Bayes Estimators for the Shape Parameter of Pareto Type I Distribution under Generalized Square Error Loss Function

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Abstract

In this paper, we obtained Basyian estimators of the shape parameter of the Pareto type I distribution using Bayian method under Generalized square error loss function and Quadratic loss function. In order to get better understanding of our Bayesian analysis we consider non-informative prior for the shape parameter Using Jeffery prior Information as well as informative prior density represented by Exponential distribution. These Bayes estimators of the shape parameter of the Pareto type I distribution are compared with some classical estimators such as, the Maximum likelihood estimator (MLE), the Uniformly minimum variance unbiased estimator (UMVUE), and the Minimum mean squared error (MinMSE) estimator according to Monte-Carlo simulation study. The performance of these estimators is compared by employing the mean square errors (MSE's).

Key words: Pareto distribution; Maximum likelihood estimator; Uniformly minimum variance unbiased estimator; Minimum mean squared error; Bayes estimator; Generalized square error loss function; Quadratic loss function; Jeffery prior; Exponential prior.

1. Introduction

The Pareto distribution is named after the economist Vilfredo Pareto (1848-1923), this distribution is first used as a model for distributing incomes of model for city population within a given area, failure model in reliability theory [7], and a queuing model in operation research [12].

A random variable X is said to follow the two parameters of Pareto distribution if its pdf is given by[2]:

$$f(t,\theta) = \frac{\theta\alpha^{\theta}}{t^{\theta+1}} \qquad t \ge \alpha, \quad \alpha > 0, \quad \theta > 0 \qquad (1)$$

Where t is a random variable, θ and α are the shape and scale parameters respectively.

The cumulative distribution function of Pareto distrcibution type I is given by [9]

$$F(x) = 1 - \left(\frac{\alpha}{t}\right)^{\theta} \qquad t \ge \alpha, \alpha > 0, \quad \theta > 0 \qquad (2)$$

Therefore, the reliability function is given as follows [2]:

$$R(t) = P_r(T > t)$$

$$= \int_t^{\infty} f(t,\alpha,\theta)dt$$

$$= \int_t^{\infty} \frac{\theta\alpha^{\theta}}{u^{\theta+1}} du = \theta\alpha^{\theta}\left[\frac{1}{\theta t^{\theta}}\right]$$

$$R(t) = \left(\frac{\alpha}{t}\right)^{\theta} \quad , \ t \ge \alpha, \ \theta > 0, \quad \alpha > 0 \qquad (3)$$

Inferences of Pareto distribution have been studied by many authors, Quandt (1966) [8]has obtained different estimators for the parameters of Pareto distribution using the method of maximum likelihood, method of least square and quantile method and discussed their properties.

Charek, D. J. in (1985) [4] walked a comparison of estimation techniques for the three parameters of Pareto distribution and studied the estimator Properties.

Rytgaard in (1990)[9] estimated the shape parameter and scale parameter of Pareto distribution by using Maximum Likelihood method and moments method and used the simulation style. The researcher found that the maximum likelihood method is the better.

Giorgi, G.M. and Crescenzi, M.(2001)[6] estimated the shape parameter of Pareto distribution-type I by using Bayes estimator under squared loss function and, as prior distributions, the truncated Erlang and the translated exponential one.

Set of Bayesian methods and concluded that the proposed methods are better than the standard Bayes and also suggested a method was linking between the usual methods and Bayesian.

Ertefaie, A. and Parsian A.(2005)[5] estimated the parameters of Pareto distribution by using Bayes estimators under LINEX loss function when the scale parameter is known and both scale and shape parameters are unknown.

Singh G., B.P., S. K., U., and R.D.(2011)[10] estimated the shape parameter of classical Pareto distribution under prior information in the form of a point guess value when the scale parameter is unknown.

Al-Athari M.(2011)[1] discussed the estimate of the parameter for Double Pareto distribution and, this study contracted with maximum likelihood, the method of moments and Bayesian using Jeffery's prior and the extension of Jeffery's prior information and, based on the results of the simulation, the maximum likelihood and Bayesian method with Jeffery's prior are found to be the best with respect to MSE.

2. Some Classical Estimators of Shape Parameter

In this section, we obtain some classical estimators of the shape parameter for the Pareto distribution represented by Maximum likelihood estimator, Uniformly Minimum Variance Unbiased Estimator and Minimum mean square error estimator.

