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BEAM DYNAMICS IN A DOUBLE RF SYSTEM

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ABSTRACT

The addition of a higher harmonic RF system to the main system allows a control of the synchrotron frequency, the spread in synchrotron frequency and the bunch length. Adjustment of the higher harmonic system so as to reduce the slope of the RF wave to zero at the bunch centre leads to a longer bunch and a greatly increased spread in synchrotron frequency. This increases the Landau camping against longitudinal coupled bunch instabilities. The motion of single particles in this highly non linear potential is calculated numerically as well as analytically (by making some approximations). The dependence of the synchrotron frequency on amplitude and the forms of the synchrotron oscillations and the RF bucket are calculated. Finally the bunch shape and the distribution of particles in Q_S are calculated for electron bunches.

1. CALCULATION OF THE RF PARAMETERS

The voltage seen by the beam with a double RF system is (refer to Fig.1) :

$$V(\phi) = V_o \left\{ \sin(\phi+\phi_s) + k \sin(n\phi+n\phi_n) \right\} \tag{1}$$

where

V_o = the peak voltage of the fundamental RF
kV_o = the peak voltage of the higher harmonic RF
nf_{RF} = the higher harmonic frequency (f_{RFN})
ϕ_s = the stable phase angle relative to the fundamental RF waveform
ϕ_n = the stable phase angle relative to the higher harmonic waveform

The equation of synchrotron motion is :

$$\ddot{\phi} + \frac{\Omega_o^2}{V_o \cos \phi_s} \left\{ V(\phi) - V_r \right\} = 0 \tag{2}$$

where V_r = the voltage loss per turn due to synchrotron radiation
and Ω_o = the synchrotron frequency for small amplitude oscillations in a single RF system with peak voltage V_o and stable phase angle ϕ_s

Integration of equation (2) gives :

$$\left(\frac{\dot{\phi}}{\Omega_o}\right)^2 + \frac{2\, Y^2(\phi,\phi_s)}{\cos \phi_s} = \left(\frac{\hat{\dot{\phi}}}{\Omega_o}\right)^2 = H \tag{3}$$

where

$$Y^2(\phi,\phi_s) = \frac{1}{V_o} \int_0^{\phi} \left\{ V(\phi) - V_r \right\} d\phi \tag{4}$$

and $\hat{\dot{\phi}}$ refers to the peak value along a given trajectory.

To maximise the bunch length, the first derivative of $V(\phi)$ should vanish at the centre of the bunch[1,2], and in order to avoid having a second region of phase stability close by, the second derivative of $V(\phi)$ must also vanish. These two conditions give :

$$n\, k \cos n\, \phi_n = - \cos \phi_s \quad ; \quad n^2\, k \sin n\, \phi_n = - \sin \phi_s \tag{5}$$

In the general case and for a given n the RF parameters (eqn.(1)) are defined by V_o, ϕ_s, k, and ϕ_n. These parameters are evaluated from the following conditions :

(i) The voltage gain per turn of the synchronous particle is equal to the total loss per turn.

(ii) The momentum acceptance of the RF bucket must be large enough to provide the required quantum lifetime.

and (iii) The two conditions given by eqn.(5) must be satisfied.

Fig.1 shows the variation of $V(\phi)$, $\dot{\phi}$ and the potential function for the proposed LEP[3] machine at 86.11 GeV.

