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BAYES ESTIMATION IN A MIXTURE INVERSE GAUSSIAN MODEL

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Abstract. In this paper a mixture model involving the inverse Gaussian distribution and its length biased version is studied from a Bayesian view-point. Using proper priors, the Bayes estimates of the parameters of the model are derived and the results are applied on the aircraft data of Proschan (1963, *Technometrics*, **5**, 375–383). The posterior distributions of the parameters are expressed in terms of the confluent-hypergeometric function and the modified Bessel function of the third kind. The integral involved in the expression of the estimate of the mean is evaluated by numerical techniques.

Key words and phrases: Length biased, Birnbaum Saunders model, Lindley approximation.

1. Introduction

The inverse Gaussian distribution (IG) is a positively skewed distribution that provides an interesting and useful alternative in reliability studies, to the Weibull, lognormal, gamma and other similar distributions. For various applications of the IG, the reader is referred to a book by Chhikara and Folks (1989). Recently a length biased inverse Gaussian distribution (LBIG) has been studied by Akman and Gupta (1992) and Gupta and Akman (1995*a*). Applications of the length biased distributions have been made in biomedical areas such as family history and disease, early detection of disease, survival and intermediate events and latency periods of AIDS due to blood transfusion. Some applications of length biased sampling in life length studies are described in Bluementhal (1967) and Schaeffer (1972). A review of length biased distributions and its applications is contained in Gupta and Kirmani (1990).

This paper deals with a random variable X_p whose distribution is a mixture of IG and LBIG as follows.

Let the pdf of X_p be

$$f_p(x) = (1-p)f_X(x) + pf_X^*(x), \qquad 0 \le p \le 1 \tag{1.1}$$

where

$$f_X(x) = \begin{cases} (\lambda/2\pi x^3)^{1/2} \exp\{-\lambda(x-\mu)^2/2\mu^2 x\}, & x > 0,\ \lambda > 0,\ \mu > 0 \\ 0; & \text{otherwise} \end{cases}$$

and

$$f_X^*(x) = x f_X(x)/\mu, \qquad \text{where} \quad 0 < \mu = E(X) < \infty;$$

see Jorgensen *et al.* (1991).

The model (1.1) represents a rich family of distributions for different values of p. Apart from the special cases, $p = 0$ and $p = 1$ the case $p = \frac{1}{2}$ yields Birnbaum and Saunder's (1969) model which was derived from a model of fatigue crack growth. We shall call it a mixture inverse Gaussian distribution (MIG). The MIG has been investigated by Gupta and Akman (1995*b*) from the view point of reliability. More specifically, they examined the nature of its failure rate and mean residual life function. They also studied the maximum likelihood estimation of the parameters and that of the reliability function.

Presently we consider the Bayes estimation of the parameters of the MIG. A Bayesian analysis of the IG was presented by Banerjee and Bhattacharyya (1976, 1979) using both improper and proper priors for (ψ, λ) where $\psi = 1/\mu$. In their approach it turned out that the posterior mean of $1/\psi$ does not exist and, therefore, the Bayes estimate of μ could not be obtained. To overcome this difficulty, Betro and Rotondi (1991) presented Bayesian results for the IG by considering proper priors which enabled them to derive Bayes estimates for μ. The Bayes estimation of the reliability for the IG and Birnbaum-Saunder's model has been studied by Padgett (1981, 1982).

We reparameterize the model (1.1) by defining $\phi = \lambda/\mu$ and derive the Bayes estimates of all the three parameters μ, ϕ and p, by taking proper priors, as has been done by Betro and Rotandi (1991) for IG. The posterior distributions of the parameters are expressed in terms of the confluent hypergeometric function and the modified Bessel function of the third kind. The integral involved in the expression of the estimate of μ is evaluated using (i) Monte-Carlo integration and (ii) iteration technique. In addition, another method, which uses Lindley's approximation, of obtaining Bayes estimates of μ is presented. Finally, the aircraft data of Proschan (1963) was used to evaluate the Bayes estimates of the parameters.

