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Bayesian Estimation and Prediction for Exponentiated Generalized Inverted Kumaraswamy Distribution Based on Dual Generalized Order Statistics

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, the shape parameters, reliability and hazard rate functions of the exponentiated generalized inverted Kumaraswamy distribution are estimated using Bayesian approach. The Bayes estimators are derived under the squared error loss function and the linear-exponential loss function based on dual generalized order statistics. Credible intervals for the parameters, reliability and hazard rate functions are obtained. The Bayesian prediction (point and interval) for a future observation of the exponentiated generalized inverted Kumaraswamy distribution is obtained based on dual generalized order statistics. All results are specialized to lower record values and a numerical study is presented. Moreover, the theoretical results are applied on three real data sets.

Keywords: Exponentiated generalized distributions; Bayesian estimation; dual generalized order statistics; exponentiated generalized inverted Kumaraswamy distribution.

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1 Introduction

In recent years, various generalized models have been proposed and their flexibilities over their baseline distributions when applied to real life data have been established. Interest have been increased among statisticians by adding one parameter or more to a baseline distribution; to provide great flexibility in modeling data in several applied areas such as reliability, engineering, economics, environmental sciences, finance and medical.

Many authors focused on the exponentiated distributions and its applications; for example, Nadarajah and Kotz [1], Ali et al. [2], Silva et al. [3], Lemonte et al. [4], Elgarhy and Shawki [5] and Rather and Subramanian [6].

Cordeiro et al. [7] proposed a class of distributions as an extension of the exponentiated type distribution which can be widely applied in many areas of biology and engineering. Given a non-negative continuous random variable *T*, the *cumulative distribution function* (cdf) of the exponentiated generalized class of distribution is defined by

$$F(t|\theta_1,\theta_2) = \left[1-(1-G(t))^{\theta_2}\right]^{\theta_1}, \quad \theta_1,\theta_2 > 0\,, \tag{1}$$

where θ_1 and θ_2 are additional shape parameters, the corresponding *probability density function* (pdf) for (1) is given by;

$$f(t|\theta_1,\theta_2) = \theta_1\theta_2\, g(t)\big(1-G(t)\big)^{\theta_2-1}\left[1-(1-G(t))^{\theta_2}\right]^{\theta_1-1}, \quad \theta_1,\theta_2 > 0. \tag{2}$$

Abd AL-Fattah et al. [8] introduced the *inverted Kumaraswamy* (IKum) distribution and studied some of its properties. The *maximum likelihood* (ML) and Bayes estimators, confidence, credible intervals for the parameters, *reliability function* (rf) and *hazard rate function* (hrf) of the IKum distribution based on Type II censored samples are obtained. The cdf and pdf are given, respectively, by

$$G(t|\theta_3,\theta_4) = \left(1-(1+t)^{-\theta_3}\right)^{\theta_4}, \quad t > 0;\ \theta_3,\theta_4 > 0, \tag{3}$$

and

$$g(t|\theta_3,\theta_4) = \theta_3\theta_4(1+t)^{-(\theta_3+1)}\left(1-(1+t)^{-\theta_3}\right)^{\theta_4-1}, \quad t > 0;\ \theta_3,\theta_4 > 0\,. \tag{4}$$

Fatima et al. [9] proposed the exponentiated IKum distribution; they derived some statistical properties of this distribution. They used the ML method to estimate the parameters. Mohie El-Din and Abu-Moussa [10] estimated the unknown parameters of the IKum distribution based on general progressive Type II censored data using ML and Bayesian methods. ZeinEldin et al. [11] introduced the Type I half-logistic IKum distribution and derived some statistical properties for it. Also, the methods of the ML, least squares, weighted least squares estimation and Cramer-von Mises minimum distance estimation are used to estimate the parameters of the Type I half-logistic IKum distribution.

AL-Dayian et al. [12] obtained the ML and Bayes estimators of the parameters, rf and hrf from IKum distribution based on dual generalized order statistics (dgos). Usman and ul Haq [13] introduced the Marshall-Olkin extended IKum distribution; this generalization has some known sub models such as the Beta Type II, Lomax and Fisk distribution.

Assuming *T* is a random variable distribution as exponentiated generalized IKum (EG-IKum) distribution with shapes parameters, $\underline{\theta} = (\theta_1,\theta_2,\theta_3,\theta_4)' > 0$ denoted by T~EG-IKum ($\underline{\theta}$). Substituting (3) in (1), then the cdf and pdf can be obtained as follows:

$$F(t|\underline{\theta}) = \left[1 - \left(1 - \left(1 - (1+t)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2}\right]^{\theta_1}, \quad t > 0,\ \theta_j > 0,,\ j = 1,2,3,4, \tag{5}$$

and

$$f(t|\underline{\theta}) = \prod_{j=1}^{4} \theta_j\,(1+t)^{-(\theta_3+1)}\left(1-(1+t)^{-\theta_3}\right)^{\theta_4-1}\left(1-\left(1-(1+t)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2-1}$$
$$\times\left[1-\left(1-\left(1-(1+t)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2}\right]^{\theta_1-1}, \quad t > 0,\ \theta_j > 0. \tag{6}$$

The rf, hrf and *reversed hazard rate function* (rhrf) are given, respectively, by

$$R(t|\underline{\theta}) = 1 - \left[1 - \left(1 - \left(1 - (1+t)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2}\right]^{\theta_1}, \quad t > 0,\ \theta_j > 0, \tag{7}$$

$$h(t|\underline{\theta}) = \frac{\prod_{j=1}^{4}\theta_j\,(1+t)^{-(\theta_3+1)}\left(1-(1+t)^{-\theta_3}\right)^{\theta_4-1}}{1-[1-(1-(1-(1+t)^{-\theta_3})^{\theta_4})^{\theta_2}]^{\theta_1}}$$
$$\times\left(1-\left(1-(1+t)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2-1}\Bigg[1$$
$$-\left(1-\left(1-(1+t)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2}\Bigg]^{\theta_1-1}, \tag{8}$$

and

$$rh(t|\underline{\theta}) = \prod_{j=1}^{4} \theta_j\,(1+t)^{-(\theta_3+1)}\left(1-(1+t)^{-\theta_3}\right)^{\theta_4-1}\left(1-\left(1-(1+t)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2-1}$$
$$\times\left[1-\left(1-\left(1-(1+t)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2}\right]^{-1}, \quad t > 0,\ \theta_j > 0. \tag{9}$$

Burkschat et al. [14] studied the dgos that enables a common approach to descending ordered random variables as reversed ordered order statistics, lower record models and lower Pfeifer records.

