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Abstract

This paper introduces a numerically-efficient technique based on the Bayesian Compressive Sampling (BCS) for the design of maximally-sparse linear arrays. The method is based on a probabilistic formulation of the array synthesis and it exploits a fast relevance vector machine (RVM) for the problem solution. The proposed approach allows the design of linear arrangements fitting desired power patterns with a reduced number of non-uniformly spaced active elements. The numerical validation assesses the effectiveness and computational efficiency of the proposed approach as a suitable complement to existing state-of-the-art techniques for the design of sparse arrays.

Key words: Array synthesis, sparse arrays, bayesian compressive sampling, relevance vector machine.

1 Introduction

Synthesizing antenna arrays with a minimum number of elements is a problem of high importance in those applications (e.g., satellite communications, radars, biomedical imaging, acoustics, and remote sensing) where the weight, the consumption, and the hardware/software complexity of the radiating device have a strong impact on the whole cost of the overall system [1][2].

Non-uniform arrangements have potential advantages with respect to uniform layouts [3] such as (*a*) significantly increased resolution (i.e, decreased mainlobe width) [4], (*b*) sidelobe level control/reduction [5], and (*c*) enhanced efficiency in dealing with physically-constrained geometries (e.g., conformal architectures) [6]. However, sparsening array elements has the main drawback of reducing the control of the beam shape [1]-[7] and several approaches for the design and optimization of sparse arrangements have been proposed in the last 50 years [1]-[30] to properly address such an issue.

Dealing with beam shape control, two different problems are usually considered in the state-of-the-art literature [20]: (*I*) the minimization of the peak sidelobe level (PSL) by determining a fixed set of N element positions over an aperture and sometimes the corresponding weights; (*II*) the synthesis of a maximally-sparse array(1) radiating a desired pattern. A wide set of methods concerned with *Problem I* [2] has been investigated including random approaches [11][15], dynamic programming [12], FIR-filter design [16], stochastic optimization methods [17][18][20][24][27][28], analytical techniques [22][30], and hybrid algorithms [25][29], as well. On the contrary, *Problem II* has received less attention and few methods have been developed [2][3][13][14][19][20][21][23][26]. Because of the limitations of available computers, first attempts relied on techniques requiring as few computational resources as possible such as the steepest descent method [13] and the iterative least-square technique [14]. However, those approaches have strong limitations as, for example, the need to *a-priori* know the number of active elements of the array and the aperture size [13][14]. In order to overcome these drawbacks, a technique exploiting the simplex search was developed in [3] to find the sparsest array match-

(1) An array with the minimum number of active elements, P, over a lattice (regular or irregular) of N positions.

ing a given reference pattern. Moreover, a mixed linear programming approach was introduced in [19] with the same aim. Further developments ranging from a recursive inversion algorithm based on the Legendre transform [21][26] up to the use of a stochastic optimizer based on the simulated annealing technique [20] or a generalized Gaussian quadrature approach [23] have been successively analyzed. More recently, *Problem II* has been solved by means of an innovative technique based on the Matrix Pencil Method (MPM) [7]. Thanks to its efficiency, the MPM generally outperforms other synthesis techniques in terms of convergence speed and array performances [7]. Despite its effectiveness, such an approach presents some limitations:

1. the locations d_p, $p = 1, ..., P$, of the P active elements of the array are proportional to the complex values of the non-zero roots of the generalized eigenvalue problem described in [7]. Consequently, *unphysical* complex solutions (i.e., $d_p \in \mathbb{C}$) can be generated [7] and an approximation [i.e., $d_p^{MPM} = \Re\left(d_p\right)$] is required (p. 2957 - [7]) whose impact on the array performances cannot be *a-priori* estimated nor neglected;

2. no requirements on the element positions [7] can be stated. Thus, no geometrical regularity or user-desired geometric features on the synthesized array can be *a-priori* enforced;

3. the method may fail in synthesizing/matching shaped beam patterns because of the imaginary parts of d_p, $p = 1, ..., P$ are not usually negligible (p. 2958 - [7]).

This paper is aimed at proposing an innovative, flexible, and computationally-efficient complement to the existing synthesis methods that solve *Problem II*. The method, based on the Bayesian Compressive Sampling (BCS) [31], is devoted to find the maximally-sparse array with the highest *a-posteriori* probability to match a user-defined reference pattern. Towards this end, an efficient BCS solver exploiting a fast relevance vector machine (RVM) algorithm [31] is adopted.

The outline of the paper is as follows. Section 2 is aimed at mathematically formulating the synthesis problem and describing an algorithm for minimizing a suitable cost function that depends on the degree of sparseness of the array and the mismatch between the desired power pattern and the actual one. Section 3 provides a selected set of numerical results to validate the proposed approach as well as to compare its performances with state-of-the-art techniques. Finally, some conclusions are drawn (Sect. 4).

2 Mathematical Formulation

2.1 BCS Formulation

Let us consider a symmetric linear arrangement of $M = 2 \times N - \chi$ ($\chi = 0$ if an even number of elements is at hand, $\chi = 1$ otherwise) isotropic elements, $w_n \in \mathbb{R}$ being the real excitation of the n-th element pair ($n = 1, ..., N$). The synthesis problem is that of finding the set of array weights such that (*a*) the radiated pattern is sufficiently close to a given reference one, $E_{REF}(u)$, and (*b*) the number P of *active* (i.e., $w_n = w_{-n} = \delta_{np} w_p$, $p = 1, .., P$, δ_{np} being the Kronecker function) array elements is as small as possible [3]. Towards this end, the BCS formulation is considered and similarly to [3] the following assumptions are taken into account: (*a*) the reference pattern is approximated in an arbitrary set of K angular positions u_k, $k = 1, ..., K$, within the visible range ($u_k \in [-1, 1]$); (*b*) the set of P active positions are constrained to a large, but finite, user-chosen set of M (i.e., $M \gg P$) candidate locations not necessarily belonging to a regular lattice. Mathematically, the problem can be formulated as follows

> ***Synthesis Problem*** **-** *Given a set of K samples of the reference pattern, $\mathbf{E}_{REF} \in \mathbb{R}^K$, and a fidelity factor ε find the set of array weights, $\mathbf{w}$, which is maximally sparse subject to $\|\mathbf{E}_{REF} - \mathbf{E}\|^2 \leq \varepsilon$*

where $\|\cdot\|$ is the ℓ_2-norm, $\mathbf{E}_{REF} \triangleq [E_{REF}(u_1), ..., E_{REF}(u_K)]^H$, $\mathbf{w} \triangleq [w_1, ..., w_N]^H$, $\mathbf{E} \triangleq [E(u_1), ..., E(u_K)]^H$ whose k-th entry is given by $E(u_k) = \sum_{n=1}^{N} \nu_n w_n cos\left[\frac{2\pi d_n u_k}{\lambda}\right]$, λ being the wavelength, d_n the distance of the n-th location from the array center ($d_1 = 0$ if $\chi = 1$), and ν_n is the Neumann's number [9] defined as $\nu_n = 2 - \chi$ if $n = 1$, and $\nu_n = 2$ otherwise.
The synthesized pattern samples $\mathbf{E}$ can be then expressed as

$$\mathbf{E} = \Psi \mathbf{w} \tag{1}$$

where $\Psi \in R^{K \times N}$ and its (k, n)-th element is given by $\psi(k, n) = \nu_n cos\left[\frac{2\pi d_n u_k}{\lambda}\right]$.

To recast the problem at hand as a BCS problem, the following three steps are necessary. Let us first rewrite the ℓ_2-norm constraint ($\left\|\mathbf{E}_{REF}-\mathbf{E}\right\|^2 \leq \varepsilon$) as[(2)] [34]

$$\mathbf{E}_{REF} - \Psi\mathbf{w} = \mathbf{e} \tag{2}$$

where $\mathbf{e} = [e_1, ..., e_K]^T$ is a zero mean Gaussian error vector [31][33][34] with an user-defined variance σ^2 proportional to the mismatching with the reference pattern (i.e., $\sigma^2 \propto \varepsilon$). Then, let us model $\mathbf{E}_{REF}$ through a Gaussian likelihood model

$$p\left(\mathbf{E}_{REF}\right|\left[\mathbf{w},\sigma^2\right]) = \frac{1}{\left(2\pi\sigma^2\right)^{\frac{K}{2}}} \exp\left(-\frac{1}{2\sigma^2}\left\|\mathbf{E}_{REF}-\Psi\mathbf{w}\right\|^2\right) \tag{3}$$

to recast the original problem as the following linear regression one with sparseness constraints ($LRSC$)

> ***LRSC Problem -*** *Given* $\mathbf{E}_{REF} \in \mathbb{R}^K$ *find* $\mathbf{w}$ *and* σ^2 *which maximize the a-posteriori probability* $p\left(\mathbf{w},\sigma^2\left|\mathbf{E}_{REF}\right.\right)$ *subject to the constraint that* $\mathbf{w}$ *is maximally-sparse*

Finally, the sparseness of $\mathbf{w}$ [33][34] is enforced. As regards the Bayesian formulation, such a task is accomplished by introducing a sparseness prior[(3)] over $\mathbf{w}$ [31]. Hereinafter, the Gaussian hierarchical prior [32][33][34] is invoked

$$p\left(\mathbf{w}\right|\mathbf{a}) = \frac{\prod_{n=1}^{N}\sqrt{a_n}\exp\left(-\frac{a_n w_n^2}{2}\right)}{\left(2\pi\right)^{\frac{N}{2}}} \tag{4}$$

where $\mathbf{a} \triangleq [a_1, ..., a_N]$ and a_n ($n = 1, ..., N$) is the n-th independent hyperparameter controlling the strength of the prior over w_n [32]. To fully specify (4), the hyperpriors over $\mathbf{a}$ [i.e., $p(\mathbf{a})$] and σ^2 [i.e., $p\left(\frac{1}{\sigma^2}\right)$] have to be defined. The Gamma distributions are here considered [32]

$$p(\mathbf{a}) = \prod_{n=1}^{N} G\left(a_n\right|\alpha_1,\alpha_2) \tag{5}$$

and

(2) It is worth pointing out that Eq. (2) and the ℓ_2-norm constraint are mathematically equivalent [34].

(3) In Bayesian inference, a *prior* represents the *a-priori* knowledge about an unknown quantity in probabilistic terms.

$$p\left(\frac{1}{\sigma^2}\right) = G\left(\left.\frac{1}{\sigma^2}\right|\alpha_3, \alpha_4\right) \tag{6}$$

where α_i ($i = 1, ..., 4$) is the i-th *scale prior*, $G\left(a_n\right|\alpha_1, \alpha_2) \triangleq \frac{\alpha_2^{\alpha_1} a_n^{\alpha_1 - 1} e^{-\alpha_2 a}}{\Gamma(\alpha_1)}$, and $\Gamma(\alpha_1) \triangleq \int_0^\infty t^{\alpha_1 - 1} e^{-t} \mathrm{d}t$ is the gamma function [32]. Thanks to (4), (5), and (6), the original synthesis problem can be finally formulated as

> ***BCS Problem*** **-** *Given* $\mathbf{E}_{REF} \in \mathbb{R}^K$*, find* $\mathbf{w}_{BCS}$*,* $\mathbf{a}_{BCS}$*, and* σ^2_{BCS} *which maximize* $p\left(\left[\mathbf{w}, \mathbf{a}, \sigma^2\right] |\mathbf{E}_{REF}\right)$.

2.2 BCS Solver - The RVM Procedure

In order to solve the *BCS Problem* by determining the unknown parameters $\mathbf{w}_{BCS}$, $\mathbf{a}_{BCS}$, and σ^2_{BCS}, the RVM method [32][31] is applied. Towards this end, let us consider that the posterior over all unknowns can be expressed as

$$p\left(\left[\mathbf{w}, \mathbf{a}, \sigma^2\right] |\mathbf{E}_{REF}\right) = p\left(\mathbf{w}\left|\left[\mathbf{E}_{REF}, \mathbf{a}, \sigma^2\right]\right.\right) p\left(\left[\mathbf{a}, \sigma^2\right] |\mathbf{E}_{REF}\right). \tag{7}$$

Moreover, because of (3) and (4), the posterior distribution over $\mathbf{w}$

$$p\left(\mathbf{w}\left|\left[\mathbf{E}_{REF}, \mathbf{a}, \sigma^2\right]\right.\right) = \frac{p\left(\mathbf{E}_{REF}\right|\left[\mathbf{w}, \sigma^2\right]) p\left(\mathbf{w}\right|\mathbf{a})}{p\left(\mathbf{E}_{REF}\right|\left[\mathbf{a}, \sigma^2\right])} \tag{8}$$

turns out to be equal to the following multivariate Gaussian distribution [34]

$$p\left(\mathbf{w}\left|\left[\mathbf{E}_{REF}, \mathbf{a}, \sigma^2\right]\right.\right) = \frac{1}{(2\pi)^{\frac{N+1}{2}} \sqrt{det\left(\Sigma\right)}} \exp\left\{-\frac{\left(\mathbf{w} - \mu\right)^H \left(\Sigma\right)^{-1} \left(\mathbf{w} - \mu\right)}{2}\right\} \tag{9}$$

where the posterior mean and the covariance are given by $\mu = \frac{\Sigma \Psi^H \mathbf{E}_{REF}}{\sigma^2}$ and $\Sigma = \left(\frac{\Psi^T \Psi}{\sigma^2} + A\right)^{-1}$, respectively, being $A \triangleq diag\left(a_1, ..., a_N\right)$.

