{-1}) + zu^2(q-q^{-1})}{(1-u^2q)(1-u^2q^{-1})}.$$

It has degree 1 and its root is $w_1 = \frac{1-u^2q^{-1}}{1-u^2q}$ which specializes at $(1+q+q^2)^{-1}$ for $u^2 = v = q^2$. This is the same as in Example 5.14 above. It means w_1 is not a pole of

$$q\frac{Q_1(zq^{-2},u)}{Q_1(z,u)} + q^{-2}u^{-2}\frac{1-z^{-1}}{1-z^{-1}q^{-2}}\frac{Q_1(zq^2,u)}{Q_1(z,u)}.$$

5.8. **Polynomiality of Drinfeld's Cartan elements.** Let us now give the second application of our main results.

By Proposition 5.5, Corollary 5.10 implies:

Theorem 5.17. *Let $f_i(z) \in \mathbb{C}[[z]]$ be the eigenvalue of $T_i(z)$ on a highest weight vector of W (for all $i \in I$). Then on $(W)_\lambda$ the operator*

$$(f_i(z))^{-1}T_i(z) \in (End((W)_\lambda))[z]$$

is a polynomial in z of degree $ht_i(\omega - \lambda)$.

Note that this is compatible with Proposition 5.8 which gives the polynomiality, up to $f_i(z)$, for the eigenvalues (the result here is much stronger as we get the polynomiality also for the off-diagonal elements).

As an illustration, let us check this result by hand in the most elementary not trivial case. Suppose that $\dot{\mathfrak{g}} = sl_2$. Let

$$V = L(Y_{1,1}) \otimes L(Y_{1,1}).$$

An highest weight vector of V is an eigenvector of $T_1(z)$. Let $g(z)$ be the eigenvalue. We study the action on the 2-dimensional space V_0. $g(z)^{-1}T_1(z)$ has a unique eigenvalue on V_0 which is $(1-zq^{-1})$. Let us study the representation as in [H4, Examples 2.2, 3.3]. From the relation $[x^+_{1,-1}, x^-_{1,0}] = -\phi^-_{1,-1}/(q-q^{-1})$, we get $h_{1,-1} = q^2 f_0 f_1 - f_1 f_0$. Let w^+ be a highest weight vector of $L(Y_{1,1})$. Then $(w^+, w^- = f_1 w^+)$ is a basis of $L(Y_{1,1})$. We use analogous notation $(v^+, v^- = f_1 v^+)$ for a second copy of $L(Y_{1,1})$. Then $(w^- \otimes v^+, q^{-2} w^- \otimes v^+ + w^+ \otimes w^-)$ is a basis of V_0. In this basis, for $m > 0$ the action of $h_{1,-m}$ is the matrix $\begin{pmatrix} \frac{[m]_q}{m}(1-q^{-2m}) & a_m \\ 0 & \frac{[m]_q}{m}(1-q^{-2m}) \end{pmatrix}$ where $a_m \in \mathbb{C}^*$ and $a_1 = q - q^{-4}$. For $r \geq 0$, the vector $x^-_{1,-r}.(w^+ \otimes v^+)$ in this basis has the form $\begin{pmatrix} \lambda_r \\ \mu_r \end{pmatrix}$ where $\lambda_r, \mu_r \in \mathbb{C}$ and $\lambda_0 = 0$, $\mu_0 = 1$. We have the relation $[h_{1,-m}, x^-_{1,-r}] = -\frac{[2m]_q}{m} x^-_{1,-m-r}$ for $m > 0$, $r \geq 0$. Since $h_{1,-m}.(w^+ \otimes v^+) = \frac{2[m]_q}{m}(w^+ \otimes v^+)$, it implies

$$\begin{cases} -\frac{[m]_q}{m}(1+q^{-2m})\lambda_r + a_m \mu_r = -\frac{[2m]_q}{m}\lambda_{m+r}, \\ [m]_q(1+q^{-2m})\mu_r = [2m]_q \mu_{m+r}. \end{cases}$$

The second equation with $m=1$ implies $\mu_r = q^{-r}$ for $r \geq 0$. Then the first equation with $m=1$ reads $-q^r\lambda_r + \frac{q^2-q^{-3}}{q+q^{-1}} = -q^{1+r}\lambda_{1+r}$ which implies $\lambda_r = q^{-r} r \frac{q^{-3}-q^2}{q+q^{-1}}$ for $r \geq 0$. Now the first equation with $r=0$ gives $a_m = -[2m]_q q^{-m} \frac{q^{-3}-q^2}{q+q^{-1}}$. Hence the image of $g(z)^{-1}T_1(z)$ in the basis is

$$(1-zq^{-1})\exp\left(\sum_{m>0} z^m \frac{\begin{pmatrix} 0 & -[2m]_q q^{-m}\frac{q^{-3}-q^2}{q+q^{-1}} \\ 0 & 0 \end{pmatrix}}{[m]_q(q^m+q^{-m})}\right)$$

$$= (1-zq^{-1})\begin{pmatrix} 1 & -\sum_{m>0} z^m \frac{[2m]_q q^m}{[m]_q(q^m+q^{-m})}\frac{q^{-3}-q^2}{q+q^{-1}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-zq^{-1} & z\frac{q-q^{-4}}{q+q^{-1}} \\ 0 & 1-zq^{-1} \end{pmatrix}.$$

It is a polynomial of degree 1 in z.

5.9. **Plan of the proof of Theorem 5.9.** We first establish (Theorem 6.1) a grading of positive prefundamental representations $R^+_{i,a}$ which has good compatibility properties with the action of $\mathbb{Z}$-graded elements in $U_q(\mathfrak{b})$. Section 6 is entirely devoted to this grading and the proof of its existence. We believe that the study of this grading will be interesting independently of the applications it finds in this paper.

Then in Section 7.1 we recall the factorization of the universal R-matrix. We give the proof of Proposition 5.5 in Section 7.2. Then we explain why it suffices to consider the case that W is a tensor product of fundamental representations with ℓ-weight spaces of dimension 1 (or of dimension at most 2 for $\dot{\mathfrak{g}}$ of type E_8). This is crucial to control the action of the Cartan factor of the universal R-matrix. By using the grading of Theorem 6.1, we can also control the action of the positive and negative parts, as we explain in Section 7.

6. A grading of positive prefundamental representations

To prove Theorem 5.9, we first establish the existence of a certain grading on positive prefundamental representations with nice properties with respect to the action of Drinfeld generators (Theorem 6.1). We believe this grading is also of independent interest.

6.1. **The grading.** Let us fix $i \in I, a \in \mathbb{C}^*$. The main result of this section is the following.

Theorem 6.1. *There exists a grading by finite-dimensional vector spaces*

$$R_{i,a}^+ = \bigoplus_{m \in \mathbb{Z}} (R_{i,a}^+)_m$$

such that

(1) for $m \geq 0$ and $x \in U_q(\mathfrak{b})^-$ of degree $r > 0$, we have

$$x((R_{i,a}^+)_m) \subset (R_{i,a}^+)_{m-r}.$$

(2) for $j \in I$, $r, m \geq 0$ we have

$$\phi_{j,r}^+(R_{i,a}^+)_m \subset (R_{i,a}^+)_{m-r} \text{ if } j \neq i,$$
$$(\phi_{i,r}^+ + a\phi_{i,r-1}^+ + \cdots + a^r \phi_{i,0}^+)(R_{i,a}^+)_m \subset (R_{i,a}^+)_{m-r},$$

(3) For $j \in I$ and $r \geq 0$ we have

$$x_{j,r}^+((R_{i,a}^+)_m) \subset (R_{i,a}^+)_{m-r} + (R_{i,a}^+)_{m-r+\delta_{i,j}}.$$

This result will be proved in this section.

