Journal of Advances in Mathematics and Computer Science

36(3): 11-32, 2021; Article no.JAMCS.66720
ISSN: 2456-9968
(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Bayesian Estimation and Prediction Based on Constant Stress-Partially Accelerated Life Testing for Topp Leone-Inverted Kumaraswamy Distribution

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2021/v36i330344
Editor(s):
(1) Dr. Burcu Gurbuz, Uskudar University, Turkey.
Reviewers:
(1) Mohamed Saber Eliwa (M. S. Eliwa), Mansoura University, Egypt.
(2) Adam, M.B., Universiti Putra Malaysia, Malaysia.
Complete Peer review History: http://www.sdiarticle4.com/review-history/66720

Original Research Article

Received 27 January 2021
Accepted 02 April 2021
Published 17 April 2021

Abstract

Accelerated life testing or partially accelerated life tests is very important in life testing experiments because it saves time and cost. Partially accelerated life tests are used when the data obtained from accelerated life tests cannot be extrapolated to usual conditions. This paper proposes, constant–stress partially accelerated life test using Type II censored samples, assuming that the lifetime of items under usual condition have the Topp Leone-inverted Kumaraswamy distribution. The Bayes estimators for the parameters, acceleration factor, reliability and hazard rate function are obtained. Bayes estimators based on informative priors is derived under the balanced square error loss function as a symmetric loss function and balanced linear exponential loss function as an asymmetric loss function. Also, Bayesian prediction (point and bounds) is considered for a future observation based on Type-II censored under two samples prediction. Numerical studies are given and some interesting comparisons are presented to illustrate the theoretical results. Moreover, the results are applied to real data sets.

Keywords: Topp Leone-inverted Kumaraswamy distribution; censored samples; balanced; square error; LINEX loss functions; Bayesian two-sample prediction; MCMC.

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1 Introduction

Rapid developments, improvements of the high technology, consumer's demands for highly reliable products and competitive markets have placed pressure on manufacturers to deliver products with high quality and reliability. In life testing, it is very difficult to estimate the time of failure for modern high reliability products such as electronics, power cables, metal fatigue, insulating materials, laser, airplane parts, aerospace vehicles, etc.; since these types of products are not likely to fail under usual operating conditions in the relatively short time available for test. For this reason, *accelerated life testing* (ALT) or *partially accelerated life testing* (PALT) are preferred to be used in manufacturing industries to obtain enough failure data in a short period of time and necessary to study its relationship with external stress variables. Such testing could save much time, man power, material sources and money. The stress can be applied in different ways like constant stress, step stress and progressive stress among others.

In constant stress where under one higher than usual stress level, each specimen is run at a constant stress level. In practical use, most products run at constant stress as a constant stress test mimics actual use, it is simple and has a lot of advantages over time-dependent stress loadings because most of real products are operated at a constant-stress condition, see Nelson [1]. For more details about ALT, see Bai and Chung [2], Balakrishnan and Han [3], AL-Dayian et al. [4], also Basak and Balakrishnan [5] among others.

In ALT the main assumption is that a life-stress relationship is known or can be assumed so that the data obtained from accelerated conditions can be extrapolated to usual conditions. In some cases, such relationship cannot be known or assumed. So, PALT are often used in such cases.

In a constant stress-*PALT* (CS-PALT), each test item is run at a constant stress under either usual use condition or accelerated condition only until the test is terminated, and the analysis of PALT has been extensively studied in recent years, [see Bai et al. [6], Ismail [7], Hassan et al. [8], Hyun and Lee [9], EL-Sagheer [10] and Behairy et al. [11]].

The Bayesian method has certain accurate advantages when the sample size is small. When little knowledge about the prior information is available, one can derive the objective Bayes estimates based on non-informative priors, such as the Jeffreys prior, For more details, [see Jeffreys [12], Xu and Tang [13] and Guan et al. [14]]. Recently many authors have studied PALT from the Bayesian view point, such as Jaheen et al. [15] and EL-Sagheer [10]. Also, few studies have considered prediction assuming PALT such as Abushal and AL-Zaydi [16] and Prakash and Singh [17].

The accuracy of parametric statistical inference and modeling of datasets largely depends on how well the probability distribution fits the given dataset once it has met all distributional assumptions. Several studies have been carried out on statistical distributions in the quest to generate distributions with more desirable and flexible properties that can model real-life datasets of varying shapes of density and failure rate functions. Currently, most studies are focused on developing new families that are generalizations of existing distributions to provide better fit to the modeling of data. These families of distributions are constructed by either compounding two or more distributions or adding one or more parameters to the baseline model. Behairy et al. [18] introduced the *Topp Leone-inverted Kumaraswamy* (TL-IK) distribution as a composite distribution of *Topp Leone* (TL(θ)) and *inverted Kumaraswamy* (IK(a,b)) distributions. It's denoted by TL-IK (a,b,θ), its *cumulative distribution function* (cdf) and *probability density function* (pdf) are given, respectively, by:

$$F(x;\underline{\vartheta}) = [\varphi(a)]^{\theta b}\{2-[\varphi(a)]^{b}\}^{\theta}, \qquad 0<x<\infty,\ \underline{\vartheta}>\underline{0}, \tag{1}$$

where

$\underline{\vartheta} = (a,b,\theta)'$ are shape parameters, and

$$\varphi(a) = 1-(1+x)^{-a}, \tag{2}$$

and the pdf corresponding to (1) is given by

$$f(x;\underline{\vartheta}) = 2ab\theta(1+x)^{-(a+1)}[\varphi(a)]^{\theta b-1}\{1-[\varphi(a)]^b\}\{2-[\varphi(a)]^b\}^{\theta-1}, x>0, \underline{\vartheta}>\underline{0}. \quad (3)$$

The *reliability function* (rf) and *hazard rate function* (hrf) are given, respectively, by:

$$R(x_0;\underline{\vartheta}) = P(X \geq x_0) = 1-[\varphi(a)]^{\theta b}\{2-[\varphi(a)]^b\}^{\theta}, \ x_0>0, \underline{\vartheta}>\underline{0}, \quad (4)$$

and

$$h(x_0;\underline{\vartheta}) = \frac{f(x_0;\underline{\vartheta})}{1-F(x_0;\underline{\vartheta})} = \frac{2ab\theta(1+x)^{-(a+1)}[\varphi(a)]^{\theta b-1}\{1-[\varphi(a)]^b\}\{2-[\varphi(a)]^b\}^{\theta-1}}{1-[\varphi(a)]^{\theta b}\{2-[\varphi(a)]^b\}^{\theta}}, x_0>0 \ \underline{\vartheta}>\underline{0}. \quad (5)$$

TL-IK (a,b,θ) distribution contains some special well-known distributions in lifetime, such as the TL-Lomax (Pareto Type II), the TL-log-logistic (Fisk), the Lomax and the log-logistic (Fisk) distributions. Behairy et al. [18] derived some transformed distributions such as the TL-exponentiated Weibull, TL-exponentiated Burr Type XII, TL-Kumaraswamy Dagum and TL-Kumaraswamy-inverse Weibull, among several others. They studied the properties of this distribution; they obtained the stress-strength reliability, moments, moment generating and quantile functions of the TL-IK distribution. Also, they derived the *maximum likelihood* (ML) estimators, asymptotic variances and covariance matrix of the ML estimators and confidence intervals for the parameters; also they obtained the ML two-sample predictors for the future observation based on Type-II censored data.

This paper is organized as follows: in Section 2, a description of the model and basic assumptions are presented. In Section 3, Bayesian point estimation and *credible intervals* (CIs) for the unknown parameters, the acceleration factor, rf and hrf of the TL-IK distribution for CS-PALT based on Type II censored data are obtained based on the *balanced square error loss* (BSEL) and *balanced linear exponential loss* (BLL) functions. Bayesian prediction (point and bounds) for a future observation based on two-sample prediction are considered in Section 4. In Section 5 numerical illustration is presented. Finally, some general conclusions are introduced in Section 6.

2 Model Description and Assumptions

In this section, the description of the model and basic assumptions are presented. Total items are divided randomly into two samples of size $n(1-\pi)$ and $n\pi$, respectively, where π is the sample proportion. The first sample is allocated to usual conditions and the other is assigned to accelerated conditions. Each test item of every sample is run without changing the test condition until reaching the censoring number.

Assumptions

The lifetimes X_i, $i = 1, \ldots, n(1-\pi)$ of items allocated to usual conditions, are independent and identically distributed (i.i.d) random variables.

The lifetimes Y_j , $j = 1, \ldots, n\pi$ of items allocated to accelerated conditions, are i.i.d random variables.

The lifetimes X_i and Y_j are mutually statistically independent.

In this study, the lifetimes of test items are assumed to have a TL-IK distribution. The pdf of an item at usual conditions is given by (3).