As for the second term on the right-hand side of (7), the delta-function approximation is used [32] to model the hyperparameter posterior

$$p\left(\left[\mathbf{a}, \sigma^2\right] |\mathbf{E}_{REF}\right) \approx \delta\left(\mathbf{a}_{BCS}, \sigma^2_{BCS}\right) \tag{10}$$

where $\mathbf{a}_{BCS}$ and σ^2_{BCS} are the most probable values, $(\mathbf{a}_{BCS}, \sigma^2_{BCS}) = \arg\max_{\mathbf{a},\sigma^2} \{p\left([\mathbf{a}, \sigma^2] \,|\mathbf{E}_{REF}\right)\}$, also called hyperparameter posterior *modes*. In order to determine their values, let us consider that

$$p\left(\left[\mathbf{a}, \sigma^2\right] |\mathbf{E}_{REF}\right) \propto p\left(\mathbf{E}_{REF} \left| \left[\mathbf{a}, \sigma^2\right]\right.\right) p\left(\mathbf{a}\right) p\left(\sigma^2\right) \tag{11}$$

and let us assume uniform scale priors. Then, $p\left(\sigma^2\right)$ and $p\left(\mathbf{a}\right)$ become constant values [32] and the maximization of (11) is equivalent to maximize the term $p\left(\mathbf{E}_{REF}\right| \mathbf{a}, \sigma^2)$, whose logarithm is given by [32]

$$\mathcal{L}\left(\mathbf{a}, \sigma^2\right) \triangleq \log\left[p\left(\mathbf{E}_{REF}\right| \mathbf{a}, \sigma^2\right)\right] = -\frac{1}{2}\left[N \log 2\pi + \log|C| + \mathbf{E}^H_{REF} C^{-1} \mathbf{E}_{REF}\right] \tag{12}$$

where $C = \sigma^2 I + \Psi A^{-1} \Psi^T$. It is worthwhile to point out that it is not possible to perform the maximization of the “marginal likelihood” (12) in an exact fashion, but a *type-II maximum likelihood* procedure [34] can be profitably exploited for determining an iterative re-estimation of $(\mathbf{a}_{BCS}, \sigma^2_{BCS})$. Such a technique, whose Matlab implementation is available in [35], is summarized in the Appendix.

Finally, by substituting (9) and (10) in (7), one obtains that

$$p\left(\left[\mathbf{w}, \mathbf{a}, \sigma^2\right] |\mathbf{E}_{REF}\right) \approx p\left(\mathbf{w} \left| \left[\mathbf{E}_{REF}, \mathbf{a}, \sigma^2\right]\right.\right)\Big\rfloor_{(\mathbf{a},\sigma^2)=\left(\mathbf{a}_{BCS}, \sigma^2_{BCS}\right)}. \tag{13}$$

The posterior over all unknowns results a multivariate Gaussian function (9) only depending on the unknown set $\mathbf{w}$ once $(\mathbf{a}_{BCS}, \sigma^2_{BCS})$ have been determined. Therefore, the value of $\mathbf{w}_{BCS} = \arg\max_{\mathbf{w}} \{p\left([\mathbf{w}, \mathbf{a}, \sigma^2] \,|\mathbf{E}_{REF}\right)\}$ turns out to be equal to the posterior mean of $p\left(\mathbf{w}\,|[\mathbf{E}_{REF}, \mathbf{a}, \sigma^2]\right)\rfloor_{(\mathbf{a},\sigma^2)=\left(\mathbf{a}_{BCS}, \sigma^2_{BCS}\right)}$ given by

$$\mathbf{w}_{BCS} = \mu\rfloor_{(\mathbf{a},\sigma^2)=\left(\mathbf{a}_{BCS}, \sigma^2_{BCS}\right)}. \tag{14}$$

2.3 BSC Synthesis Method - Algorithmic Implementation

The algorithmic implementation of the BCS-based pattern synthesis consists of the following steps:

1. *Input Phase* - Set the reference pattern $E_{REF}(u)$, the grid of admissible locations (d_n; $n = 1, ..., N$), the set of pattern sampling points (u_k; $k = 1, ..., K$), the target variance σ^2 of the error term $\mathbf{e}$, and its *initial estimate* σ_0^2 for the sequential solver of the RVM algorithm (see the Appendix);

2. *Matrix Definition* - Fill the entries of the matrices $\mathbf{E}_{REF}$, Ψ, $\mathbf{e}$, and $\hat{\mathbf{E}}_{REF} = \mathbf{E}_{REF} + \mathbf{e}$;

3. *Hyperparameter Posterior Modes Estimation* - Find $(\mathbf{a}_{BCS}, \sigma^2_{BCS})$ by maximizing (12) as described in the Appendix;

4. *Array Weights Estimation* - Find $\mathbf{w}_{BCS}$ by (14);

5. *Output Phase* - Return the estimated array weights, $\mathbf{w}_{BCS}$, the number of active array elements, $P_{BCS} = -\chi + 2\left\|\mathbf{w}_{BCS}\right\|_0$[(4)], and the corresponding hyperparameter modes $(\mathbf{a}_{BCS}, \sigma^2_{BCS})$.

Starting from an user-required pattern $E_{REF}(u)$ (i.e., its sampled representation $\mathbf{E}_{REF}$), the control parameters of the synthesis process are the following variables: (*a*) d_n, $n = 1, ..., N$; (*b*) u_k, $k = 1, ..., K$; (*c*) σ^2, and (*d*) σ_0^2. Consequently, it is possible to synthesize arbitrary reference patterns specifying the pattern matching accuracy (*c*) and the sequential solver initialization (*d*). Moreover, the BCS method allows one to enforce pattern constraints within the whole or in a subset of the visible range (*b*) as well as to set suitable geometrical features of the array arrangement (*a*).

3 Numerical Analysis and Assessment

This section is devoted to numerically assess potentialities and limitations of the proposed BCS approach for the design of sparse linear arrays. The numerical analysis is carried out by considering a set of representative/benchmark reference patterns to evaluate the effectiveness and

[(4)] In this paper $\left\|\mathbf{x}\right\|_0$ is the ℓ_0-norm of $\mathbf{x}$ (i.e., the number of non-zero elements of $\mathbf{x}$).

reliability of the BCS in approximating a user-desired pattern. In order to evaluate the "degree of optimality" of the array designs, the following metrics and pattern descriptors are used: the matching error ξ defined as(5)

$$\xi \triangleq \frac{\int_0^1 |E_{REF}(u) - E(u)|^2 \, \mathrm{d}u}{\int_0^1 |E_{REF}(u)|^2 \, \mathrm{d}u}, \tag{15}$$

the aperture length L, the mean inter-element spacing $\Delta L = \frac{L}{P-1}$, and the minimum spacing $\Delta L_{min} = min_{p=1,..,P-1} \{|d_{p+1} - d_p|\}$.

3.1 BCS Sensitivity Analysis

As a first numerical experiment, the synthesis of a non-uniform array matching a Dolph-Chebyshev pattern [2] is considered. A broadside Dolph-Chebyshev pattern with $L = 9.5\lambda$ and $PSL = -20$ [dB] is assumed as reference. Let us notice that such a pattern can be synthesized through a uniform array with $P_{UNI} = 20$ $\frac{\lambda}{2}$-spaced elements. The BCS synthesis has been carried out by sampling $E_{REF}(u)$ at K points ($u_k \in [0, 1]$, $u_k = \frac{k-1}{K-1}$, $k = 1, ..., K$) and assuming the following grid of admissible locations

$$d_n = \frac{L\,(n-1)}{2\,(N-1)},\ n = 1, ..., N. \tag{16}$$

Figure 1(*a*) describes the BCS results by reporting the matching error ξ versus the number of active elements P_{BCS} for different values of the control parameters: $K = \{5, ..., 25\}$, $\sigma^2 \in [10^{-5}, 1]$, $\sigma_0^2 \in [10^{-5}, 1]$, and $N \in [5,\ 5 \times 10^4]$. The Pareto front of the solution set in the plane (ξ, P_{BCS}) is indicated, as well. As it can be observed, different BCS trade-off solutions are obtained with accuracy and element number in the range $\xi \in [10^{-6},\ 2]$ and $P_{BCS} \in [5, 20]$, respectively. By comparing the patterns related to three representative points of the Pareto front (i.e., $P_{BCS} = \{8, 14, 20\}$) with the reference one [Fig. 1(*b*)], it turns out that the solution with $P_{BCS} = 8$ elements provides a very poor matching ($\xi = 2.91 \times 10^{-1}$), while a reliable reconstruction ($\xi = 0.99 \times 10^{-4}$) is yielded choosing the solution having $P_{BCS} = 14$ [Fig. 1(*b*)] with a non-negligible saving of array elements with respect to the $\frac{\lambda}{2}$-spaced uniform array (i.e.,

(5) Only $u \in [0, 1]$ is considered in the definition of ξ for symmetry reasons.

$\frac{P_{BCS}}{P_{UNI}} = 0.7$). As a general by-product, it results that a value of the accuracy index around the threshold $\xi = 10^{-4}$ identifies an optimal trade-off BCS solution, whereas lower ξ values usually require more radiating elements [$P_{BCS} = 20$, $\xi = 2.03 \times 10^{-6}$ - Fig. 1(*b*)] without significant/relevant improvements in the matching of the reference pattern. As regards the resulting layouts, it is worth pointing out that the optimal BCS array ($P_{BCS} = 14$) has an aperture and an excitation displacement [Fig. 1(*c*)] close to those of the uniform array. This proves the effective non-uniform sampling of the ideal current distribution affording $E_{REF}(u)$. Otherwise, different apertures [e.g., $L_{BCS}\rfloor_{P=8} = 6.2\lambda$ vs. $L_{BCS}\rfloor_{P=14} = 9.5\lambda$] and weights [Fig. 1(*c*)] are synthesized in correspondence with greater values of ξ. As for the element arrangement, a positive feature of the BCS arrays is the enlarged inter-element spacing with respect to the corresponding uniform array [Fig. 1(*c*)] despite the closely-spaced admissible locations [Eq. (16)].

In order to provide a deeper understanding about the sensitivity of the BCS performances on the control parameters, Figures 2 and 3 summarize the results of a comprehensive numerical analysis. More specifically, the matching error has been evaluated as a function of K, or σ_0^2, or σ^2, or N by setting the other parameters to the values used to obtain the optimal trade-off with $P_{BCS} = 14$ (i.e., $K = 15$, $\sigma^2 = 10^{-2}$, $\sigma_0^2 = 2.0 \times 10^{-3}$, $N = 501$). For completeness, the behavior of P_{BCS} has been reported, as well. As expected [Fig. 2(*a*)], the pattern matching improves as the number of samples K of $E_{REF}(u)$ increases. However, ξ does not further decreases beyond a threshold value ($K = 15$) slightly above the Nyquist threshold ($K_{Nyquist} = 11$) even though the corresponding number of array elements P_{BCS} still grows. A sampling value K between $K_{Nyquist}$ and $1.5K_{Nyquist}$ turns out to be a reliable choice as confirmed by the behaviour of the plots of $|E_{BCS}(u) - E_{MPM}(u)|^2$ for $K = \{7, 15, 24\}$ [Fig. 3(*a*)], as well. Indeed, the lowest value of K gives the poorest fitting [$\xi\rfloor_{K=7} = 0.91$ - Fig. 3(*a*)], while satisfactory reconstructions are obtained when $K > K_{Nyquist}$ ($\xi\rfloor_{K=15} = 0.99 \times 10^{-4}$). A further increment of K only marginally enhances the accuracy [$\xi\rfloor_{K=24} = 0.98 \times 10^{-4}$ - Fig. 3(*a*)].