Given $x_1, x_2, \dots, x_n$ a random sample of size n from Pareto distribution, we consider estimation using method of Maximum likelihood as follows:

$$L(\theta; t_1, t_2, \dots, t_n) = f(t_1;\theta).f(t_2;\theta).\dots.f(t_n;\theta) = \prod_{i=1}^{n} f(t_n;\theta)$$

$$L(t_1, \dots, t_n|\theta) = \frac{\theta^n \alpha^{n\theta}}{\prod_{i=1}^{n} t_i^{\theta+1}} = \theta^n \alpha^{n\theta} e^{-(\theta+1)\sum \ln t}$$

$$= \theta^n e^{n\theta \ln\alpha} e^{-(\theta+1)\sum \ln t}$$

Taking the logarithm for the likelihood function, so we get the function:

$$\ln L(\theta; t_1, \dots, t_n) = n \ln\theta + n\theta\ln\alpha - (\theta+1)\sum_{i=1}^{n} \ln t_i$$

The partial derivative for the log-likelihood function with respect to θ and then equating to zero we have:

$$\frac{\partial[\ln L(\theta; t_1, \dots, t_n)]}{\partial\theta} = \frac{n}{\theta} + n\ln\alpha - \sum_{i=1}^{n} \ln t_i = 0$$

The MLE of θ denoted by $\hat{\theta}_{MLE}$ is:

$$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} \ln t_i - n\ln\alpha} = \frac{n}{\sum_{i=1}^{n} \ln\left(\frac{t_i}{\alpha}\right)} \quad (4)$$

The Pareto distribution belongs to the exponential family of distribution [2] as the density function (1) can be written:

$$f(x,\theta) = \theta e^{\theta \ln\alpha - (\theta+1)\ln t} = \theta e^{-\ln t} e^{-\theta \ln\left(\frac{t}{\alpha}\right)}$$

Hence,

$$a(\theta) = \theta \ , b(t) = e^{-\ln t}, \ c(\theta) = -\theta \ , \quad d(t) = \ln\left(\frac{t}{\alpha}\right)$$

Therefore, statistic $P = \sum_{i=1}^{n} \ln\left(\frac{t_i}{\alpha}\right)$ is a complete sufficient statistic for θ.

It is easy to show that, statistic P is distributed as Gamma distribution with parameters n and θ as follows:
If $T \sim \text{Pareto}\ (\alpha,\theta)$, then:

$\ln\left(\frac{t_i}{\alpha}\right) \sim \text{Exponential}(\theta) \Rightarrow P = \sum_{i=1}^{n} \ln\left(\frac{t_i}{\alpha}\right) \sim$ Gamma (n,θ), with the density

$$g(p) = \frac{\theta^n}{\Gamma(n)} p^{n-1} e^{-\theta p} \qquad , \ p \geq 0 \ , \theta > 0$$

$$E\left(\frac{1}{\sum_{i=1}^{n} \ln\left(\frac{t_i}{\alpha}\right)}\right) = E\left(\frac{1}{P}\right) = \int_0^\infty \frac{\theta^n p^{n-2} e^{-\theta p}}{\Gamma(n)} dp$$

$$E\left(\frac{1}{P}\right) = \frac{\theta}{n-1} \qquad (5)$$

Hence, $\frac{n-1}{P}$ is unbiased estimator for θ, and P represents a complete sufficient statistics for θ. Thus, by theorem of Lehmann-Scheffe, the UMVUE of θ denoted by $\hat{\theta}_{UMVUE}$ is given by: [8]

$$\hat{\theta}_{UMVUE} = \frac{n-1}{P} \qquad (6)$$

Minimum Mean Squared Error (MinMSE) estimator can be found in the class of estimators of the form $\frac{c}{P}$. Therefore

$$MSE_\theta\left(\frac{c}{P}\right) = E\left[\left(\frac{c}{P} - \theta\right)^2\right]$$

$$= c^2 E\left[\left(\frac{1}{P}\right)^2\right] - 2c\theta E\left(\frac{1}{P}\right) + \theta^2$$