Let $T_{(1,n,m,k)}, T_{(2,n,m,k)}, \dots T_{(n,n,m,k)}$ be *n* dgos from an absolutely cdf with corresponding pdf. Hence, the joint pdf has the form

$$f_{T_{(1,n,m,k)},T_{(2,n,m,k)},\dots T_{(n,n,m,k)}}\left(t_{(1)},\dots,t_{(n)}\right) =$$
$$k\left(\prod_{l=1}^{n-1}\gamma_l\right)\left[\prod_{i=1}^{n-1}\left(F\left(t_{(i)}\right)\right)^m f\left(t_{(i)}\right)\right]\left(F\left(t_{(n)}\right)\right)^{k-1} f\left(t_{(n)}\right), \tag{10}$$

where $F^{-1}(1) \geq t_{(1)} \dots \geq t_{(n)} \geq F^{-1}(0), n \in N,\ k \geq 1,\ m_1, \dots, m_{n-1} = m,$

$m \in \mathbb{R}$ and $\gamma_r = k + (n-r)(m+1) \geq 1$, for all $1 \leq r \leq n$.

The plots of the pdf, hrf and rhrf of the EG-IKum are given, respectively, in Figs. 1-3. Fig. 1 shows the flexibility of the density function; where the curves of the pdf are unimodel curves, approximately symmetric and negative skewed for different values of shape parameters. In Fig. 2, the EG-IKum represents most major hazard shapes: increasing, decreasing and unimodal failure rates. From Fig. 3 one can observe that the curves of the rhrf at all the parameter values are monotonically decreasing.

The paper is organized as follows: In Section 2, the Bayes estimators of the parameters, rf and hrf based on dgos under squared error (SE) and linear exponential (LINEX) loss functions of the EG-IKum distribution are derived. Bayesian prediction (point and interval) for a future observation of the EG-IKum distribution are obtained based on dgos in Section 3. Also, a numerical study is presented in Section 4 to illustrate the theoretical results developed in this paper.

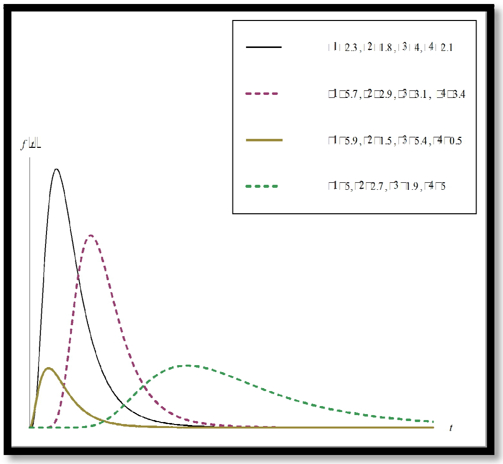

Fig. 1. The plots of the probability density function

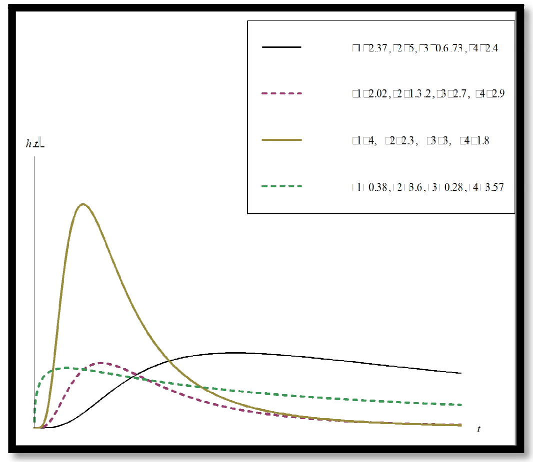

Fig. 2. The plots of the hazard rate function

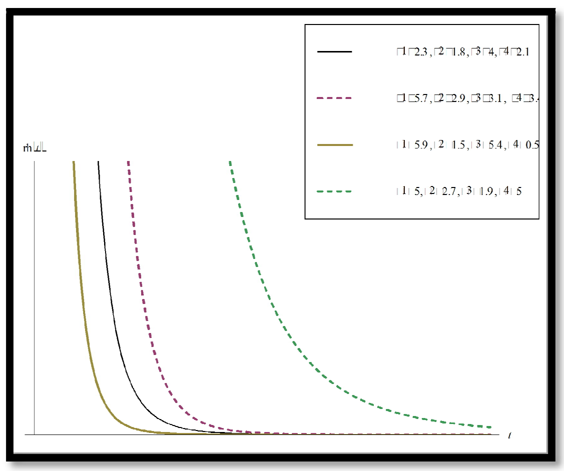

Fig. 3. The plots of the reversed hazard rate function

2 Bayesian Estimation for Exponentiated Generalized Inverted Kumaraswamy Distribution

This section is devoted to estimate the parameters, rf and hrf of the EG-IKum distribution based on dgos using the Bayesian approach, under SE and LINEX loss functions. Also the credible intervals are obtained. Suppose that $T_{(1,n,m,k)}, T_{(2,n,m,k)}, \dots, T_{(n,n,m,k)}$ are n dgos from the EG-IKum distribution, then the likelihood function can be derived by substituting (5) and (6) in (10) as follows:

$$L(\underline{\theta}|\underline{t}) \propto \prod_{j=1}^{4} \theta_j^n \prod_{i=1}^{n} (1+t_i)^{-(\theta_3+1)} \left(1-(1+t_i)^{-\theta_3}\right)^{\theta_4-1} \left(1-\left(1-(1+t_i)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2-1}$$
$$\times \prod_{i=1}^{n-1} \left[1-\left(1-\left(1-(1+t_i)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2}\right]^{\theta_1(m+1)-1} \times \left[1-\left(1-\left(1-(1+t_n)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2}\right]^{\theta_1 k-1}. \quad (11)$$