Concerning the sensitivity to σ^2, the integral error has small variations for $\sigma^2 < 10^{-2}$, while it sharply increases afterwards [Fig. 2(*b*)] as pointed out by the plots of $|E_{BCS}(u) - E_{REF}(u)|^2$

in correspondence with a set of representative values of of σ^2 (i.e., $\sigma^2 = \{10^{-5}, 10^{-2}, 1\}$) [Fig. 3(*b*)]. More sparse arrays are synthesized in correspondence with larger values of σ^2 at the expense of higher ξ values [Fig. 2(*b*)]. Good tradeoffs between accuracy and element reduction then arise by setting $\sigma^2 \in [10^{-3}, 10^{-1}]$. Such an outcome indicates that the BCS performances are significantly less sensitive to σ^2 than to K. As a matter of fact, a reduction of ξ of about one order in magnitude requires a variation of K of about $10 - 20\%$ [Fig. 2(*a*)], while the same effect holds true for a variation of σ^2 of more than two orders in magnitude [Fig. 2(*b*)]. Similar deductions can be drawn from the behaviour of the integral error versus σ_0^2. Moreover, the matching error increases almost monotonically with σ_0^2, whereas low P_{BCS} values are obtained within the range $\sigma_0^2 \in [5.0 \times 10^{-4}, 5.0 \times 10^{-2}]$ [Fig. 2(*c*)]. Such a range can be also assumed as reference guideline since smaller σ_0^2 values only marginally improve the matching accuracy [$\sigma_0^2 = 10^{-5}$, $\xi = 4.29 \times 10^{-5}$ - Fig. 3(*c*)], while higher values do not allow reliable syntheses [$\sigma_0^2 = 1$, $\xi = 0.1$ - Fig. 3(*c*)].

Finally, the plots in Figure 2(*d*) are concerned with the sensitivity of the BCS on N. By analyzing the behaviour of P_{BCS}, it comes out that great care must be exercised on the choice of N to obtain a sparse array matching with a good accuracy the reference one. A good receipt coming also from other heuristic analyses suggests to choose $N \in \left[5 \times \frac{L}{\lambda}; 100 \times \frac{L}{\lambda}\right]$.

3.2 BCS Assessment - Synthesis of Broadside Patterns

The second set of experiments is aimed at assessing in a more exhaustive fashion the performances of the BCS when dealing with broadside patterns. More specifically, Dolph-Chebyshev reference patterns with $L \in \{9.5\lambda, 14.5\lambda, 19.5\lambda\}$ and $PSL \in \{-20, -30, -40\}$ [dB] have been used and the Pareto fronts of the BCS solutions are shown in Fig. 4(*a*). As expected, wider apertures require more elements to reach the accuracy threshold $\xi = 10^{-4}$ (e.g., $P_{BCS}\rfloor_{\frac{L}{\lambda}=9.5} = 14$, $P_{BCS}\rfloor_{\frac{L}{\lambda}=14.5} = 20$, and $P_{BCS}\rfloor_{\frac{L}{\lambda}=19.5} = 36$). On the contrary, P_{BCS} does not generally change when varying the peak sidelobe level (e.g., $P_{BCS}\rfloor_{PSL=-20\,dB} = P_{BCS}\rfloor_{PSL=-30\,dB} = P_{BCS}\rfloor_{PSL=-40\,dB} = 26$). The BCS method allows a saving of about $30 - 35\%$ of the array elements with respect to the corresponding uniformly $\frac{\lambda}{2}$-spaced array still keeping a very accurate pattern matching (i.e., $\xi < 10^{-4}$) [Tab. I]. This implies an increasing of the average inter-element distance ($\frac{\Delta L}{\lambda/2} \in [1.46, 1.56]$) and, usually, of the minimum spacing between adjacent elements ($\frac{\Delta L_{\min}}{\lambda/2} \in [1.25, 1.56]$ except for the case with $L = 19.5\lambda$ and $PSL = -30$ [dB]). On the other hand, the array aperture only slightly reduces (e.g., $\frac{L_{BCS}}{L_{UNI}} = 0.995$ when $L = 19.5\lambda$ and $PSL = -30$ [dB]) since it controls the mainlobe pattern matching.

As far as the "shape" of the BCS Pareto front is concerned [Fig. 4(*a*)], the plot of the matching error shows a step-like behaviour whatever the array aperture and PSL conditions. Moreover, it exists a threshold value of P_{BCS} below which the BCS cannot provide an accurate matching for a given $E_{REF}(u)$. For example, the case $L = 19.5\lambda$ - $PSL = -30$ [dB] shows that ξ decreases of more than two orders in magnitude passing from $P_{BCS} = 24$ to $P_{BCS} = 26$. This is visually pointed out in Fig. 4(*c*) where the plots of $|E_{BCS}(u)|^2$ for $P_{BCS} = \{24, 26\}$ are compared to the reference pattern.

Such a behaviour is further confirmed by the results in Fig. 4(*b*) where Taylor patterns [1] with transition index $T = 6$ and different sizes (i.e., $L \in \{9.5\lambda, 14.5\lambda, 19.5\lambda\}$) and PSLs (i.e., $PSL \in \{-20, -30, -40\}$ [dB]) are taken into account. Also in this case, a small variation of P_{BCS} ($P_{BCS} = 24 \to 26$) leads to a significant improvement of the reconstruction accuracy ($\xi\rfloor_{P_{BCS}=24} = 8.11 \times 10^{-3} \to \xi\rfloor_{P_{BCS}=26} = 3.13 \times 10^{-5}$). The reliable solutions with $\xi < 10^{-4}$ provide also for Taylor syntheses an accurate matching of the reference pattern with negligible errors confined to very low sidelobes, far from the mainlobe [see the inset of Fig. 4(*d*)], which do not contain relevant portions of the radiated power.

As for the element saving with respect to the $\frac{\lambda}{2}$-spaced arrangement, the values in Tab. I confirm that $\frac{P_{BCS}}{P_{UNI}} \in [0.65,\ 0.70]$ as well as the conclusion drawn for the Dolph-Chebyshev patterns on the distribution of the array elements (i.e., $1.43 \leq \frac{\Delta L}{\lambda/2} \leq 1.55$). Concerning the computational issues, the BCS turns out to be very efficient ($t_{BCS} < 0.35$ [s] - Tab. I) whatever the broadside reference pattern, despite the non-optimized implementation of the Matlab code.

In order to complete the analysis of the performance of the BCS approach when dealing with broadside patterns, comparisons with state-of-the-art techniques have been carried out, as well. Towards this purpose, the MPM approach [7][(6)] has been considered because of its efficiency and the enhanced matching accuracy compared to similar methods such as the Prony technique

[(6)] A MATLAB implementation of the MPM has been used for the numerical tests (`mpencil` function - *http://www.mathworks.se/matlabcentral/index.html*) by setting the default parameters as suggested in [7].

[7]. The results from the analysis of different Dolph-Chebyshev references are summarized in Fig. 5 where the plots of ξ versus P for both BCS and MPM[7] arrays are shown. Let us consider the test case characterized by a reference pattern with $PSL = -30$ [dB] defined over a linear aperture of length $L = 9.5\lambda$ [Fig. 5(*a*)]. In such a case, the MPM provides a more accurate fitting than the BCS whatever the number of array elements (e.g., $P = 12$: $\xi\rfloor_{BCS} = 7.02 \times 10^{-3}$ vs. $\xi\rfloor_{MPM} = 1.04 \times 10^{-4}$ [7]) and the BCS generally requires a larger P to satisfy the condition $\xi \leq 10^{-4}$ ($P_{BCS} = 14 \rightarrow \xi\rfloor_{BCS} = 2.62 \times 10^{-5}$ vs. $P_{MPM} = 13$ $\rightarrow \xi\rfloor_{BCS} = 2.76 \times 10^{-6}$). The BCS performances come closer to those of the MPM as L increases [$L = 14.5\lambda$ - Fig. 5(*b*) and $L = 19.5\lambda$ - Fig. 5(*c*)] and sometimes the BCS outperforms the MPM in terms of fitting index for both small and large values of P [Figs. 5(*b*)-5(*c*)]. Moreover and with reference to Figs. 5(*c*)-5(*e*), it results that the efficiency of the BCS enhances when PSL reduces. As a matter of fact, the MPM overcomes the BCS when $L = 19.5\lambda$ and $PSL = -20$ [dB] [Fig. 5(*d*)], while $\xi_{BCS} < \xi_{MPM}$ for the aperture $L = 19.5\lambda$ with $PSL = -40$ [dB] [Fig. 5(*e*)] as also pictorially pointed out by the plots of $E_{MPM}(u)$ and $E_{BCS}(u)$ synthesized with the corresponding $P = 26$-element arrangement [inset of Fig. 5(*e*)]. As it can be observed, the BCS properly matches the reference pattern within the entire visible range, while the MPM accuracy worsen near the mainlobe and in the far sidelobes.

Similar conclusions hold true when dealing with Taylor reference patterns. The behavior of ξ versus P (Fig. 6) still indicates that the MPM outperforms the BCS concerning the minimum P to reach the matching threshold $\xi = 10^{-4}$ when dealing with small arrays and high PSLs [$P_{MPM} = 12 \rightarrow \xi_{MPM} = 9.89 \times 10^{-5}$ vs. $P_{BCS} = 14 \rightarrow \xi_{BCS} = 7.82 \times 10^{-5}$ - Fig. 6(*a*)], while the BCS betters the MPM performance for larger L with low peak sidelobe levels [$P_{MPM} = 26 \rightarrow \xi_{MPM} = 2.38 \times 10^{-4}$ vs. $P_{BCS} = 26 \rightarrow \xi_{BCS} = 3.62 \times 10^{-5}$ - Fig. 6(*e*)]. This is further confirmed by the patterns of the optimal trade-off solutions displayed in the insets of the pictures of Fig. 6.

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[7] Please notice that only the MPM arrays with SVD-truncation parameter below 10^{-3} have been reported in order to guarantee an accurate pattern matching [7].

3.3 BCS Assessment - Synthesis of Shaped Patterns

In order to evaluate the flexibility of the proposed approach, numerical tests concerned with shaped patterns have been also performed. The first experiment deals with the reconstruction of flat top patterns defined over an aperture of $L = 4.5\lambda$ with different PSLs as in [36]. The plots of ξ as a function of P show that neither the MPM nor the BCS is able to reduce the number of array elements of the uniform array (being 0.6λ its inter-element distance) synthesized in [36] still keeping a good accuracy, although the BCS [$P_{BCS} = 10 \rightarrow \xi_{BCS} = 4.55 \times 10^{-6}$ - Fig. 7(*a*)] reduces the array aperture with respect to [36] ($\frac{L_{BCS}}{L} < 0.97$ - Tab. II). On the contrary, the MPM defines wider arrangements ($\frac{L_{MPM}}{L} = 1.74$), as shown in Fig. 7(*d*), without yielding a good matching with the reference patterns ($\xi_{MPM} > 2.5 \times 10^{-3}$ - Tab. II). The enhanced accuracy of the BCS is also pointed out by the plots of $E_{REF}(u)$, $E_{BCS}(u)$, and $E_{MPM}(u)$ in the insets of Figs. 7(*a*)-7(*c*) related to the arrays with $P_{BCS} = P_{MPM} = 10$. For completeness, the distributions of the array excitations along the array extension are given in Fig. 7(*d*). As it can be observed and also predicted in [7], the worsening of the performances of the MPM is mainly due to the errors in estimating the element positions caused by the non-negligible values of the imaginary parts of the non-zero roots of the associated eigenvalue problem.

The second experiment considers as reference the *Woodward* pattern with $L = 8.5\lambda$ analyzed in [37]. The plots of ξ versus P show that the BCS faithfully reconstructs the reference pattern synthesizing an array of $P_{BCS} = 12$ elements [$\xi_{BCS} = 2.79 \times 10^{-5}$ - Fig. 8(*a*)] with a reduction of about $\frac{1}{3}$ of the array elements with respect to the uniform layout ($P_{UNI} = 18$). As a side effect of the approximation, the optimal BCS trade-off slightly improves the PSL of the reference pattern ($P_{BCS} = 12 \rightarrow PSL_{BCS} = -20.2$ [dB] vs. $PSL_{UNI} = -20$ [dB] - Tab. III), as well. On the contrary, both the MPM synthesis in [37] and the MPM pattern generated with $P_{MPM} = 12$ elements do not provide an accurate fitting [$P_{MPM} = 12 \rightarrow \xi_{MPM} = 4.02 \times 10^{-3}$ - Fig. 8(*a*)], unless using more antenna elements (e.g., $P_{MPM} = 14$), and significantly worsen the PSL ($P_{MPM} = 12 \rightarrow PSL_{MPM} = -13.2$ [dB]) as highlighted by the plots of the associated patterns [Fig. 8(*b*)]. For completeness, the behaviour of the array excitations and the corresponding figures of merit are reported in Fig. 8(*c*) and Tab. III, respectively. As for the computational costs, the BCS still retains the numerical efficiency proved in synthesizing broadside patterns (Tab. III).