The pdf and cdf for an item tested at accelerated conditions are given by:

$$f(y;\underline{\Phi}) = 2ab\theta\beta(1+\beta y)^{-(a+1)}[\varphi(a\beta)]^{\theta b-1}\{1-[\varphi(a\beta)]^b\}\{2-[\varphi(a\beta)]^b\}^{\theta-1},$$
$$y>0;\ \underline{\vartheta}>\underline{0};\ \beta>1, \quad (6)$$

where

$$\underline{\Phi} = (\underline{\vartheta}, \beta)', \underline{\vartheta} = (a, b, \theta)', \ \varphi(a\beta) = 1 - (1 + \beta y)^{-a}, \tag{7}$$

$Y = \beta^{-1}X$, β is the acceleration factor which is the ratio of the mean life at usual condition to that at accelerated condition and $\beta > 1$.

and

$$F(y; \underline{\Phi}) = [\varphi(a\beta)]^{\theta b}\{2 - [\varphi(a\beta)]^b\}^{\theta}, \qquad y > 0;\ \underline{\vartheta} > \underline{0};\ \beta > 1. \tag{8}$$

The rf and hrf for an item tested at accelerated conditions are as follows:

$$R(y_0; \underline{\Phi}) = 1 - [\varphi(a\beta)]^{\theta b}\{2 - [\varphi(a\beta)]^b\}^{\theta}, \qquad y_0 > 0;\ \underline{\vartheta} > \underline{0};\ \beta > 1, \tag{9}$$

and

$$h(y_0; \underline{\Phi}) = \frac{2ab\theta\beta(1+\beta y)^{-(a+1)}[\varphi(a\beta)]^{\theta b-1}\{1-[\varphi(a\beta)]^b\}\{2-[\varphi(a\beta)]^b\}^{\theta-1}}{1-[\varphi(a\beta)]^{\theta b}\{2-[\varphi(a\beta)]^b\}^{\theta}}, y_0 > 0;\ \underline{\vartheta} > \underline{0}; \beta > 1. \tag{10}$$

3 Bayesian Estimation

In this section, the Bayes point and CIs estimation for the parameters, rf and hrf of TL-IK distribution for CS-PALT based on Type II censored sample are derived. The BSEL and BLL loss functions are used.

Bayesian statistics is an effective tool for solving some inference problems in a situation when the available sample is small for more complex statistical analysis to be applied. The lack of information may be offset (up to a certain point) by using the Bayesian approach, as it enables us to utilize more sources of information. Besides the sample data, so-called prior information may be included into the analysis.

In Bayesian analysis, the unknown parameters are considered as random variables and use the prior information about the unknown parameters is considered. The posterior information is obtained to estimate the behavior of the products under usual conditions. The integrations obtained in these cases are complicated, therefore numerical results are performed using *Markov Chain Monte Carlo* (MCMC) method.

Considering, the failure times consist of r^{th} smallest lifetimes $x_{(1)} \le x_{(2)} \le \cdots \le x_{(r)}$ out of a random sample of $n(1-\pi)$ lifetimes $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n(1-\pi))}$ under usual conditions and $y_{(1)} \le y_{(2)} \le \cdots \le y_{(r)}$ out of a random sample of $n\pi$ lifetimes

$Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n\pi)}$ at accelerated conditions, respectively.

The *likelihood function* (LF) for $\{(x_{(i)})\colon\ i = 1, \ldots, n(1-\pi)\}$ at usual conditions is given by

$$L_u(\underline{\vartheta}|\underline{x}) \propto \left[\prod_{i=1}^{r} f(x_{(i)}|\underline{\vartheta})\right]\left[R(x_{(r)}|\underline{\vartheta})\right]^{n(1-\pi)-r}, \tag{11}$$

where $\underline{x} = x_{(1)}, x_{(2)}, \ldots, x_{[n(1-\pi)-r]}$, $\underline{\vartheta} = (a, b, \theta)'$ and $f(x_{(i)}|\underline{\vartheta})$, $R(x_{(r)}|\underline{\vartheta})$ are given by (3) and (4), respectively.

The LF for $\{(y_{(j)})\colon j = 1, \ldots, n\pi)\}$ at accelerated conditions is given by

$$L_g\left(\underline{\Phi}\middle|\underline{y}\right) \propto \left[\prod_{j=1}^{r} f(y_{(j)}|\underline{\Phi})\right]\left[R(y_{(r)}|\underline{\Phi})\right]^{n\pi-r}, \tag{12}$$

where $\underline{y} = y_{(1)}, y_{(2)}, \ldots, y_{(n\pi-r)}$, $\underline{\Phi} = (\underline{\vartheta}, \beta)'$, $f(y_{(j)}|\underline{\Phi})$ and $R(y_{(r)}|\underline{\Phi})$ are given by (6) and (9), respectively.
Let c_u and c_g be the number of censored items at usual and accelerated conditions, respectively, where

$$c_u = n(1-\pi) - r \quad \text{and} \quad c_g = n\pi - r. \tag{13}$$

Substituting (3) and (4) into (11), and substituting (6) and (9) into (12), hence the LF according to CS-PALT, for $\{(x_{(i)}), (y_{(j)}): i = 1, \ldots, n(1-\pi), j = 1, \ldots, n\pi\}$, can be written as:

$$\begin{aligned} L\left(\underline{\Phi}; \underline{x}, \underline{y}\right) \propto\ & (2ab\theta)^{2r}\beta^r\{\prod_{i=1}^{r}\left(1+x_{(i)}\right)^{-a}[\varphi_i(a)]^{\theta b-1}\{1-[\varphi_i(a)]^b\} \\ & \times\{2-[\varphi_i(a)]^b\}^{\theta-1}\}\left\{1-[\varphi_r(a)]^{\theta b}\{2-[\varphi_r(a)]^b\}^{\theta}\right\}^{c_u} \\ & \times\{\prod_{j=1}^{r}\left(1+\beta y_{(j)}\right)^{-(a+1)}\left[\varphi_j(a\beta)\right]^{\theta b-1}\left\{1-\left[\varphi_j(a\beta)\right]^b\right\} \\ & \times\left\{2-\left[\varphi_j(a\beta)\right]^b\right\}^{\theta-1}\}\left\{1-[\varphi_r(a\beta)]^{\theta b}\{2-[\varphi_r(a\beta)]^b\}^{\theta}\right\}^{c_g}, \end{aligned} \tag{14}$$

where

$$\begin{cases} \underline{\Phi} = \left(\underline{\vartheta}, \beta\right)', \varphi_i(a) = 1-\left(1+x_{(i)}\right)^{-a}, \varphi_r(a) = 1-\left(1+x_{(r)}\right)^{-a}, \\ \quad \varphi_j(a\beta) = 1-\left(1+\beta y_{(j)}\right)^{-a},\ \varphi_r(a\beta) = 1-\left(1+\beta y_{(r)}\right)^{-a}. \end{cases} \tag{15}$$

The conjugate informative priors are assumed as prior distributions for the parameters. Further that, the elements of the parameters vector $\underline{\Phi} = \left(\underline{\vartheta}, \beta\right)'$ are assumed to be independent and each has gamma distribution, that $\Phi_i \sim$ gamma (α_i, γ_i) and α_i, γ_i are the hyper-parameters of the prior distribution for $i = 1, \ldots, 4$. Then the joint prior distribution of all the unknown parameters has a joint pdf given by

$$\pi\left(\underline{\Phi}\right) \propto \prod_{i=1}^{4} \Phi_i^{\alpha_i - 1}\exp(-\gamma_i\Phi_i), \qquad \beta > 1;\ (\underline{\vartheta}_j, \alpha_i, \gamma_i) > 0, \tag{16}$$

where $\Phi_1 = a, \Phi_2 = b, \Phi_3 = \theta$ and $\Phi_4 = \beta, \underline{\vartheta} = (a, b, \theta)'$.

Combining the LF in (14) and the joint prior distribution given by (16), then the joint posterior distribution for $\underline{\Phi}$ can be obtained as follows:

$$\begin{aligned} \pi\left(\underline{\Phi}\middle|\underline{x}, \underline{y}\right) =\ & A2^{2r}(ab\theta)^{2r}\beta^r\Phi_i^{\alpha_i-1}\exp(-\gamma_i\Phi_i)\{\textstyle\prod_{i=1}^{r}\left(1+x_{(i)}\right)^{-a}[\varphi_i(a)]^{\theta b-1} \\ & \times\{1-[\varphi_i(a)]^b\}\{2-[\varphi_i(a)]^b\}^{\theta-1}\}\left\{1-[\varphi_r(a)]^{\theta b}\{2-[\varphi_r(a)]^b\}^{\theta}\right\}^{c_u} \\ & \times\{\textstyle\prod_{j=1}^{r}\left(1+\beta y_{(j)}\right)^{-(a+1)}\left[\varphi_j(a\beta)\right]^{\theta b-1}\left\{1-\left[\varphi_j(a\beta)\right]^b\right\} \\ & \times\left\{2-\left[\varphi_j(a\beta)\right]^b\right\}^{\theta-1}\}\left\{1-[\varphi_r(a\beta)]^{\theta b}\{2-[\varphi_r(a\beta)]^b\}^{\theta}\right\}^{c_g}, \end{aligned} \tag{17}$$

where $\underline{\Phi}, \varphi_i(a), \varphi_r(a), \varphi_j(a\beta)$ and $\varphi_r(a\beta)$ are given by (15) and A is the normalizing constant which can be determined as follows:

$$\int_{\underline{\Phi}} \pi\left(\underline{\Phi}\middle|\underline{x}, \underline{y}\right) d\underline{\Phi} = 1, \tag{18}$$

where

$$\int_{\underline{\Phi}} = \int_a\int_b\int_\theta\int_\beta \quad \text{and} \quad d\underline{\Phi} = da\ db\ d\theta\ d\beta. \tag{19}$$

3.1 Bayesian estimation based on balanced loss functions

Zellner [19] introduced the class of *balanced loss function* (BLF). An extended class of BLF was introduced by Jozani et al. [20] with the following form:

$$L^*(\theta,\theta^*) = \omega l(\hat{\theta}_0,\ \theta) + (1-\omega)\ l(\theta,\theta^*), \tag{20}$$

where $l(\theta,\theta^*)$ is an arbitrary loss function, when $\hat{\theta}_0$ is a chosen estimator of θ and the weight $\omega\epsilon[0,1]$. The BLF can be specialized to various choices of loss functions such as the squared error, absolute error, entropy, *linear exponential* (LINEX) loss functions, also the BLF generalizes the SEL function. Ahmadi et al. [21] suggested the BSEL function by substituting $l(\theta,\theta^*) = (\theta^*-\theta)^2$ in (20), hence the BSEL function has the following form:

$$L_1^*(\theta,\theta^*) = \omega\left(\theta^*-\hat{\theta}_0\right)^2 + (1-\omega)(\theta^*-\theta)^2,$$

the corresponding Bayes estimator; $\tilde{u}_{\mathrm{BSEL}}$, of a function; $u\ (\theta)$, using BSEL function is given by

$$\tilde{u}_{\mathrm{BSEL}} = \omega\tilde{u}_{\mathrm{ML}} + (1-\omega)\tilde{u}_{\mathrm{SEL}}, \tag{21}$$

where $\tilde{u}_{\mathrm{ML}}$ is the ML estimator of $u\ (\theta)$ and $\tilde{u}_{\mathrm{SEL}}$ is the Bayes estimator using SEL function.