The likelihood function can be rewritten as

$$L(\underline{\theta};\underline{t}) \propto \prod_{i=1}^{n} \delta_i\,(u_i)^{-1} \prod_{j=1}^{4} \theta_j^{n}\, e^{\theta_1[(m+1)\sum_{i=1}^{n-1} ln(u_i)+kln(u_n)]}, \tag{12}$$

where $\delta_i = (1+t_i)^{-(\theta_3+1)}\left(1-(1+t_i)^{-\theta_3}\right)^{\theta_4-1}\left(1-\left(1-(1+t_i)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2-1}$, (13)

and

$$u_i = \left[1-\left(1-\left(1-(1+t_i)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2}\right] \text{ and } u_n = \left[1-\left(1-\left(1-(1+t_n)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2}\right]. \tag{14}$$

2.1 Point estimation

In this subsection, the Bayes estimators for the parameters, rf and hrf based on dgos under SE and LINEX loss functions of the EG-IKum distribution are obtained.

Let $\theta_1,\ \theta_2,\ \theta_3$ and θ_4 are independent random variables with gamma prior distribution with the pdf as follows:

$$\pi(\theta_j) = \frac{d_j^{c_j}}{\Gamma(c_j)}\theta_j^{c_j-1}e^{-d_j\theta_j}, \qquad \theta_j, d_j, c_j > 0,\ j = 1,2,3,4,$$

where c_j, d_j are the hyper parameters which are assumed to be known.

A joint prior density function of $\underline{\theta} = (\theta_1,\theta_2,\theta_3,\theta_4)'$ is given by

$$\pi(\underline{\theta}) \propto \prod_{j=1}^{4}\theta_j^{c_j-1}e^{-d_j\theta_j}. \tag{15}$$

The joint posterior density can be derived by using (12) and (15) as follows:

$$\pi(\underline{\theta}|\underline{t}) \propto L(\underline{\theta}|\underline{t})\pi(\underline{\theta}),$$

$$\pi(\underline{\theta}|\underline{t}) = \frac{1}{\varphi}\prod_{j=1}^{4}\theta_j^{c_j+n-1}e^{-d_j\theta_j}\prod_{i=1}^{n}\delta_i\,(u_i)^{-1}e^{\theta_1[(m+1)\sum_{i=1}^{n-1} ln(u_i)+kln(u_n)]}, \tag{16}$$

where φ^{-1} is the normalizing constant .

2.1.1 Bayesian estimation under the squared error loss function

Under SE loss function, the Bayes estimators of the parameters $\underline{\theta}$ are given by their marginal posterior expectations as shown below:

$$\theta^*_{j(SE)} = E(\theta_j|\underline{t}) = \int_{\underline{\theta}} \theta_j\pi(\underline{\theta}|\underline{t})\, d\underline{\theta},\quad j = 1,2,3,4, \tag{17}$$

where $\pi(\underline{\theta}|\underline{t})$ is given in (16).

The Bayes estimators of the rf and hrf under SE loss function can be obtained as follows:

$$R^*_{(SE)}(t) = E(R(t)|\underline{t}) = \int_{\underline{\theta}} R(t)\pi(\underline{\theta}|\underline{t})\, d\underline{\theta},\quad j = 1,2,3,4, \tag{18}$$

where $R(t)$ is given in (7),

and

$$h^*_{(SE)}(t) = E\left(h(t)\middle|\underline{t}\right) = \int_{\underline{\theta}} h(t)\pi\left(\underline{\theta}\middle|\underline{t}\right)\, d\underline{\theta}\,, \qquad j = 1,2,3,4, \tag{19}$$

where $h(t)$ is given in (8).

2.1.2 Bayesian estimation under the linear exponential loss function

Under the LINEX loss function, the Bayes estimators for the shape parameters $\underline{\theta}$ are given, respectively, by

$$\theta^*_{j(LNX)} = \frac{-1}{v}\ln E\left(e^{-v\theta_j}\middle|\underline{t}\right), \tag{20}$$

where

$$E\left(e^{-v\theta_j}\middle|\underline{t}\right) = \int_{\underline{\theta}} e^{-v\theta_j}\pi\left(\underline{\theta}\middle|\underline{t}\right)\, d\underline{\theta}. \quad j = 1,2,3,4,$$

where $\pi\left(\underline{\theta}\middle|\underline{t}\right)$ is given in (16).

Also, the Bayes estimators for the rf and hrf based on dgos can be derived as follows:

$$R^*_{(LNX)}(t) = \frac{-1}{v}\ln E\left(e^{-vR(t)}\middle|\underline{t}\right), \text{ where } E\left(e^{-vR(t)}\middle|\underline{t}\right) = \int_{\underline{\theta}} e^{-vR(t)}\pi\left(\underline{\theta}\middle|\underline{t}\right)\, d\underline{\theta}\,, \tag{21}$$

and

$$h^*_{(LNX)}(t) = \frac{-1}{v}\ln E\left(e^{-vh(t)}\middle|\underline{t}\right). \quad \text{where } E\left(e^{-vh(t)}\middle|\underline{t}\right) = \int_{\underline{\theta}} e^{-vh(t)}\pi\left(\underline{\theta}\middle|\underline{t}\right)\, d\underline{\theta}\,. \tag{22}$$

2.2 Credible interval

The 100 (1- ω) % credible interval for $\underline{\theta}$ is $\left(L(\underline{t}), U(\underline{t})\right)$,

where

$$P\left[L(\underline{t}) < \underline{\theta} < U(\underline{t})\middle|\underline{t}\right] = \int_{L(\underline{t})}^{U(\underline{t})} \pi\left(\underline{\theta}\middle|\underline{t}\right) d\underline{\theta} = 1 - \omega.$$