Remark 6.2. *The condition (2) for $r = 0$ implies that the grading is compatible with weight decomposition:*

$$(R_{i,a}^+)_\lambda = \bigoplus_{m \in \mathbb{Z}} (R_{i,a}^+)_\lambda \cap (R_{i,a}^+)_m \text{ for } \lambda \in \mathfrak{t}^*.$$

It also implies that for $j \in I$, $r > 0$ we have

$$(h_{j,r} - \lambda_{j,r})(R_{i,a}^+)_m \subset (R_{i,a}^+)_{m-r} \text{ where } \lambda_{j,r} = \frac{\delta_{i,j} a^r}{r(q_i - q_i^{-1})}.$$

Up to a shift, we can assume that an ℓ-lowest weight vector of $R_{i,a}^+$ has degree 0 and that $(R_{i,a}^+)_m = 0$ for $m < 0$.

6.2. **Root vectors.** Let us remind results from [Be, Da1] where the root vectors $E_\alpha \in U_q(\mathfrak{b})$, $F_\alpha \in U_q(\mathfrak{b}^-)$ are constructed for

$$\alpha \in \Phi_+^{Re} = \Phi_0^+ \sqcup \{\beta + m\delta | m > 0, \beta \in \Phi_0\}.$$

Here Φ_0 (resp. Φ_0^+) is the set of roots (resp. positive roots) of $\dot{\mathfrak{g}}$ and δ is the standard imaginary root of $\mathfrak{g}$. For example, we have for $i \in I$, $m \geq 0$ and $r > 0$:

$$E_{m\delta+\alpha_i} = x_{i,m}^+, \quad E_{r\delta-\alpha_i} = -k_i^{-1} x_{i,r}^-, \quad F_{m\delta+\alpha_i} = x_{i,-m}^-, \quad F_{r\delta-\alpha_i} = -x_{i,-r}^+ k_i.$$

We will consider the subalgebras

$$U_q(\mathfrak{b}^-)^{+,0} = \mathfrak{t} \otimes U_q(\mathfrak{b}^-)^+ \subset U_q(\mathfrak{b}^-)^{\geq 0} = U_q(\mathfrak{b}^-)^0 \otimes U_q(\mathfrak{b}^-)^+,$$
$$U_q(\mathfrak{b})^{-,0} = U_q(\mathfrak{b})^- \otimes \mathfrak{t} \subset U_q(\mathfrak{b})^{\leq 0} = U_q(\mathfrak{b})^- \otimes U_q(\mathfrak{b})^0.$$

The $\mathbb{Z}$-grading of $U_q(\mathfrak{g})$ induces $\mathbb{Z}$-gradings on $U_q(\mathfrak{b}^-)^{+,0}$ and $U_q(\mathfrak{b})^{-,0}$.

The subalgebra of $U_q(\mathfrak{b})$ (resp. $U_q(\mathfrak{b}^-)$) generated by the $E_{-\alpha+r\delta}$ (resp. $F_{-\alpha+r\delta}$) for $\alpha \in \Phi_0^+$, $r > 0$ and by $\mathfrak{t}$ is $U_q(\mathfrak{b})^{-,0}$ (resp. $U_q(\mathfrak{b}^-)^{+,0}$). Note that we have

$$\deg(E_{-\alpha+r\delta}) = -\deg(F_{-\alpha+r\delta}) = r.$$

6.3. **Examples for the grading.** The examples in this section are given as an illustration and are not used for the main results of the paper.

First, for $\dot{\mathfrak{g}} = sl_2$, we can check directly that we can choose

$$(R_{1,a}^+)_m = \mathbb{C}.v_m^*, \quad m \geq 0.$$

Indeed,

(1) for $m \geq 0$, we have $x_{1,1}^-((R_{1,a}^+)_m) = (R_{1,a}^+)_{m-1}$ and $x_{1,r}^-((R_{1,a}^+)_m) = \{0\}$ if $r > 1$.

(2) For $r \geq 0$, $(\phi_{1,r}^+ + a\phi_{1,r-1}^+ + \cdots + a^r\phi_{1,0}^+) = a^{r-1}(\delta_{r\neq 0}\phi_{1,1}^+ + a\phi_{1,0}^+) = \delta_{r,0}k_1$ on $R_{1,a}^+$.

(3) We have $x_{1,0}^+((R_{1,a})_m) = (R_{1,a})_{m+1}$ and $x_{1,r}^+((R_{1,a})_m) = \{0\}$ for $r > 0$.

This can be generalized to the case $N_i = 1$. In this special case, there is a simple proof thanks to the following result.

Theorem 6.3. [HJ] *Suppose $N_i = 1$. Let $j \in I$, $r > 0$, $\alpha \in \Phi_0^+$.*

(1) $\phi_{j,\delta_{i,j}+r}^+$ acts by 0 and $k_i^{-1}\phi_{i,1}^+$ is a scalar operator on $R_{i,a}^+$.

(2) $x_{j,r}^+$ acts by 0 on $R_{i,a}^+$.

(3) If $\alpha(\alpha_i^\vee) = 0$, then $E_{-\alpha+r\delta}$ acts by 0 on $R_{i,a}^+$.

(4) If $\alpha(\alpha_i^\vee) = 1$, then $E_{-\alpha+(r+1)\delta}$ acts by 0 on $R_{i,a}^+$.

Remark 6.4. *Precisely, the statement is proved in* [HJ, Section 7.2] *for $L_{i,a}^+$ by giving an asymptotic construction of $L_{i,a}^+$. The same construction works for $\overline{L}_{i,a}^+$. By using $\hat{\omega}$, this gives also an asymptotic construction of $R_{i,a}^+$. Hence the result.*

Corollary 6.5. *If $N_i = 1$, there is $p \in \mathbb{Z}$ such that for any $m \in \mathbb{Z}$,*

$$(R_{i,a}^+)_m = \bigoplus_{\{\alpha \in Q^+ | \alpha(\alpha_i^\vee) = m+p\}} (R_{i,a}^+)_\alpha.$$

Proof. Let p be the degree of a lowest weight vector. Let $w \in (R_{i,a}^+)_\alpha$ be a non zero weight vector. Then by Section 6.2 there is a non zero lowest weight vector of the form $E_{-\alpha_{i_1}+r_1\delta} \cdots E_{-\alpha_{i_N}+r_N\delta} w$. Hence $m = r_1 + \cdots + r_N + p$ and $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_N}$. But by Theorem 6.3, $r_1 = \cdots = r_N = 1$ and $\alpha_{i_1}(\alpha_i^\vee) = \cdots = \alpha_{i_N}(\alpha_i^\vee) = 1$. So $m = N + p = \alpha(\alpha_i^\vee) + p$. □

In general, the statements of Theorem 6.3 and Corollary 6.5 do not hold (see the following example). The reason is that the asymptotic construction in [HJ] do not work in these cases. That is why we give a completely different proof in the next subsections.