Similar conclusions can be also drawn when considering wider reference apertures. For example, with reference to a Woodward reference pattern with $L = 19.5\lambda$ [Fig. 9(*a*)], the BCS yields an accurate approximation with less elements than the MPM ($P_{BCS} = 26$ vs. $P_{MPM} = 28$). Moreover, the accuracy of the MPM significantly worsens when using the same number of active elements of the BCS solution [$P = 26$ - $\xi_{MPM} = 4.81 \times 10^{-2}$, $PSL_{MPM} = -3.6$ [dB] vs. $\xi_{BCS} = 3.52 \times 10^{-5}$, $PSL_{BCS} = -17.4$ - Tab. IV and Fig. 9(*b*)]. As for the array arrangement, the BCS provides a more widely-spaced design characterized by the following parameters: $\frac{\Delta L_{\min}}{\lambda/2} = 0.975$ and $\frac{\Delta L}{\lambda/2} = 1.56$ (Tab. IV).

3.4 BCS Assessment - Constrained Synthesis

This section is devoted to assess the reliability of the BCS approach in solving constrained synthesis problems (i.e., matching a reference pattern under some explicit geometric and/or radiation constraints). Towards this aim, the synthesis of a Dolph-Chebyshev pattern with $L = 19.5\lambda$ and $PSL = -30$ [dB] under different synthesis constraints has been addressed.

The first test case has been formulated by enforcing the pattern matching constraints in the angular region $u_k \notin [u_m, u_M]$, being $u_m = 0.5$ and $u_M = 0.6$. As desired, the pattern of the optimal BCS trade-off solution ($\xi = 3.71 \times 10^{-5}$ - Tab. V) fits in a faithful way the reference one within the constrained region as well as in the transition regions close to the unconstrained angular range [Fig. 10(*b*)]. It is also of interest to observe that the distribution of the array excitations of the BCS synthesis and those of the uniform array quite significantly differ [Fig. 10(*a*)].

To further verify the efficiency of the BCS to include pattern constraints in the synthesis process without affecting the reliability of the matching in the remaining portion of the pattern, the constraint has been moved in another region of the visible range by setting $u_m = 0.8$ and $u_M = 1.0$. As expected, the trade-off pattern carefully matches the reference in the constrained region ($\xi = 6.81 \times 10^{-5}$ - Tab. V), while uncontrolled lobes appear for $u > 0.8$ [Fig. 11(*b*)]. The use of a directive element [e.g., a $cos(\theta)$ radiating element] might then enable the control of the sidelobes in the whole visible region [Fig. 11(*b*)] with a significant saving of active elements in comparison with the uniform array synthesizing the entire Dolph pattern ($P_{BCS} = 21$ vs. $P_{UNI} = 40$).

The last part of the numerical assessment is aimed at analyzing the capability of the BCS approach to also take into account geometrical constraints. Towards this end and considering the same reference pattern of the previous experiments, two different aperture-blockage problems have been defined: (*i*) $d_n \notin [5.3\lambda, 6.5\lambda]$ and (*ii*) $d_n \notin [0.0\lambda, 1.0\lambda]$. The plots of the synthesized trade-off arrangements assess the effectiveness and reliability of the BCS technique in constraining the element positions to desired locations [Fig. 12(*a*) and 13(*a*)], while designing sparse arrangements ($\Delta L > \lambda/2$ - Tab. V) with reduced apertures ($L_{BCS} < 19.47$), as well. It is also worthwhile to point out that, notwithstanding the non-negligible reduction of the admissible spatial region for the array elements (more than 10% in both cases), the $E_{BCS}(u)$ pattern matches the reference $E_{REF}(u)$ with a great care [Fig. 12(*b*) and Fig. 13(*b*)] as confirmed by the values of the matching index [(*i*) $\xi = 5.82 \times 10^{-6}$ and (*ii*) $\xi = 4.81 \times 10^{-5}$ - Tab. V].

4 Conclusions

In this paper, the BCS has been applied to the synthesis of sparse arrays with desired radiation properties. The pattern matching problem has been properly reformulated in a suitable Bayesian framework and successively solved with a fast solver. An extensive numerical validation has been carried out dealing with different reference patterns, array sizes, and constraints to assess the feasibility and reliability of the BCS approach as well as its efficiency, flexibility, and accuracy. Selected comparisons with state-of-the-art techniques have highlighted the advantages and limitations of the BCS synthesis in terms of sensitivity on control parameters, performances, and computational complexity. The proposed technique has shown the following main features:

- several tradeoffs solutions can be easily obtained by means of simple modifications of the control parameters (σ^2, u_k, d_n, and σ_0^2) (Sect. 3.1);
- BCS favorably compares with state-of-the-art techniques such as the MPM [7] in terms of accuracy, array sparseness, and computational burden when matching reference broadside patterns (Sect. 3.2);

- on average the number of active elements in a BCS array turns out to be smaller than the corresponding uniform arrangement ($P_{BCS} \approx 0.7 \div 0.65\, P_{UNI}$) still providing a high accuracy in matching the reference pattern (i.e., $\xi \leq 10^{-4}$);
- BCS usually outperforms MPM when dealing with shaped beampatterns (Sect. 3.3);
- application-specific constraints on either the radiation pattern or the geometrical characteristics of the array can be easily and efficiently taken into account (Sect. 3.4).

Subjects of future researches will be the analysis of the mutual coupling effects in the presence of realistic array elements as well as an enhanced exploitation of directive elements. Further extensions, out-of-the-scope of the present paper, will concern with complex excitations and non-symmetric layouts.

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Appendix

- *Sequential Solver for the Maximization of* $\mathcal{L}\left(\mathbf{a}, \sigma^2\right)$

The marginal likelihood maximization algorithm proposed in [34] is hereinafter customized to deal with user-defined pattern matching problems. Starting from the knowledge of $\mathbf{E}_{REF}$ and Ψ, the following sequence is iteratively (r being the iteration index) applied:

1. **Initialization (**$r = 0$**) -** Set $\left[\sigma^2\right]^{(r)} = var\left[\mathbf{E}_{REF}\right] \times \sigma_0^2$ and the n-th entry of the diagonal matrix $A^{(r)} \triangleq diag\left(a_1^{(r)}, ..., a_N^{(r)}\right)$ as follows

$$a_n^{(r)} = \frac{\left\|\psi_n\right\|^4}{\left\|\psi_n^T \mathbf{E}_{REF}\right\|^2 - \left[\sigma^2\right]^{(r)} \left\|\psi_n\right\|^2} \tag{17}$$

if $n = \hat{n}$ and $a_n^{(r)} = \infty$ otherwise, $\hat{n}$ and ψ_n being randomly picked integers within $[1, N]$ and the n-th column of Ψ, respectively;

2. **Update** - Evaluate $\Sigma^{(r)} = \Sigma\left(A^{(r)}, [\sigma^2]^{(r)}\right)$ and $\mu^{(r)} = \mu\left(A^{(r)}, [\sigma^2]^{(r)}\right)$ to compute the *sparsity* factors $s_n^{(r)} = \psi_n^T C_{-n}^{-1} \psi_n$, $n = 1, ..., N$ and the *quality* factors $z_n^{(r)} = \psi_n^T C_{-n}^{-1} \mathbf{E}_{REF}$, $n = 1, ..., N$ where $C_{-n} = C - a_n^{-1} \psi_n \psi_n^T$;

3. **Candidate Basis Vector Evaluation** - Select the r-th candidate basis vector[(8)] ψ_n, $n = r$, and compute $\Theta_n^{(r)} = \left(z_n^{(r)}\right)^2 - s_n^{(r)}$. If $\Theta_n^{(r)} > 0$, then update the value of $a_n^{(r)}$ by means of (17), otherwise set $a_n^{(r)} = \infty$;

4. **Convergence Check** - Compute the value of $\Theta_n^{(r)}$ $\forall n \in 1, ..., N$. If $\Theta_n^{(r)} \leq \tau$ $\forall n$ (τ being the *tolerance factor* usually set to 10^{-8} [35]), then terminate. Otherwise, update the iteration index ($r \leftarrow r + 1$) and go to step 2.

[(8)] Please refer to [34] for a review of the strategies for candidate selection.

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FIGURE CAPTIONS

- **Figure 1.** BCS *Sensitivity Analysis* (*Dolph-Chebyshev:* $L = 9.5\lambda$, $PSL = -20$ dB) - Plot of the representative points of a set of BCS solutions in the (ξ, P_{BCS}) plane (*a*). Power patterns (*b*) and corresponding layouts (*c*) of the reference and of a set of representative BCS arrays.

- **Figure 2.** BCS *Sensitivity Analysis* (*Dolph-Chebyshev:* $L = 9.5\lambda$, $PSL = -20$ dB) - Behaviours of ξ and P_{BCS} versus (*a*) K, (*b*) σ^2, (*c*) σ_0^2, and (*d*) N.

- **Figure 3.** BCS *Sensitivity Analysis* (*Dolph-Chebyshev:* $L = 9.5\lambda$, $PSL = -20$ dB) - Plots of $|E_{REF}(u) - E_{BCS}(u)|^2$ of representative BCS solutions computed at different values of (*a*) K, (*b*) σ^2, (*c*) σ_0^2, and (*d*) N.

- **Figure 4.** BCS *Assessment* (*Broadside Pattern Synthesis*) - Pareto fronts in the (ξ, P_{BCS}) plane (*a*)(*b*) and power patterns (*c*)(*d*) of representative BCS solutions when matching (*a*)(*c*) Dolph-Chebyshev and (*b*)(*d*) Taylor reference patterns.

- **Figure 5.** BCS *Assessment* (*Broadside Pattern Synthesis*) - Representative points in the (ξ, P) plane of BCS and MPM solutions synthesized when matching the reference *Dolph-Chebyshev* patterns characterized by: (*a*) $L = 9.5\lambda$ - $PSL = -30$ [dB], (*b*) $L = 14.5\lambda$ - $PSL = -30$ [dB], (*c*) $L = 19.5\lambda$ - $PSL = -30$ [dB], (*d*) $L = 19.5\lambda$ - $PSL = -20$ [dB], and (*e*) $L = 19.5\lambda$ - $PSL = -40$ [dB].

- **Figure 6.** BCS *Assessment* (*Broadside Pattern Synthesis*) - Representative points in the (ξ, P) plane of BCS and MPM solutions synthesized when matching the reference *Taylor* patterns characterized by: (*a*) $L = 9.5\lambda$ - $PSL = -30$ [dB], (*b*) $L = 14.5\lambda$ - $PSL = -30$ [dB], (*c*) $L = 19.5\lambda$ - $PSL = -30$ [dB], (*d*) $L = 19.5\lambda$ - $PSL = -20$ [dB], and (*e*) $L = 19.5\lambda$, $PSL = -40$ [dB].

- **Figure 7.** BCS *Assessment* (*Shaped Pattern Synthesis:* $L = 5.4\lambda$ [36]) - Representative points in the (ξ, P) plane of BCS and MPM solutions synthesized when matching the reference *Shaped* patterns [36] characterized by: (*a*) $PSL = -20$ dB, (*b*) $PSL = -30$ [dB], and (*c*) $PSL = -40$ [dB]. Array excitations (*d*).

- **Figure 8.** BCS *Assessment* (*Flat-Top Pattern Synthesis:* $L = 8.5\lambda$ [37]) - Representative points in the (ξ, P) plane of BCS and MPM solutions (*a*), optimal trade-off beampatterns (*b*), and associated array excitations (*c*).

- **Figure 9.** BCS *Assessment* (*Flat Top Pattern Synthesis:* $L = 19.5\lambda$) - Representative points in the (ξ, P) plane of BCS and MPM solutions (*a*), optimal trade-off beampatterns (*b*), and associated array excitations (*c*).

- **Figure 10.** BCS *Assessment* [*Constrained Synthesis* - *Dolph-Chebyshev:* $L = 19.5\lambda$, $u_k \notin (0.45, 0.55)$] - Array excitations (*a*) and power patterns (*b*).