Taking the partial derivative with respect to c on the both sides and then equating to zero we have:

$$\frac{\partial}{\partial c} MSE_\theta\left(\frac{c}{P}\right) = 2cE\left[\left(\frac{1}{P}\right)^2\right] - 2\theta E\left(\frac{1}{P}\right) = 0$$

$$c = \frac{\theta E\left(\frac{1}{P}\right)}{E\left(\frac{1}{P}\right)^2} \qquad (7)$$

$$E\left[\left(\frac{1}{P}\right)^2\right] = \int_0^\infty \left(\frac{1}{P}\right)^2 \frac{\theta^n}{\Gamma(n)} p^{n-1} e^{-\theta p} dp$$

$$= \frac{\theta^2}{(n-1)(n-2)} \qquad (8)$$

After substitution (5), (8) into (7), we have

$$c = \frac{\frac{\theta^2}{n-1}}{\frac{\theta^2}{(n-1)(n-2)}}$$

Therefore, we get c = n-2 and hence

$$\hat{\theta}_{MinMSE} = \frac{n-2}{P} \qquad (9)$$

Is the minimum mean square error estimator for θ.

Now, we'll derive the MSE for three classical estimators to compare them theoretically according to MSE, as follows:

$$MSE(\hat{\theta}_{MLE}) = MSE_{\theta}(\tfrac{n}{P}) = E\left[\left(\tfrac{n}{P} - \theta\right)^2\right]$$

$$MSE(\hat{\theta}_{MLE}) = var\left(\frac{n}{P}\right) + \left[E\left(\frac{n}{P}\right) - \theta\right]^2 \qquad (10)$$

$$var\left(\frac{n}{P}\right) = n^2 var\left(\frac{1}{P}\right)$$

$$var\left(\frac{1}{P}\right) = E\left[\left(\frac{1}{P}\right)^2\right] - \left[E\left(\frac{1}{P}\right)\right]^2 = \frac{\theta^2}{(n-1)^2(n-2)} \qquad (11)$$

$$\left[E\left(\frac{n}{P}\right) - \theta\right]^2 = \left[E\left(\frac{n}{P}\right)\right]^2 + 2\theta E\left(\frac{n}{P}\right) + \theta^2$$

$$\left[E\left(\frac{n}{P}\right) - \theta\right]^2 = \frac{\theta^2}{(n-1)^2} \qquad (12)$$

Substituting (11) and (12) into (10) gives

$$MSE(\hat{\theta}_{MLE}) = \frac{\theta^2(n+2)}{(n-1)(n-2)} \qquad (13)$$

Hence, MSE for UMVUE and MinMSE are obtained by the same way, as follows:

$$MSE(\hat{\theta}_{UMVUE}) = E\left[\left(\frac{n-1}{P} - \theta\right)^2\right]$$

$$= (n-1)^2 var\left(\tfrac{1}{P}\right) + \left[E\left(\tfrac{n-1}{P}\right) - \theta\right]^2 \qquad (14)$$

Recall that,

$var\left(\frac{1}{P}\right) = \frac{\theta^2}{(n-1)^2(n-2)}$ and $\left(\frac{n-1}{P}\right)$ is unbiased estimator for θ,

Then,

$$MSE(\hat{\theta}_{UMVUE}) = \frac{(n-1)^2\theta^2}{(n-1)^2(n-2)} + [\theta - \theta]^2 \qquad (15)$$

$$MSE(\hat{\theta}_{UMVUE}) = \frac{\theta^2}{(n-2)} \qquad (16)$$

$$MSE(\hat{\theta}_{MinMSE}) = E\left[\left(\frac{n-2}{P} - \theta\right)^2\right] = (n-2)^2 var\left(\frac{1}{P}\right) + \left[E\left(\frac{n-2}{P}\right) - \theta\right]^2 \qquad (17)$$

$$MSE(\hat{\theta}_{MinMSE}) = \frac{(n-2)\theta^2}{(n-1)^2} + \frac{\theta^2}{(n-1)^2} = \frac{\theta^2}{(n-1)} \qquad (18)$$

From (13), (16), (18), we find that:

$MSE(\hat{\theta}_{MinMSE}) \leq MSE(\hat{\theta}_{UMVUE}) \leq MSE(\hat{\theta}_{MLE})$

Now, we can say that, Minimum Mean Squared Error (MinMSE) is the best estimator among the Maximum

Likelihood Estimator (MLE), and the Uniformly Minimum Variance Unbiased Estimator (UMVUE), while the Maximum Likelihood Estimator is the worse among these three estimators.