Example 6.6. *Let us consider the B_2-case with $i = 1$ the node satisfying $N_1 = 2$. Let v be a lowest weight vector of $V = R_{1,1}^+$. By* [HJ, Theorem 6.4], *we have $dim(V_{2\alpha_1+\alpha_2}) = 3$. If the statement of theorem 6.3 held for V, we would have*

$$V_{2\alpha_1+\alpha_2} = \mathbb{C}(x_{1,0}^+)^2 x_{2,0}^+.v \oplus \mathbb{C}x_{1,0}^+ x_{2,0}^+ x_{1,0}^+.v \oplus \mathbb{C}x_{2,0}^+(x_{1,0}^+)^2.v.$$

But $(x^+_{1,0})^2 x^+_{2,0}.v = 0$*, contradiction. The statement of Corollary 6.5 does not hold neither. Consider a grading such that* v *has degree* 0*. The weight spaces of weight* 0, α_1, $2\alpha_1$, $\alpha_1+\alpha_2$ *are of dimension* 1*, generated respectively by* v, $x^+_{1,0}.v$, $(x^+_{1,0})^2.v$, $(x^+_{2,0}x^+_{1,0}).v$*, and have respective degree* 0, 1, 2, 1*.* $V_{2\alpha_1+\alpha_2}$ *is generated by* $v_1 = (x^+_{1,0}x^+_{2,0}x^+_{1,0}).v$, $v_2 = x^+_{2,0}(x^+_{1,0})^2.v$ *and* $v_3 = x^+_{2,1}(x^+_{1,0})^2.v = -x^+_{1,1}x^+_{2,0}x^+_{1,0}.v$*. By construction,* v_3 *has degree* 1 *and* v_2 *has degree* 2*. Since* $x^-_{1,1}(v_1 + \mathbb{C}v_2 + \mathbb{C}v_3) \subset (V)_1$*, there are* $\lambda, \mu \in \mathbb{C}$ *such that* $v_1 + \lambda v_2 + \mu v_3$ *has degree* 2*. Hence* $V_{2\alpha_1+\alpha_2} \cap (V)_1$ *has dimension* 1 *and* $V_{2\alpha_1+\alpha_2} \cap (V)_2$ *has dimension* 2*.*

6.4. **Coproduct and root vectors.** Let $\alpha \in \Phi_0^+$ and $r > 0$. Set $k_\alpha = \prod_{1\leq i\leq n} k_i^{\alpha(\omega_i^\vee)}$ We have [Da1, Theorem 4, (3)]:

$$\Delta(F_{-\alpha+r\delta}) \in F_{-\alpha+r\delta} \otimes k_\alpha + \sum_{\beta\in\Phi_0^+, p>0} U_q(\mathfrak{b}^-) \otimes (U_q(\mathfrak{b}^-)F_{-\beta+p\delta}). \tag{6.26}$$

This gives the factor k_α in the decomposition of $F_{-\alpha+r\delta}$ in $U_q(\mathfrak{b}^-)^+ \otimes \mathfrak{t}$:

$$F_{-\alpha+r\delta} \in U_q(\mathfrak{b}^-)^+ k_\alpha \text{ and } E_{-\alpha+r\delta} \in U_q(\mathfrak{b})^- k_\alpha^{-1}. \tag{6.27}$$

This last point also follows from [Da2, Proposition 9.3]. Now let $i \in I$ and $r > 0$. The Q-grading of $U_q(\mathfrak{g})$ induces a Q-grading of $U_q(\mathfrak{b}^-)$. Let $U_q(\mathfrak{b}^-)_+$ (resp. $U_q(\mathfrak{b}^-)_-$) be the subalgebra of $U_q(\mathfrak{b}^-)$ of elements of positive (resp. negative) Q-degree. Then we have

$$\Delta(h_{i,-r}) \in h_{i,-r} \otimes 1 + 1 \otimes h_{i,-r} + (U_q(\mathfrak{b}^-))_- \otimes (U_q(\mathfrak{b}^-))_+. \tag{6.28}$$

6.5. **Drinfeld relations.** Let us give examples of relations between Drinfeld generators that we will use:

$$[x^+_{j,-m}, x^-_{i,0}] = \delta_{i,j}\frac{\delta_{m,0}k_i - \phi^-_{i,-m}}{q_i - q_i^{-1}} \text{ for } m \in \mathbb{Z},\ i, j \in I, \tag{6.29}$$

and for $m \geq 0$, $p \in \mathbb{Z}$, $i, j \in I$:

$$\phi^-_{i,-m}x^+_{j,p} = -\sum_{0\leq l\leq m-1} q_i^{-lC_{i,j}} x^+_{j,p-l-1}\phi^-_{i,-m+l+1} + \sum_{0\leq l\leq m} q_i^{-(l+1)C_{i,j}} x^+_{j,p-l}\phi^-_{i,-m+l}. \tag{6.30}$$

We will also use the following technical result.

Lemma 6.7. *Let* $i \in I$, $\alpha \in \Phi_0^+$, $r > 0$*. Then we have a decomposition in* $U_q(\mathfrak{b}^-)$

$$x^-_{i,0}F_{-\alpha+r\delta} = q^{(\alpha,\alpha_i)}F_{-\alpha+r\delta}x^-_{i,0} + \sum_{-r\leq p\leq 0} a_p(\phi^-_{i,p} + \phi^-_{i,p+1} + \cdots + \phi^-_{i,0})k_\alpha + ak_ik_\alpha$$

where $a \in U_q(\mathfrak{b}^-)^{>0}$ *has* $\mathbb{Z}$*-degree* $-r$ *and* $a_p \in U_q(\mathfrak{b}^-)^{>0}$ *is a sum of elements of* $\mathbb{Z}$*-degree* $-r-p$ *or* $-r-p+1$*.*

Proof. We will compute the decomposition of $F_{-\alpha+r\delta}x^-_{i,0}$ in (3.15) by using the full quantum loop algebra $U_q(\mathfrak{g})$ and the decomposition (2.4). Indeed, by (6.27), $F_{-\alpha+r\delta}$ is a product x^+k_α where x^+ has degree $-r$ and is an algebraic combination of the $x^+_{j,-m}$ ($j \in I$, $0 \leq m \leq r$). We first have $F_{-\alpha+r\delta}x^-_{i,0} = q^{-\lambda}x^+x^-_{i,0}k_\alpha$ where

$$\lambda = \sum_{j\in I}\alpha(\omega_j^\vee)d_jC_{j,i} = \sum_{j\in I}\alpha(\omega_j^\vee)(\alpha_i,\alpha_j) = (\alpha,\alpha_i).$$

The relations (6.29) imply

$$[x^+, x_{i,0}^-] \in \mathcal{A}k_i\mathcal{A} + \sum_{-r\leq m\leq 0} \mathcal{A}\phi_{i,m}^-\mathcal{A}.$$

where $\mathcal{A} = \mathbb{C}[x_{j,p}^+]_{j\in I, 0\leq p\leq r}$. Then the relations (6.30) imply

$$[x^+, x_{i,0}^-] \in \mathcal{A}k_i + \sum_{-r\leq m\leq 0} \mathcal{A}\phi_{i,m}^-.$$

Hence

$$F_{-\alpha+r\delta}x_{i,0}^- = q^{-\lambda}x_{i,0}^-F_{-\alpha+r\delta} - \sum_{-r\leq m\leq 0} q^{-\lambda}b_m\phi_{i,m}^-k_\alpha - q^{-\lambda}ak_ik_\alpha, \tag{6.31}$$

where $b_m \in \mathcal{A}$ has $\mathbb{Z}$-degree $-r-m$ and $a\in\mathcal{A}$ has $\mathbb{Z}$-degree r. Note that $\mathcal{A}$ is not contained in $U_q(\mathfrak{b}^-)$. But we have $\mathcal{A}\subset U_q(\mathfrak{g})^+$. So (6.31) is the decomposition of $F_{-\alpha+r\delta}x_{i,0}^-$ in (2.4). But $F_{-\alpha+r\delta}x_{i,0}^- \in U_q(\mathfrak{b}^-)$ and the decomposition in (3.15) is unique. Hence, for degree reason, we have $a\in U_q(\mathfrak{b}^-)^+$ and $b_m\in U_q(\mathfrak{b}^-)^+$ for $m\leq 0$. Now (6.31) can be rewritten as in the Lemma, with

$$a_p = b_p - b_{p-1} \text{ for } -r\leq p\leq 0$$

where we set $b_{-r-1}=0$. □

Example 6.8. *For example, in the case $\dot{\mathfrak{g}} = sl_2$, we have for $r>0$, $F_{r\delta-\alpha_1} = -x_{1,-r}^+k_1$ and*

$$x_{1,0}^-F_{r\delta-\alpha_1} = q^2F_{r\delta-\alpha_1}x_{1,0}^- + \phi_{1,-r}^-k_1.$$

6.6. **Tensor product of ℓ-weight vectors.** By using (6.28), we prove exactly as in [H4, Proposition 3.2] the following.