Also, if $l(\theta,\theta^*) = e^{v(\theta^*-\theta)} - v(\theta^*-\theta) - 1$, is substituted in (20), the BLL function is

$$L_2^*(\theta^*,\theta) = \omega[e^{v(\theta^*-\hat{\theta}_0)} - v\left(\theta^*-\hat{\theta}_0\right) - 1] + (1-\omega)[e^{v(\theta^*-\theta)} - v(\theta^*-\theta) - 1],$$

then the Bayes estimator using the BLL function of θ takes the form

$$\hat{\hat{\theta}}(\mathrm{x}) = -\frac{1}{v}In\left[\omega e^{-v\hat{\theta}_0} + (1-\omega)E\left(e^{-v\theta}\middle|x\right)\right], \tag{22}$$

where $v \neq 0$ is the shape parameter.

The estimator of a function, using BLF is actually a mixture of the ML estimator of the function and the Bayes estimators using any loss function. Other estimators, such as the least squares estimator may replace the ML estimator. Many authors obtained the Bayes estimators for different other distributions using the symmetric and asymmetric BLF, [see AL-Hussaini and Hussein [22], Al-Zahrani and Al-Sobhi [23], Abushal and AL-Zaydi [16]].

In this paper, the BSEL and BLL functions are considered to obtain the Bayes estimators for the unknown parameters and the acceleration factor β of TL-IK($\underline{\vartheta}$, β) distribution.

3.2 Bayesian estimation under balanced squared error loss function

Let $\underline{\mathrm{X}}$ and $\underline{\mathrm{Y}}$ are censored samples of size *r* obtained from a life-test on n items (Type II censored sample) whose lifetimes have a TL-IK ($\underline{\vartheta}$, β) distribution, and the BSEL function is defined by (21).

(a) Bayesian estimation for the parameters

From (17) and (21) the Bayes estimators for the parameters under BSEL function can be derived, respectively, as follows:

$$\tilde{\Phi}_{\mathrm{i}_{\mathrm{BSEL}}} = \omega\hat{\Phi}_{\mathrm{iML}} + (1-\omega)\int_{\underline{\Phi}}\Phi_{\mathrm{i}}\pi\left(\Phi_{\mathrm{i}}\middle|\underline{\mathrm{x}},\underline{\mathrm{y}}\right)\ \mathrm{d}\underline{\Phi}, \tag{23}$$

where $\pi\left(\Phi_{\mathrm{i}}\middle|\underline{\mathrm{x}},\underline{\mathrm{y}}\right)$ is given by (17), and $\int_{\underline{\Phi}}$, $\mathrm{d}\underline{\Phi}$, are given by (19).
One can obtain the Bayes estimators for the parameters under the BSEL function, by substituting Φ_{i} by $a, b, \theta\ or\ \beta$ in (23).

(b) Bayesian estimation for the rf and hrf

The Bayes estimators for the rf and hrf; under the BSEL function, assuming that the parameters are unknown, are given below

The Bayes estimator for the rf is

$$\tilde{R}_{BSEL}(y) = \omega\hat{R}_{ML}(y) + (1-\omega)\int_{\underline{\Phi}} R(y)\pi\left(\underline{\Phi}\middle|\underline{x},\underline{y}\right)\, d\underline{\Phi}. \tag{24}$$

The Bayes estimator for the hrf is

$$\tilde{h}_{BSEL}(y) = \omega\hat{h}_{ML}(y) + (1-\omega)\int_{\underline{\Phi}} h(y)\pi\left(\underline{\Phi}\middle|\underline{x},\underline{y}\right)\, d\underline{\Phi}, \tag{25}$$

where $\hat{R}_{ML}(y)$ and $\hat{h}_{ML}(y)$ are the estimators for $R(y)$ and $h(y)$ using the ML method and $\hat{\Phi}_{iML}$ are the ML estimators of $a, b, \theta\ or\ \beta$.

3.3 Bayesian estimation under balanced linear exponential loss function

Let $\underline{X}$ and $\underline{Y}$ are censored samples of size r obtained from a life-test on n items (Type II censored sample) whose lifetimes have a TL-IK $(\underline{\vartheta}, \beta)$ distribution and the BLL function is defined by (22).

(a) Bayesian estimation for the parameters

From (17) and (22) the Bayes estimators for the parameters under BLL function can be derived, respectively, as follows:

$$\hat{\hat{\Phi}}_{i_{BLL}} = \frac{-1}{v} In\{\omega \exp\left(-v\hat{\Phi}_{iML}\right) + (1-\omega)\int_{\underline{\Phi}} \exp(-\Phi_i v)\, \pi\left(\Phi_i\middle|\underline{x},\underline{y}\right)\, d\underline{\Phi}. \tag{26}$$

One can obtain the Bayes estimators of the parameters under BLL function, by substituting Φ_i by a, b, θ or β in (26).

(b) Bayesian estimation for the rf and hrf

The Bayes estimators of the rf and hrf based on BLL function when the parameters are unknown are derived below

The Bayes estimator of the rf is given by

$$\hat{\hat{R}}_{BLL}(y) = \frac{-1}{v} In\{\omega \exp\left(-v\hat{R}_{ML}(y)\right) + (1-\omega)\int_{\underline{\Phi}} \exp(-R(y)v)\, \pi\left(\Phi_i\middle|\underline{x},\underline{y}\right)\, d\underline{\Phi}\}. \tag{27}$$

The Bayes estimator of the hrf is given by

$$\hat{\hat{h}}_{BLL}(y) = \frac{-1}{v} \ln\left\{\omega \exp\left(-v\hat{h}_{ML}(y)\right) + (1-\omega)\int_{\underline{\Phi}} \exp(-h(y)v)\, \pi\left(\Phi_i\middle|\underline{x},\underline{y}\right)\, d\underline{\Phi}\right\}, \tag{28}$$

where $\hat{R}_{ML}(y)$ and $\hat{h}_{ML}(y)$ are the estimators of $R(y)$ and $h(y)$ using the ML method and $\hat{\Phi}_{iML}$ are the ML estimators of a, b, θ or β.

3.4 Credible intervals

In general $\left(L(\underline{x}, \underline{y}) < \Phi_i < U(\underline{x}, \underline{y})\right)$ are 100(1-τ) % CIs for Φ_i ,

where $\Phi_i = a, b, \theta$ or β if

$$P\left((L(\underline{x}\,,\underline{y}) < \Phi_i < U(\underline{x}\,,\underline{y}))\middle|\underline{x}\,,\underline{y}\right) = \int_{L(\underline{x},\underline{y})}^{U(\underline{x},\underline{y})} \pi\left(\Phi_i\middle|\underline{x}\,,\underline{y}\right)\, d\Phi_i = 1-\tau, \tag{29}$$

The lower and upper bounds $\left[L(\underline{x}\,,\underline{y}),\ U(\underline{x}\,,\underline{y})\right]$ can be obtained by evaluating,

$$P(\Phi_i > L(\underline{x}\,,\underline{y})\left|\underline{x}\,,\underline{y}\right) = 1-\frac{\tau}{2} \qquad \text{and} \qquad P(\Phi_i > U(\underline{x}\,,\underline{y})\left|\underline{x}\,,\underline{y}\right) = \frac{\tau}{2}.$$

4 Bayesian Prediction Based on Two-Sample Prediction

Prediction for future events on the basis of the past and present knowledge is a fundamental problem of statistics, arising in many contexts.

Assuming that $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(r)}$ are the first *r* ordered life times in a random sample of n components (Type II censoring) whose failure times are independent and identically distributed as random variables Y`s having the pdf for an item tested at accelerated conditions is given by (6) which is an informative sample, and that $T_{(1)}, T_{(2)}, \ldots, T_{(m)}$ is a future independent random sample (of size m) from the same distribution. Our aim is to predict a statistic in the future sample based on the informative sample [see Kaminsky and Rhodin [24], Valiollahi et al. [25], Abushal and AL-Zaydi [16], Ateya and Mohammed [26] and Raqab et al. [27]].