3. Standard Bayes Estimator

In this section, we used a two loss functions as following:

a) Generalized Square Error Loss Function (GS).[11]

Al-Nasser and Saleh (2006) suggested the Generalized Square Error Loss Function in estimating the scale parameter and the Reliability function for Weibull distribution, which introduced as follows:

$$L_1(\theta,\hat{\theta}) = (\sum_{j=0}^{k} a_j\theta^j)(\hat{\theta}-\theta)^2 \quad ,k = 0,1,2,3,\dots. \qquad (19)$$

Where a_j , $j = 0,1,2,3,\dots.k$ is a constant.

b) Quadratic Loss Function (QLF).[3]

DeGroot (1970) discussed different types of loss functions and obtained the Bayes estimates under these loss functions. He proposed the Quadratic loss function which is asymmetric loss function defined for the positive values of the parameter. If $\hat{\theta}$ is an Quadratic Loss Function $L_2(\theta,\hat{\theta})$ will be,

$$L_2(\theta,\hat{\theta}) = \left(\frac{\theta-\hat{\theta}}{\theta}\right)^2 \qquad (20)$$

4. Prior and Posterior Distribution:

In this study we consider informative as well as non-informative prior:

4.1 Bayes Estimator Using Jeffery Prior Information [5]:

Let us assume that θ has non-informative prior density defined as using Jeffrey prior information $g(\theta)$ which given by:

$g_1(\theta) \propto \sqrt{I(\theta)}$

Where $I(\theta)$ represented Fisher information which defined as follows:

$$I(\theta) = -nE\left(\frac{\partial^2 \ln f}{\partial\theta^2}\right)$$

Hence,

$$g_1(\theta) = b\sqrt{-nE\left(\frac{\partial^2 \ln f(t;\theta)}{\partial\theta^2}\right)} \qquad (21)$$

$$\ln f(t;\theta) = \ln\theta + \theta\ln\alpha - (\theta+1)\ln t$$

$$\frac{\partial \ln f}{\partial\theta} = \frac{1}{\theta} + \ln\alpha - \ln t$$

$$\frac{\partial^2 \ln f}{\partial\theta^2} = -\frac{1}{\theta^2}$$

Hence, we get:

$$E\left(\frac{\partial^2 \ln f(t;\theta)}{\partial\theta^2}\right) = -\frac{1}{\theta^2}$$

After substitution into (21) we find that:

$$g_1(\theta) = \frac{b}{\theta}\sqrt{n} \qquad \theta > 0 \qquad (22)$$

The posterior density function is:

$$h_1(\theta|t_1,\dots,t_n) = \frac{g_1(\theta)L(\theta;t_1,\dots,t_n)}{\int_0^\infty g_1(\theta)L(\theta;t_1,\dots,t_n)\,d\theta} \qquad (23)$$

$$h_1(\theta|t,\dots,t_n) = \frac{\theta^n e^{n\theta\ln\alpha} e^{-(\theta+1)\sum\ln x}\frac{c}{\theta}\sqrt{n}}{\int_0^\infty \theta^n e^{n\theta\ln\alpha} e^{-(\theta+1)\sum\ln x}\frac{c}{\theta}\sqrt{n}\,d\theta} \quad \text{, where } \alpha \text{ constant}$$

$$= \frac{\theta^{n-1}e^{-\theta P}}{\int_0^\infty \theta^{n-1}e^{-\theta P}d\theta}$$

Hence, the posterior density function of θ with Jeffery prior is:

$$h_1(\theta|t,\dots,t_n) = \frac{P^n\theta^{n-1}e^{-\theta P}}{\Gamma(n)} \qquad (24)$$

The posterior density is recognized as the density of the Gamma distribution:

$\theta \sim \text{Gamma}\ (n, P)$.

With:

$$E(\theta) = \frac{n}{P} \quad , \quad ver(\theta) = \frac{n}{P^2}$$

To obtain Bayes estimator under Generalized Square Error Loss Function (GS) with Jeffery prior, Recall that, the Generalized Square Error Loss Function (GS) is:

$$l_1(\hat\theta,\theta) = (\sum_{j=0}^{k} a_j\theta^j)(\hat\theta-\theta)^2 \quad , k = 0,1,2,3,\dots.$$

$$l_1(\hat\theta,\theta) = (a_0 + a_1\theta + \cdots + a_k\theta^k)(\hat\theta-\theta)^2 \qquad (25)$$