Then a $100(1-\omega)\%$ credibility interval for θ_j based on dgos is $\left(L_j(\underline{x}), U_j(\underline{x})\right)$, where

$$P\left[\theta_j > L_j(\underline{t})\middle|\underline{t}\right] = \int_{L_j(\underline{t})}^{\infty} \pi\left(\underline{\theta}\middle|\underline{t}\right) d\theta_j = 1 - \frac{\omega}{2}, \quad j = 1,2,\dots,4, \tag{23}$$

and

$$P\left[\theta_j > U_j(\underline{t})\middle|\underline{t}\right] = \int_{U_j(\underline{t})}^{\infty} \pi\left(\underline{\theta}\middle|\underline{t}\right) d\theta_j = \frac{\omega}{2}, \quad j = 1,2,\dots,4. \tag{24}$$

3 Prediction Based on Dual Generalized Order Statistics

In this section the Bayesian prediction (point and interval) is considered for a future observation of the EG-IKum distribution based on dgos under SE and LINEX loss functions.

The marginal pdf of r^{th} dgos $T\,(r,n,m,k), 1 \le r \le n$ is given by

$$f^{(r,n,m,k)}\left(t_{(r)}\right) = \frac{C_{r-1}}{(r-1)!}\left[F\left(t_{(r)}\right)\right]^{\gamma_r-1} f(t_{(r)}) g_m^{r-1}(F\left(t_{(r)}\right)) \,, \tag{25}$$

where $C_{r-1} = \prod_{l=1}^{r}\gamma_l\,,\; g_m(t) = h_m(t) - h_m(1),\; x \in [0,1)$,

$$h_m(t) = \begin{cases} -\frac{1}{m+1}t^{m+1}\;, & m \ne -1\,, \\ -\ln t\,, & m = -1\;. \end{cases} \tag{26}$$

(See Khan and Khan) [15]

Let $T(1,n,m,k),\dots,T(r,n,m,k)$ be a dgos of size n with the pdf $f(t;\theta)$ and suppose $Y\left(1,n_y,m_y,k_y\right),\dots,Y\left(r_y,n_y,m_y,k_y\right), k_y > 0, m_y \in \mathbb{R}$ is a second unobserved dgos of size n_y. Using (5), (6), (25) and (26), the density function of the dgos $Y_{(s)}$ can be obtained just by replacing $t_{(r)}$ with $y_{(s)}$ as follows:

$$f\left(y_{(s)}|\underline{\theta}\right) = \frac{\prod_{j=1}^{4}\theta_j\, C_{s-1}}{(s-1)!}\left(1+y_{(s)}\right)^{-(\theta_3+1)}\left(1-\left(1+y_{(s)}\right)^{-\theta_3}\right)^{\theta_4-1}$$
$$\times\left(1-\left(1-\left(1+y_{(s)}\right)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2-1}\left[u_{y_{(s)}}\right]^{\theta_1\gamma_s-1} g_{m_y}^{s-1}\left(F\left(y_{(s)}\right)\right), \tag{27}$$

where

$$u_{y_{(s)}} = \left[1-\left(1-\left(1-\left(1+y_{(s)}\right)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2}\right],$$

$$C_{s-1} = \prod_{l=1}^{s}\gamma_l\,,\; g_M(y_s) = h_M(y_s) - h_M(1),\; \gamma_s = k_y + \left(n_y - s\right)\left(m_y+1\right),\; for\ all\ 1 \le s \le n_y,$$

$$g_{m_y}^{s-1}\left(F\left(y_{(s)}\right)\right) = \begin{cases} \frac{1}{\left(m_y+1\right)^{s-1}}\left[1-\left[u_{y_{(s)}}\right]^{\theta_1\left(m_y+1\right)}\right]^{s-1} & , \quad m_y \ne -1\,, \\ \left[-\ln\left(u_{y_{(s)}}\right)^{\theta_1}\right]^{s-1} & , \quad m_y = -1\,. \end{cases} \tag{28}$$

For the future sample of size n_y, let $Y_{(s)}$ denotes the s^{th} ordered life time, $1 \le s \le n_y$, the pdf of the dgos $Y_{(s)}$ from EG-IKum distribution is derived by substituting (28) in (27).

Case one: for $m_y \ne -1$

$$f_1\left(y_{(s)}|\underline{\theta}\right) = \frac{\prod_{j=1}^{4}\theta_j\, C_{s-1}}{\left(m_y+1\right)^{s-1}(s-1)!}\left(1+y_{(s)}\right)^{-(\theta_3+1)}\left(1-\left(1+y_{(s)}\right)^{-\theta_3}\right)^{\theta_4-1}$$
$$\times\left(1-\left(1-\left(1+y_{(s)}\right)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2-1}\left[u_{y_{(s)}}\right]^{\theta_1\gamma_s-1}\left[1-\left[u_{y_{(s)}}\right]^{\theta_1\left(m_y+1\right)}\right]^{s-1}.$$

Using the binomial expansion, one obtains

$$f_1\left(y_{(s)}\middle|\underline{\theta}\right)=\frac{\prod_{j=1}^{4}\theta_j\,C_{s-1}}{\left(m_y+1\right)^{s-1}(s-1)!}\left(1+y_{(s)}\right)^{-(\theta_3+1)}\left(1-\left(1+y_{(s)}\right)^{-\theta_3}\right)^{\theta_4-1}$$
$$\times\left(1-\left(1-\left(1+y_{(s)}\right)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2-1}\sum_{\varsigma=0}^{s-1}(-1)^{\varsigma}\binom{s-1}{\varsigma}\left[u_{y_{(s)}}\right]^{\theta_1(\gamma_s+\varsigma(m_y+1))-1},$$

let

$$\xi=\frac{C_{s-1}}{\left(m_y+1\right)^{s-1}(s-1)!}\ ,\ \eta_\varsigma=(-1)^{\varsigma}\binom{s-1}{\varsigma}\ ,\ \text{and}\ \varpi_\varsigma=\left(\gamma_s+\varsigma(m_y+1)\right).$$