For the future sample of size m, let $T_{(s)}$ denotes the s^{th} order statistic, $1 \le s \le m$, then he pdf of $T_{(s)}$ is given by

$$f_{s:m}\left(t_{(s)}\right) = D(s) f\left(t_{(s)}|\underline{\Phi}\right)\left[F\left(t_{(s)}|\underline{\Phi}\right)\right]^{s-1}\left[1-F\left(t_{(s)}|\underline{\Phi}\right)\right]^{m-s} \quad , \quad t_{(s)} > 0,$$
$$\underline{\Phi} = \left(\underline{\vartheta},\beta\right)', D(s) = \frac{1}{B(s,m-s+1)}, s = 1,2,3,\ldots,m, \tag{30}$$

using the binomial expansion theorem for $\left[1-F\left(t_{(s)}|\underline{\Phi}\right)\right]^{m-s}$, yields

$$f_{s:m}\left(t_{(s)}\right) = D(s) f\left(t_{(s)}\right) \sum_{l_7=0}^{m-s} (-1)^{l_7}\binom{m-s}{l_7}\left[F\left(t_{(s)}\right)\right]^{s+l_7-1}, \tag{31}$$

and substituting (6) and (8) into (31), then one can obtain the pdf of s^{th} order statistic for an item tested at accelerated conditions:

$$h\left(t_{(s)}|\underline{\Phi}\right) = 2ab\theta\beta D(s) \sum_{l_7=0}^{m-s} \delta\left(1+\beta t_{(s)}\right)^{-(a+1)}\left[\varphi_s(a\beta)\right]^{\theta b(s+l_7)-1}\left\{1-\left[\varphi_s(a\beta)\right]^b\right\}$$
$$\times\left\{2-\left[\varphi_s(a\beta)\right]^b\right\}^{\theta(s+l_7)-1}, t_{(s)} > 0; \beta > 1; \left(\underline{\vartheta} > \underline{0}\right), \tag{32}$$

where

$$\sum_{l_7=0}^{m-s} \delta = \sum_{l_7=0}^{m-s} (-1)^{l_7}\binom{m-s}{l_7}, \underline{\vartheta} = (a,b,\theta)' \text{ and } \varphi_s(a\beta) = 1-\left(1+\beta t_{(s)}\right)^{-a}. \tag{33}$$

Assuming that the parameters $\underline{\Phi}$ are unknown and independent, then the *Bayesian predictive density* (BPD) of $T_{(s)}$ given $\underline{x}\,,\underline{y}$; based on the informative prior, can be obtained, as follows:

$$h\left(t_{(s)}\middle|\underline{x},\underline{y}\right) = \int_{\underline{\Phi}} h\left(t_{(s)}|\underline{\Phi}\right)\pi\left(\underline{\Phi}\middle|\underline{x},\underline{y}\right) d\underline{\Phi}, \tag{34}$$

where

$\pi\left(\underline{\Phi}\middle|\underline{x},\underline{y}\right)$, $\int_{\underline{\Phi}} d\underline{\Phi}$ and $h\left(t_{(s)}|\underline{\Phi}\right)$ are given by (17), (19) and (32), respectively.

Substituting (17) and (32) into (34), then the BPD of $T_{(s)}$ given $\underline{x}\,,\underline{y}$ is given by

$$h\left(t_{(s)}\middle|\underline{x},\underline{y}\right) = A2^{2r+1}D(s)\sum_{l_7=0}^{m-s}\delta\int_{\underline{\Phi}}\Phi_i^{\alpha_i}\exp(-\gamma_i\Phi_i)\,(ab\theta)^{2r}\beta^r\left(1+\beta t_{(s)}\right)^{-(a+1)}$$
$$\times\left[\varphi_s(a\beta)\right]^{\theta b(s+l_7)-1}\left\{1-\left[\varphi_s(a\beta)\right]^b\right\}\left\{2-\left[\varphi_s(a\beta)\right]^b\right\}^{\theta(s+l_7)-1}$$
$$\times\left\{\prod_{i=1}^{r}\left(1+x_{(i)}\right)^{-a}\left[\varphi_i(a)\right]^{\theta b-1}\left\{1-\left[\varphi_i(a)\right]^b\right\}\left\{2-\left[\varphi_i(a)\right]^b\right\}^{\theta-1}\right\}$$
$$\times\left\{1-\left[\varphi_r(a)\right]^{\theta b}\left\{2-\left[\varphi_r(a)\right]^b\right\}^{\theta}\right\}^{c_u}\left\{\prod_{j=1}^{r}\left(1+\beta y_{(j)}\right)^{-(a+1)}\left[\varphi_j(a\beta)\right]^{\theta b-1}\right.$$
$$\left.\times\left\{1-\left[\varphi_j(a\beta)\right]^b\right\}\left\{2-\left[\varphi_j(a\beta)\right]^b\right\}^{\theta-1}\right\}$$
$$\times\left\{1-\left[\varphi_r(a\beta)\right]^{\theta b}\left\{2-\left[\varphi_r(a\beta)\right]^b\right\}^{\theta}\right\}^{c_g}d\underline{\Phi}, \quad (35)$$

where $\varphi_i(a)$, $\varphi_r(a)$, $\varphi_j(a\beta)$ and $\varphi_r(a\beta)$ are given by (15), $\int_{\underline{\Phi}} d\underline{\Phi}$, $D(s)$, $\varphi_s(a\beta)$ and $\sum_{l_7=0}^{m-s}\delta$ are defined in (19), (30) and (33), respectively, and α_i, γ_i are the hyper-parameters of the prior distribution for $i = 1, \ldots, 4$. A is the normalizing constant which is given by (18).

4.1 Point predictor

Based on Type II censoring, the Bayes point predictor is considered under two types of loss functions BSEL function, as a symmetric loss function and BLL loss function, as an asymmetric loss function.

(a) Balanced squared error loss function

The *Bayes predictor* (BP) for the future observation $T_{(s)}$, under BSEL function can be derived using the following equation

$$\hat{t}_{(s)(BSEL)} = \omega\hat{t}_{(s)(ML)} + (1-\omega)E\left(t_{(s)}\middle|\underline{x},\underline{y}\right), \quad (36)$$

where $\hat{t}_{(s)(ML)}$ is the ML predictor for the future observation $t_{(s)}$ and $E\left(t_{(s)}\middle|\underline{x},\underline{y}\right)$ can be obtained using

$$E\left(t_{(s)}\middle|\underline{x},\underline{y}\right) = \int_{\underline{\Phi_*}} t_{(s)}\,h\left(t_{(s)}\middle|\underline{x},\underline{y}\right)d\underline{\Phi_*}$$
$$= A2^{2r+1}D(s)\sum_{l_7=0}^{m-s}\delta\int_{\underline{\Phi_*}}\Phi_i^{\alpha_i}\exp(-\gamma_i\Phi_i)\,(ab\theta)^{2r}\beta^r\,t_{(s)}\left(1+\beta t_{(s)}\right)^{-(a+1)}$$
$$\times\left[\varphi_s(a\beta)\right]^{\theta b(s+l_7)-1}\left\{1-\left[\varphi_s(a\beta)\right]^b\right\}\left\{2-\left[\varphi_s(a\beta)\right]^b\right\}^{\theta(s+l_7)-1}$$
$$\times\left\{\prod_{i=1}^{r}\left(1+x_{(i)}\right)^{-a}\left[\varphi_i(a)\right]^{\theta b-1}\left\{1-\left[\varphi_i(a)\right]^b\right\}\left\{2-\left[\varphi_i(a)\right]^b\right\}^{\theta-1}\right\}$$
$$\times\left\{1-\left[\varphi_r(a)\right]^{\theta b}\left\{2-\left[\varphi_r(a)\right]^b\right\}^{\theta}\right\}^{c_u}\left\{\prod_{j=1}^{r}\left(1+\beta y_{(j)}\right)^{-(a+1)}\left[\varphi_j(a\beta)\right]^{\theta b-1}\right.$$
$$\left.\times\left\{1-\left[\varphi_j(a\beta)\right]^b\right\}\left\{2-\left[\varphi_j(a\beta)\right]^b\right\}^{\theta-1}\right\}$$
$$\times\left\{1-\left[\varphi_r(a\beta)\right]^{\theta b}\left\{2-\left[\varphi_r(a\beta)\right]^b\right\}^{\theta}\right\}^{c_g}d\underline{\Phi_*}, \quad (37)$$

where

$$\int_{\underline{\Phi_*}} = \int_{t_{(s)}}\int_{\beta}\int_{\theta}\int_{b}\int_{a} \text{ and } d\underline{\Phi_*} = da\,db\,d\theta\,d\beta\,dt_{(s)}. \quad (38)$$

(b) Balanced linear exponential loss function

The BP for the future observation $T_{(s)}$, under BLL function can be obtained using the following equation:

$$\hat{t}_{(s)(BLL)} = \frac{-1}{v}In\left\{\omega\exp\left(-v\hat{t}_{(s)(ML)}\right) + (1-\omega)E\left(\exp\left(-v\,t_{(s)}\right)\middle|\underline{x},\underline{y}\right)\right\}, \quad (39)$$

where $\hat{t}_{(s)(ML)}$ is the ML predictor for the future observation $t_{(s)}$ and