Then the Risk function under the Generalized Square Error Loss Function is denoted by $R_{GS}(\hat\theta,\theta)$is:

$$R_{GS}(\hat\theta,\theta) = E[l_1(\hat\theta,\theta)]$$

$$= \int_0^\infty l_1(\hat\theta,\theta)h_1(\theta|\underline{t})d\theta$$

$$= \int_0^\infty (a_0 + a_1\theta + \cdots + a_k\theta^k)(\hat\theta^2 - 2\hat\theta\theta + \theta^2)h_1(\theta|\underline{t})d\theta$$

$$= a_0\hat\theta^2 - 2a_0\hat\theta E(\theta|\underline{t}) + a_0E(\theta^2|\underline{t}) + a_1\hat\theta^2E(\theta|\underline{t}) - 2a_1\hat\theta E(\theta^2|\underline{t}) + a_1E(\theta^3|\underline{t}) + \cdots + a_k\hat\theta^2E(\theta^k|\underline{t}) - 2a_k\hat\theta E(\theta^{k+1}|\underline{t}) + a_kE(\theta^{k+2}|\underline{t})$$

Taking the partial derivative for $R_{GS}(\hat\theta,\theta)$ with respect to $\hat\theta$ and setting it equal to zero yields

$$\hat\theta = \frac{a_0E(\theta|\underline{t}) + a_1E(\theta^2|\underline{t}) + \cdots + a_kE(\theta^{k+1}|\underline{t})}{a_0 + a_1E(\theta|\underline{t}) + \cdots + a_kE(\theta^k|\underline{t})} \qquad (26)$$

Since $\theta \sim \Gamma(n,P)$ and $E(\theta) = \frac{n}{P}$, $var(\theta) = \frac{n}{P^2}$ .Then,

$$\hat\theta = \frac{a_0\frac{n}{P} + a_1\frac{(n+1)n}{P^2} + \cdots + a_k\frac{(n+k)(n+k-1)\dots..(n+1)n}{P^{k+1}}}{a_0 + a_1\frac{n}{P} + \cdots + a_k\frac{(n+k-1)(n+k-2)\dots..(n+1)n}{P^k}}$$

Therefore, the Bayes estimator for θ of Pareto distribution under Generalized square error loss function with Jeffery prior denoted by $\hat\theta_{JGS}$is:

$$\hat\theta_{JGS} = \frac{\sum_{j=0}^{k} a_j\frac{\Gamma(n+1+j)}{P^{j+1}\Gamma(n)}}{\sum_{j=0}^{k} a_i\frac{\Gamma(n+j)}{P^j\Gamma(n)}} \qquad (27)$$

Now, we derive Bayes estimator using Quadratic Loss function, where

$$l_2(\hat{\theta},\theta) = \left(\frac{\theta-\hat{\theta}}{\theta}\right)^2 = \left(1-\frac{\hat{\theta}}{\theta}\right)^2$$

The Risk function under the Quadratic Loss function is denoted by $R_Q(\hat{\theta},\theta)$:

$$R_Q(\hat{\theta},\theta) = E(1-\frac{\hat{\theta}}{\theta})^2 = \int_0^\infty (1-\frac{\hat{\theta}}{\theta})^2 h_1(\theta|\underline{t})d\theta$$

$$\frac{\partial R_Q(\hat{\theta},\theta)}{\partial\hat{\theta}} = 2\int_0^\infty (1-\frac{\hat{\theta}}{\theta})\left(-\frac{1}{\theta}\right)h_1(\theta|\underline{t})d\theta$$

Let: $\frac{\partial R_Q(\hat{\theta},\theta)}{\partial\hat{\theta}} = 0$

$$\Rightarrow \hat{\theta}\int_0^\infty \frac{1}{\theta^2}h_1(\theta|\underline{t})d\theta - \int_0^\infty \frac{1}{\theta}h_1(\theta|\underline{t})d\theta = 0$$

Hence,

$$\hat{\theta} = \frac{E(\frac{1}{\theta})}{E(\frac{1}{\theta^2})} \tag{28}$$

$$E\left(\frac{1}{\theta}\right) = \int_0^\infty \frac{P^n\theta^{n-2}e^{-\theta P}d\theta}{\Gamma(n)} = \frac{P}{(n-1)} \tag{29}$$

$$E\left(\frac{1}{\theta^2}\right) = \int_0^\infty \frac{P^n\theta^{n-3}e^{-\theta P}d\theta}{\Gamma(n)} = \frac{P^2}{(n-1)(n-2)} \tag{30}$$

Substituting (29) and (30) into (28), gives

$$\hat{\theta}_{JQ} = \frac{\frac{P}{(n-1)}}{\frac{P^2}{(n-1)(n-2)}} = \frac{n-2}{P}$$

It is obvious that, Bayes estimator under Quadratic loss function with Jeffery prior is equivalent to the MinMSE estimator.