Then

$$f_1\left(y_{(s)}\middle|\underline{\theta}\right)=\ \xi\prod_{j=1}^{4}\theta_j\left(1+y_{(s)}\right)^{-(\theta_3+1)}\left(1-\left(1+y_{(s)}\right)^{-\theta_3}\right)^{\theta_4-1}$$

$$\times\left(1-\left(1-\left(1+y_{(s)}\right)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2-1}\sum_{\varsigma=0}^{s-1}\eta_\varsigma\left[u_{y_{(s)}}\right]^{\theta_1\varpi_\varsigma-1},\qquad y_{(s)}>0\ ;\ \ \theta_j>0. \tag{29}$$

Case two: for $m_y=-1$

$$f_2\left(y_{(s)}\middle|\underline{\theta}\right)=\frac{\prod_{j=1}^{4}\theta_j\,k_y^s\,\theta_1^{\ s-1}}{(s-1)!}\left(1+y_{(s)}\right)^{-(\theta_3+1)}\left(1-\left(1+y_{(s)}\right)^{-\theta_3}\right)^{\theta_4-1}$$
$$\times\left(1-\left(1-\left(1+y_{(s)}\right)^{-\theta_3}\right)^{\theta_4}\right)^{\theta_2-1}\left[u_{y_{(s)}}\right]^{\theta_1k_y-1}\left[-\ln\left(u_{y_{(s)}}\right)\right]^{s-1},y_{(s)}>0;\theta_j>0. \tag{30}$$

The *Bayesian predictive density* (BPD) function can be derived as follows:

$$f\left(y_{(s)}\middle|\underline{t}\right)=\int_{\underline{\theta}}\ f\left(y_{(s)}\middle|\underline{\theta}\right)\pi\left(\underline{\theta}\middle|\underline{t}\right)d\underline{\theta}, \tag{31}$$

where $\pi\left(\underline{\theta}\middle|\underline{t}\right)$ is the posterior density function of $\underline{\theta}$ and $f\left(y_{(s)}\middle|\underline{\theta}\right)$ is the pdf of $y_{(s)}$.

Substituting (16), (29) and (30) in (31), the BPD of $y_{(s)}$ given $\underline{t}$ is obtained as given below

Case one: for $m_y\neq-1$

$$f_1\left(y_{(s)}\middle|\underline{t}\right)=\int f_1\left(y_{(s)}\middle|\underline{\theta}\right)\pi\left(\underline{\theta}\middle|\underline{t}\right)d\underline{\theta}. \tag{32}$$

Case two: for $m_y=-1$

$$f_2\left(y_{(s)}\middle|\underline{t}\right)=\int f_2\left(y_{(s)}\middle|\underline{\theta}\right)\pi\left(\underline{\theta}\middle|\underline{t}\right)d\underline{\theta}. \tag{33}$$

3.1 Point prediction

The *Bayes predictor* (BP) of the future dgos $Y_{(s)}$ can be derived under SE and LINEX loss functions as follows:

Case one: for $m_y\neq-1$

The BP of the future dgos $Y_{(s)}$ can be obtained under SE loss function using (32) as given below

$$y^{*}_{(s)(SE)} = E\left(y_{(s)}\middle|\underline{t}\right) = \int_{0}^{\infty} y_{(s)}\ f_1\left(y_{(s)}\middle|\underline{t}\right)\ dy_{(s)}. \tag{34}$$

The BP of the future dgos $Y_{(s)}$ can be derived under LINEX loss function using (32) as follows:

$$y^{*}_{(s)(LNX)} = \frac{-1}{\upsilon} \ln E \int_{0}^{\infty} e^{-\upsilon y_{(s)}}\ f_1\left(y_{(s)}\middle|\underline{t}\right)\ dy_{(s)}. \tag{35}$$

Case one: for $m_y = -1$

The BP of the future dgos $Y_{(s)}$ can be obtained under SE loss function using (33) as given below

$$y^{*}_{(s)(SE)} = E\left(y_{(s)}\middle|\underline{t}\right) = \int_{0}^{\infty} y_{(s)}\ f_2\left(y_{(s)}\middle|\underline{t}\right)\ dy_{(s)}. \tag{36}$$

The BP of the future dgos $Y_{(s)}$ can be derived under LINEX loss function using (33) as follows:

$$y^{*}_{(s)(LNX)} = \frac{-1}{\upsilon} \ln E \int_{0}^{\infty} e^{-\upsilon y_{(s)}}\ f_2\left(y_{(s)}\middle|\underline{t}\right)\ dy_{(s)}. \tag{37}$$

3.2 Interval prediction

The *Bayesian predictive bounds* (BPB) of the future dgos $Y_{(s)}$ can be obtained using (32) and (33) as follows:

Case one: for $m_y \neq -1$

$$P\left[Y_{(s)} > L_{(s)}(\underline{t})\middle|\underline{t}\right] = \int_{L_{(s)}(\underline{t})}^{\infty} \int_{\underline{\theta}} f_1\left(y_{(s)}\middle|\underline{t}\right) d\underline{\theta}\, dy_{(s)} = 1 - \frac{\omega}{2}, \tag{38}$$

and

$$P\left[Y_{(s)} > U_{(s)}(\underline{t})\middle|\underline{t}\right] = \int_{U_{(s)}(\underline{t})}^{\infty} \int_{\underline{\theta}} f_1\left(y_{(s)}\middle|\underline{t}\right) d\underline{\theta}\, dy_{(s)} = \frac{\omega}{2}. \tag{39}$$

Case two: for $m_y = -1$

$$P\left[Y_{(s)} > L_{(s)}(\underline{t})\middle|\underline{t}\right] = \int_{L_{(s)}(\underline{t})}^{\infty} \int_{\underline{\theta}} f_2\left(y_{(s)}\middle|\underline{t}\right) d\underline{\theta}\, dy_{(s)} = 1 - \frac{\omega}{2}, \tag{40}$$

and

$$P\left[Y_{(s)} > U_{(s)}(\underline{t})\middle|\underline{t}\right] = \int_{U_{(s)}(\underline{t})}^{\infty} \int_{\underline{\theta}} f_2\left(y_{(s)}\middle|\underline{t}\right) d\underline{\theta}\, dy_{(s)} = \frac{\omega}{2}. \tag{41}$$

BPB can be obtained by solving the previous equations numerically.