$E\left(\exp(-v\, t_{(s)})\,\middle|\,\underline{x},\underline{y}\right)$ can be obtained as given below

$$E\left(\exp(-v\, t_{(s)})\,\middle|\,\underline{x},\underline{y}\right) = \int_{t_{(s)}} \exp(-v\, t_{(s)})\, h\left(t_{(s)}\,\middle|\,\underline{x},\underline{y}\right)\, dt_{(s)}$$

$$= A2^{2r+1}D(s)\sum_{l_7=0}^{m-s}\delta\int_{\underline{\Phi_*}}\Phi_i^{\alpha_i}(ab\theta)^{2r}\beta^r\exp\left[-\left(\gamma_i\Phi_i+v\,t_{(s)}\right)\right]$$
$$\times\left(1+\beta t_{(s)}\right)^{-(a+1)}\left[\varphi_s(a\beta)\right]^{\theta b(s+l_7)-1}\left\{1-\left[\varphi_s(a\beta)\right]^b\right\}$$
$$\times\left[2-\left(\varphi_s(a\beta)\right)^b\right]^{\theta(s+l_7)-1}\{\prod_{i=1}^{r}\left(1+x_{(i)}\right)^{-a}\left[\varphi_i(a)\right]^{\theta b-1}$$
$$\times\left\{1-\left[\varphi_i(a)\right]^b\right\}\left\{2-\left[\varphi_i(a)\right]^b\right\}^{\theta-1}\}$$
$$\times\left\{1-\left[\varphi_r(a)\right]^{\theta b}\left\{2-\left[\varphi_r(a)\right]^b\right\}^{\theta}\right\}^{c_u}\{\prod_{j=1}^{r}\left(1+\beta y_{(j)}\right)^{-(a+1)}$$
$$\times\left[\varphi_j(a\beta)\right]^{\theta b-1}\left\{1-\left[\varphi_j(a\beta)\right]^b\right\}\left\{2-\left[\varphi_j(a\beta)\right]^b\right\}^{\theta-1}\}$$
$$\times\left\{1-\left[\varphi_r(a\beta)\right]^{\theta b}\left\{2-\left[\varphi_r(a\beta)\right]^b\right\}^{\theta}\right\}^{c_g}d\underline{\Phi_*}, \tag{40}$$

where

$\int_{\underline{\Phi_*}}$ and $d\underline{\Phi_*}$ are defined in (38), $s = 1, 2, 3, \dots, m$, and α_i, γ_i are the hyper-parameters of the prior distribution for $i = 1, \dots, 4$.

4.2 Bayesian predictive bounds

In general $P\left(L_{(s)}(\underline{x},\underline{y}) < T_{(s)} < U_{(s)}(\underline{x},\underline{y})\,\middle|\,\underline{x},\underline{y}\right)$ is a 100(1-τ) % *Bayesian predictive bounds* (BPB) for any future observation $T_{(s)}$, can be obtained by

$$P\left(L_{(s)}(\underline{x},\underline{y}) < T_{(s)} < U_{(s)}(\underline{x},\underline{y})\,\middle|\,\underline{x},\underline{y}\right) = \int_{L(\underline{x},\underline{y})}^{U(\underline{x},\underline{y})} h\left(t_{(s)}\,\middle|\,\underline{x},\underline{y}\right)\, dt_{(s)} = 1-\tau.$$

A 100(1-τ) % BPB *lower limit* (LL) and *upper limit* (UL) for the future observation $T_{(s)}$ is obtained by solving the following two nonlinear equations,

$$P\left(T_{(s)} \geq L(\underline{x},\underline{y})\,\middle|\,\underline{x},\underline{y}\right) = 1-\frac{\tau}{2} \quad \text{and} \quad P\left(T_{(s)} \geq U(\underline{x},\underline{y})\,\middle|\,\underline{x},\underline{y}\right) = \frac{\tau}{2}. \tag{41}$$

Then the BPB for $T_{(s)}$ can be derived as follows:

$$P\left(T_{(s)} \geq q_s\,\middle|\,\underline{x},\underline{y}\right) = \int_{q_s}^{\infty} h\left(t_{(s)}\,\middle|\,\underline{x},\underline{y}\right)\, dt_{(s)},$$

Using (35), and substituting q_s by $L_{(s)}(\underline{x},\underline{y})$ and $U_{(s)}(\underline{x},\underline{y})$ then solving two nonlinear equations simultaneously, one obtains

$$
\begin{aligned}
P\left(T_{(s)} \geq q_s \middle| \underline{x}, \underline{y}\right) = {} & A2^{2r+1}D(s) \sum_{l_7=0}^{m-s} \delta \int_{q_s}^{\infty} \int_{\underline{\Phi}} \Phi_i^{\alpha_i} \exp\left(-\gamma_i \Phi_i\right) (ab\theta)^{2r} \beta^r \\
& \times \left(1 + \beta t_{(s)}\right)^{-(a+1)} \left[\varphi_s(a\beta)\right]^{\theta b(s+l_7)-1} \left\{1 - \left[\varphi_s(a\beta)\right]^b\right\} \\
& \times \left\{2 - \left[\varphi_s(a\beta)\right]^b\right\}^{\theta(s+l_7)-1} \{\prod_{i=1}^{r} \left(1 + x_{(i)}\right)^{-a} \left[\varphi_i(a)\right]^{\theta b-1} \\
& \times \left\{1 - \left[\varphi_i(a)\right]^b\right\} \left\{2 - \left[\varphi_i(a)\right]^b\right\}^{\theta-1} \} \\
& \times \left\{1 - \left[\varphi_r(a)\right]^{\theta b} \left\{2 - \left[\varphi_r(a)\right]^b\right\}^{\theta}\right\}^{c_u} \{\prod_{j=1}^{r} \left(1 + \beta y_{(j)}\right)^{-(a+1)} \\
& \times \left[\varphi_j(a\beta)\right]^{\theta b-1} \left\{1 - \left[\varphi_j(a\beta)\right]^b\right\} \left\{2 - \left[\varphi_j(a\beta)\right]^b\right\}^{\theta-1} \} \\
& \times \left\{1 - \left[\varphi_r(a\beta)\right]^{\theta b} \left\{2 - \left[\varphi_r(a\beta)\right]^b\right\}^{\theta}\right\}^{c_g} d\underline{\Phi} dt_{(s)}, \qquad (42)
\end{aligned}
$$

where $s = 1, 2, 3, \ldots, m$.

Remark

If $s = 1$, $s = m$ and $s = \frac{m+1}{2}$ in (36) and (39), one can predict the minimum observable; $T_{(1)}$, the maximum observable; $T_{(m)}$, and the median observable if m is odd; $T_{\left(\frac{m+1}{2}\right)}$, under BSEL and BLL functions, respectively.

5 Numerical Illustration

This section aims to investigate the precision of the theoretical results of estimation and prediction on the basis of the simulated and real data.

5.1 Simulation algorithm

In this subsection, a simulation study is conducted to illustrate the performance of the presented Bayes estimates on the basis of the generated data from the TL-IK (a, b, θ) distribution considering the CS-PALT. The *estimated risks* (ERs) and CIs for the parameters, rf and hrf under Type II censoring are computed. Also, the Bayes two-sample predictors (point and bounds) are calculated. Simulation studies are performed using R programming language for illustrating the obtained results.

The steps of the simulation procedure based on Type II censored data are as follows:

a. For given values of $\underline{\vartheta_i}$, random samples of size n are generated from the TL-IK (a, b, θ) distribution.
b. For given values of $\underline{\vartheta_j}$, random samples of size n are generated from the TL-IK (a, b, θ) distribution.

- The transformation between the uniform distribution and TL-IK distribution is obtained as follows:

$$
x = \left[\left(1 - (u)^{\frac{1}{b}}\right)^{-\frac{1}{a}} - 1\right]\left(1 - \sqrt{1 - (u)^{\frac{1}{\theta}}}\right), \quad 0 < u < 1.
$$

c. For each sample size x_i`s, are sorted, such that $x_1 \leq x_2 \leq \cdots \leq x_{n(1-\pi)}$.
d. For each sample size y_j`s, are sorted, such that $y_1 \leq y_2 \leq \cdots \leq y_{n\pi}$.
e. Considering two different proportions π of the sample items allocated to accelerated conditions, $\pi = 20\%$ and $\pi = 30\%$, under Type-II censored data.
f. The number of failures r are chosen to be less than $n(1 - \pi)$ and $n\pi$.
g. All the previous steps are repeated N times, where N represents a fixed number of simulated samples, where N =10000 is the number of repetitions.

- Simulation studies have been performed using (R programming language) for illustrating the theoretical results of estimation problem. The performance of the resulting estimators of the acceleration, shape parameters and acceleration factor have been considered in terms of their averages and ERs, where
$\overline{\widehat{\Phi}_i} = \frac{\sum_{j=1}^{N} \widehat{\Phi}_i^j}{N}$, $\Phi_i = a, b, \theta$ and β and ER= $\frac{\sum_{j=1}^{N}(\text{estimate}-\text{true value})^2}{N}$.
- The Bayes predictors (point and bound) for a future observation from the TL-IK distribution based on Type II censored data are computed for the two-sample case.
- Simulation results of the Bayes estimates under SEL function and LINEX loss function are displayed in Tables 1-4, where the samples of size (n=30, 60, 100), are used. For each sample size, the censoring level is 10% and the chosen parameters value are selected as (Case 1 $a = 1.2,\ b = 0.9, \theta = 0.8,\ \beta = 1.,\ y_0 = 0.1, \pi = 0.20$ and $\pi = 0.30$) , (Case 2, $a = 1.2,\ b = 0.9, \theta = 0.8,\ \beta = 1.6,\ y_0 = 0.1, \pi = 0.30$) and (Case 3, $a = 1.2,\ b = 0.9, \theta = 0.8,\ \beta = 2.1,\ y_0 = 0.1, \pi = 0.30$)
- The set of hyper parameters is ($\alpha_i = 10, 0.5, 0.1, 0.5,\ c_i = 10, 10, 20, 15$), for $i = 1, \ldots, 4$ respectively.
- Gamma distribution is used as informative prior.

h. Simulation results of the Bayes estimates for the parameters and acceleration factor under BSEL and BLL functions are computed, from (36) and (39), according to the MCMC method, and presented in Table 5, where the samples of size (n=30, $\pi = 0.20$ and $r = 0.1$).

i. The Bayes two-sample point predictors for the future observation $t_{(s)}$, under SEL and LINEX loss functions are computed using (37) and (40). Also, by solving (41) one can get the 95% BPB for the s^{th} order statistics in a future Type-II censored sample.