2. Bayes estimation

The model (1.1) can be written as

$$(2.1) \qquad f(x \mid \mu, \lambda, p) = (\lambda/2\pi x^3)^{1/2} \exp\{-\lambda(x-\mu)^2/2\mu^2 x\}(1 - p + px/\mu), \qquad x > 0,\ \lambda, \mu > 0.$$

It is convenient to rewrite (2.1) as

$$(2.2) \qquad f(x \mid \mu, \phi, p) = \left(\frac{\phi\mu}{2\pi}\right)^{1/2} x^{-3/2} e^{\phi} \exp\left\{-\frac{\phi}{2}\left(\frac{x}{\mu} + \frac{\mu}{x}\right)\right\}(1 - p + px/\mu)$$

where $\phi = \frac{\lambda}{\mu}$. Given a random sample $\boldsymbol{x} = \{x_1, \dots, x_n\}$ from (2.2), the likelihood can be written as

$$(2.3)\quad L(\mu, \phi, p \mid x) = (2\pi)^{-n/2} \prod_{i=1}^{n} x_i^{-3/2} \phi^{n/2} \mu^{n/2} e^{n\phi} \exp\left\{ \frac{-n\phi}{2} \left(\frac{\bar{x}}{\mu} + \mu \tilde{x} \right) \right\} \cdot \prod_{i=1}^{n} (1 - p + p x_i / \mu)$$

where $\tilde{x} = \frac{1}{n} \sum_{i=1}^{n} x_i^{-1}$. Assume that a prior information about μ, ϕ and p is summarized in their joint density $\pi(\mu, \phi, p) = \pi(\mu \mid \phi)\pi(\phi)\pi(p)$ where the conditional density of μ given ϕ is given by an IG density

$$(2.4)\quad \pi(\mu \mid \phi) = \left(\frac{\eta\phi\omega}{2\pi} \right)^{1/2} e^{\phi\omega} \mu^{-3/2} \exp\left\{ -\frac{\phi\omega}{2} \left(\frac{\eta}{\mu} + \frac{\mu}{\eta} \right) \right\}, \qquad \eta > 0.$$

The marginal density of ϕ is given by a Gamma density

$$(2.5)\quad \pi(\phi) = \frac{a^{\gamma}}{\Gamma(\gamma)} \phi^{\gamma - 1} e^{-a\phi}, \qquad \gamma > 0,\ a > 0,$$

and the marginal density of p is given by a Beta density

$$(2.6)\quad \pi(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{\mathbf{B}(\alpha, \beta)}, \qquad \alpha > 0,\ \beta > 0.$$

Then the posterior distribution of (μ, ϕ, p) has the density function,

$$(2.7)\quad \pi(\mu, \phi, p \mid x) = C \phi \frac{\nu - 1}{2} \mu^{(n-3)/2} \exp\left\{ -\phi \left(\frac{\nu_1}{2\mu} - \nu_2 + \frac{\nu_3 \mu}{2} \right) \right\} \cdot \prod_{i=1}^{n} (1 - p + p x_i / \mu) p^{n(\alpha - 1)} (1-p)^{n(\beta-1)}$$

where C is the normalizing constant, $\nu = n + 2\gamma$, $\nu_1 = n\bar{x} + \omega\eta$, $\nu_2 = n + \omega - a$ and $\nu_3 = n\tilde{x} + \omega/\eta$. Note that the domain of ν_2 is all real numbers, while the domains of ν, ν_1 and ν_3 are all positive real numbers. The density function in, (2.7) can be rewritten as

$$(2.8)\quad \begin{aligned} \pi &= \pi(\mu, \phi, p \mid x) \\ &= \phi^{(\nu-1)/2} \exp\left\{ -\phi \left(\frac{\nu_1}{2\mu} - \nu_2 + \nu_3 \frac{\mu}{2} \right) \right\} \sum_{k=0}^{n} \mu^{(n-3)/2-k} S_k \\ &\quad \cdot p^{na-n+k} (1-p)^{n\beta - k} \end{aligned}$$

where,

$$S_0 = 1,\ S_1 = \sum_{i} X_i,\ S_2 = \sum_{i<j} x_i x_j,\ S_3 = \sum_{i<j<k} x_i x_j x_k,\ \dots,\ S_n = \prod_{i=1}^{n} x_i.$$

