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BAYES SEQUENTIAL ESTIMATION OF A POISSON RATE: A DISCRETE TIME APPROACH[1]

BY BRADLEY NOVIC
Carnegie-Mellon University

This paper provides explicit solutions to the problem of estimating the arrival rate λ of a Poisson process using a Bayes sequential approach. The loss associated with estimating λ by d is assumed to be of the form $(\lambda - d)^2\lambda^{-p}$ and the cost of observation includes both a time cost and an event cost. A discrete time approach is taken in which decisions are made at the end of time intervals having length t. Limits of the procedures as t approaches zero are discussed and related to the continuous time Bayes sequential procedure.

1. Introduction. Suppose that one observes a continuous-time Poisson process in order to estimate its arrival rate λ using a Bayes sequential approach. The observation cost is assumed to be c_1 per unit time and c_2 per event observed where c_1 and c_2 are nonnegative constants. The loss associated with estimating λ by d is assumed to be of the form $L(\lambda, d) = (\lambda - d)^2\lambda^{-p}$ where we will discuss values of p in the interval $0 \leqslant p \leqslant 3$. Thus the total loss resulting from estimating λ by d after having observed the process for time t during which X_t events occurred is $L(\lambda, d) + c_1 t + c_2 X_t$. Prior information about λ is assumed to be represented by a gamma distribution.

In this paper a discrete time approach is considered in which the process is observed continuously but decisions are made at the end of time intervals having length t. The optimal Bayes sequential decision procedure for estimating λ is determined explicitly. Furthermore, the solutions will be presented in a simple form which makes them appealing for practical applications. This approach contrasts with one taken by Shapiro and Wardrop [6] in which decisions can only be made at the time an event occurs. El-Sayyad and Freeman [4] have also attacked the problem using both approaches and various loss and cost functions. Shapiro and Wardrop [7] have also solved the problem in continuous time using the notion of "monotone case" for continuous time problems and employing Dynkin's identity.

If the prior gamma distribution over λ has density function $g(\lambda) = \Gamma(\alpha)^{-1}\beta^{\alpha}\lambda^{\alpha-1}e^{-\beta\lambda}$ and one observes X_t events in time t, the posterior distribution at time t will also be a gamma distribution with parameters $(\alpha + X_t, \beta + t)$. It will be convenient then to represent the results of experimentation by a plot of the posterior parameters. The optimal stopping region will be determined using backward induction; see, e.g., DeGroot (1970) or Chow, Robbins and Siegmund (1971).

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2. Observation cost proportional to observation time ($c_2 = 0$). Define $\rho_0(\alpha, \beta)$ as the risk, or minimum expected loss, of estimating λ without any observation of the process when (α, β) are the parameters of the prior distribution. Define $\rho_i(\alpha, \beta, t)$ as the risk, or minimum expected total loss, of the optimal procedure when i sampling intervals of length t are available. Also define $\rho^*(\alpha, \beta, t)$ as the risk of the optimal procedure in the infinite horizon case. Then

$$(2.1) \qquad \rho_1(\alpha, \beta, t) = \min\{\rho_0(\alpha, \beta), E[\rho_0(\alpha + X_t, \beta + t)] + c_1 t\} \qquad \text{and}$$
$$\rho_i(\alpha, \beta, t) = \min\{\rho_0(\alpha, \beta), E[\rho_{i-1}(\alpha + X_t, \beta + t, t)] + c_1 t\}$$

where the expected value is taken with respect to the marginal distribution of X_t. The functions ρ_i determine the optimal procedure in a finite horizon problem. The role they play here will be their use in determining the nature of the function $\rho^*(\alpha, \beta, t)$. Since $\rho^*(\alpha, \beta, t) = \min\{\rho_0(\alpha, \beta), \mathrm{E}[\rho^*(\alpha + X_t, \beta + t, t)] + c_1 t\}$, the optimal stopping region in the (α, β) plane is $\{(\alpha, \beta)|\rho^*(\alpha, \beta, t) = \rho_0(\alpha, \beta)\}$ and the continuation region is $\{(\alpha, \beta)|\rho^*(\alpha, \beta, t) < \rho_0(\alpha, \beta)\}$.