- **Figure 11.** BCS *Assessment* (*Constrained Synthesis* - *Dolph-Chebyshev:* $L = 19.5\lambda$, $u_k \notin (0.8, 1.0]$) - Array excitations (*a*) and power patterns when using isotropic or directive elements (*b*).

- **Figure 12.** BCS *Assessment* [*Constrained Synthesis* - *Dolph-Chebyshev:* $L = 19.5\lambda$, $d_n \notin (5.3\lambda, 6.5\lambda)$] - Array excitations (*a*) and power patterns (*b*).

- **Figure 13.** BCS *Assessment* [*Constrained Synthesis* - *Dolph-Chebyshev:* $L = 19.5\lambda$, $d_n \notin (0.0\lambda, 1.0\lambda)$] - Array excitations (*a*) and power patterns (*b*).

TABLE CAPTIONS

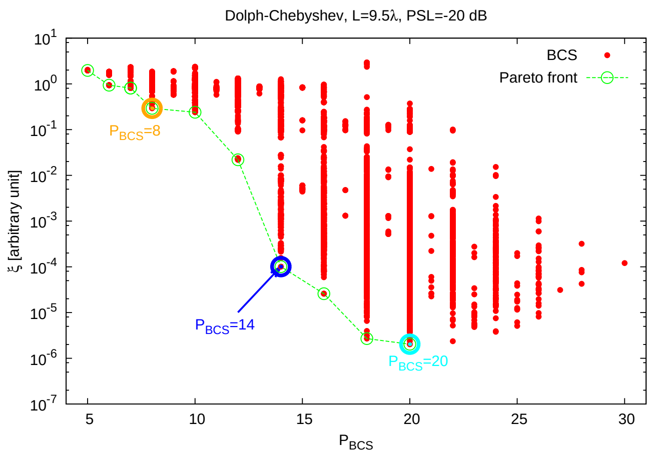
- **Table I.** BCS *Assessment* (*Broadside Pattern Synthesis*) - Array performance indexes.

- **Table II.** BCS *Assessment* (*Shaped Pattern Synthesis:* $L = 5.4\lambda$ [36]) - Array performance indexes.

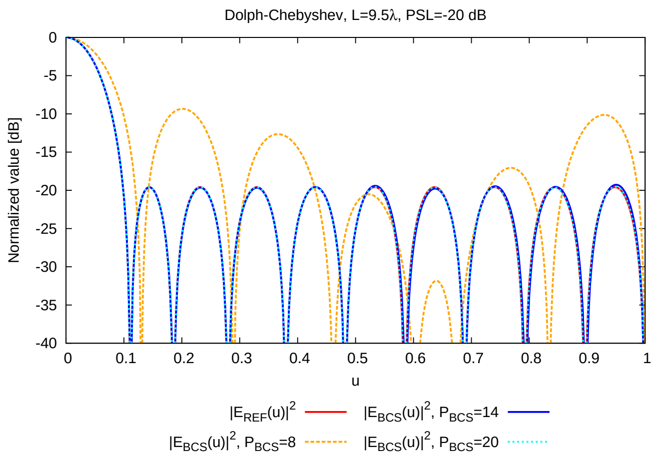
- **Table III.** BCS *Assessment* (*Shaped Pattern Synthesis:* $L = 8.5\lambda$ [37]) - Array performance indexes.

- **Table IV.** BCS *Assessment* (*Shaped Pattern Synthesis:* $L = 19.5\lambda$) - Array performance indexes.

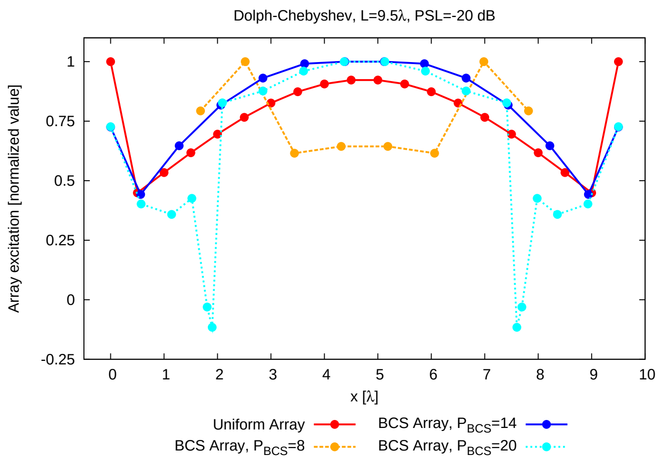
- **Table V.** BCS *Assessment* (*Constrained Synthesis* - *Dolph-Chebyshev:* $L = 19.5\lambda$) - Array performance indexes.



(a)



(b)



(c)

Figure 1 - G. Oliveri et *al.*, “Bayesian Compressive Sampling for...”

Figure 2 - G. Oliveri et *al.*, "Bayesian Compressive Sampling for..."

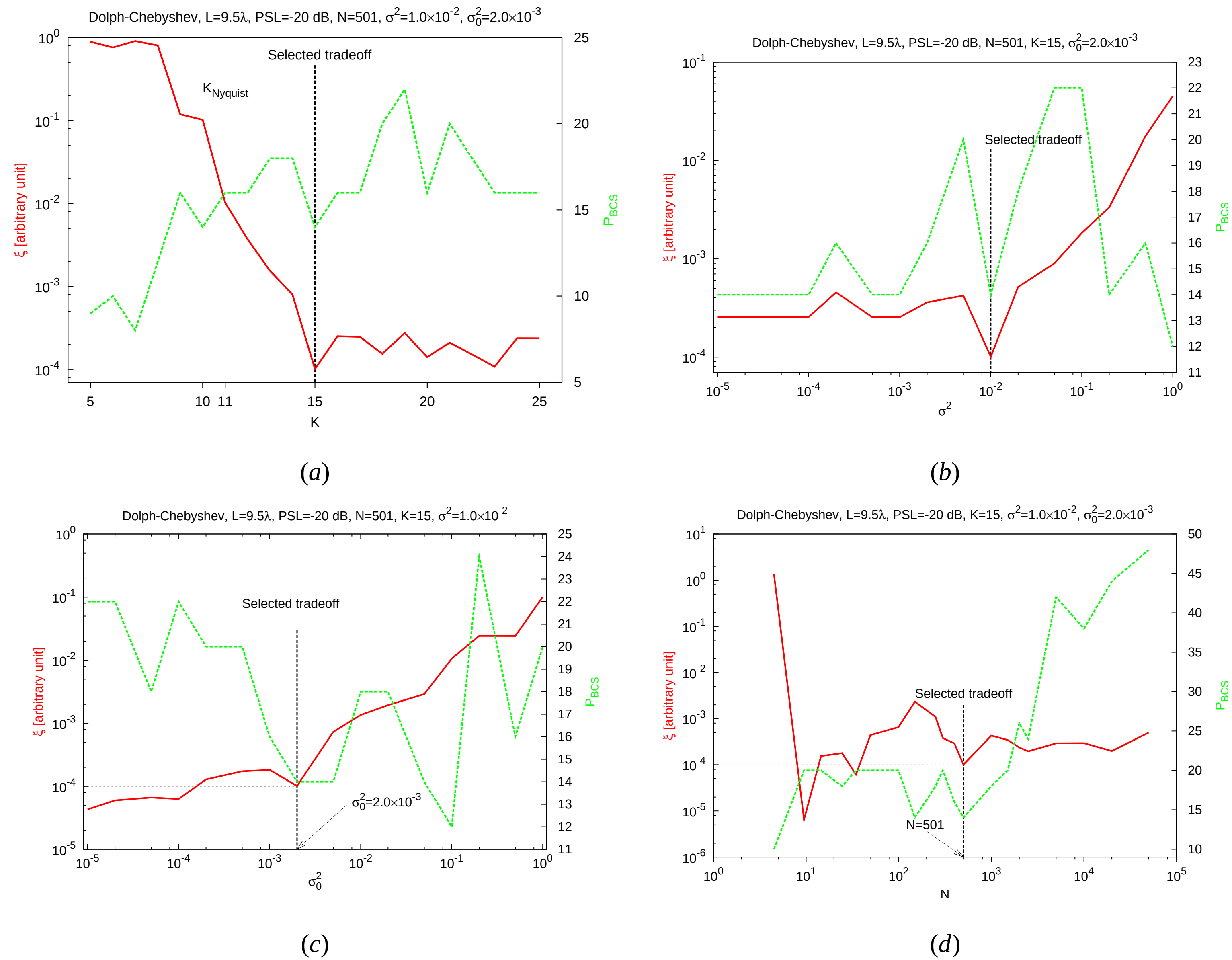
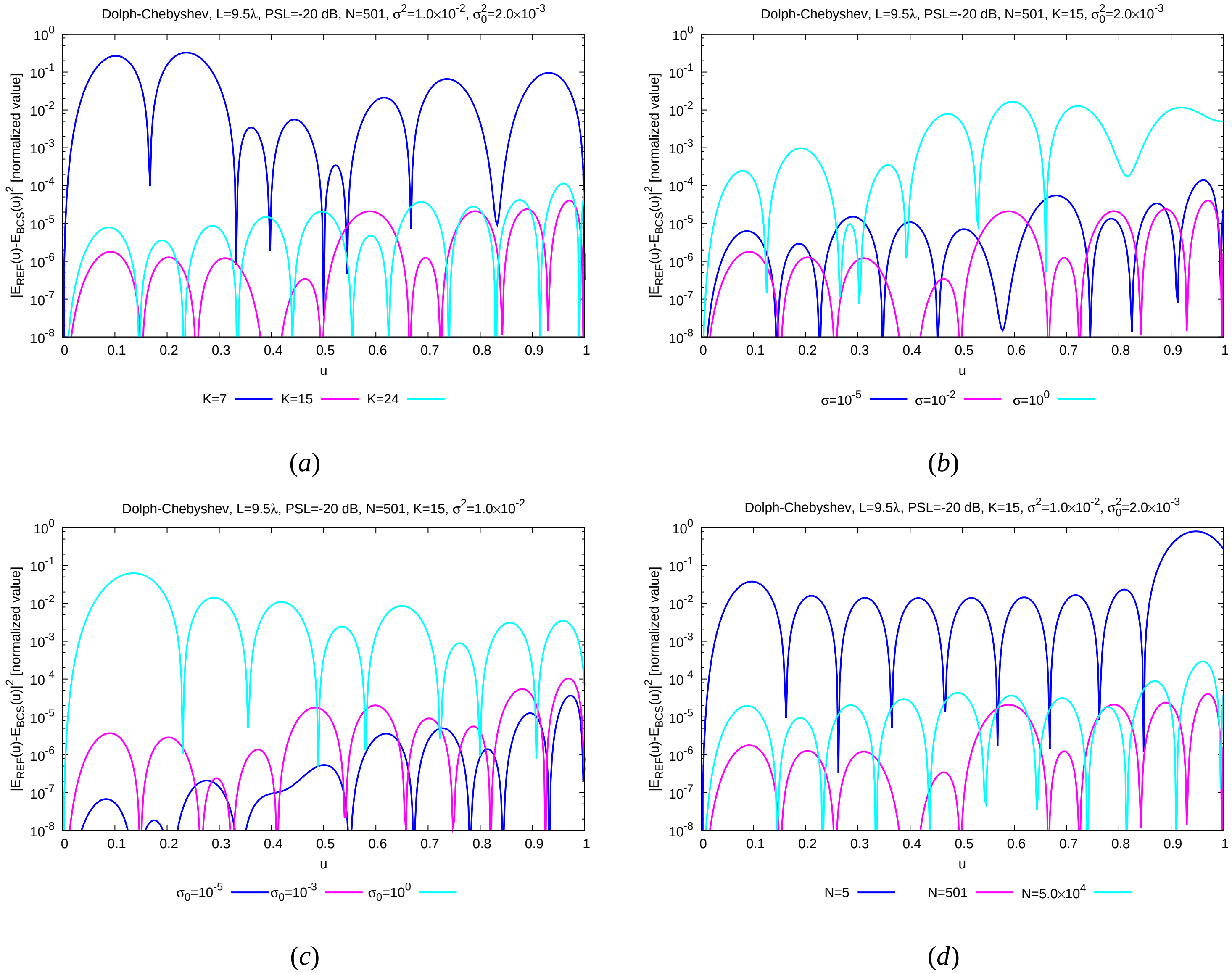
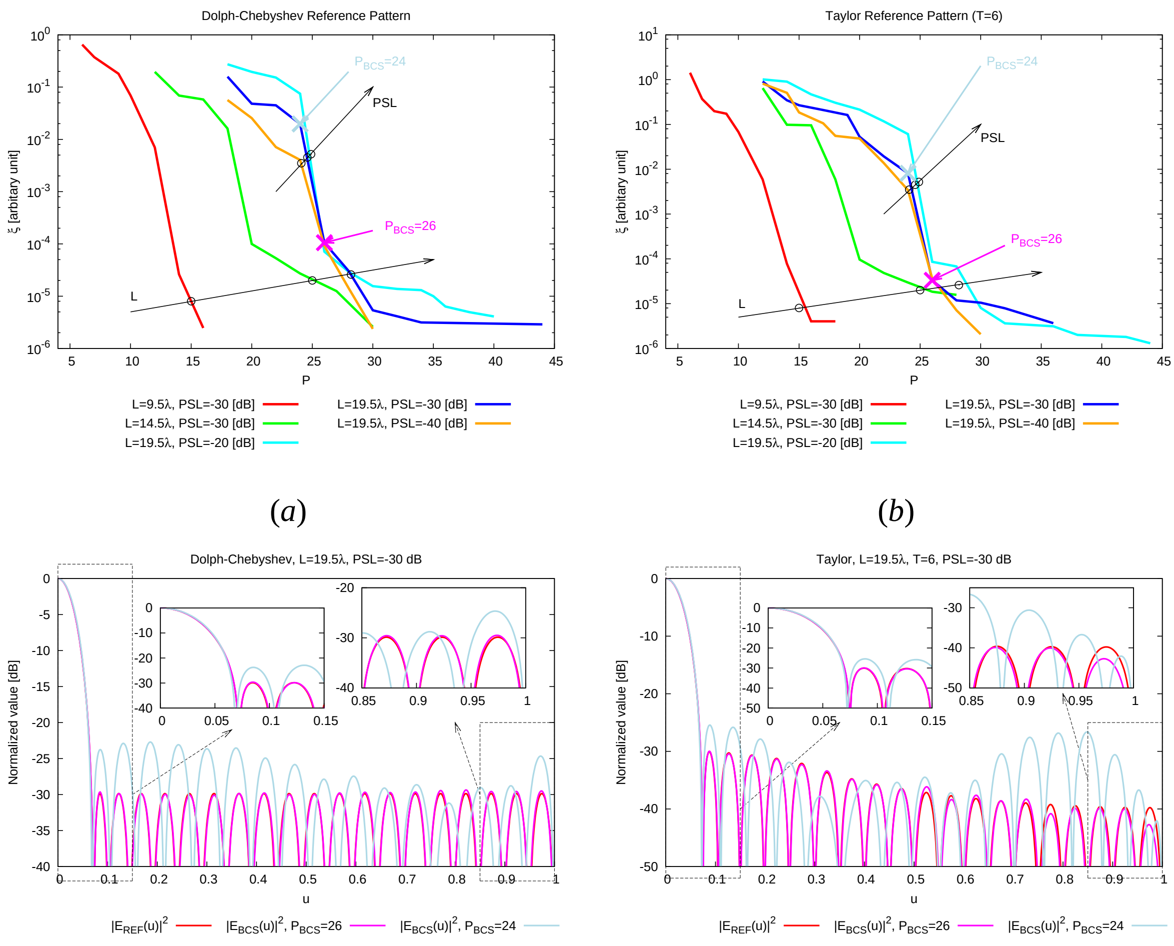