- Bayes predictive points and bounds for some order statistics of future sample based on informative prior are displayed in Table 7.

5.2 Some applications

This subsection aims to demonstrate how the proposed method can be used in practice. Two real lifetime data sets are used for this purpose. The TL-IK (a, b, θ) distribution is fitted to the two real data using Kolmogorov-Smirnov goodness of fit test through the R programming language.

Application 1

The data is given by Murthy et al. [28] which refers to the time between failures for a repairable item: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86 and 1.17.

Application 2

The second application is given by Dumonceaux and Antle [29]. With respect to the maximum flood level (in millions of cubic feet per second) for the Susquehanna River at Harrisburg, Pennsylvania. Each number is the maximum flood level for a four year period, the first, 0.654, being for the period 1890-1893, and the last, 0.265, being for the period 1966-1969. The data is 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.3235 0.269, 0.740, 0.418, 0.412 0.494, 0.416, 0.338, 0.392, 0.484, 0.265.

The Kolmogorov–Smirnov goodness of fit test is applied to check the validity of the fitted model. The p values are given, respectively, by 0.5860 and 0.6465. The p value given in each case showed that the model fits the data very well.

Table 6 indicates that the Bayes estimates and *standard errors* (SEs) based on informative prior. Also, Bayes point predictors and bounds of some order statistics for future sample based on informative prior are displayed in Table 8.

5.3 Concluding remarks

a. It is observed from Tables 1-4 that the ERs perform better when the sample size increases.
b. Tables 1-4 indicate that the intervals of the parameters become narrower as the sample size increases.
c. It is noticed from Table 5, that the Bayes estimates under the BSEL and BLL functions have the smallest ERs regard to their corresponding ML estimates.
d. From Table 7 and 8, one can observe that the length of the interval of the first order statistic is smaller than the length of the interval of the last order statistic. Also, the BPB include the predictive values (between the LL and UL).
e. From the results of Tables 1-4, one can notice that the acceleration factor when β increases, ERs decrease.
f. Tables 1 and 2 show that when the proportion π of the sample items allocated to accelerated conditions decreases, the ERs of the parameters and acceleration factor, perform better.

6 General Conclusion

For products having a high reliability, the test of product life under usual conditions often requires a long period of time. So ALT or PALT is used to facilitate estimating the reliability of the unit in a short period of time. In ALT test items are run only at accelerated conditions, in some cases, such relationship cannot be known or assumed. Therefore, PALT are often used in such cases, in PALT they are run at both usual and accelerated conditions. This study proposed a CS-PALT in the case of Type II censoring, where each test item is run at a constant stress under either usual use condition or accelerated condition until the test is terminated. It is assumed that the lifetime of test units has the TL-IK distribution. The results of the Bayes estimators are presented based on Type II censored sample for the shape parameters, the acceleration factor, rf and hrf of TL-IK distribution. The Bayes estimators are derived under BSEL and BLL functions. The BLF is a mixture of Bayesian and non-Bayesian estimates. Bayes point and bound prediction for a new observation of CS-PALT from TL-IK distribution, based on Type II censored sample are derived. Informative prior are obtained. The numerical computations are carried out to illustrate the performance of the procedure. Moreover, the results are applied on real data sets.

In general, it is noticed that the ERs perform better when the weight ω decreases. It appears using informative prior, the ERs of the estimated parameters, the acceleration factor and the interval length decrease when the sample size increases. The length of the interval of the first order statistic is smaller than the length of the interval of the last order statistic.

Bayesian estimation under different types of loss functions such as general entropy and precautionary loss functions for estimating the parameters of TL-IK distribution would be useful as a basis for further research in distribution theory.

Acknowledgements

The authors would like to thank the Referees and the Editor for their helpful comments and suggestions which led to improvements of an earlier version of this article.

Competing Interests

Authors have declared that no competing interests exist.

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Appendix

Table 1. The averages, ERs, and 95% CIs for the parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\theta}, \boldsymbol{\beta}$, rf and hrf, using informative prior under SEL function based on Type II censoring (N=10000, $\boldsymbol{\pi = 0.20}$, $\boldsymbol{\pi = 0.30}$ and r=0.1n) (Case 1, $\boldsymbol{a = 1.2}$, $\boldsymbol{b = 0.9}, \boldsymbol{\theta = 0.8}$, $\boldsymbol{\beta = 1.1}$, $\boldsymbol{y_0 = 0.1}$)

n	*r*	π	Parameters, rf and hrf	Averages	ERs	UL	LL	length
30	3	0.20	a	1.20069	1.71573E-6	1.20309	1.19896	0.00413
			b	0.90029	1.23862E-6	0.90214	0.89808	0.00406
			θ	0.79869	2.39626E-6	0.80056	0.79806	0.00399
			β	1.10187	4.25719E-6	1.10300	0.79658	0.00346
			$R(y_0)$	0.66750	1.86108E-6	0.66887	0.66581	0.00306
			$h(y_0)$	5.56872	2.19477E-6	5.57002	5.56720	0.00281
		0.30	a	1.20253	7.46159E-6	1.204179	1.19984	0.00434
			b	0.89940	4.89754E-6	0.90092	0.89678	0.00414
			θ	0.79878	8.91100E-6	0.80042	0.79615	0.00427
			β	1.09671	1.25554E-5	1.09956	1.09509	0.00447
			$R(y_0)$	0.66635	2.44978E-6	0.66832	0.66418	0.00415
			$h(y_0)$	5.56500	9.34675E-6	5.56739	5.56138	0.00601
60	6	0.20	a	1.19948	5.75155E-7	1.20019	1.19815	0.00204
			b	0.90004	4.79194E-7	0.90149	0.89896	0.00254
			θ	0.79924	8.60564E-7	0.80013	0.79799	0.00213
			β	1.10008	4.49504E-7	1.10105	1.09828	0.00278
			$R(y_0)$	0.66775	5.35514E-7	0.66842	0.66586	0.00256
			$h(y_0)$	5.56697	5.16821E-7	5..56790	5.56567	0.00223
		0.30	a	1.20205	4.89366E-6	1.20310	1.19992	0.00318
			b	0.89823	3.84152E-6	0.90009	0.89661	0.00348
			θ	0.80214	6.06279E-6	0.80348	0.99472	0.00401
			β	1.09791	3.27049E-6	1.10017	1.09636	0.00381
			$R(y_0)$	0.66751	2.33865E-6	0.66902	0.66555	0.00347
			$h(y_0)$	5.56512	6.78460E-6	5.56725	5.56326	0.00399
100	10	0.20	a	1.19943	4.15306E-7	1.19996	1.19879	0.00117
			b	0.89977	2.23804E-7	0.90036	0.89862	0.00174
			θ	0.79993	3.44116E-7	0.80098	0.79860	0.00238
			β	1.09957	3.13264E-7	1.10018	1.09881	0.00137
			$R(y_0)$	0.66821	3.06646E-7	0.66912	0.66694	0.00219
			$h(y_0)$	5.5674	2.89840E-7	5.56833	5.56615	0.00218
		0.30	a	1.20119	2.08919E-6	1.20271	1.19964	0.00307
			b	0.89861	3.24453E-6	0.90013	0.89616	0.00397
			θ	0.80145	3.04991E-6	0.80255	0.79911	0.00344
			β	1.10093	1.50486E-6	1.10199	1.09919	0.00280
			$R(y_0)$	0.66705	2.20431E-6	0.66837	0.66508	0.00329
			$h(y_0)$	5.56963	4.89307E-6	5.57042	5.56754	0.00288

Table 2. The averages, ERs, and 95% CIs for the parameters $\boldsymbol{a, b, \theta, \beta}$, rf and hrf, using informative prior under LINEX loss function based on Type II censoring (N=10000, $\boldsymbol{\pi = 0.20}$, $\boldsymbol{\pi = 0.30}$ and r=0.1n) (Case 1, $\boldsymbol{a = 1.2}$, $\boldsymbol{b = 0.9}$, $\boldsymbol{\theta = 0.8}$, $\boldsymbol{\beta = 1.1}$, $\boldsymbol{v = -2}$, $\boldsymbol{y_0 = 0.1}$)