The stopping risk $\rho_0(\alpha, \beta)$ associated with the Bayes estimate $d = \beta^{-1}(\alpha - p)$ is given by $\rho_0(\alpha, \beta) = \beta^{P-2}\Gamma(\alpha - p + 1)/\Gamma(\alpha)$, where we assume $\alpha > p$. By (2.1), $\rho_1(\alpha, \beta, t) = \min\{\rho_0(\alpha, \beta), \rho_0(\alpha, \beta) - [t\beta^{p-2}\Gamma(\alpha - p + 1)/(\beta + t)\Gamma(\alpha) - c_1 t]\}$. Thus, the optimal procedure when at most one sampling period is available for observation is characterized by the following stopping region $R_{s,t}$ and continuation region $R_{c,t}$:

$$(2.2) \qquad R_{s,t} = \{(\alpha, \beta)|\Gamma(\alpha)/\Gamma(\alpha - p + 1) \geqslant \beta^{p-2}/c_1(\beta + t), \alpha > p, \beta > 0\}$$
$$R_{c,t} = \{(\alpha, \beta)|\Gamma(\alpha)/\Gamma(\alpha - p + 1) < \beta^{p-2}/c_1(\beta + t), \alpha > p, \beta > 0\}.$$

3. Characterization of the optimal stopping region for $1 \leqslant p < 3$; $c_2 = 0$. Note that for $1 \leqslant p \leqslant 2$, $\Gamma(\alpha)/\Gamma(\alpha - p + 1)$ is nondecreasing in α and $\beta^{p-2}/c_1(\beta + t)$ is decreasing in β. This is also true for $2 < p < 3$ if $\beta > t(p - 2)/(3 - p)$. In either case once the point (α, β) enters $R_{s,t}$, the posterior parameters $(\alpha + X_t, \beta + t)$ must remain there. Consider any point $(\alpha, \beta) \in R_{s,t}$. Then $\rho_1(\alpha, \beta, t) = \rho_0(\alpha, \beta)$ and $\rho_1(\alpha + X_t, \beta + t, t) = \rho_0(\alpha + X_t, \beta + t)$ for all t and X_t. Hence it follows from (2.1) by induction that $\rho_n(\alpha, \beta, t) = \rho_0(\alpha, \beta)$ for any n and $(\alpha, \beta) \in R_{s,t}$. Furthermore, it follows from [3], page 296, that $\lim_{n\to\infty}\rho_n(\alpha, \beta, t) = \rho^*(\alpha, \beta, t)$. Hence, $\rho^*(\alpha, \beta, t) = \rho_0(\alpha, \beta)$ for $(\alpha, \beta) \in R_{s,t}$ and $\rho^*(\alpha, \beta, t) < \rho_0(\alpha, \beta)$ in $R_{c,t}$. This proves that $R_{s,t}$ and $R_{c,t}$ are the optimal stopping and continuation regions for the infinite horizon problem when $1 \leqslant p < 3$.

One can partition the continuation region $R_{c,t}$ into a number of disjoint regions such that in each region the optimal procedure is explicitly described. The kth region is defined as:

$$(3.1) \qquad R_k = \{(\alpha, \beta)|(\beta + kt)^{p-2}/c_1[\beta + (k + 1)t] \leqslant \Gamma(\alpha)/\Gamma(\alpha - p + 1)$$
$$< [\beta + (k - 1)t]^{p-2}/c_1(\beta + kt), \alpha > p, \beta > 0\}.$$

The optimal procedure in the kth region is to observe the process for at least one more period. No more than k periods will ever be needed, however, since the posterior parameters must lie in $R_{s,t}$ after k further sampling periods. Since the prior parameters must belong to one of the above regions for some $k = k_0 < \infty$, there is an upper bound $k_0 t$ on the total observation time.

Note that the optimal procedure was derived by realizing that for certain values of p monotonicity arises and the myopic rule is optimal. When $c_2 = 0$ we only find this to be true for $1 \leqslant p < 3$, and for the case in which $2 < p < 3$ we also need to choose t such that the prior parameter β is greater than $t(p-2)/(3-p)$. If this condition is not met, one can partition $R_{s,t}$ into $R_1 \cup R_2 \cup R_3$ where $R_1 = \{(\alpha, \beta) | \Gamma(\alpha)/\Gamma(\alpha - p + 1) \geqslant \beta^{p-3}/c_1, \alpha > p, \beta > 0\}$, $R_2 = \{(\alpha, \beta) | \beta^{p-2}/c_1(\beta + t) \leqslant \Gamma(\alpha)/\Gamma(\alpha - p + 1) < \beta^{p-3}/c_1, \alpha > p, \beta > t(p-2)/(3-p)\}$, and $R_3 = \{(\alpha, \beta) | \beta^{p-2}/c_1(\beta + t) \leqslant \Gamma(\alpha)/\Gamma(\alpha - p + 1) < \beta^{p-3}/c_1, \alpha > p, \beta \leqslant t(p-2)/(3-p)\}$. In $R_1 \cup R_2$ it is optimal to stop sampling and in $R_{c,t}$ one should continue. The optimal procedure in R_3 is unknown, however.