4.2 Bayes Estimator under Exponential Prior Distribution

Assuming that θ has informative prior as Exponential prior which takes the following form:

$$g_2(\theta) = \frac{1}{\lambda}e^{-\frac{\theta}{\lambda}} \quad , \ \theta,\lambda > 0 \tag{31}$$

Since, the posterior distribution of θ is:

$$h_2(\theta|\underline{t}) = \frac{L(t_1,\dots,t_n|\theta)g_2(\theta)}{\int_0^\infty L(t_1,\dots,t_n|\theta)g_2(\theta)d\theta} \tag{32}$$

$$h_2(\theta|\underline{t}) = \frac{\theta^n e^{-(\theta+1)P}\frac{1}{\lambda}e^{-\frac{\theta}{\lambda}}}{\int_0^\infty \theta^n e^{-(\theta+1)P}\frac{1}{\lambda}e^{-\frac{\theta}{\lambda}}d\theta}$$

$$h_2(\theta|\underline{t}) = \frac{(P+\frac{1}{\lambda})^{n+1}\theta^n e^{-\theta(P+\frac{1}{\lambda})}}{\Gamma(n+1)} \tag{33}$$

To obtain the Bayes Estimator using Generalized Square Error Loss Function (GS), recall that

$$\hat{\theta} = \frac{a_0 E(\theta|\underline{t}) + a_1 E(\theta^2|\underline{t}) + \cdots + a_k E(\theta^{k+1}|\underline{t})}{a_0 + a_1 E(\theta|\underline{t}) + \cdots + a_k E(\theta^k|\underline{t})}$$

Since $\theta \sim \Gamma\left(n+1, P+\frac{1}{\lambda}\right)$ and $E(\theta) = \frac{n+1}{P+\frac{1}{\lambda}}$, $var(\theta) = \frac{n+1}{\left(P+\frac{1}{\lambda}\right)^2}$

$$E(\theta^k) = \frac{(n+k)(n+k-1)\ldots.(n+1)}{\left(P+\frac{1}{\lambda}\right)^K} \quad (34)$$

After substituting, we get

$$\hat{\theta} = \frac{a_0\left\{\frac{n+1}{P+\frac{1}{\lambda}}\right\} + a_1 \frac{(n+2)(n+1)}{\left(P+\frac{1}{\lambda}\right)^2} + \cdots + a_k \frac{(n+k)(n+k-1)\ldots..(n+1)}{(P+\frac{1}{\lambda})^{k+1}}}{a_0 + a_1\left\{\frac{n+1}{P+\frac{1}{\lambda}}\right\} + \cdots + a_k \frac{(n+k)(n+k-1)\ldots.(n+1)}{(P+\frac{1}{\lambda})^k}}$$

So, the Bayes estimator of θ under Generalized Square Error loss function denoted by $\hat{\theta}_{EGS}$ is:

$$\hat{\theta}_{EGS} = \frac{\sum_{j=0}^{k} a_j \frac{\Gamma(n+2+j)}{(P+\frac{1}{\lambda})^{j+1}\,\Gamma(n+1)}}{\sum_{j=0}^{k} a_j \frac{\Gamma(n+1+j)}{(P+\frac{1}{\lambda})^{j}\,\Gamma(n+1)}} \quad (35)$$

To obtain the Bayes Estimator using Quadratic loss function (GS) with Exponential prior,

$$E\left(\frac{1}{\theta}\right) = \int_0^\infty \frac{1}{\theta} h_2(\theta|\underline{t})d\theta$$

$$E\left(\frac{1}{\theta}\right) = \frac{(P+\frac{1}{\lambda})}{n}\int_0^\infty \frac{(P+\frac{1}{\lambda})^n \theta^{n-1} e^{-\theta(P+\frac{1}{\lambda})} d\theta}{\Gamma(n)}$$

$$E\left(\frac{1}{\theta}\right) = \frac{(P+\frac{1}{\lambda})}{n} \quad (36)$$

$$E\left(\frac{1}{\theta^2}\right) = \int_0^\infty \frac{1}{\theta^2} h_2(\theta|\underline{t})d\theta \quad (37)$$