Figure 3 - G. Oliveri et *al.*, "Bayesian Compressive Sampling for..."





(*a*) (*b*)

(*c*) (*d*)

Figure 4 - G. Oliveri et *al.*, "Bayesian Compressive Sampling for..."

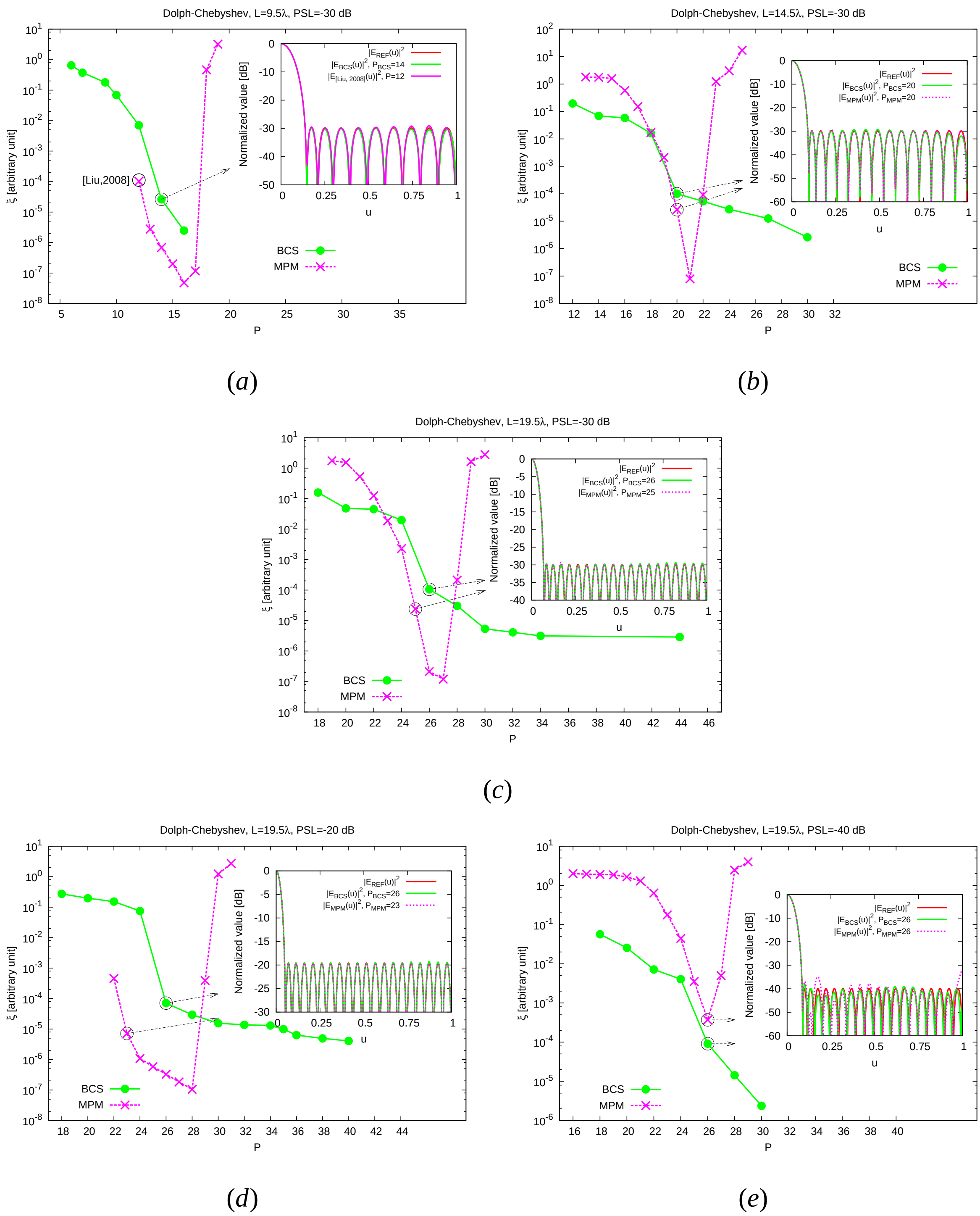


Figure 5 - G. Oliveri et *al.*, “Bayesian Compressive Sampling for...”

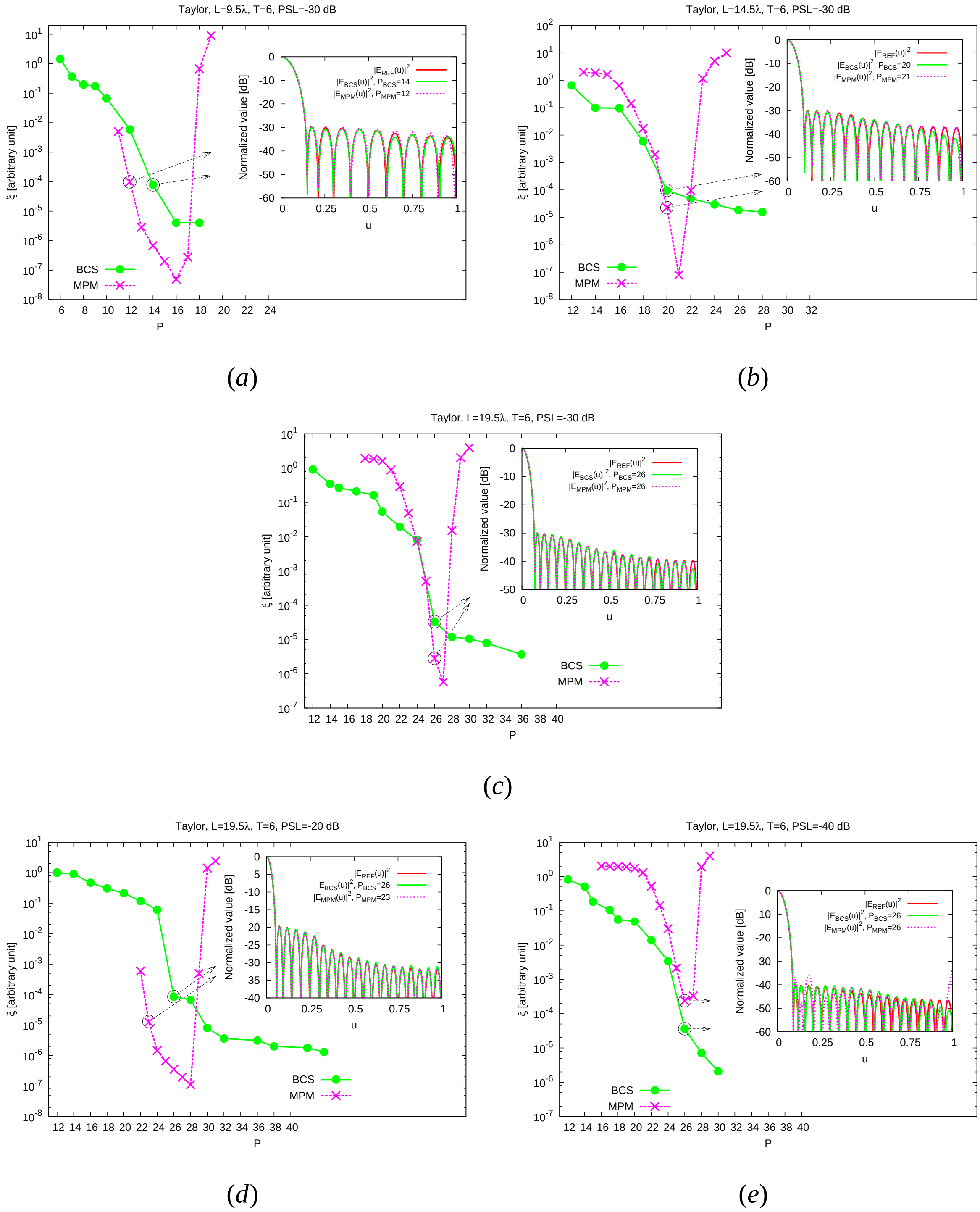


Figure 6 - G. Oliveri et *al.*, “Bayesian Compressive Sampling for...”