n	*r*	π	Parameters, rf and hrf	Averages	ERs	UL	LL	length
30	3	0.20	a	1.19925	1.91903E-6	1.20126	1.19731	0.00396
			b	0.89843	3.03733E-6	0.89989	0.89725	0.00263
			θ	0.79887	2.22817E-6	0.80035	0.79723	0.00311
			β	1.10076	1.07387E-6	1.10154	1.09908	0.00245
			$R(y_0)$	0.66736	1.42236E-6	0.66845	0.66572	0.00273
			$h(y_0)$	5.5666	1.80872E-6	5.56791	5.56446	0.00345
		0.30	a	1.19660	1.36493E-5	1.19906	1.19379	0.00526
			b	0.89752	4.45489E-6	0.90005	0.89427	0.00578
			θ	0.80246	7.14589E-6	0.80395	0.79972	0.00405
			β	1.09766	6.62505E-6	1.09985	1.09956	0.00401
			$R(y_0)$	0.67091	7.93992E-6	0.67212	0.66814	0.00397
			$h(y_0)$	5.56741	3.14186E-6	5.56549	5.56929	0.00379
60	6	0.20	a	1.19922	9.13066E-7	1.20005	1.19764	0.00241
			b	0.90068	7.01373E-7	0.90132	0.89931	0.00201
			θ	0.80035	5.62288E-7	0.80149	0.79914	0.00235
			β	1.09931	8.53230E-7	1.10029	1.09789	0.00231
			$R(y_0)$	0.66790	4.30056E-7	0.66858	0.66657	0.00201
			$h(y_0)$	5.56781	9.28006E-7	5.56963	5.56642	0.00321
		0.30	a	1.19942	7.34786E-6	1.20106	1.19733	0.00373
			b	0.90131	2.54556E-6	0.90279	0.89954	0.00326
			θ	0.79839	3.27576E-6	0.79980	0.79705	0.00276
			β	1.10239	6.46466E-6	1.10345	1.09979	0.00366
			$R(y_0)$	0.66689	5.22675E-6	0.66891	0.66535	0.00355
			$h(y_0)$	5.56599	2.33193E-6	5.56775	5.56434	0.00341
100	10	0.20	a	1.19985	4.39619E-7	1.20108	1.19848	0.00259
			b	0.90028	3.43473E-7	0.90135	0.89863	0.00272
			θ	0.79969	3.52695E-7	0.80055	0.79861	0.00193
			β	1.09971	2.21895E-7	1.10027	1.09875	0.00151
			$R(y_0)$	0.66830	3.88078E-7	0.66922	0.66712	0.00209
			$h(y_0)$	5.56788	5.39676E-7	5.56888	5.56629	0.00259
		0.30	a	1.20185	4.25730E-6	1.20351	1.19978	0.00372
			b	0.89828	1.78002E-6	0.89985	0.89675	0.00308
			θ	0.80097	1.23511E-6	0.80176	0.79975	0.00201
			β	1.10148	3.55717E-6	1.10272	1.09992	0.00351
			$R(y_0)$	0.66715	2.13786E-6	0.66854	0.66505	0.00349
			$h(y_0)$	5.56647	1.51223E-6	5.56786	5.56521	0.00265

Table 3. The averages, ERs, and 95% CIs for the parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\theta}, \boldsymbol{\beta}$, rf and hrf, using informative prior under SEL and LINEX loss functions based on Type II censoring (N=10000, $\boldsymbol{\pi = 0.30}$ and r=0.1n) (Case 2, $\boldsymbol{a = 1.2}$, $\boldsymbol{b = 0.9}$, $\boldsymbol{\theta = 0.8}$, $\boldsymbol{\beta = 1.6}$, $\boldsymbol{y_0 = 0.1}$)

n	***r***	***π***	**Parameters, rf and hrf**	**Averages**	**ERs**	**UL**	**LL**	**length**
			a	1.20185	4.19554E-6	1.20308	1.19994	0.00315
			b	0.89884	2.95029E-6	0.90072	0.89668	0.00404
30	3	0.30	θ	0.80229	6.38164E-6	0.80367	0.79984	0.00383
			β	1.59694	1.11331E-5	1.59964	1.59481	0.00483
			$R(y_0)$	0.66687	3.37533E-6	0.66869	0.66513	0.00356
			$h(y_0)$	5.56548	4.67443E-6	5.56678	5.56374	0.00304
			a	1.19946	3.92873E-6	1.20047	1.19765	0.00282
			b	0.90092	1.42068E-6	0.90263	0.89937	0.00326
60	6	0.30	θ	0.79898	6.30215E-6	0.80059	0.79731	0.00328
			β	1.59828	3.76032E-6	1.60014	1.59704	0.00309
			$R(y_0)$	0.66929	1.78518E-6	0.67067	0.66761	0.00307
			$h(y_0)$	5.56899	2.53353E-6	5.56973	5.56758	0.00215
			a	1.19919	7.58094E-7	1.19979	1.19832	0.00148
			b	0.89935	7.66351E-7	0.90025	0.89777	0.00248
100	10	0.30	θ	0.80051	9.10878E-7	0.80159	0.79898	0.00261
			β	1.60088	9.90337E-7	1.60149	1.59969	0.00179
			$R(y_0)$	0.66907	8.51166E-7	0.66968	0.66812	0.00156
			$h(y_0)$	5.56804	9.79175E-7	5.56909	5.56721	0.00188

LINEX loss function , $\boldsymbol{v = -2}$

n	***r***	***π***	**Parameters, rf and hrf**	**Averages**	**ERs**	**UL**	**LL**	**length**
			a	1.19823	3.97739E-6	1.20009	1.19667	0.00342
			b	0.90094	1.38509E-6	0.90189	0.89943	0.00246
30	3	0.30	θ	0.79757	7.04014E-6	0.79956	0.79596	0.00361
			β	1.60160	3.29579E-6	1.60286	1.59983	0.00302
			$R(y_0)$	0.66639	4.35257E-6	0.66831	0.66483	0.00349
			$h(y_0)$	5.56878	2.03036E-6	5.56974	5.56734	0.00239
			a	1.19911	1.30179E-6	1.20049	1.19721	0.00327
			b	0.89925	2.92363E-6	0.90035	0.89828	0.00208
60	6	0.30	θ	0.80116	2.07227E-6	0.80253	0.79987	0.00266
			β	1.59898	1.25002E-6	1.59980	1.59813	0.00167
			$R(y_0)$	0.66973	2.56587E-6	0.67042	0.66820	0.00221
			$h(y_0)$	5.56742	2.02632E-6	5.56831	5.56613	0.00218
			a	1.20033	7.66650E-7	1.20182	1.19904	0.00278
			b	0.89967	8.53064E-7	0.90039	0.89861	0.00178
100	10	0.30	θ	0.79941	7.99417E-7	0.80043	0.79823	0.00220
			β	1.60038	8.19701E-7	1.60117	1.59953	0.00118
			$R(y_0)$	0.66762	8.59290E-7	0.66830	0.66631	0.00199
			$h(y_0)$	5.56631	9.61814E-7	5.56719	5.56528	0.00191

Table 4. The averages, ERs, and 95% CIs for the parameters a, b, θ, β, rf and hrf, using informative prior under SEL and LINEX loss functions based on Type II censoring (N=10000, $\pi = 0.30$ and r=0.1n) (Case 3, $a = 1.2,\ b = 0.9, \theta = 0.8,\ \beta = 2.1,\ \ y_0 = 0.1$)

n	*r*	*π*	Parameters, rf and hrf	Averages	ERs	UL	LL	length
30	3	0.30	a	1.19828	3.77733E-6	1.199691	1.19645	0.00324
			b	0.89958	1.18722E-6	0.90147	0.89791	0.00357
			θ	0.80212	5.10159E-6	0.80328	0.80003	0.00325
			β	2.10156	3.08468E-6	2.10296	2.09969	0.00327
			$R(y_0)$	0.66741	1.25416E-6	0.66847	0.66586	0.00261
			$h(y_0)$	5.56937	4.28065E-6	5.57053	5.56724	0.00328
60	6	0.30	a	1.19959	6.38645E-7	1.20040	1.19812	0.00228
			b	0.90079	9.57495E-7	0.90215	0.89988	0.00227
			θ	0.80070	7.74498E-7	0.80186	0.99976	0.00210
			β	2.10021	7.74848E-7	2.10188	2.09891	0.00297
			$R(y_0)$	0.66862	5.69206E-7	0.66966	0.66724	0.00241
			$h(y_0)$	5.56827	9.66754E-7	5.56912	5.56665	0.00247
100	10	0.30	a	1.20042	4.72605E-7	1.20132	1.19937	0.00196
			b	0.90006	6.08802E-7	0.90089	0.89887	0.00202
			θ	0.80030	2.81219E-7	0.80107	0.79947	0.00161
			β	2.10009	2.92664E-7	2.10097	2.09888	0.00209
			$R(y_0)$	0.66828	1.62255E-7	0.66907	0.66728	0.00179
			$h(y_0)$	5.56805	5.25446E-7	5.56876	5.56702	0.00174

LINEX loss function , $v = -2$

n	*r*	*π*	Parameters, rf and hrf	Averages	ERs	UL	LL	length
30	3	0.30	a	1.20183	3.86056E-6	1.20273	1.19979	0.00293
			b	0.89957	1.06856E-6	0.90069	0.89739	0.00329
			θ	0.80116	1.64752E-6	0.80199	0.79973	0.00227
			β	2.09929	1.68902E-6	2.10104	2.09739	0.00364
			$R(y_0)$	0.66966	3.99184E-6	0.67203	0.66749	0.00455
			$h(y_0)$	5.56953	1.10143E-6	5.57138	5.56752	0.00386
60	6	0.30	a	1.20063	9.32755E-7	1.20183	1.19932	0.00252
			b	0.90056	9.16252E-7	0.90185	0.89946	0.00239
			θ	0.79931	9.41378E-7	0.80029	0.79774	0.00256
			β	2.09926	7.69666E-7	2.09991	2.09770	0.00221
			$R(y_0)$	0.66884	7.13959E-7	0.66967	0.66705	0.00262
			$h(y_0)$	5.56765	6.77502E-7	5.56900	5.56610	0.00289
100	10	0.30	a	1.19983	5.12842E-7	1.20042	1.19905	0.00137
			b	0.89987	2.27616E-7	0.90059	0.89881	0.00178
			θ	0.80031	3.69448E-7	0.80109	0.79929	0.00180
			β	2.10046	3.91024E-7	2.10122	2.09963	0.00160
			$R(y_0)$	0.66858	2.28936E-7	0.66912	0.66752	0.00159
			$h(y_0)$	5.56713	3.63259E-7	5.56789	5.56617	0.00172