Substituting (2-108) in (2-131) we get

$$E\left(\frac{1}{\theta^2}\right) = \frac{(P+\frac{1}{\lambda})^2}{n(n-1)}\int_0^\infty \frac{(P+\frac{1}{\lambda})^{n-1} \theta^{n-2} e^{-\theta(P+\frac{1}{\lambda})} d\theta}{\Gamma(n-1)}$$

$$E\left(\frac{1}{\theta^2}\right) = \frac{(P+\frac{1}{\lambda})^2}{n(n-1)}$$

Substituting into (28), we get

$$\hat{\theta} = \frac{\frac{(P + \frac{1}{\lambda})}{n}}{\frac{(P + \frac{1}{\lambda})^2}{n(n-1)}}$$

So, the Bayes estimator of θ under Quadratic loss function, denoted by $\hat{\theta}_{EQ}$ is:

$$\hat{\theta}_{EQ} = \frac{n-1}{(P + \frac{1}{\lambda})} \quad (38)$$

5. Simulation study

In our simulation study, we generated I = 2500 samples of size n = 20, 50, and100 from Pareto distribution to represent small, moderate and large sample size with the several values of shape parameter, θ =0.5, 1.5 and 2.5. We chose two values of λ for the Exponential prior (λ =0.5, 3).

In this section, Monte-Carlo simulation study is performed to compare the methods of estimation using mean square Errors (MSE's), where

$$MSE(\hat{\theta}) = \frac{\sum_{i=1}^{I} (\hat{\theta}_i - \theta)^2}{I}$$

The results of the simulation study for estimating the shape parameter (θ) of Pareto distribution when the scale parameter (α) is known, are summarized and tabulated in tables (1), (2) and (3) which contain the Expected values and MSE's for estimating the shape parameter, and we have observed that:

1. Table (1), shows that, the performance of Bayes estimator under Quadratic loss function with exponential prior (λ=0. 5) is the best estimator comparing to other estimators for all sample sizes.
2. From table (2), we notice the performance of Bayes estimator under Generalized Square error loss function (k=1) with exponential prior (λ=0. 5), then Bayes estimator under Generalized Square (k=2) error loss function with exponential prior (λ=0. 5).
3. Tables (3), shows that, the performance of Bayes estimator under Generalized Square error loss function (k=2) with exponential prior (λ=0. 5) is the best estimator comparing to other estimators for all sample sizes. Followed by Bayes estimator under Generalized Square error loss function (k=1) with exponential prior (λ=0.5), for all sample sizes.
4. It is observed that, MSE's of all estimators of shape parameter is increasing with the increase of the shape parameter value with all sample sizes.

5. In general, we conclude that, in situation involving estimation of parameter of Pareto type I distribution under different loss functions using exponential prior distribution with small value of $\lambda(\lambda = 0.5)$ is more appropriate than each of using Jeffery prior distribution or large relatively, value of $\lambda(\lambda = 3)$ for all sizes of samples.
6. Finally for all parameter values, an obvious reduction in MSE is observed with the increase in sample size.

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Table (1): The Expected Values and (MSE) of the Different Estimators for Pareto Distribution where $\theta = 0.5$

Estimator		Criteria \ n	20	50	100
MinMSE		EXP.	0.4763792	0.4896486	0.4950124
		MSE	0.0136495	0.0052249	0.0025757
BJ(GS1), K=1		EXP.	0.5473866	0.5169127	0.5084979
		MSE	0.0199013	0.0060398	0.0027758
BJ(GS2), K=2		EXP.	0.5602370	0.5216054	0.5107820
		MSE	0.0226894	0.0063997	0.0028616
BJ(Qu.)		EXP.	0.4763792	0.4896486	0.4950124
		MSE	0.0136495	0.0052249	0.0025757
BE(GS1) K=1	$\lambda = 0.5$	EXP.	0.5434793	0.5163561	0.5083599
		MSE	0.0173370	0.0057662	0.0027163
	$\lambda = 3$	EXP.	0.5687681	0.525326	0.5126857
		MSE	0.0233826	0.0065398	0.0028995
BE(GS2) K=2	$\lambda = 0.5$	EXP.	0.5555686	0.5209476	0.5106209
		MSE	0.0196999	0.0061053	0.0027996
	$\lambda = 3$	EXP.	0.5818041	0.5300539	0.5148900
		EXP.	0.0267770	0.0069828	0.0030050
BE(Qu.)	$\lambda = 0.5$	MSE	0.4762662	0.4896519	0.4950116
		EXP.	0.0121879	0.0050095	0.0025232
	$\lambda = 3$	MSE	0.4981992	0.4981209	0.4992151
		EXP.	0.0140328	0.0052617	0.0025859
Best Estimator			BE(Qu.) $\lambda = 0.5$	BE(Qu.) $\lambda = 0.5$	BE(Qu.) $\lambda = 0.5$