Note that when $p = 1$ the optimal stopping region becomes

$$R_{s,t} = \left\{(\alpha, \beta) | \beta \geqslant \tfrac{1}{2}(t^2 + 4/c_1)^{\frac{1}{2}} - \tfrac{t}{2}, \alpha > 1\right\}.$$

If (α_0, β_0) are the parameters of the prior distribution, the optimal procedure is a fixed time procedure in which one samples for exactly $\tau = \max\{0, [\![\tfrac{1}{2}(t^2 + 4/c_1)^{\frac{1}{2}} + \tfrac{t}{2} - \beta_0]\!]\}$ units of time, where $[\![\cdot]\!]$ is the greatest integer function. Note when $t = 1$ (ordinary Poisson sampling), that the optimal procedure is equivalent to taking a sequential random sample of exactly $\tau = \max\{0, [\![\tfrac{1}{2}(1 + 4/c_1) + \tfrac{1}{2} - \beta_0]\!]\}$ Poisson random variables.

When $p = 2$ the optimal stopping region becomes

$$R_{s,t} = \left\{(\alpha, \beta) | \beta \geqslant \frac{1}{c_1(\alpha - 1)} - t, \alpha > 2, \beta > 0\right\}.$$

The simple form of the boundary is appealing for practical applications.

4. The optimal procedure for $c_2 > 0$. The cost function for observation time t has the form $c(X_t) = c_1 t + c_2 X_t$. Thus $\rho_1(\alpha, \beta, t) = \min\{\rho_0(\alpha, \beta), \mathrm{E}[\rho_0(\alpha + X_t, \beta + t) + c(X_t)]\} = \min\{\rho_0(\alpha, \beta), \rho_0(\alpha, \beta) - [t\beta^{p-2}\Gamma(\alpha - p + 1)/(\beta + t)\Gamma(\alpha) - c_1 t - c_2 t\alpha/\beta]\}$. Hence the optimal procedure when at most one sampling period is available is characterized by the following stopping region $\mathcal{C}_{s,t}$ and continuation region $\mathcal{C}_{c,t}$:

$$\mathcal{C}_{s,t} = \left\{(\alpha, \beta) | 1 - \frac{c_1\Gamma(\alpha)(\beta + t)}{\Gamma(\alpha - p + 1)\beta^{p-2}} - \frac{c_2\Gamma(\alpha + 1)(\beta + t)}{\Gamma(\alpha - p + 1)\beta^{p-1}} \leqslant 0, \alpha > p, \beta > 0\right\}$$

(4.1)

$$\mathcal{C}_{c,t} = \left\{(\alpha, \beta) | 1 - \frac{c_1\Gamma(\alpha)(\beta + t)}{\Gamma(\alpha - p + 1)\beta^{p-2}} - \frac{c_2\Gamma(\alpha + 1)(\beta + t)}{\Gamma(\alpha - p + 1)\beta^{p-1}} > 0, \alpha > p, \beta > 0\right\}.$$

Note that for $1 < p < 2$ and $\beta > t(p-1)/(2-p)$ the sum

$$\frac{c_1\Gamma(\alpha)(\beta+t)}{\Gamma(\alpha-p+1)\beta^{p-2}} + \frac{c_2\Gamma(\alpha+1)(\beta+t)}{\Gamma(\alpha-p+1)\beta^{p-1}}$$

is nondecreasing in α and β. It follows that once $\mathcal{C}_{s,t}$ is entered by posterior parameters it cannot be escaped. By the same arguments given for the case in which $c_2 = 0$, $\mathcal{C}_{s,t}$ is the optimal stopping region and $\mathcal{C}_{c,t}$ the optimal continuation region when $1 \leqslant p < 2$ and $\beta > t(p-1)/(2-p)$.