Integrating (2.8) with respect to μ,

$$\int \pi d\mu = \sum_{k=0}^{n} 2\phi^{(\nu-1)/2} S_k p^{n\alpha-n+k}(1-p)^{n\beta-k} e^{\phi\nu_2}(\nu_1/\nu_3)^{(n-1-2k)/4} \cdot K_{(n-1)/2-k}(\phi\sqrt{\nu_1\nu_3}),$$

where $\nu_1, \nu_3 > 0$ and K_λ denotes the modified Bessel function of the third kind with index λ. Similarly

$$(2.9)\quad \iint \pi d\mu d\phi = \sum_{k=0}^{n} 2S_k p^{n\alpha-n+k}(1-p)^{n\beta-k}(\nu_1/\nu_3)^{(n-1-2k)/4} \cdot \frac{\sqrt{\pi}(2\sqrt{\nu_1\nu_3})^{(n-1)/2-k}}{(\sqrt{\nu_1\nu_3}-\nu_2)^{(\nu+n)/2-k}} \frac{\Gamma\left(\frac{\nu+n}{2}-k\right)\Gamma\left(\frac{\nu-n}{2}+k+1\right)}{\Gamma\left(\frac{\nu}{2}+1\right)} \cdot F\left(\frac{\nu+n}{2}-k, \frac{n}{2}-k; \frac{\nu}{2}+1; \frac{-\sqrt{\nu_1\nu_3}-\nu_2}{\sqrt{\nu_1\nu_3}-\nu_2}\right),$$

where $\frac{\nu+1}{2} > |\frac{n-1}{2}-k|$; for all $k = 0, \ldots, n$, $\sqrt{\nu_1\nu_3}-\nu_2 > 0$ and $F(a,b;c;d)$ is the confluent hypergeometric function, see Gradshteyn and Ryzhik (1980). Finally,

$$(2.10)\quad \frac{1}{C} = \iiint \pi d\mu d\phi dp = \sum_{k=0}^{n} 2\sqrt{\pi} S_k (\nu_1/\nu_3)^{(n-1-2k)/4} \frac{(2\sqrt{\nu_1\nu_3})^{(n-1)/2-k}}{(\sqrt{\nu_1\nu_3}-\nu_2)^{(\nu+n)/2-k}} \cdot \frac{\Gamma\left(\frac{\nu+n}{2}+k\right)\Gamma\left(\frac{\nu-n}{2}+k+1\right)}{\Gamma\left(\frac{\nu}{2}+1\right)} \cdot F\left(\frac{\nu+n}{2}-k, \frac{n}{2}-k; \frac{\nu}{2}+1; \frac{-\sqrt{\nu_1\nu_3}-\nu_2}{\sqrt{\nu_1\nu_3}-\nu_2}\right) \cdot \mathbf{B}(n\alpha-n+k+1, n\beta-k+1)$$

$$\text{with}\quad \alpha > 1-\frac{1}{n} \text{ and } \beta > 1-\frac{1}{n}.$$

Bayesian estimation of μ, ϕ and p is easily achieved by evaluating the posterior marginal of μ, ϕ and p, namely

$$(2.11)\quad \pi(p \mid x) = C \cdot \sum_{k=0}^{n} 2S_k p^{n\alpha-n+k}(1-p)^{n\beta-k}(\nu_1/\nu_3)^{(n-1-2k)/4} \cdot \frac{\sqrt{\pi}(2\sqrt{\nu_1\nu_3})^{(n-1)/2-k}}{(\sqrt{\nu_1\nu_3}-\nu_2)^{(\nu+n)/2-k}} \frac{\Gamma\left(\frac{\nu+n}{2}-k\right)\Gamma\left(\frac{\nu-n}{2}+k+1\right)}{\Gamma\left(\frac{\nu}{2}+1\right)}$$

$$\cdot F\left(\frac{\nu+n}{2}-k, \frac{n}{2}-k; \frac{\nu}{2}+1; \frac{-\sqrt{\nu_1\nu_3}-\nu_2}{\sqrt{\nu_1\nu_3}-\nu_2}\right).$$