Proposition 6.9. [H4] *Let V_1, V_2 in category $\overline{\mathcal{O}}$ and consider an l-weight vector*

$$w = \left(\sum_\alpha w_\alpha\otimes v_\alpha\right) + \left(\sum_\beta w'_\beta\otimes v'_\beta\right) \in V_1\otimes V_2$$

satisfying the following conditions.

(i) The v_α (resp. v'_β) are ℓ-weight (resp. weight) vectors of ℓ-weight $\mathbf{\Psi}_\alpha$ (resp. weight ω_β).

(ii) For any β, there is an α satisfying $\omega_\beta > \varpi(\mathbf{\Psi}_\alpha)$.

(iii) For any α, we have $\sum_{\{\alpha'|\omega_{\alpha'}=\omega_\alpha\}} w_{\alpha'}\otimes v_{\alpha'}\neq 0$.

Then the ℓ-weight of w is the product of one $\mathbf{\Psi}_\alpha$ by an l-weight of V_1.

6.7. **Proof of Theorem 6.1.** We can assume $a=1$. By using the twisting by $\hat{\omega}$, we can work with $\overline{L}_{i,1}^+$. It will be important for our proof as we will use that $U_q(\mathfrak{b}^-)^-$ as, in opposition to $U_q(\mathfrak{b})^-$, is generated by a family of Drinfeld generators (see (3.14)). We do not need such a property for $U_q(\mathfrak{b}^-)^+$ as we have the coproduct formulas (6.26). The conditions to be proved become for $m\geq 0$, $j\in I$, $r\leq 0$:

(1) for $x\in U_q(\mathfrak{b}^-)^+$ of degree $r<0$, $x((\overline{L}_{i,1}^+)_m)\subset(\overline{L}_{i,1}^+)_{m+r}$.

(2) $\phi_{j,r}^-(\overline{L}_{i,1}^+)_m \subset (\overline{L}_{i,1}^+)_{m+r}$ if $j\neq i$ and $(\phi_{i,r}^- + \phi_{i,r+1}^- + \cdots + \phi_{i,0}^-)(\overline{L}_{i,1}^+)_m \subset (\overline{L}_{i,1}^+)_{m+r}$.

(3) $x_{j,r}^-((\overline{L}_{i,1}^+)_m) \subset (\overline{L}_{i,1}^+)_{m+r} + (\overline{L}_{i,1}^+)_{m+r+\delta_{i,j}}$.

Let v (resp. v') be an ℓ-highest weight vector of $\overline{L}_{i,1}^+$ (resp. of $\tilde{L}(Y_{i,q_i^{-1}})$). Let

$$V = U_q(\mathfrak{b}^-).(v \otimes v') \subset \overline{L}_{i,1}^+ \otimes \tilde{L}(Y_{i,q_i^{-1}})$$

Then we have a surjective morphism of $U_q(\mathfrak{b}^-)$-modules

$$\phi : V \to \overline{L}_{i,q_i^2}^+.$$

Let $V' = \overline{L}_{i,1}^+ \otimes v'$. From (6.26), (6.28), V' is a $U_q(\mathfrak{b}^-)^{\geq 0}$-module and

$$\chi_q(V') = \chi_q(\overline{L}_{i,q_i^2}^+).$$

Let us prove that $V' \subset V$. Consider an ℓ-weight vector w of V whose ℓ-weight is an ℓ-weight of $\overline{L}_{i,q_i^2}^+$. If w is not in V', in a decomposition of w as in Proposition 6.9, we would have some terms $w_\alpha \otimes v_\alpha$ with v_α ℓ-weight vector of $\tilde{L}(Y_{i,q_i^{-1}})$ which is not in $\mathbb{C}.v'$. But by [FM, Lemma 6.1, Remark 6.2] (and its proof), we have

$$\tilde{\chi}_q(\tilde{L}(Y_{i,q_i^{-1}})) \in 1 + A_{i,1}^{-1}\mathbb{Z}[A_{j,b}^{-1}]_{j\in I, b\in\mathbb{C}^*}.$$

Hence the ℓ-weight $\mathbf{\Psi}_\alpha$ of v_α would be in $([\overline{\omega}_i]^{-1}Y_{i,q_i^{-1}})A_{i,1}^{-1}\mathbb{Z}[A_{j,b}^{-1}]_{j\in I,b\in\mathbb{C}^*}$. Contradiction as by Theorem 4.1, $A_{i,1}^{-1}$ is not a factor of the ℓ-weights occurring in $\tilde{\chi}_q(\overline{L}_{i,q_i^2}^+)$. Moreover, $\overline{L}_{i,1}^+$ and $\overline{L}_{i,q_i^{-2}}^+$ have the same character, so it implies that $V' \subset V$.

Now we may consider the restriction of ϕ to V'. From our discussion, it is an isomorphism of $U_q(\mathfrak{b}^-)^{\geq 0}$-module. It induces a linear isomorphism

$$\tilde{\phi} : \overline{L}_{i,1}^+ \to \overline{L}_{i,q_i^2}^+.$$

As the pullback of $\overline{L}_{i,,q_i^2}^+$ by $\tau_{q_i^2}$ is $\overline{L}_{i,1}^+$, there is a unique linear isomorphism $\tau : \overline{L}_{i,q_i^2}^+ \to \overline{L}_{i,1}^+$ satisfying $\tau(g.x) = q_i^{-2m} g.\tau(x)$ for any $x \in \overline{L}_{i,q_i^2}^+$, $g \in U_q(\mathfrak{b}^-)$ of $\mathbb{Z}$-degree m and such that $\Phi(v) = v$ where

$$\Phi = \tau \circ \tilde{\phi} : \overline{L}_{i,1}^+ \to \overline{L}_{i,1}^+.$$

We have for $j \in I$, $r > 0$ and $\alpha \in \phi_0^+$

$$\tilde{\phi} F_{-\alpha+r\delta} = F_{-\alpha+r\delta}\tilde{\phi} \quad , \quad \tilde{\phi}\phi_j^-(z) = \phi_j^-(z)\left(\frac{1-z^{-1}}{1-z^{-1}q_i^2}\right)^{\delta_{i,j}} \tilde{\phi}.$$

Hence Φ is a linear automorphism of $\overline{L}_{i,1}^+$ which commutes with the k_j and

$$\Phi F_{-\alpha+r\delta} = q_i^{2r} F_{-\alpha+r\delta}\Phi \quad , \quad \Phi\phi_j^-(z)(1-z^{-1})^{-\delta_{i,j}} = \phi_j^-(q_i^{-2}z)(1-(zq_i^{-2})^{-1})^{-\delta_{i,j}}\Phi.$$

For $i = j$, the last equation can be rewritten as ($r \geq 0$):

$$\Phi(\phi_{i,-r}^- + \phi_{i,-r+1}^- + \cdots + \phi_{i,0}^-) = q_i^{2r}(\phi_{i,-r}^- + \phi_{i,-r+1}^- + \cdots + \phi_{i,0}^-)\Phi.$$

The weight spaces of $\overline{L}_{i,1}^+$ are stable by Φ. Let us prove by induction on the height of α that $\Phi_{-\alpha}$ is diagonalizable on $(\overline{L}_{i,1}^+)_{-\alpha}$ with eigenvalues of the form q_i^{-2m}, with $m \geq 0$ integer. For $\alpha = 0$ it is clear by construction. In general, there is a finite family $(\alpha_1, r_1), \cdots, (\alpha_R, r_R)$

with the $\alpha_j \in \Phi_0^+$, $r_j > 0$ such that the intersection of the $\mathrm{Ker}(F_{-\alpha_j+r_j\delta}) \cap (\overline{L}_{i,1}^+)_{-\alpha}$ is zero. By the induction hypothesis, there is $M \geq 0$ such that the polynomial

$$P(X) = \prod_{0\leq m\leq M}(X - q_i^{-2m})$$

satisfies $P(\Phi) = 0$ on $\bigoplus_{1\leq j\leq R}(\overline{L}_{i,1}^+)_{-\alpha+\alpha_j}$. Let $r = Max_j(r_j)$ and consider the polynomial

$$Q(X) = \prod_{0\leq m\leq M+r}(X - q_i^{-2m}).$$

For $1 \leq j \leq R$ we have

$$Q(\Phi q_i^{-2r_j})F_{-\alpha_j+r_j\delta} = F_{-\alpha_j+r_j\delta}Q(\Phi).$$

But $P(X)$ divides $Q(Xq_i^{-2r_j})$ and so $Q(\Phi q_i^{-2r_j}) = 0$ on $\bigoplus_{1\leq j\leq R}(\overline{L}_{i,1}^+)_{-\alpha+\alpha_j}$. Hence the operator $F_{-\alpha_j+r_j\delta}Q(\Phi)$ is zero on $(\overline{L}_{i,1}^+)_{-\alpha}$. Since this is true for any j, we get $Q(\Phi) = 0$ on $(\overline{L}_{i,1}^+)_{-\alpha}$ and the result.