If we let $c_1 = 0$ the above arguments hold additionally for $0 \leqslant p \leqslant 1$ and all t. This follows by noting that

$$\frac{c_2\Gamma(\alpha+1)(\beta+t)}{\Gamma(\alpha-p+1)\beta^{p-1}}$$

is increasing in α and β for $0 \leqslant p \leqslant 1$. The optimal stopping and continuation regions in this case are

$$\mathfrak{D}_{s,t} = \left\{(\alpha,\beta)|1 - \frac{c_2\Gamma(\alpha+1)(\beta+t)}{\Gamma(\alpha-p+1)\beta^{p-1}} \leqslant 0, \alpha > p, \beta > 0\right\} \quad \text{and}$$

$$(4.2)$$

$$\mathfrak{D}_{c,t} = \left\{(\alpha,\beta)|1 - \frac{c_2\Gamma(\alpha+1)(\beta+t)}{\Gamma(\alpha-p+1)\beta^{p-1}} > 0, \alpha > p, \beta > 0\right\}.$$

5. Convergence to continuous time. One can use the discrete time approach as an approximation to a continuous time problem, the approximation improving as the length t of the interval is suitably decreased. In viewing continuous time as a limiting version of the discrete time problem as $t \to 0$, one obtains the stopping regions

$$R_s = \lim_{t\to 0} R_{s,t} = \{(\alpha,\beta)|\Gamma(\alpha)/\Gamma(\alpha-p+1)$$

$$(5.1) \qquad \geqslant \beta^{p-3}/c_1, \alpha > p, \beta > 0\},$$

$$\mathcal{C}_s = \lim_{t\to 0}\mathcal{C}_{s,t} = \{(\alpha,\beta)|1 - c_1\beta^{3-p}\Gamma(\alpha)/\Gamma(\alpha-p+1)$$

$$- c_2\beta^{2-p}\Gamma(\alpha+1)/\Gamma(\alpha-p+1) \leqslant 0, \alpha > p, \beta > 0\},$$

$$\mathfrak{D}_s = \lim_{t\to 0}\mathfrak{D}_{s,t} = \{(\alpha,\beta)|1 - c_2\beta^{2-p}\Gamma(\alpha+1)/\Gamma(\alpha-p+1)$$

$$\leqslant 0, \alpha > p, \beta > 0\},$$

which are the corresponding optimum boundaries derived by Shapiro and Wardrop in their continuous time approach. It can be shown that as $t \to 0$ the Bayes risk of the optimum discrete time procedure converges to the risk of the continuous time Bayes sequential procedure, a result needed to rigorously prove that the limit of the optimal discrete time solution yields the solution to the limiting problem.

When $p = 3$ and $c_2 = 0$, or when $p = 2$ and $c_2 > 0$, an exact solution to the discrete time problem was not obtainable. However, one can identify in each case a stopping region, a continuation region, and a region separating the above two in which the optimal procedure is unknown. It should be noted that as the length t of the sampling interval decreases, the size of the region within which the procedure is unknown decreases. Taking the limit as $t \to 0$ this region vanishes and one again obtains the optimum boundaries for the continuous time problem given by Shapiro and Wardrop.

When $p = 1$ and $c_2 = 0$ the optimal procedure is again a fixed time procedure with optimum stopping region

$$R_s = \left\{(\alpha, \beta) | \beta \geqslant (c_1)^{-\frac{1}{2}}, \alpha > 1\right\}.$$

If (α_0, β_0) is the prior parameter point, the optimal procedure in this case is to sample for exactly $\tau = \max\{0, (c_1)^{-\frac{1}{2}} - \beta_0\}$ units of time.

In contrast, the optimal procedure when $p = 3$ and $c_2 = o$ is an "inverse" sampling scheme with optimum stopping region.

$$R_s = \{(\alpha, \beta) | (\alpha - 1)(\alpha - 2) \geqslant 1/c_1, \alpha > 3, \beta > 0\}.$$

Hence, the optimal procedure in this case is to sample until exactly k events occur, where k is the smallest nonnegative integer such that $(\alpha_0 + k - 1)(\alpha_0 + k - 2) \geqslant 1/c_1$.

When $p = 2$ the optimal procedure is characterized by the stopping region

$$R_s = \left\{(\alpha, \beta) | \beta \geqslant \frac{1}{c_1(\alpha - 1)}, \alpha > 2, \beta > 0\right\}.$$

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Department of Statistics
Carnegie-Mellon University
Schenley Park
Pittsburgh, Pennsylvania 15213