Thus the Bayes estimate of p can be obtained from (2.11) as

$$\tilde{p} = E(p) = \int_0^1 p\pi(p \mid x)dp \tag{2.12}$$

$$= C\sum_{k=0}^{n} 2\sqrt{\pi}S_k(\nu_1/\nu_3)^{(n-1-2k)/k}\frac{(2\sqrt{\nu_1\nu_3})^{(n-1)/2-k}}{(\sqrt{\nu_1\nu_3}-\nu_2)^{(\nu+n)/2-k}}$$

$$\cdot\frac{\Gamma\left(\frac{\nu+n}{2}-k\right)\Gamma\left(\frac{\nu-n}{2}+k+1\right)}{\Gamma\left(\frac{\nu}{2}+1\right)}$$

$$\cdot F\left(\frac{\nu+n}{2}-k, \frac{n}{2}-k; \frac{\nu}{2}+1, \frac{-\sqrt{\nu_1\nu_3}-\nu_2}{\sqrt{\nu_1\nu_3}-\nu_2}\right)$$

$$\cdot \mathbf{B}(n\alpha-n+k+2, n\beta-k+1),$$

where $\alpha > 1-2/n$ and $\beta > 1-1/n$. Similarly,

$$\pi(\phi \mid x) = C\iint \pi d\mu dp \tag{2.13}$$

$$= C\sum_{k=0}^{n} 2\phi^{(\nu-1)/2}S_k e^{\phi\nu_2}(\nu_1/\nu_3)^{(n-1-2k)/4}K_{(n-1)/2-k}(\phi\sqrt{\nu_1\nu_3})$$

$$\cdot \mathbf{B}(n\alpha-n+k+1, n\beta-k+1)$$

and

$$\tilde{\phi} = E(\phi) = \int \phi\pi(\phi \mid \boldsymbol{x})d\phi \tag{2.14}$$

$$= C\sum_{k=0}^{n} 2S_k(\nu_1/\nu_3)^{(n-1-2k)/4}\mathbf{B}(n\alpha-n+k+1, n\beta-k+1)$$

$$\cdot\frac{\sqrt{\pi}(2\sqrt{\nu_1\nu_3})^{(n-1-2k)/2}}{(\sqrt{\nu_1\nu_3}-\nu_2)^{(\nu+n)/2+1-k}}\frac{\Gamma\left(\frac{\nu+n}{2}+1-k\right)\Gamma\left(\frac{\nu-n}{2}+2+k\right)}{\Gamma\left(\frac{\nu}{2}+2\right)}$$

$$\cdot F\left(\frac{\nu+n}{2}+1-k, \frac{n}{2}-k; \frac{\nu}{2}+2; \frac{-\sqrt{\nu_1\nu_3}-\nu_2}{\sqrt{\nu_1\nu_3}-\nu_2}\right),$$

where $\frac{\nu+3}{2} > |\frac{n-1}{2}-k|$, $k = 0, 1, \ldots, n$ and $\sqrt{\nu_1\nu_3}-\nu_2 > 0$. In the same way

$$\pi(\mu \mid x) = C\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\left(\frac{\nu_1}{2\mu}-\nu_2+\nu_3\frac{\mu}{2}\right)^{(\nu+1)/2}} \tag{2.15}$$

$$\cdot\sum_{k=0}^{n}\mu^{(n-3-2k)/2}S_k\mathbf{B}(n\alpha-n+k+1, n\beta-k+1),$$

where $\alpha > 1 - \frac{1}{n}$ and $\beta > 1 - \frac{1}{n}$. Thus

$$\begin{aligned}(2.16)\quad \tilde{\mu} &= E(\mu) \\ &= C2^{(\nu+1)/2} I_\mu \sum_{k=0}^{n} \Gamma\left(\frac{\nu+1}{2}\right) S_k \mathbf{B}(n\alpha - n + k + 1, n\beta - k + 1),\end{aligned}$$

where

$$(2.17)\qquad I_\mu = \int_0^\infty \frac{\mu^{(n-1)/2-k+(\nu+1)/2}}{(\nu_1 - 2\nu_2\mu + \nu_3\mu^2)^{(\nu+1)/2}} d\mu.$$

The integral I_μ is evaluated by (i) Monte-Carlo integration and (ii) iteration technique described in the appendix. A comparison between the two computational techniques is also discussed in the Appendix.