For $m > 0$, we can define $(\overline{L}_{i,1}^+)_m$ as the eigenspace of Φ of eigenvalue q_i^{-2m}.

Let us prove (3). For $r < 0$ and $j \in I$, $[h_{j,r} - \lambda_{j,r}, x_{j,0}^-] = [h_{j,r}, x_{j,0}^-]$ is a non zero multiple of $x_{j,r}^-$. Hence, it suffices to prove the result for $x_{j,0}^-$. If $j \neq i$, we have $x_{j,0}^-.v' = 0$ and so $x_{j,0}^-\Phi = \Phi x_{j,0}^-$. Hence the result. Now suppose that $j = i$. Consider a weight vector $w \in (\overline{L}_{i,1}^+)_m$. We prove the result by induction on the height of the weight of w. For $m = 0$ is follows from the case $\dot{\mathfrak{g}} = sl_2$. In general, by Section 6.2, there is $F_{-\alpha+r\delta}$ ($\alpha \in \Phi_0^+$, $r > 0$) such that $F_{-\alpha+r\delta}x_{i,0}^-.w \neq 0$. By the induction hypothesis

$$x_{i,0}^-F_{-\alpha+r\delta}.w \in (\overline{L}_{i,1}^+)_{m-r} + (\overline{L}_{i,1}^+)_{m-r+1}.$$

Let λ, a_p, a, k as in Lemma 6.7. By the result above, we have for $p \leq 0$

$$a_p(\phi_{i,p}^- + \phi_{i,p+1}^- + \cdots + \phi_{i,0}^-)kw \subset a_p(\overline{L}_{i,1}^+)_{m+p} \subset (\overline{L}_{i,1}^+)_{m-r} + (\overline{L}_{i,1}^+)_{m-r+1}$$

as $a_p \in U_q(\mathfrak{b}^-)^{>0}$ is a sum of elements of $\mathbb{Z}$-degree $-r-p$ or $-r-p+1$. So

$$F_{-\alpha+r\delta}x_{i,0}^-.w \in (\overline{L}_{i,1}^+)_{m-r} + (\overline{L}_{i,1}^+)_{m-r+1}$$

and $x_{i,0}^-.w \in (\overline{L}_{i,1}^+)_m + (\overline{L}_{i,1}^+)_{m+1}$.

To conclude, let us prove that the $(\overline{L_{i,1}}^+)_m$ are finite-dimensional. First let us prove by induction on $m \geq 0$ that

$$(\overline{L}_{i,1}^+)_m = \bigoplus_{\{\alpha|\alpha(\alpha_i^\vee)\leq mN_i\}}(\overline{L}_{i,1}^+)_m \cap (\overline{L}_{i,1}^+)_{-\alpha}.$$

This is clear if $m = 0$. For $m > 0$, let $w \in (\overline{L}_{i,1}^+)_m$ of weight $-\alpha$. Then there is $F_{-\beta+r\delta}$ such that $F_{-\beta+r\delta}w \neq 0$. We have $F_{-\alpha+r\delta}w \in (\overline{L}_{i,1}^+)_{m-r} \cap (\overline{L}_{i,1}^+)_{-\alpha+\beta}$. Hence, by induction hypothesis, $(-\beta+\alpha)(\alpha_i^\vee) \leq (m-r)N_i$. But we have $0 \leq \beta(\alpha_i^\vee) \leq N_i$. So

$$\alpha(\alpha_i^\vee) \leq N_i + (m-r)N_i \leq mN_i$$

as $r > 0$. If m is fixed, by using the following Lemma 6.10 there is a finite number of weights α of $\overline{L}^+_{i,1}$ such that $\alpha(\alpha_i^\vee) \leq mN_i$. Hence the result.

Lemma 6.10. *Let $i \in I$, $a \in \mathbb{C}^*$ and $M \geq 0$. There is a finite number of weights α of $\overline{L}^+_{i,a}$ such that $\alpha(\alpha_i^\vee) \leq M$.*

Proof. By construction, the character of $\overline{L}^+_{i,a}$ is the character of $L^-_{i,a}$ and it does not depend on a. It is proved in [HJ, Theorem 6.1] that $\chi(L^-_{i,1})$ is the limit of $\tilde{\chi}(W^{(i)}_{k,1})$ when $k \to +\infty$. By (4.16), a weight α satisfying $\alpha(\alpha_i^\vee) \leq M$ is a weight of $\overline{L}^+_{i,a}$ only if it is a weight of $W^{(i)}_{M,1}$. Hence the result. □

Remark 6.11. *In the proof, if instead of $\overline{L}^+_{i,q_i^{-2}}$ we had used $\overline{L}^+_{i,q_i^{-2k}}$ with $k \geq 2$, we would get the same grading.*

7. End of the proof of Theorem 5.9

In this section we finish the proof of Theorem 5.9. We first recall the factorization of the universal R-matrix and show that it implies Proposition 5.5. Then we establish the main reduction for the proof of Theorem 5.9 in Section 7.3: it suffices to consider certain distinguished tensor products of fundamental representations.

7.1. **Factorization of the universal R-matrix.** The universal R-matrix has a factorization [Da1]

$$\mathcal{R} = \mathcal{R}^+\mathcal{R}^0\mathcal{R}^-\mathcal{R}^\infty,$$

where $\mathcal{R}^\pm \in U_q(\mathfrak{b})^\pm \hat{\otimes} U_q(\mathfrak{b}^-)^\mp$,

$$\mathcal{R}^0 = \exp\left(-\sum_{m>0, i,j\in I} \frac{(q_i - q_i^{-1})(q_j - q_j^{-1}) m \tilde{B}_{i,j}(q^m)}{(q-q^{-1})[m]_q} h_{i,m} \otimes h_{j,-m}\right),$$

and $\mathcal{R}^\infty = q^{-t_\infty}$ where $t_\infty \in \dot{\mathfrak{h}} \otimes \dot{\mathfrak{h}}$ is the canonical element (for the standard invariant symmetric bilinear form as in [Da1]), that is, if we denote formally $q = e^h$, then $\mathcal{R}^\infty = e^{-ht_\infty}$.

For a variable x, the q-exponential in x is a formal power series $\exp_{q^p}(x) = \sum_{r\geq 0} \frac{x^r}{[r]'_{q^p}!}$ where $p \in \mathbb{Z}$ and $[r]'_v! = \prod_{1\leq s\leq r} \frac{v^{2s}-1}{v^2-1} = v^{\frac{r(r-1)}{2}}[r]_v!$ for $r \geq 0$.

$\mathcal{R}^+$ (resp. $\mathcal{R}^-$) is a product of q-exponentials of a multiple of $E_{\alpha+m\delta} \otimes F_{\alpha+m\delta}$ with $m \geq 0$, $\alpha \in \Phi_0^+$ (resp. with $m > 0$, $\alpha \in \Phi_0^-$).