Table 5. Estimates and ERs of the parameters, rf and hrf under BSEL and BLL functions based on informative prior (N=10000, $n = 30$, $\pi = 0.20$ and r=0.1n) ($a = 1.2$, $b = 0.9$, $\theta = 0.8$, $\beta = 1.6$, $y_0 = 0.1$)

Estimate	ω=0"BSEL"	ω=0.2	ω=0.4	ω=0.6	ω=0.8	ω=1"MLE"
$\hat{a}$	1.20033	1.24166	1.28149	1.31992	1.36005	1.40000
ER ($\hat{a}$)	1.60091E-7	3.88953E-7	5.58599E-7	8.75478E-7	2.90848E-6	8.17608E-6
$\hat{b}$	0.90218	0.93976	0.98005	1.02032	1.05998	1.10000
ER ($\hat{b}$)	2.63566E-7	3.85805E-7	5.80240E-7	9.66499E-7	1.67064E-6	5.87707E-6
$\hat{\theta}$	0.80204	0.90045	1.00042	1.09997	1.19971	1.30000
ER ($\hat{\theta}$)	1.96747E-7	3.00819E-7	6.38532E-7	8.18610E-7	2.60889E-6	3.11248E-6
$\hat{\beta}$	1.60027	1.62102	1.63899	1.65942	1.67994	1.70000
ER ($\hat{\beta}$)	2.61161E-7	3.76574E-7	5.14337E-7	8.15323E-7	1.07969E-6	7.41239E-6
$\hat{R}(y_0)$	0.66806	0.73501	0.80018	0.86704	0.93298	0.99921
ER	2.75752E-7	2.85655E-6	5.7897E-7	8.81109E-7	2.09044E-6	5.85526E-6
$\hat{h}(y_0)$	5.56739	4.46051	3.34935	2.24064	1.13209	0.02325
ER	1.45967E-7	3.81817E-7	4.39773E-7	9.95013E-7	1.51111E-6	7.35209E-6
BLL function, $\upsilon = -2$						
Estimate	**ω=0"BSEL"**	**ω=0.2**	**ω=0.4**	**ω=0.6**	**ω=0.8**	**ω=1"MLE"**
$\hat{a}$	1.20069	1.23920	1.27967	1.32024	1.36019	1.40000
ER ($\hat{a}$)	2.00465E-7	3.97014E-7	4.33159E-7	8.28531E-7	1.25379E-6	6.99748E-6
$\hat{b}$	0.90122	0.94034	0.97923	1.02023	1.06001	1.10000
ER ($\hat{b}$)	8.03701E-8	3.23174E-7	4.06385E-7	7.00235E-7	2.70726E-6	6.80547E-6
$\hat{\theta}$	0.80002	0.90169	0.99905	1.10013	1.19942	1.30000
ER ($\hat{\theta}$)	2.31238E-7	3.64184E-7	4.41942E-7	9.19464E-7	1.15306E-6	6.54806E-6
$\hat{\beta}$	1.60036	1.61793	1.63925	1.65975	1.67984	1.70000
ER ($\hat{\beta}$)	2.34321E-7	3.68024E-7	3.92602E-7	9.54879E-7	1.54728E-6	1.06934E-5
$\hat{R}(y_0)$	0.66831	0.73593	0.80009	0.86728	0.93347	0.99921
ER	1.93155E-7	3.00485E-7	6.88979E-7	6.97544E-7	1.07195E-6	7.25789E-6
$\hat{h}(y_0)$	5.56737	4.45879	3.35003	2.24130	1.13212	0.02325
ER	1.28345E-7	3.02446E-7	4.30584E-7	8.19115E-7	1.03427E-6	4.67539E-6

Table 6. Bayes estimates for the parameters, rf and hrf and their standard errors for the real data sets based on Type II censoring (r=0.1n, $\pi = 0.20, = 0.50$, $y_0 = 0.1$)

Application I									
	n	***r***	**Parameters, rf and hrf**	**Estimates**	**SE**		**Parameters, rf and hrf**	**Estimates**	**SE**
$\pi = 0.20$	30	3	a	1.26989	2.29675E-4	$\pi = 0.50$	a	1.26895	0.00010
			b	1.22040	7.89338E-5		b	1.22034	0.00012
			θ	2.40068	1.40375E-4		θ	2.39938	0.00012
			β	1.50064	1.39394E-4		β	1.49970	0.00010
			$R(y_0)$	0.99033	1.63626E-4		$R(y_0)$	0.99001	0.00024
			$h(y_0)$	0.41337	7.57679E-5		$h(y_0)$	0.41243	0.00019

Table 6. Continued

	LINEX loss function , $v = -2$							
	n	***r***	**Parameters, rf and hrf**	**Estimates**	**SE**		**Parameters, rf and hrf**	**Estimates**	**SE**
$\pi = 0.20$	30	3	a	1.26867	1.8051E-4	$\pi = 0.50$	a	1.26899	0.00012
			b	1.22149	7.8722E-5		b	1.22010	0.00019
			θ	2.39949	8.1134E-5		θ	2.39815	0.00022
			β	1.49914	9.0367E-5		β	1.50268	0.00027
			$R(y_0)$	0.99233	8.0545E-5		$R(y_0)$	0.99157	0.00011
			$h(y_0)$	0.41228	1.2035E-4		$h(y_0)$	0.41248	0.00014
	Application II								
	n	***r***	**Parameters, rf and hrf**	**Estimates**	**SE**		**Parameters, rf and hrf**	**Estimates**	**SE**
$\pi = 0.20$	30	3	a	0.92145	3.43902E-4	$\pi = 0.50$	a	0.92271	0.00030
			b	1.40051	9.79770E-5		b	1.39945	0.00012
			θ	1.29968	2.27600E-4		θ	1.29929	0.00023
			β	2.08111	1.69078E-4		β	2.08270	0.00038
			$R(y_0)$	0.97281	9.47695E-5		$R(y_0)$	0.97495	0.00028
			$h(y_0)$	0.87894	1.43772E-4		$h(y_0)$	0.87901	0.00018
	LINEX loss function , $v = -2$								
	n	***r***	**Parameters**	**Estimates**	**SE**		**Parameters, rf and hrf**	**Estimates**	**SE**
$\pi = 0.20$	30	3	a	0.92067	2.26217E-4	$\pi = 0.50$	a	0.91900	0.00019
			b	1.39971	9.69652E-5		b	1.39802	0.00017
			θ	1.29939	7.65505E-5		θ	1.30019	0.00016
			β	2.07985	1.49296E-4		β	2.07934	0.00011
			$R(y_0)$	0.97206	9.87284E-5		$R(y_0)$	0.97237	0.00010
			$h(y_0)$	0.87775	1.02901E-4		$h(y_0)$	0.87779	0.00018

Table 7. Bayes point predictors and bounds for a new observation from a future sample based on Type II censoring under two-sample prediction
(N=10000, *n*=30, r=0.1n, $a = 1.1$, $b = 1.2, \theta = 2.1$, $\beta = 1.2$)

	$\pi = 0.20$					$\pi = 0.30$			
s	$\hat{t}_{(s)}(SEL)$	**UL**	**LL**	**Length**	**s**	$\hat{t}_{(s)}(SEL)$	**UL**	**LL**	**length**
1	0.78025	0.78145	0.77860	0.00285	1	0.78792	0.79039	0.78738	0.00301
15	0.97961	0.97955	0.97612	0.0034	15	0.99806	1.00004	0.99642	0.00362
25	1.29064	1.29264	1.28889	0.00375	25	1.29817	1.30028	1.29594	0.00435
	LINEX loss function , $v = -2$								
	$\pi = 0.20$					$\pi = 0.30$			
s	$\hat{t}_{(s)}$(LINEX)	**UL**	**LL**	**Length**	**s**	$\hat{t}_{(s)}$(LINEX)	**UL**	**LL**	**length**
1	0.77984	0.78095	0.77843	0.00252	1	0.78968	0.79047	0.78778	0.00269
15	0.97960	0.98079	0.97776	0.00304	15	1.00093	1.00239	0.99909	0.00331
25	1.28818	1.28982	1.28638	0.00344	25	1.29724	1.29939	1.29544	0.00395

Table 8. Bayes point predictors and bounds for a future observation from a future sample for real data sets based on Type II censoring under two-sample prediction (r=0.1n, $\pi = 0.20,\ v = -2$)

	Application I					Application II			
s	$\hat{t}_{(s)}(SEL)$	UL	LL	Length	s	$\hat{t}_{(s)}(SEL)$	UL	LL	Length
1	0.11055	0.11173	0.10939	0.00234	1	0.26438	0.26564	0.26235	0.00328
15	1.23264	1.23378	1.23137	0.00241	10	0.39452	0.39628	0.39188	0.00439
25	2.36861	2.36986	2.36505	0.00482	15	0.48475	0.48800	0.48345	0.00455
	Application I					**Application II**			
s	$\hat{t}_{(s)}$**(LINEX)**	**UL**	**LL**	**Length**	**s**	$\hat{t}_{(s)}$**(LINEX)**	**UL**	**LL**	**Length**
1	0.11062	0.11155	0.10933	0.00221	1	0.26419	0.26551	0.26283	0.002686
15	1.23289	1.23422	1.23147	0.00275	10	0.39767	0.39914	0.39639	0.00274
25	2.36892	2.37030	2.36705	0.00325	15	0.48359	0.48487	0.48165	0.003227

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