4 Numerical Results

This section aims to illustrate the theoretical results of the Bayes estimates and Bayes predictors under SE and LINEX loss functions. Numerical results are presented for EG-IKum distribution based on lower record values through a simulation study and some applications.

4.1 Simulated example

In this subsection, a simulation study is conducted to illustrate the performance of the Bayes estimates and predictors for different sample sizes of lower record values and for different population parameter values from EG-IKum distribution. The performance was evaluated through a measurement of accuracy. In order to study the precision and variation the estimates, it is convenient to use the *estimated risks* (ERs) based on lower record values.

I. Bayesian estimation

The lower record values can be obtained as a special case from dgos by setting $m = -1, k = 1$, the estimation results obtained in Sections 2 can be specialized to lower records. The averages for the Bayes estimates of θ_j, where, $j = 1,2,3,4$, are evaluated. Also, rf and hrf and their average estimates, ERs are computed based on lower record values according to the following steps:

a. For given values of $\theta_j, j = 1,2,3,4$, random samples of size *n* is generated from EG-IKum distribution observing that if U is uniform distribution (0,1), then $t_{ij} = \left[(1-(1-(1-(U_{ij})^{\frac{1}{\theta_1}})^{\frac{1}{\theta_2}})^{\frac{1}{\theta_4}})^{-\frac{1}{\theta_3}} - 1\right]$, is EG-IKum ($\underline{\theta}$) distribution.
b. For each sample size *n*, consider that the first observation is the first lower record value t_1 then denote it by R_1 and the second observation t_2 denote it by R_2 ; which is smaller than the maximum $(t_1 > t_2)$ record and if $t_1 \leq t_2$ ignore it and repeat until you get a sample of record values (*Rv*).
c. The averages for the Bayes estimates of the parameters, rf and hrf under SE and LINEX loss functions are computed; at specified number of surviving units with population parameter values θ_j and hyper parameters of the prior distribution, the computations are performed using R programming language.
d. Tables 1 and 2 present the Bayes averages under SE and LINEX loss functions of the parameters and their ERs and credible intervals based on lower record values for different population parameter values for $\theta_1 = (0.8, 0.2), \theta_2 = (0.6, 0.3), \theta_3 = (1.2, 0.4)$ and $\theta_4 = (1.5,\ 0.7)$ based on records of size *Rv* =3, 5, 7 and *number of replications* (NR) = 10000.
e. Table 3 displays the Bayes averages and 95% credible intervals of the rf and hrf at $t_0 = 0.5, 1, 2$ from EG-IKum distribution based on lower record values for different samples of records of size *Rv*= 3,7 and NR = 10000.

II. Bayesian prediction

The BP for the future lower record values can be obtained from the above results of the dgos when $m = -1$, $k = 1, m_y = -1$ and $k_y = 1$, as given below:

a. Determining the value of s, $1 \leq s \leq n_y$, which is the index of the future unobserved lower record value from the second sample.
b. Using (36), (37), (40) and (41), the BP for the future lower record is calculated under SE and LINEX loss functions, respectively.
c. Table 6 displays the BP and 95% credible intervals for the future lower record values of $Y_{(s)}$ from EG-IKum distribution, where $Rv = 6, \theta_1 = 0.8$, $\theta_2 = 0.5, \theta_3 = 1.2$ and $\theta_4 = 0.7$.

4.2 Applications

In this subsection, three applications to real data sets are provided to illustrate the importance of the EG-IKum distribution based on lower records and to demonstrate how the proposed method can be used in practice. Tables 4 and 5 display the Bayes estimates of the parameters, rf, hrf and standard error for the real data based on lower records. The BP and 95% credible intervals for the future lower record values $Y_{(s)}$ from the three real data are shown in Table 7.

To check the validity of the fitted model, Kolmogorov-Smirnov goodness of fit test is performed for each data set and the p values in each case indicates that the model fits the data very well.

Table 1. Bayes averages, estimated risks and credible intervals based on lower records
$(\theta_1 = 0.8, \theta_2 = 0.6, \theta_3 = 1.2, \theta_4 = 1.5, \text{NR} = 10000)$

Rv	Loss functions	Estimators	Average	ER	LL	UL	Length
3	SE	θ_1^*	0.8013	0.0021	0.7993	0.8994	0.0032
		θ_2^*	0.6020	0.0020	0.5999	0.6034	0.0036
		θ_3^*	1.1983	0.0016	1.1968	1.2001	0.0033
		θ_4^*	1.4987	0.0013	1.4957	1.5004	0.0047
	LINEX	θ_1^*	0.7979	0.0030	0.7960	0.7999	0.0039
		θ_2^*	0.5983	0.0019	0.5958	0.6007	0.0048
		θ_3^*	1.2007	0.0006	1.1997	1.2013	0.0015
		θ_4^*	1.4994	0.0007	1.4971	1.5006	0.0035
5	SE	θ_1^*	0.7991	0.0014	0.7973	0.7994	0.0028
		θ_2^*	0.6005	0.0009	0.5986	0.6018	0.0032
		θ_3^*	1.1991	0.0009	1.1975	1.2000	0.0025
		θ_4^*	1.4992	0.0009	1.4969	1.5005	0.0037
	LINEX	θ_1^*	0.7978	0.0029	0.7963	0.7993	0.0030
		θ_2^*	0.5994	0.0008	0.5974	0.6003	0.0029
		θ_3^*	1.1994	0.0006	1.1978	1.2000	0.0022
		θ_4^*	1.4993	0.0005	1.4980	1.4980	0.0021
7	SE	θ_1^*	0.8000	0.0004	0.7991	0.8002	0.0013
		θ_2^*	0.5994	0.0008	0.5975	0.6005	0.0019
		θ_3^*	1.1992	0.0007	1.1980	1.1998	0.0023
		θ_4^*	1.4994	0.0006	1.4977	1.5005	0.0018
	LINEX	θ_1^*	0.8003	0.0005	0.7994	0.8007	0.0013
		θ_2^*	0.5990	0.0008	0.5981	0.5998	0.0017
		θ_3^*	1.1998	0.0003	1.1988	1.2006	0.0018
		θ_4^*	1.4994	0.0005	1.4981	1.5003	0.0018