Another method of obtaining a Bayes estimate of μ, which uses Lindley's approximation, can be used as described below.

Let $L(\theta) = \ln f(x_1, x_2, \ldots, x_n \mid \theta) = \ln \prod_{i=1}^{n} \ell(x_i \mid \theta)$ denote the log of the likelihood function and $\rho(\theta) = \ln g(\theta)$ denote the log of the prior density. Let $u(\theta)$ be an arbitrary function of θ. Then a Bayes estimate of $u(\theta)$ involves the ratio of integrals of the form

$$(2.18)\qquad I(x_1, x_2, \ldots, x_n) = \frac{\int u(\theta) e^{L(\theta)+\rho(\theta)} d\theta}{\int e^{L(\theta)+\rho(\theta)} d\theta}.$$

Here θ may be vector valued.

Lindley approximation. Suppose n is sufficiently large so that $L(\theta)$ defined above concentrates around a unique maximum likelihood estimator $\hat{\theta} = \hat{\theta}(x_1, x_2, \ldots, x_n)$ for $\theta = (\theta_i)$, $p \times 1$, $\hat{\theta} = (\hat{\theta}_i)$. Then $I(\cdot)$ defined in (2.18) is expressible approximately as

$$\begin{aligned}(2.19)\quad & I(x_1, x_2, \ldots, x_n) \\ &\approx u(\hat{\theta}) + \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \left[\frac{\partial^2 u(\theta)}{\partial\theta_i \partial\theta_j} \Big|_{\theta=\hat{\theta}} \right. \\ &\qquad \left. + \left\{ 2\frac{\partial u(\theta)}{\partial\theta_i} \Big|_{\theta=\hat{\theta}} \right\} \left\{ \frac{\partial \rho(\theta)}{\partial\theta_j} \Big|_{\theta=\hat{\theta}} \right\} \right] \hat{\sigma}_{ij} \\ &+ \left\{ \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{l=1}^{p} \frac{\partial^3 L(\theta)}{\partial\theta_i \partial\theta_j \partial\theta_l} \Big|_{\theta=\hat{\theta}} \cdot \left\{ \frac{\partial u(\theta)}{\partial\theta_k} \Big|_{\theta=\hat{\theta}} \right\} \hat{\sigma}_{ij} \hat{\sigma}_{kl} \right\}\end{aligned}$$

where $\hat{\sigma}_{ij}$ denotes the (i,j) element of $\hat{\Sigma} \equiv (\hat{\sigma}_{ij})$, for $\hat{\Sigma}^{-1} = \hat{\Lambda} = (\hat{\lambda}_{ij})$ and $\hat{\lambda}_{ij} = -\frac{\partial^2 L(\theta)}{\partial\theta_i \partial\theta_j} |_{\theta=\hat{\theta}}$. For details, see Lindley (1980) and Press (1989). In our case, for computational convenience, we shall assume $p = 0$ and so $\theta = (\mu, \phi)$, $u(\boldsymbol{\theta}) = \mu$

and $\hat{\boldsymbol{\theta}} = (\bar{X}, \bar{X}V)$ where $V = \sum_{i=1}^{n}(\frac{1}{x_i} - \frac{1}{\bar{x}})$. Also

$$L(\boldsymbol{\theta}) = -\frac{n}{2}\ln 2\pi - \frac{3}{2}\ln\left(\prod_{i=1}^{n} x_i\right) + \frac{n}{2}\ln\phi + \frac{n}{2}\ln\mu + n\phi - \frac{n\phi}{2}\left(\frac{\bar{x}}{\mu} + \mu\bar{x}\right) \tag{2.20}$$

and

$$\rho(\boldsymbol{\theta}) = \frac{1}{2}\ln\left(\frac{n\phi\omega}{2\pi}\right) + \phi\omega - \frac{3}{2}\ln\mu - \frac{\phi\omega}{2}\left(\frac{\eta}{\mu} + \frac{\mu}{\eta}\right) + \gamma\ln a - \ln\Gamma(\gamma) + (\gamma - 1)\ln\phi - a\phi. \tag{2.21}$$