Example 7.1. *In the case $\dot{\mathfrak{g}} = sl_2$, we have*

$$\mathcal{R}^+ = \prod_{m\geq 0} exp_q\left((q^{-1} - q) x^+_{1,m} \otimes x^-_{1,-m}\right), \ \mathcal{R}^- = \prod_{m>0} exp_q\left((q-q^{-1}) k_1^{-1} x^-_{1,m} \otimes x^+_{1,-m} k_1\right),$$

$$\mathcal{R}^0 = exp\left(-(q-q^{-1}) \sum_{m>0} \frac{m}{[m]_q (q^m + q^{-m})} h_{1,m} \otimes h_{1,-m}\right).$$

7.2. **Proof of Proposition 5.5.** Let $i \in I$. Since each $F_{\alpha+m\delta}$ ($m > 0, \alpha \in \Phi_0^-$) acts by 0 on the lowest weight vector of $R_{i,1}^+$, only $\mathcal{R}^0$ and $\mathcal{R}^\infty$ contribute to the specialization of $t_{R_{i,1}^+}(z,u)$. Let us replace in $\mathcal{R}^0$ each $h_{l,m}$ ($m > 0, l \in I$) by $-z^m \frac{\delta_{i,l}}{m(q_i - q_i^{-1})}$, that is we take the trace on $R_{i,1}^+$. We get

$$\exp\left(\sum_{j \in I, m>0} z^m \frac{\tilde{B}_{i,j}(q^m)[d_j]_q}{[m]_q} h_{j,-m}\right) = \exp\left(\sum_{m>0} z^m \frac{\tilde{h}_{i,-m}}{[d_i]_q[m]_{q_i}}\right) = T_i(z)$$

as $[d_j]_q[m]_{q_j} = [m]_q[d_j]_{q^m}$ and $\tilde{B}_{i,j}(q^m) = \tilde{C}_{j,i}(q^m)/[d_i]_{q^m}$ for any $m \in \mathbb{Z}$, $j \in I$.
We have proved Proposition 5.5.

7.3. **Reduction.** It suffices to prove theorem 5.9 for W in $\mathcal{C}$ simple or standard (that is a tensor product of fundamental representations). A representation in $\mathcal{C}$ is said to be thin if its ℓ-weight spaces are of dimension 1.

Remark 7.2. *Let $i \in I$ and $a \in \mathbb{C}^*$. If $N_i = 1$, then $L(Y_{i,a})$ is simple as a $U_q(\dot{\mathfrak{g}})$-module, as shown by V. Chari* [C2] *(see also* [HJ, Remark 7.7]*). In particular, if this $U_q(\dot{\mathfrak{g}})$ fundamental representation is minuscule, then $L(Y_{i,a})$ is thin.*

Proposition 7.3. *Suppose that $\dot{\mathfrak{g}}$ is not of type E_8. Any simple object in $\mathcal{C}$ occurs as a simple composition factor of a tensor product of thin fundamental representations.*

Example 7.4. *If $\dot{\mathfrak{g}} = sl_2$, any simple object is $\mathcal{C}$ is isomorphic to a tensor product of thin KR-modules, as shown by V. Chari and A. Pressley* [CP].

Proof. This is clear for $\dot{\mathfrak{g}}$ of type A, B, C or G_2 as all fundamental representations are thin [H1].

Type D_n ($n \geq 4$): for $a \in \mathbb{C}^*$ and $i \in \{1, n-1, n\}$, the representation $L(Y_{i,a})$ is thin (for example by Remark 7.2). We have to prove the result for $L(Y_{i,a})$, $2 \leq i \leq n-2$. It can be obtained by induction on i by using the following relation for $1 \leq i \leq n-3$:

$$[L(Y_{1,a}) \otimes L(Y_{i,aq^{i+1}})] = [L(Y_{1,a}Y_{i,aq^{i+1}})] + [L(Y_{i+1,aq^i})].$$

This relation can be easily established following the proof of T-systems relations in [H2]. By [FM], it suffices to prove that dominant monomials have the same multiplicities when the q-character morphism is applied on both sides of the identity. Besides, the q-character of a fundamental representation has a unique dominant monomial (that is it is affine-minuscule) and its q-character is given by the algorithm of Mukhin and the first author [FM]. Then it is not difficult to see that the left side of the identity has 2 dominant monomials: the highest monomial $Y_{1,a}Y_{i,aq^{i+1}}$ and $Y_{1,a}Y_{i,aq^{i+1}}A_{1,aq}^{-1}A_{2,aq^2}^{-1} \cdots A_{i,q^i}^{-1} = Y_{i+1,aq^i}$. Then the identity result follows as $L(Y_{i+1,aq^i})$ is affine-minuscule and as it can be proved as in [H2, H3] that $L(Y_{1,a}Y_{i,aq^{i+1}})$ is affine-minuscule.

Type F_4: for $a \in \mathbb{C}^*$ and $i = 1, 4$, the representation $L(Y_{i,a})$ is thin [H1]. We have to prove the result for $L(Y_{2,a})$, $L(Y_{3,a})$. As above, it follows from the relations for $a \in \mathbb{C}^*$:

$$[L(Y_{1,a}) \otimes L(Y_{1,aq^4})] = [L(Y_{1,a}Y_{1,aq^4})] + [L(Y_{2,aq^2})],$$

$$[L(Y_{4,a}) \otimes L(Y_{4,aq^2})] = [L(Y_{4,a}Y_{4,aq^2})] + [L(Y_{3,aq})].$$

Type E_6: for $a \in \mathbb{C}^*$ and $i = 1, 5$, the representation $L(Y_{i,a})$ is thin (for example by Remark 7.2). By using the same arguments as above, this implies the result for $i \neq 6$. Then we have

$$[L(Y_{1,a}) \otimes L(Y_{5,aq^6})] = [L(Y_{1,a}Y_{5,aq^6})] + [L(Y_{6,aq^3})].$$

Type E_7: for $a \in \mathbb{C}^*$, the representation $L(Y_{6,a})$ is thin (for example by Remark 7.2). As above, we get the result for $i = 6, 5, 4, 3$. Now the monomial

$$Y_{1,aq^5}Y_{6,aq^{10}}^{-1} = Y_{6,a}A_{6,aq}^{-1}A_{5,aq^2}^{-1}A_{4,aq^3}^{-1}A_{3,aq^4}^{-1}A_{7,aq^5}^{-1}A_{2,aq^5}^{-1}A_{3,aq^6}^{-1}A_{4,aq^7}^{-1}A_{5,aq^8}^{-1}A_{6,aq^9}^{-1}$$

occurs in $\chi_q(L(Y_{6,a}))$. In particular, we get as above

$$[L(Y_{6,a}) \otimes L(Y_{6,aq^{10}})] = [L(Y_{6,a}Y_{6,aq^{10}})] + [L(Y_{1,aq^5})].$$

We have the result for $i = 1$. As $[L(Y_{1,aq^5}) \otimes L(Y_{1,aq^7})] = [L(Y_{1,aq^5}Y_{1,aq^7})] + [L(Y_{2,aq^6})]$, this also implies that $L(Y_{2,aq^6})$ is a composition factor of

$$L(Y_{6,a}) \otimes L(Y_{6,aq^{10}}) \otimes L(Y_{6,aq^2}) \otimes L(Y_{6,aq^{12}})$$

and we have the result for $i = 2$. We conclude as

$$[L(Y_{6,a}) \otimes L(Y_{1,aq^7})] = [L(Y_{6,a}Y_{1,aq^7})] + [L(Y_{7,aq^4})].$$