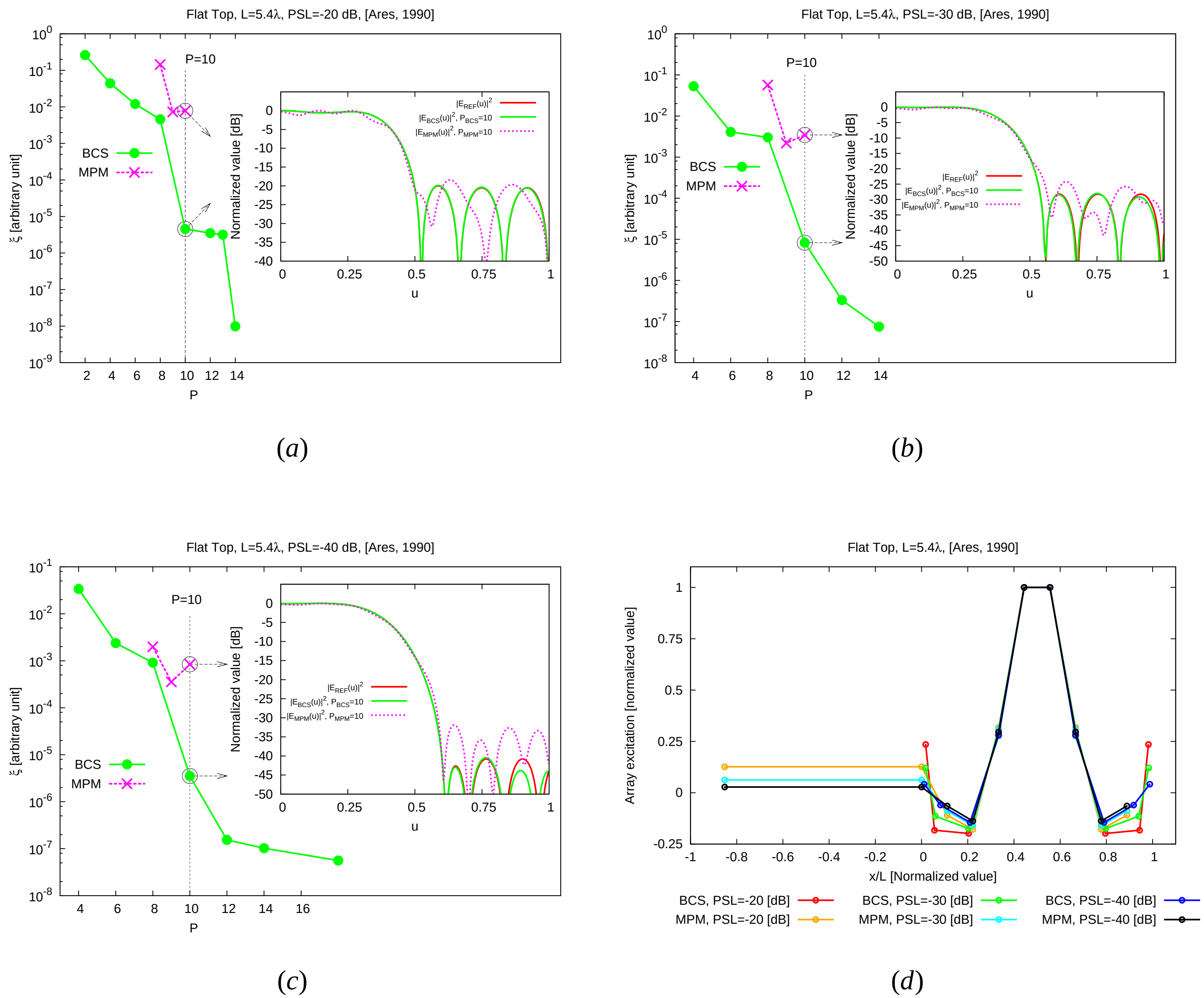
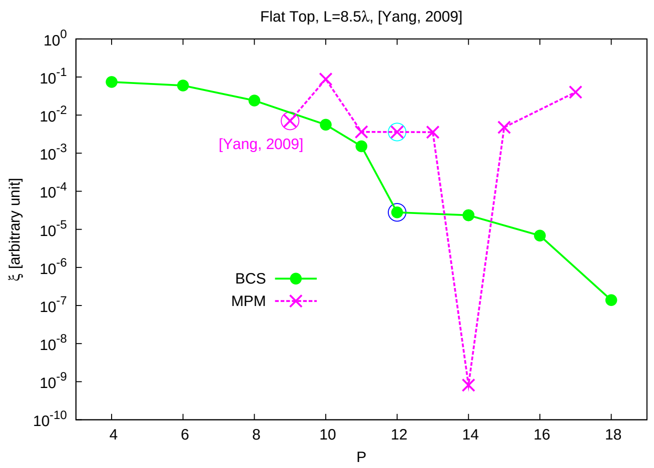
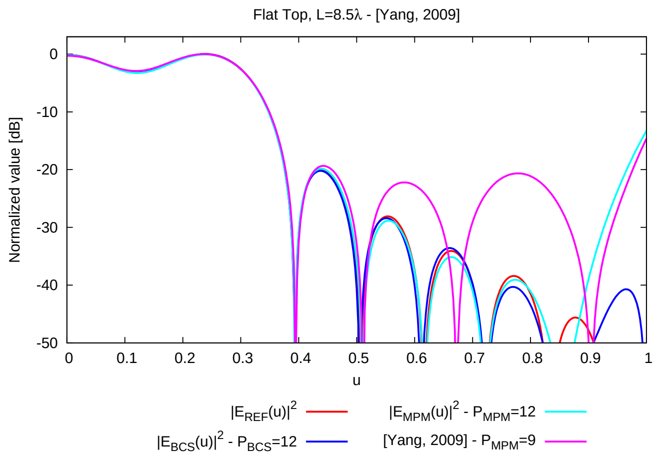


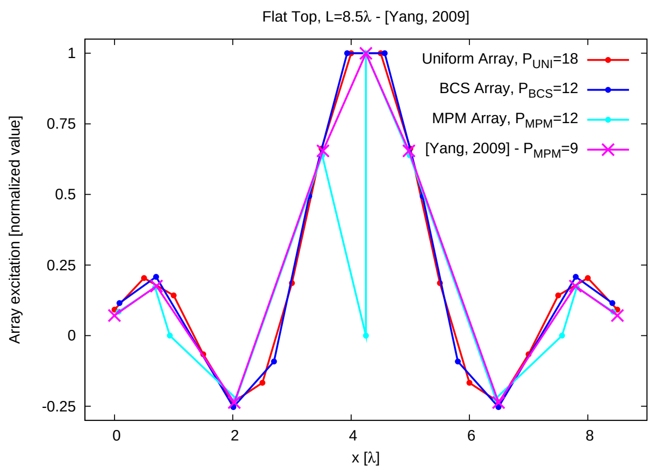
Figure 7 - G. Oliveri et al., "Bayesian Compressive Sampling for..."



(*a*)

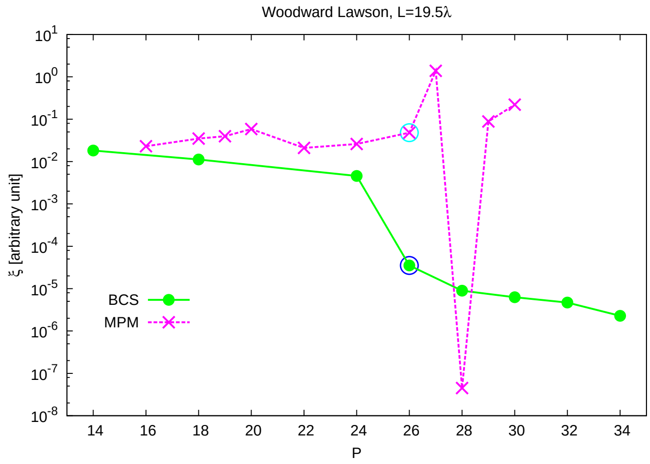


(*b*)

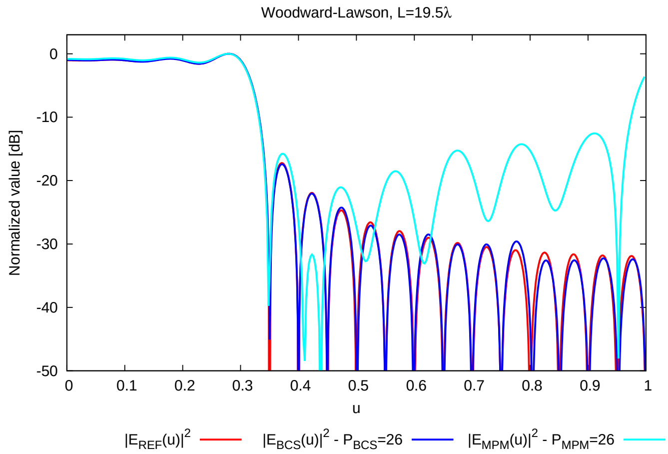


(*c*)

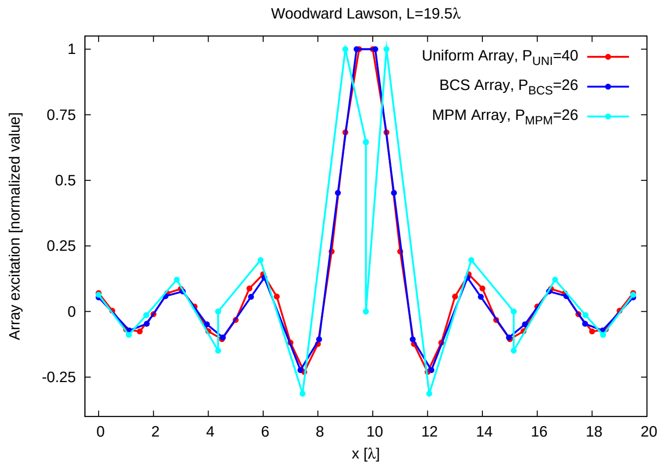
Figure 8 - G. Oliveri et *al.*, "Bayesian Compressive Sampling for..."



(*a*)

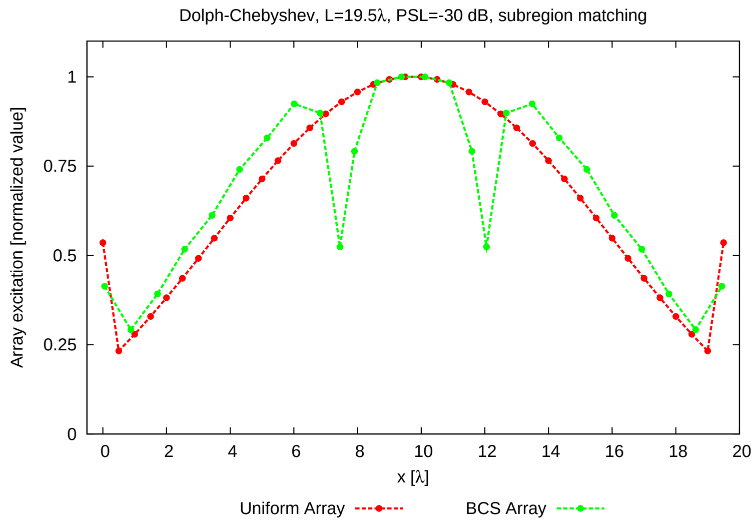


(*b*)

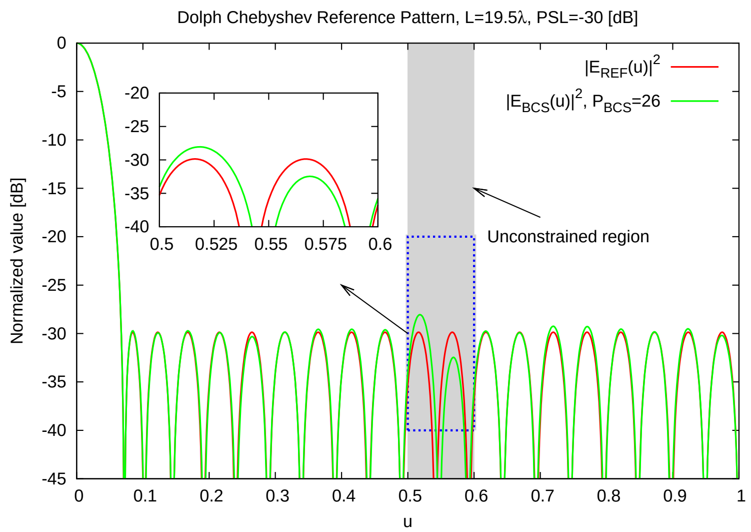


(*c*)

Figure 9 - G. Oliveri et *al.*, "Bayesian Compressive Sampling for..."