These will yield

$$\begin{aligned} I(x_1, x_2, \ldots, x_n) = \hat{\mu} &+ \frac{\partial}{\partial\mu}\rho(\theta)\hat{\sigma}_{11} + \frac{\partial\rho(\theta)}{\partial\phi}\hat{\sigma}_{12} \\ &+ \frac{1}{2}\frac{\partial^3 L(\theta)}{\partial\mu^3}\hat{\sigma}_{11}^2 + \frac{3}{2}\frac{\partial^3 L(\theta)}{\partial\mu^2\partial\phi}\hat{\sigma}_{11}\hat{\sigma}_{12} \\ &+ \frac{\partial^3 L(\theta)}{\partial\phi^2\partial\mu}\left\{(\hat{\sigma}_{12})^2 + \frac{1}{2}\hat{\sigma}_{11}\hat{\sigma}_{22}\right\} + \frac{1}{2}\frac{\partial^3 L(\theta)}{\partial^3\phi}\hat{\sigma}_{12}\hat{\sigma}_{22}, \end{aligned}$$

where the derivatives are evaluated by substituting the maximum likelihood estimates for the parameters.

3. Illustration

Proschan (1963) gave the failure intervals for the air conditioning system of 13 different aircrafts of the same type. In the following we use his data on only four aircraft numbers 7907, 7915, 7916, 8044. The data are as Table 1.

Table 1.

AIRCRAFT #												
7907	194	15	41	29	33	181						
7915	359	9	12	270	603	3	104	2	438			
7916	50	254	5	283	35	12						
8044	487	18	100	7	97	5	85	91	43	230	3	130

The Bayes estimates of the above data sets are presented in Table 2.

Table 2.

AIRCRAFT #	n		$\tilde{p}$	$\tilde{\phi}$	$\tilde{\mu}$ ITERATION	MONTE-CARLO
7907	6	(i)	.18	.98	74.92	67.41
		(ii)	.34	.71	81.43	70.09
7915	9	(i)	.79	.01	29.77	17.32
		(ii)	.60	.00	39.69	21.49
7916	6	(i)	.68	.21	54.99	59.81
		(ii)	.45	.19	71.13	90.94
8044	12	(i)	.54	.02	36.40	29.32
		(ii)	.66	.02	25.71	14.17

The above results are corresponding to
(i) $a = 1, \gamma = 2, \omega = 5, \eta = 5, \alpha = 1, \beta = 1$,
(ii) $a = 1, \gamma = 2, \omega = 5, \eta = 5, \alpha = 2, \beta = 2$.
In order to obtain Bayes estimates using Lindley's approximation, we assumed p known for computational convenience. The maximum likelihood estimators (MLE's) of μ and ϕ, needed in the Lindley's approximation have been given in Gupta and Akman (1995*b*). Using these MLE's, the Bayes estimates of μ are given in Table 3.

Table 3.

AIRCRAFT #	$p = 0$		$p = 1/2$		$p = 1$	
7907	83.89	(82.17)	59.69	(54.17)	37.98	(35.71)
7915	197.52	(200)	49.08	(41.45)	11.37	(8.59)
7916	106.57	(106.5)	52.44	(43.40)	23.31	(17.68)
8044	107.52	(108)	44.31	(39.95)	19.74	(14.78)

The above results have been obtained by using the parameters of the priors as $a = 1$, $\gamma = 2$, $\omega = 5$, $\eta = 5$. The MLE's are given in parenthesis.

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Appendix

A.1 *Monte-Carlo integration*

The integral (2.17) is taken first by using the substitution $y = \frac{1}{x+1}$, $dy = -y^2 dx$ to convert the limits of integration to $(0,1)$. Then the integral is computed at 1000 uniform random variates over $(0,1)$. See Ross ((1990), p. 38) for details of the Monte-Carlo simulation technique.