Table (2): The Expected Values and (MSE) of the Different Estimators for Pareto Distribution where $\theta = 1.5$

Estimator		Criteria \ n	20	50	100
MinMSE		EXP.	1.4291400	1.4689480	1.485038
		MSE	0.1228453	0.0470273	0.023181
BJ(GS1), K=1		EXP.	1.6565990	1.5564700	1.528356
		MSE	0.1846281	0.0551474	0.0251786
BJ(GS2), K=2		EXP.	1.7183760	1.5798090	1.539836
		MSE	0.2230434	0.0603243	0.026431
BJ(Qu.)		EXP.	1.4291400	1.4689480	1.485038
		MSE	0.1228453	0.0470273	0.023181
BE(GS1) K=1	$\lambda = 0.5$	EXP.	1.4872670	1.4936660	1.497626
		MSE	0.0928410	0.0422249	0.0220095
	$\lambda = 3$	EXP.	1.6889840	1.5707460	1.535681
		MSE	0.1922915	0.0567820	0.0256258
BE(GS2) K=2	$\lambda = 0.5$	EXP.	1.5387000	1.5153790	1.504703
		MSE	0.1025327	0.0440064	0.0224988
	$\lambda = 3$	EXP.	1.7493540	1.5938970	1.547116
		MSE	0.2329843	0.0625458	0.0270325
BE(Qu.)	$\lambda = 0.5$	EXP.	1.2931650	1.4114340	1.455629
		MSE	0.1120421	0.0453318	0.0227055
	$\lambda = 3$	EXP.	1.4675230	1.4840920	1.492576
		MSE	0.1180203	0.0462615	0.0230052
Best Estimator			BE(GS1) $\lambda = 0.5$	BE(GS1) $\lambda = 0.5$	BE(GS1) $\lambda = 0.5$

Table (3): The Expected Values and (MSE) of the Different Estimators for Pareto Distribution where $\theta = 3.5$

Estimator		Criteria \ n	20	50	100
MinMSE		EXP.	2.3818980	2.4482490	2.4750620
		MSE	0.3412374	0.1306231	0.0643913
BJ(GS1), K=1		EXP.	2.7675090	2.5967140	2.5485580
		MSE	0.5167389	0.1537475	0.0700807
BJ(GS2), K=2		EXP.	2.8814260	2.6400880	2.5699490
		MSE	0.6337590	0.1696799	0.0739489
BJ(Qu.)		EXP.	2.3818980	2.4482490	2.4750620
		MSE	0.3412374	0.1306231	0.0643913
BE(GS1) K=1	$\lambda = 0.5$	EXP.	2.2701560	2.3980450	2.4488160
		MSE	0.2311243	0.1106840	0.0590988
	$\lambda = 3$	EXP.	2.7709360	2.6025570	2.5521110
		MSE	0.4773680	0.1504353	0.0694583
BE(GS2) K=2	$\lambda = 0.5$	EXP.	2.3564580	2.4367970	2.4690260
		MSE	0.2152809	0.1081323	0.0585790
	$\lambda = 3$	EXP.	2.8797880	2.6451980	2.5733250
		MSE	0.5854833	0.1663805	0.0734203
BE(Qu.)	$\lambda = 0.5$	EXP.	1.9697860	2.2638660	2.3789650
		MSE	0.4142413	0.1448410	0.0678637
	$\lambda = 3$	EXP.	2.4024360	2.4566090	2.4792320
		MSE	0.3110141	0.1261586	0.0633155
Best Estimator			BE(GS2)	BE(GS2)	BE(GS2)

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