□

Remark 7.5. *The statement is not satisfied for $\dot{\mathfrak{g}}$ of type E_8. Indeed the fundamental representation $L(Y_{i,a})$ is not thin for any i in this case. For $i = 1$, it is known by* [HN, Section 6.1.2]*. The Lie algebra for the sub Dynkin diagram $I \setminus \{7\}$ is of type D_7. The q-character of fundamental representations are known for $\dot{\mathfrak{g}}$ of type D. In particular we have the result for $2 \leq i \leq 5$. In the same way we conclude for $i = 6$ by considering $I \setminus \{1, 2, 3\}$. The Lie algebra for the sub Dynkin diagram $I \setminus \{1\}$ is of type E_7 and by using* [HN, Section 6.1.2]*, we get the result for $i = 7$. The Lie algebra for the sub Dynkin diagram $I \setminus \{1, 2\}$ is of type E_6 and by using* [HN, Section 6.1.2]*, we get the result for $i = 8$.*

Proposition 7.6. *Suppose that $\dot{\mathfrak{g}}$ is of type E_8. Any simple object in $\mathcal{C}$ occurs as a simple composition factor of a tensor product of fundamental representations $L(Y_{i,a})$ with $i = 1$.*

Proof. As above, we have the result for $1 \leq i \leq 5$. Now the monomial

$$Y_{7,aq^6}Y_{1,aq^{12}}^{-1} = Y_{1,a}A_{1,aq}^{-1}A_{2,aq^2}^{-1}A_{3,aq^3}^{-1}A_{4,aq^4}^{-1}A_{5,aq^5}^{-1}A_{8,aq^6}^{-1}A_{6,aq^6}^{-1}A_{5,aq^7}^{-1}A_{4,aq^8}^{-1}A_{3,aq^9}^{-1}A_{2,aq^{10}}^{-1}A_{1,aq^{11}}^{-1}$$

occurs in $\chi_q(L(Y_{1,a}))$. In particular, we get as above

$$[L(Y_{1,a}) \otimes L(Y_{1,aq^{12}})] = [L(Y_{1,a}Y_{1,aq^{12}})] + [L(Y_{7,aq^6})].$$

We have the result for $i = 7$. As $[L(Y_{7,aq^6}) \otimes L(Y_{7,aq^8})] = [L(Y_{7,aq^6}Y_{7,aq^8})] + [L(Y_{6,aq^7})]$, this also implies that $L(Y_{6,aq^7})$ is a composition factor of

$$L(Y_{1,a}) \otimes L(Y_{1,aq^{12}}) \otimes L(Y_{1,aq^2}) \otimes L(Y_{1,aq^{14}})$$

and we have the result for $i = 6$. We conclude as

$$[L(Y_{1,a}) \otimes L(Y_{7,aq^8})] = [L(Y_{1,a}Y_{7,aq^8})] + [Y_{8,aq^5}].$$

□

Consequently, if $\dot{\mathfrak{g}}$ is not type E_8 (resp. is of type E_8) it suffices to prove Theorem 5.9 for W tensor product of thin fundamental representations (resp. of fundamental representations $L(Y_{1,a})$).

7.4. **Proof of Theorem 5.9.** By (5.19), we can assume that $a = 1$. We use the grading of $V = R_{i,1}^+$ established in Theorem 6.1. For $j \in I$ and $r > 0$ let us consider $\overline{h}_{j,r} = h_{j,r} - \frac{\delta_{i,j}}{r(q_i - q_i^{-1})}$. Note that by Remark 6.2, $\pi_V(\overline{h}_{j,r})$ is nilpotent. We have

$$\mathcal{R}^0 = \exp\left(-\sum_{m>0,k,j\in I}\frac{(q_k - q_k^{-1})(q_j - q_j^{-1})}{(q-q^{-1})}\frac{m\tilde{B}_{k,j}(q^m)}{[m]_q}\overline{h}_{k,m}\otimes h_{j,-m}\right)(1\otimes T_i(1)).$$

This implies a factorization

$$L_V(z) = L_V^+(z)L_V^0(z)(\mathrm{Id}_V\otimes T_i(z))L_V^-(z)L_V^\infty,$$

where $L_V^\pm(z) = (\pi_V(z)\otimes Id)(\mathcal{R}^\pm)$, $L_V^\infty = q^{-(\pi_V\otimes\mathrm{Id})(t_\infty)}$ and

$$L_V^0(z) = \exp\left(-\sum_{m>0,k,j\in I}\frac{(q_k - q_k^{-1})(q_j - q_j^{-1})}{(q-q^{-1})}z^m\frac{m\tilde{B}_{k,j}(q^m)}{[m]_q}\pi_V(\overline{h}_{k,m})\otimes h_{j,-m}\right).$$

Let W be a simple object in $\mathcal{C}$. The image of $t_V(z,u)$ in $(\mathrm{End}(W))[[z, u_j^{\pm 1}]]_{j\in I}$ is a (possibly infinite) linear combination of terms of the following form (we do not include L_V^∞ which does not depend on z): the product of two factors, the first one being

$$z^R Tr_{V,u}(E_{\beta_1+s_1\delta}\cdots E_{\beta_{r'}+s_{r'}\delta}\overline{h}_{i_1,r_1}\cdots\overline{h}_{i_p,r_p}E_{q_1\delta-\gamma_1}\cdots E_{q_r\delta-\gamma_r}) \tag{7.32}$$

and the second one being

$$\pi_W(F_{\beta_1+s_1\delta}\cdots F_{\beta_{r'}+s_{r'}\delta}h_{i_1,-r_1}\cdots h_{i_p,-r_p}T_i(z)F_{q_1\delta-\gamma_1}\cdots F_{q_r\delta-\gamma_r}), \tag{7.33}$$

where $\beta_1,\cdots,\beta_{r'}\in\Phi_0^+$, $\gamma_1,\cdots,\gamma_r\in\Phi_0^+$, $i_1,\cdots,i_p\in I$, $s_1,\cdots,s_{r'}\geq 0$, $r_1,\cdots,r_p > 0$, $q_1,\cdots,q_r > 0$ and

$$R = (s_1+\cdots+s_{r'}) + (r_1+\cdots+r_p) + (q_1+\cdots+q_r).$$

Moreover, so that (7.32) is non zero, we must have for weight reason

$$\beta_1+\cdots+\beta_{r'} = \gamma_1+\cdots+\gamma_r$$

and by the conditions in Theorem 6.1

$$R \leq (\beta_1(\omega_i^\vee)+\cdots+\beta_{r'}(\omega_i^\vee)).$$

Hence we have

$$R \leq (\gamma_1(\omega_i^\vee)+\cdots+\gamma_r(\omega_i^\vee)).$$

So that (7.33) is non zero on $(W)_\lambda$, we must have for weight reason

$$(\gamma_1(\omega_i^\vee)+\cdots+\gamma_r(\omega_i^\vee)) \leq ht_i(\omega-\lambda).$$

This implies $R \leq ht_i(\omega-\lambda)$. Hence, there is a finite number of choices for the $s_1,\cdots,s_{r'}$, $r_1,\cdots,r_p$, $q_1,\cdots,q_r$. So the total sum is a finite linear combination of those terms.

Now suppose that W is simple and thin. By Proposition 5.8, the eigenvalues of $T_i(z)$ on W_λ are of the form $f(z)Q(z)$ where $f(z)$ does not depend on λ and Q is a polynomial of degree $ht_i(\omega - \lambda)$. The restriction of $(f(z))^{-1}T_i(z)$ on W_λ is a polynomial of degree $ht_i(\omega - \lambda)$. Hence each factor (7.33) is the product of $f(z)$ by a polynomial of degree $ht_i(\omega - \lambda) - r$. Hence $(f(z))^{-1}t_V(z,u)$ is a polynomial in z of degree at most $ht_i(\omega - \lambda)$.