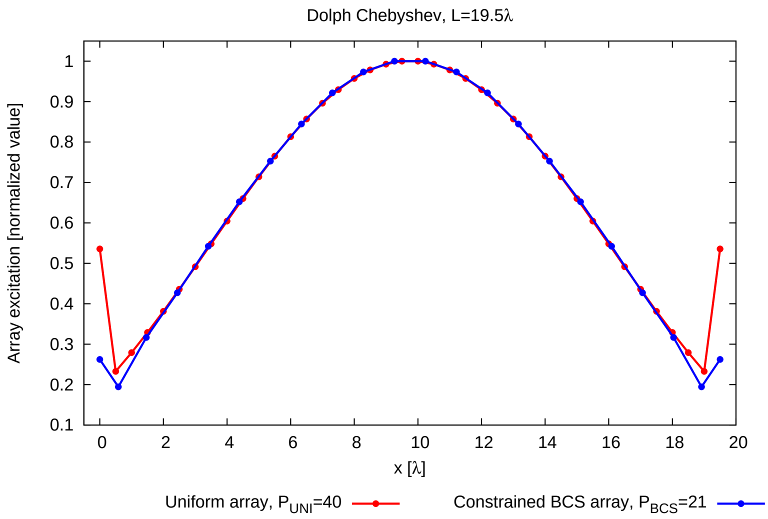


(*a*)

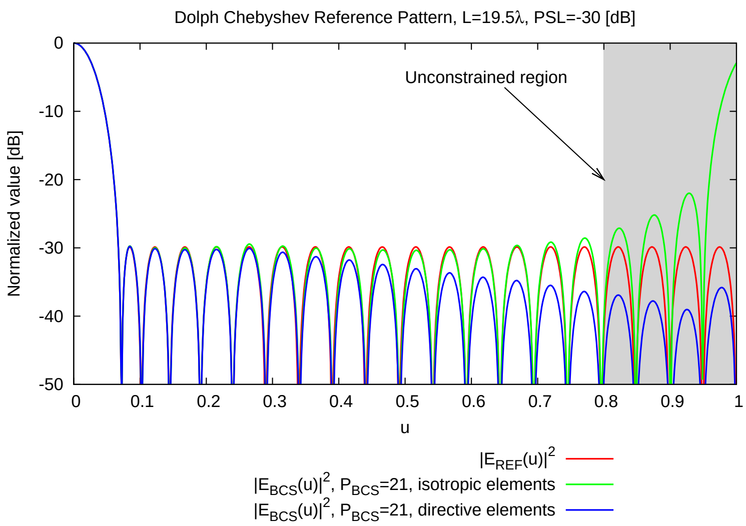


(*b*)

Figure 10 - G. Oliveri et *al.*, "Bayesian Compressive Sampling for..."

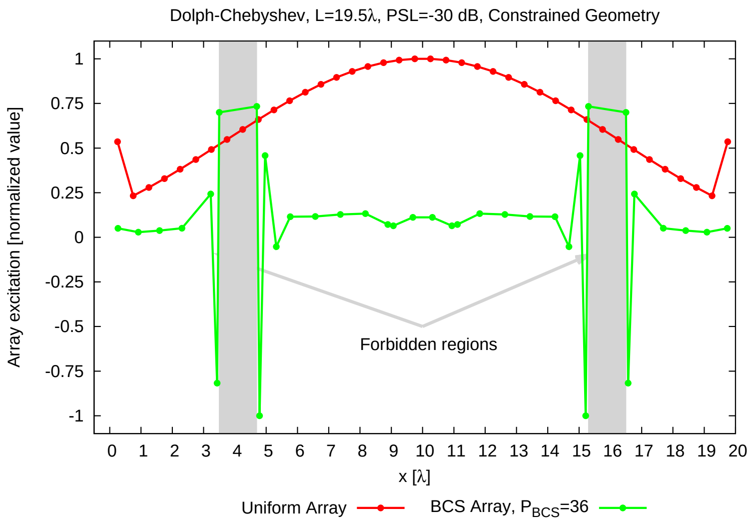


(*a*)

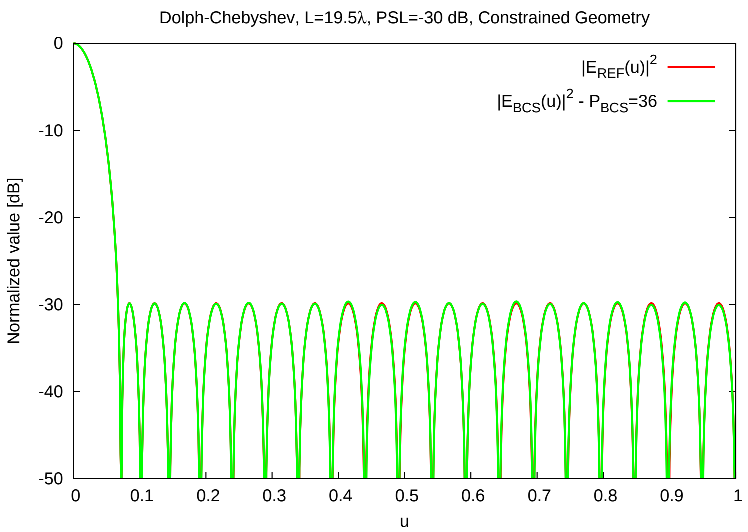


(*b*)

Figure 11 - G. Oliveri et *al.*, "Bayesian Compressive Sampling for..."

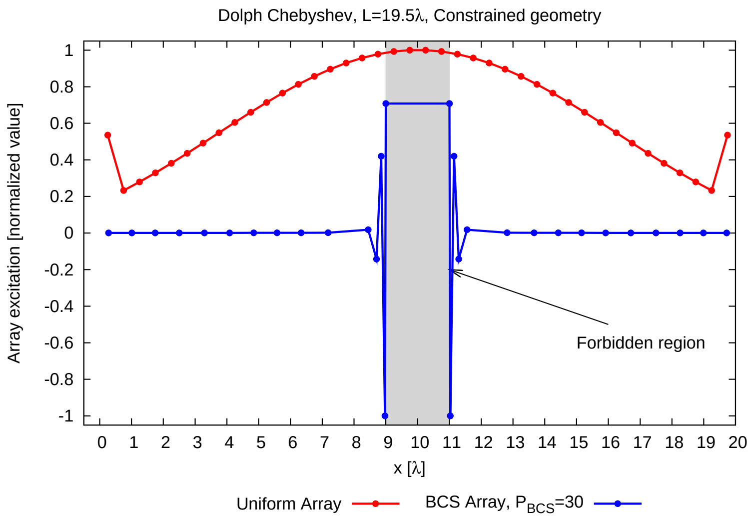


(*a*)

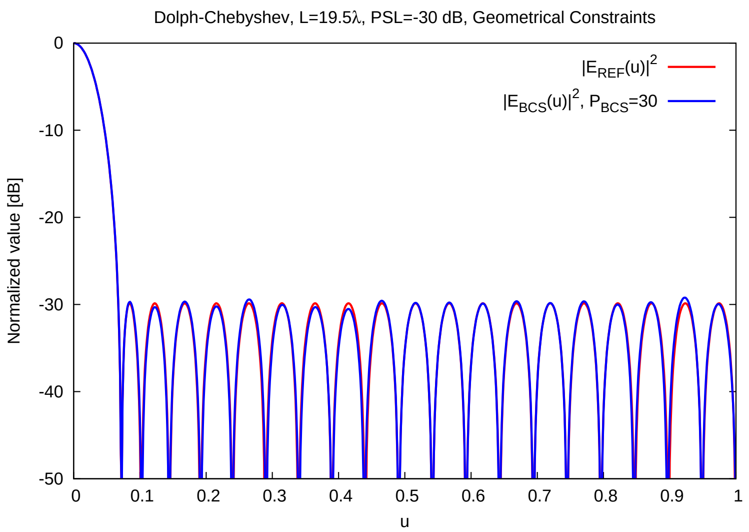


(*b*)

Figure 12 - G. Oliveri et *al.*, "Bayesian Compressive Sampling for..."



(*a*)



(*b*)

Figure 13 - G. Oliveri et *al.*, “Bayesian Compressive Sampling for...”

Table I - G. Oliveri et *al.*, "Bayesian Compressive Sampling for..."

Reference Pattern			*Uniform*		BCS					
Type	L [λ]	PSL [dB]	P_{UNI}	$\frac{L_{UNI}}{L}$	ξ [$\times 10^{-5}$]	$\frac{P_{BCS}}{P_{UNI}}$	$\frac{\Delta L_{\min}}{\lambda/2}$	$\frac{\Delta L}{\lambda/2}$	$\frac{L_{BCS}}{L}$	t [$\times 10^{-1}$s]
Dolph	9.5	−30	20	1.0	2.62	0.70	1.26	1.46	1.000	1.12
Dolph	14.5	−30	30	1.0	9.98	0.66	1.35	1.52	1.000	2.93
Dolph	19.5	−20	40	1.0	7.10	0.65	1.50	1.56	0.997	2.14
Dolph	19.5	−30	40	1.0	3.03	0.70	0.78	1.42	0.995	1.18
Dolph	19.5	−40	40	1.0	9.09	0.65	1.56	1.56	1.000	1.13
Taylor	9.5	−30	20	1.0	7.82	0.70	1.22	1.46	1.000	1.27
Taylor	14.5	−30	30	1.0	9.64	0.66	1.35	1.52	1.000	3.14
Taylor	19.5	−20	40	1.0	8.53	0.65	1.34	1.55	0.994	1.92
Taylor	19.5	−30	40	1.0	3.13	0.65	0.80	1.43	0.993	1.48
Taylor	19.5	−40	40	1.0	3.62	0.65	1.36	1.54	0.990	1.01

Reference Pattern		*Method*	*Indexes*					
L [λ]	PSL [dB]		ξ	P	$\frac{\Delta L_{\min}}{\Delta L_{UNI}}$	$\frac{\Delta L}{\Delta L_{UNI}}$	$\frac{L}{L_{UNI}}$	t [s]
5.4	-20	[36]	$-$	10	1.00	1.00	1.00	$-$
5.4	-20	BCS	4.55×10^{-6}	10	0.34	0.96	0.96	1.5×10^{-1}
5.4	-20	MPM	7.82×10^{-3}	10	0.99	1.74	1.74	3.3×10^{-2}
5.4	-30	[36]	$-$	10	1.00	1.00	1.00	$-$
5.4	-30	BCS	8.27×10^{-6}	10	0.39	0.96	0.96	1.4×10^{-1}
5.4	-30	MPM	3.45×10^{-3}	10	0.99	1.74	1.74	2.5×10^{-2}
5.4	-40	[36]	$-$	10	1.00	1.00	1.00	$-$
5.4	-40	BCS	3.53×10^{-6}	10	0.63	0.97	0.97	1.6×10^{-1}
5.4	-40	MPM	0.84×10^{-3}	10	0.99	1.74	1.74	2.9×10^{-2}

Table II - G. Oliveri et *al.*, “Bayesian Compressive Sampling for...”

	Uniform	BCS	MPM	MPM [37]
L [λ]	8.5	8.33	8.36	8.50
PSL [dB]	-20	-20.2	-13.2	-14.63
P	18	12	12	9
$\frac{P}{P_{UNI}}$	$-$	0.66	0.66	0.50
$\frac{\Delta L_{\min}}{\Delta L_{UNI}}$	$-$	1.18	< 0.01	1.42
$\frac{\Delta L}{\Delta L_{UNI}}$	$-$	1.51	1.52	2.12
$\frac{L}{L_{UNI}}$	$-$	0.980	0.984	1.00
t [s]	$-$	2.0×10^{-1}	2.8×10^{-1}	$-$
ξ	$-$	2.79×10^{-5}	4.02×10^{-3}	7.02×10^{-3}

Table III - G. Oliveri et *al.*, "Bayesian Compressive Sampling for..."

	Uniform (WLM)	BCS	MPM
L $[\lambda]$	19.5	19.5	19.5
PSL [dB]	-17.2	-17.4	-3.6
P	40	26	26
$\frac{P}{P_{UNI}}$	$-$	0.65	0.65
$\frac{\Delta L_{\min}}{\Delta L_{UNI}}$	$-$	0.975	< 0.01
$\frac{\Delta L}{\Delta L_{UNI}}$	$-$	1.56	1.56
$\frac{L}{L_{UNI}}$	$-$	1.0	1.0
t [s]	$-$	1.4×10^{-1}	3.3×10^{-1}
ξ		3.52×10^{-5}	4.81×10^{-2}

Table IV - G. Oliveri et *al.*, "Bayesian Compressive Sampling for..."

Reference Pattern		*Constraint*	BCS *Indexes*					
L [λ]	PSL [dB]		ξ	P	$\Delta L_{min}[\lambda]$	ΔL [λ]	L [λ]	t [$\times 10^{-1}$s]
19.5	-30	$u_k \notin (0.5, 0.6)$	3.71×10^{-5}	26	0.455	0.776	19.36	2.17
19.5	-30	$u_k \notin (0.8, 1)$	6.81×10^{-5}	21	0.585	0.928	19.50	1.40
19.5	-30	$d_n \notin (5.3, 6.5)$ [λ]	5.82×10^{-6}	36	0.067	0.556	19.47	1.61
19.5	-30	$d_n \notin (0, 1)$ [λ]	4.81×10^{-5}	30	0.029	0.670	19.44	1.65