Table 2. Bayes averages, estimated risks and credible intervals based on lower records ($\theta_1 = 0.2$, $\theta_2 = 0.3$, $\theta_3 = 0.4$, $\theta_4 = 0.7$ and NR = 10000)

Rv	Loss functions	Estimators	Average	ER	LL	UL	Length
3	SE	θ_1^*	0.1990	0.0060	0.1973	0.2002	0.0029
		θ_2^*	0.2985	0.0046	0.2969	0.3008	0.0040
		θ_3^*	0.3989	0.0035	0.3972	0.4003	0.0031
		θ_4^*	0.6980	0.0031	0.6964	0.6997	0.0033
	LINEX	θ_1^*	0.2027	0.0160	0.1995	0.2046	0.0051
		θ_2^*	0.3005	0.0024	0.2987	0.3015	0.0028
		θ_3^*	0.4018	0.0050	0.3999	0.4027	0.0028
		θ_4^*	0.6996	0.0013	0.6977	0.7004	0.0027
5	SE	θ_1^*	0.2007	0.0050	0.1993	2.0016	0.0023
		θ_2^*	0.2991	0.0028	0.2977	0.3000	0.0022
		θ_3^*	0.3990	0.0027	0.3979	0.4002	0.0023
		θ_4^*	0.7017	0.0028	0.6996	0.7030	0.0033
	LINEX	θ_1^*	0.1993	0.0056	0.1975	0.2007	0.0031
		θ_2^*	0.3007	0.0024	0.2991	0.3015	0.0024
		θ_3^*	0.3998	0.0017	0.3984	0.4009	0.0025
		θ_4^*	0.6988	0.0019	0.6975	0.6997	0.0022
7	SE	θ_1^*	0.1995	0.0031	0.1986	0.1997	0.0017
		θ_2^*	0.2991	0.0022	0.2983	0.2998	0.0014
		θ_3^*	0.3997	0.0013	0.3987	0.4005	0.0018
		θ_4^*	0.7001	0.0007	0.6990	0.7011	0.0021
	LINEX	θ_1^*	0.1999	0.0023	0.1987	0.2007	0.0020
		θ_2^*	0.2992	0.0022	0.2979	0.3000	0.0020
		θ_3^*	0.4006	0.0024	0.3995	0.4015	0.0021
		θ_4^*	0.7011	0.0019	0.6990	0.7022	0.0022

Table 3. Bayes averages, estimated risks and credible intervals for the rf and hrf at $t_0 = 0.5, 1, 2$, from EG-IKum distribution based on SE and LINEX loss functions for different sample size of records Rv, and repetitions NR = 10000

Rv	t_0	Loss functions	Estimators	Average	ER	LL	UL	Length
3	0.5	SE	$R^*(t_0)$	0.7802	0.0014	0.7782	0.7812	0.0030
			$h^*(t_0)$	0.4581	0.0027	0.4557	0.4596	0.0039
		LINEX	$R^*(t_0)$	0.7801	0.0012	0.7785	0.7810	0.0025
			$h^*(t_0)$	0.4594	0.0013	0.4586	0.4601	0.0015
	1	SE	$R^*(t_0)$	0.6368	0.0006	0.6356	0.6373	0.0017
			$h^*(t_0)$	0.3541	0.0082	0.3507	0.3562	0.0054
		LINEX	$R^*(t_0)$	0.6378	0.0020	0.6361	0.6388	0.0026
			$h^*(t_0)$	0.3580	0.0046	0.3560	0.3588	0.0027
	2	SE	$R^*(t_0)$	0.4751	0.0015	0.4738	0.4764	0.0025
			$h^*(t_0)$	0.2398	0.0092	0.2382	0.2418	0.0035
		LINEX	$R^*(t_0)$	0.4770	0.0038	0.4751	0.4780	0.0029
			$h^*(t_0)$	0.2401	0.0071	0.2391	0.2410	0.0019
7	0.5	SE	$R^*(t_0)$	0.7797	0.0007	0.7788	0.7804	0.0016
			$h^*(t_0)$	0.4584	0.0019	0.4569	0.4597	0.0027
		LINEX	$R^*(t_0)$	0.7799	0.0009	0.7790	0.7805	0.0014
			$h^*(t_0)$	0.4589	0.0007	0.4581	0.4596	0.0015
	1	SE	$R^*(t_0)$	0.6365	0.0005	0.6359	0.6371	0.0012
			$h^*(t_0)$	0.3577	0.0041	0.3556	0.3587	0.0031
		LINEX	$R^*(t_0)$	0.6360	0.0012	0.6351	0.6368	0.0017
			$h^*(t_0)$	0.3574	0.0033	0.3562	0.3587	0.0025
	2	SE	$R^*(t_0)$	0.4886	0.0010	0.4874	0.4894	0.0020
			$h^*(t_0)$	0.2053	0.0023	0.2045	0.2057	0.0019
		LINEX	$R^*(t_0)$	0.4891	0.0037	0.4878	0.4899	0.0037
			$h^*(t_0)$	0.2056	0.0014	0.2048	0.2064	0.0014

Table 4. Bayes estimates for the parameters and standard errors for the real data based on lower records