Now consider a tensor product $W = W_1 \otimes \cdots \otimes W_N$ of thin simple representations W_i. By using inductively the formula $(\mathrm{Id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{1,3}\mathcal{R}_{1,2}$ we have for $N \geq 2$

$$(\mathrm{Id} \otimes \Delta)^N(\mathcal{R}) = \mathcal{R}_{1,N+2}\mathcal{R}_{1,N+1}\mathcal{R}_{1,N} \cdots \mathcal{R}_{1,2}. \tag{7.34}$$

Hence we can use the same proof as above where the term inside π_V in the factor (7.32) is replaced by a product of N such terms and the factor (7.33) is replaced by a product of N such terms with π_{W_j} $(1 \leq j \leq N)$ instead of π_W.

Let us conclude the proof of Theorem 5.9 when $\dot{\mathfrak{g}}$ is not of type E_8. It suffices to prove that the degree of the polynomial in z is not only less than or equal to $ht_i(\omega-\lambda)$, but equal to it. First note that Theorem 5.9 implies Theorem 5.17, so the degree of the polynomial in Theorem 5.17 is less than or equal to the degree in Theorem 5.9. But the eigenvalues of $(f_i(z))^{-1}T_i(z)$ are exactly of degrees $ht_i(\omega - \lambda)$ by Proposition 5.8. Hence the result. For the same reason, the degree is exactly $ht_i(\omega - \lambda)$ in Corollary 5.10.

To conclude, let us prove the result when $\dot{\mathfrak{g}}$ is of type E_8. We have seen it suffices to consider the case of a tensor product of fundamental representations $W = L(Y_{1,a})$. This representation has been studied for example in [HN, Section 6.1.2][2]. It has a unique ℓ-weight space W' whose dimension is not 1: it corresponds to the monomial $m = Y_{5,aq^{14}}Y_{5,aq^{16}}^{-1}$ and it is of dimension 2. To use the same argument as above, we just have to prove the action of $(f(z))^{-1}T_i(z)$ on W' is a polynomial of degree $ht_i(\omega - \lambda)$ (with the same notation as above).

The q-character of W can be computed by using the algorithm of Mukhin and the first author [FM]. In particular the monomials in $m\mathbb{Z}[A_{5,b}^{\pm 1}]_{b\in\mathbb{C}^*}$ which occur in $\chi_q(W)$ are necessarily m, $mA_{5,aq^{15}}$, $mA_{5,aq^{15}}^{-1}$ with respective multiplicities 2, 1, 1. Besides, by setting $Y_{j,b} = 1$ for $j \neq 5$ in $mA_{5,aq^{15}} + 2m + mA_{5,aq^{15}}^{-1}$ we get $Y_{5,aq^{14}}^2 + 2Y_{5,aq^{14}}Y_{5,aq^{16}}^{-1} + Y_{5,aq^{16}}^{-2}$ which is the q-character of a simple $U_q(\hat{sl}_2)$-module of dimension 4. Let U be the subalgebra of $U_q(\mathfrak{g})$ generated by the $x_{5,m}^{\pm}$, $k_5^{\pm 1}$, $\tilde{h}_{5,r}$ $(m \in \mathbb{Z}, r \in \mathbb{Z} \setminus \{0\})$. Then U is isomorphic to $U_q(\hat{sl}_2)$. From the discussion above, W' generates a simple U-module of dimension 4 which is the sum of 3 ℓ-weight spaces of W. Let w be an highest weight vector of this module. Then $\mathbb{C}.w$ is a ℓ-weight space of W of dimension 1 and $(f(z))^{-1}T_i(z)$ is a polynomial on $\mathbb{C}.w$. But we have also $W' = \sum_{m\in\mathbb{Z}} \mathbb{C}.x_{5,m}^-.w$. By Lemma 5.7, for $i \neq 5$, $T_i(z)$ commutes with the $x_{5,m}^-$. So, we have the result on W'. Suppose that $i = 5$. Then $T_5(z) \in U[[z]]$. The result follows from Theorem 5.17 that we have already established in the case $\dot{\mathfrak{g}} = sl_2$ (in fact, this is exactly the example explained in Section 5.8).

Remark 7.7. *In the case $N_i = 1$, for example for any i for $\dot{\mathfrak{g}}$ of type A, the proof in Section 7.4 is simplified thanks to Section 6.3. Indeed, by Theorem 6.3, we have $\pi_V(E_{\alpha+m\delta}) = 0$ for ($m \geq 1$ and $\alpha \in \Phi_0^+$) or ($m > -\alpha(\alpha_i^\vee)$ and $\alpha \in \Phi_0^-$). Moreover we have a scalar action $\pi_V(h_{j,m}) = \frac{-\delta_{i,j} Id_V}{m(q_i - q_i^{-1})}$ for $m > 0$ and $j \in I$. This implies $L_V^0(z) = 1$. Consider a tensor*

[2]The representation W has dimension 249. As a $U_q(\dot{\mathfrak{g}})$-module, it has two simple constituents, one of them is the trivial representation of dimension 1.

product $W = W_1 \otimes \cdots \otimes W_N$ of thin simple representations. By using formula (7.34), we can use the same arguments as above by considering only terms of the form

$$z^{r_1+\cdots+r_N} Tr_{V,u}((E_{\beta_1^{(1)}} \cdots E_{\beta_{r_1'}^{(1)}} E_{\delta-\gamma_1^{(1)}} \cdots E_{\delta-\gamma_{r_1}^{(1)}}) \cdots (E_{\beta_1^{(N)}} \cdots E_{\beta_{r_N'}^{(N)}} E_{\delta-\gamma_1^{(N)}} \cdots E_{\delta-\gamma_{r_N}^{(N)}}))$$

$$\pi_{W_1}(F_{\beta_1^{(1)}} \cdots F_{\beta_{r_1'}^{(1)}} T_i(z) F_{\delta-\gamma_1^{(1)}} \cdots F_{\delta-\gamma_{r_1}^{(1)}}) \otimes \cdots \otimes \pi_{W_N}(F_{\beta_1^{(N)}} \cdots F_{\beta_{r_N'}^{(N)}} T_i(z) F_{\delta-\gamma_1^{(N)}} \cdots F_{\delta-\gamma_{r_N}^{(N)}})$$

where $r_1 \leq R_1, \cdots, r_N \leq R_N$, $\gamma_1^{(1)}(\alpha_i^\vee) = \cdots = \gamma_{r_N}^{(N)}(\alpha_i^\vee) = 1$ and $R_1, \cdots, R_N$ are fixed.

Example 7.8. *Let us now consider the example in Section 5.7 from the angle of the proof above (strictly speaking, the example here is essentially equivalent to Section 5.7). In the case $\dot{\mathfrak{g}} = sl_2$ with $V = R_{1,1}^+$, we have $\pi_V(x_{1,m}^+) = 0$ for $m \geq 1$ and $\pi_V(x_{1,m}^-) = 0$ for $m \geq 2$. Moreover $\pi_V(h_{1,m}) = \frac{-Id_V a^m}{m(q-q^{-1})}$ for $m > 0$. Hence*

$$L_V(z) = L_V^+(z)(Id_V \otimes T(az))L_V^-(z)(\pi_V \otimes Id)(\mathcal{R}^\infty),$$

$$L_V^+(z) = exp_q\left((q^{-1}-q)\pi_V(x_{1,0}^+) \otimes x_{1,0}^-\right) \ , \ L_V^-(z) = exp_q\left((q-q^{-1})z\pi_V(k_1^{-1}x_{1,1}^-) \otimes x_{1,-1}^+ k_1\right).$$

Hence

$$t_V(z,u) = \sum_{r\geq 0} \frac{((q-q^{-1})z)^r}{[r]_q!} \sum_{m \geq r} u^{2m} \begin{bmatrix} m \\ r \end{bmatrix}_q q^{\frac{r(3-r)}{2} - rm} (x_{1,0}^-)^r T_1(z) (x_{1,-1}^+ k_1)^r k_1^{-m}.$$

When we specialize $x_{1,-1}^+$ on a simple finite-dimensional representation W, it becomes nilpotent and the sum is finite. Then we recover the formulas as in Section 5.7.

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