A.2 *Iteration technique*

When γ is an integer, then ν is an integer too and I_μ can be expressed in an iterative way.

Indeed,

$$I(\gamma,\nu) = \int_0^\infty \frac{t^\gamma}{(\nu_1 t^2 - 2\nu_2 t + \nu_3)^{(\nu+1)/2}} dt$$
$$= \frac{1}{\nu_1(\nu-1)\nu_3^{(\nu-1)/2}} + \frac{\nu_2}{\nu_1} I(0,\theta) \quad \text{if} \quad \gamma = 1$$

and

$$= \frac{\gamma-1}{\nu_1(\nu-1)} \{\nu_1 I(\gamma,\nu) - 2\nu_2 I(\gamma-1,\nu) + \nu_3 I(\gamma-2,\nu)\}$$
$$+ \frac{\nu_2}{\nu_1} I(\gamma-1,\nu) \quad \text{if} \quad \gamma > 1$$

by which the iteration follows:

$$\text{(A.1)} \quad I(\gamma,\nu) = \frac{\nu_2}{\nu_1}\frac{\nu-2\gamma+1}{\nu-\gamma} I(\gamma-1,\nu) + \frac{\nu_3(\gamma-1)}{\nu_1(\nu-\gamma)} I(\gamma-2,\nu) \quad \text{if} \quad \gamma > 1,$$

and

$$I(1,\nu) = \frac{1}{\nu_1(\nu-1)\nu_3^{(\nu+1)/2}} + \frac{\nu_2}{\nu_1} I(0,\nu).$$

In case $d = \nu_1\nu_3 - \nu_2 \neq 0$, $I(0,\nu)$ is obtained by iteration in ν as follows:

$$\int_0^\infty \frac{1}{(\nu_1 t^2 - 2\nu_2 t + \nu_3)^{(\nu+1)/2}} dt$$
$$= \frac{1}{\nu_3} \{I(0,\nu-2) - \nu_1 I(2,\nu) + 2\nu_2 I(1,\nu)\}$$

and by (A.1)

$$I(0,\nu) = \frac{1}{d(\nu-1)} \{\nu_1(\nu-2) I(0,\nu-2) + \nu_2/\nu_3^{(\nu-1)/2}\} \quad \text{if} \quad \nu > 2$$

while by direct integration

$$I(0,1) = \begin{cases} \frac{1}{\sqrt{d}} \left\{\frac{\pi}{2} + \arctan\left(\frac{\nu_2}{d}\right)\right\} & \text{if} \quad d > 0 \\ \frac{1}{2\sqrt{|d|}} \log\left(1 - \frac{2\sqrt{|d|}}{\nu_2 + \sqrt{|d|}}\right) & \text{if} \quad d < 0 \end{cases}$$

and

$$I(0,2) = \frac{1}{d}\left\{\nu_1^{1/2} + \frac{\nu_2}{\nu_3^{1/2}}\right\}.$$

If $d = 0$, then

$$I(0,\nu) = \frac{\nu_1^{(\nu-1)/2}}{\nu(-\nu_2)^\nu}$$

where the negative sign takes into account the fact that $d = 0$ is possible only for strictly negative values of ν_2.

A.3 *Comparisons between the two methods and some comments*

1. The difference between two computation methods occurs because the Monte-Carlo method provides approximate results whereas exact results are obtained by the iteration methods.

2. Iteration method requires more computation time (and repeated calculations by the computer) while Monte-Carlo method simply evaluates the function at (sufficiently many) uniform random numbers which is quite routine.

3. Iteration method is more accurate since M-C method is an approximation and its accuracy depends on the number of uniform random numbers at which the integral is evaluated.

4. When $p \neq 0$, the model differs substantially from the $p = 0$ (IG) case, since $\frac{\mu}{x}f(x)$ vanishes when $p = 0$. In case $p \neq 0$, Lindley's approximation will have several more derivatives than in the case $p = 0$ since, for $p \neq 0$, the model becomes $(1-p)f(x) + p\frac{\mu}{x}f(x)$ which causes the difference in the approximation.

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