Application	*Rv*	Loss functions	Estimators	Estimates	Standard Errors
I	3	SE	θ_1^*	0.9009	8.88e-05
			θ_2^*	0.9001	1.01e-04
			θ_3^*	0.8990	1.04e-04
			θ_4^*	1.7008	1.12e-04
		LINEX	θ_1^*	0.9007	9.11e-05
			θ_2^*	0.8978	1.84e-04
			θ_3^*	0.9014	1.53e-04
			θ_4^*	1.7012	2.27e-04
II	4	SE	θ_1^*	0.9009	3.94e-05
			θ_2^*	1.5994	6.25e-05
			θ_3^*	0.8999	5.94e-05
			θ_4^*	6.0491	4.04e-05
		LINEX	θ_1^*	0.8996	4.54e-05
			θ_2^*	1.5994	4.33e-05
			θ_3^*	0.9009	6.57e-05
			θ_4^*	6.0501	4.97e-05
III	7	SE	θ_1^*	0.5996	5.26e-05
			θ_2^*	0.8992	8.39e-05
			θ_3^*	0.4000	6.54e-05
			θ_4^*	0.6982	2.17e-04
		LINEX	θ_1^*	0.6031	2.88e-04
			θ_2^*	0.9006	8.60e-05
			θ_3^*	0.4015	1.83e-04
			θ_4^*	0.7004	1.06e-04

Table 5. Bayes estimates for rf, hrf and standard errors from EG-IKum distribution for the real data based on lower records

Application	*Rv*	Loss functions	Estimators	Estimates	Standard Errors
I	3	SE	$R^*(t_0)$	0.9501	8.68e-05
			$h^*(t_0)$	0.3421	1.49e-04
		LINEX	$R^*(t_0)$	0.9486	8.06e-05
			$h^*(t_0)$	0.3426	8.46e-05
II	4	SE	$R^*(t_0)$	1.0004	5.38e-05
			$h^*(t_0)$	0.0003	4.46e-05
		LINEX	$R^*(t_0)$	0.9981	9.61e-05
			$h^*(t_0)$	0.0001	2.94e-05
III	7	SE	$R^*(t_0)$	0.6899	9.75e-05
			$h^*(t_0)$	0.8340	3,32e-04
		LINEX	$R^*(t_0)$	0.6914	8.67e-05
			$h^*(t_0)$	0.8344	2.53e-04

Table 6. Point predictors and 95% credible intervals for the future lower record values $y^*_{(s)}$ from EG-IKum distribution $(Rv = 6, \theta_1 = 0.8, \theta_2 = 0.5, \theta_3 = 1.2, \theta_4 = 0.7)$

s	Loss functions	$y^*_{(s)}$	LL	UL	Length
1	SE	**0.8995**	0.8977	0.9006	0.0028
	LINEX	0.9004	0.8997	0.9009	0.0012
3	SE	0.9011	0.8990	0.9018	0.0029
	LINEX	0.9009	0.8998	0.9015	0.0017
6	SE	0.9131	0.9097	0.9156	0.0059
	LINEX	0.9091	0.9074	0.9100	0.0025

Table 7. Point predictors and 95% credible intervals for the future lower record values $y^*_{(s)}$ from the three real data

Application	s	SE				LINEX			
		$y^*_{(s)(SE)}$	Credible interval			$y^*_{(s)(LNX)}$	Credible interval		
			LL	UL	Length		LL	UL	Length
I	1	1.2007	1.1989	1.2015	0.0026	1.1996	1.1986	1.2013	0.0026
	3	1.2012	1.1994	1.2027	0.0033	1.1999	1.1974	1.2014	0.0040
II	1	1.2020	1.1999	1.2038	0.0039	1.1986	1.1973	1.1995	0.0022
	3	1.2019	1.1993	1.2033	0.0040	1.2017	1.1999	1.2026	0.0027
III	1	0.9008	0.8987	0.9018	0.0030	0.8994	0.8982	0.9002	0.0020
	5	0.90213	0.8999	0.9048	0.0048	0.9013	0.8995	0.9032	0.0036

I. The first application is given by Hinkley [16]. It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The data is 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

   From the original data, one can observe that the following lower record values; 0.77, 0.47, 0.32, with p-value = 0.1866.

II. The second application is a real data set obtained from Lee and Wang [17]. It represents the remission times (in months) of a random sample of 128 bladder cancer patients. The data is 08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.2, 2.23, 0.26, 0.31, 0.73, 0.52, 4.98, 6.97, 9.02, 13.29, 0.4, 2.26, 3.57, 5.06, 7.09, 11.98, 4.51, 2.07, 0.22, 13.8, 25.74, 0.5, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 19.13, 6.54, 3.36, 0.82, 0.51, 2.54, 3.7, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 1.76, 8.53, 6.93, 0.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 3.25, 12.03, 8.65, 0.39, 10.34, 14.83, 34.26, 0.9, 2.69, 4.18, 5.34, 7.59, 10.66, 4.5, 20.28, 12.63, 0.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 6.25, 2.02, 22.69, 0.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 8.37, 3.36, 5.49, 0.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 12.02, 6.76, 0.4, 3.02, 4.34, 5.71, 7.93, 11.79, 18.1, 1.46, 4.4, 5.85, 2.02, 12.07.

   From the original data, the following lower record values are: 8, 2.09, 0.2, 0.19, with p-value = 0.2126.

III. The third application is the vinyl chloride data obtained from clean upgrading, monitoring wells in mg/L; this data set was used for Bhaumik et al. [18]. The data are: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

   From the original data, one can observe the following lower record values; 5.1, 1.2, 0.6, 0.5, 0.4, 0.2, 0.1, where the p-value = 0.1845.

4.3 Concluding remarks

- From Tables 1 and 2 one can notice that the ERs of the Bayes averages of the shape parameters decrease when the sample size of *Rv* increases. Also, the lengths of the credible intervals become narrower as the sample size of records increases.

- It is clear from Table 3 that the ERs of rf and hrf performs better when the sample size of *Rv* increases, and the lengths of the credible intervals get shorter when the sample size of *Rv* increases.

- One can observe that the ERs for the estimates of the parameters, rf and hrf under LINEX loss function have the less values than the corresponding ERs of the estimates under SE loss function.

- From Tables 6 and 7, the BP and the lengths of the BPB increase when *s* increases.

Competing Interests

Authors have declared that no competing interests exist.

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