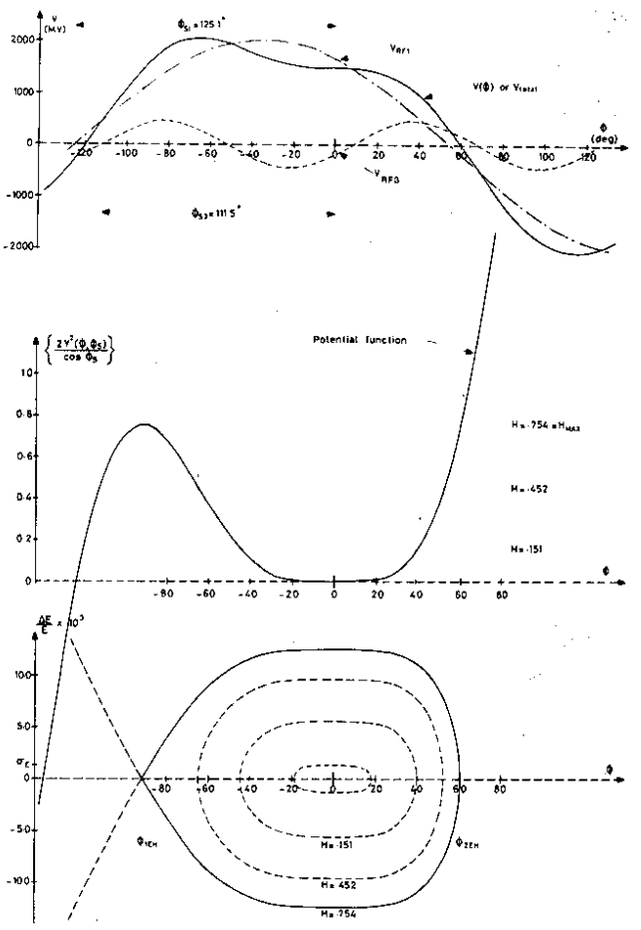

Figure 1

Double RF system for LEP.

2. DISTRIBUTION OF PARTICLES IN RF PHASE

For any RF waveform, the electron density distribution is given by[4,5,6]:

$$\psi\left(\phi, \frac{\Delta p}{p}\right) \propto \exp\left\{\frac{-e V_o Y^2 (\phi, \phi_s)}{2\pi h \eta \beta^2 E \left(\frac{\sigma_p}{p}\right)^2}\right\} \exp\left\{\frac{-\left(\frac{\Delta p}{p}\right)^2}{2\left(\frac{\sigma_p}{p}\right)^2}\right\} \qquad (6)$$

and hence the instantaneous current distribution is given by :

$$i(\phi) \propto e\, 2\pi f_{RF} \exp\left\{\frac{-e V_o Y^2 (\phi, \phi_s)}{2\pi h \eta \beta^2 E \left(\frac{\sigma_p}{p}\right)^2}\right\} \qquad (7)$$

where $\left(\frac{\sigma_p}{p}\right)$ is the rms momentum spread of the bunch (Gaussian) and E is the beam energy.

Figure 2 shows the calculated bunch distributions for a single RF system, a double RF system in which the conditions of eqn.(5) are fulfilled and for a double RF system in which the third harmonic voltage is limited to around one half of the value needed to provide zero slope of the RF waveform.

3. DISTRIBUTION OF PARTICLES IN Q_s

Associated with each point in phase space $(\dot{\phi},\phi)$ there is a corresponding value for the constant of motion H (refer to eqn.(3)). Along a trajectory of constant H the density $\psi(H)$ is constant. Inside the RF bucket each trajectory has two turning points in its ϕ motion

(ϕ_{1t}, ϕ_{2t}) ; where $\dot{\phi} = 0$). The phase space area A(H) and the periodic time T(H) of a trajectory are given by :

$$A(H) = 2 \int_{\phi_{1t}}^{\phi_{2t}} \dot{\phi}\, d\phi \qquad \text{and} \qquad T(H) = 2 \int_{\phi_{1t}}^{\phi_{2t}} \frac{d\phi}{\dot{\phi}} \tag{8}$$

The number of particles between two neighbouring trajectories is :

$$\Delta N = \psi(H)\ \Delta A(H) \tag{9}$$

Numerical evaluation of eqns (8) and (9) gives the distribution of particles as a function of the synchrotron frequency (1/T). Fig.3 shows the distributions of particles in Q_S for a series of higher harmonic voltages and for the LEP design.

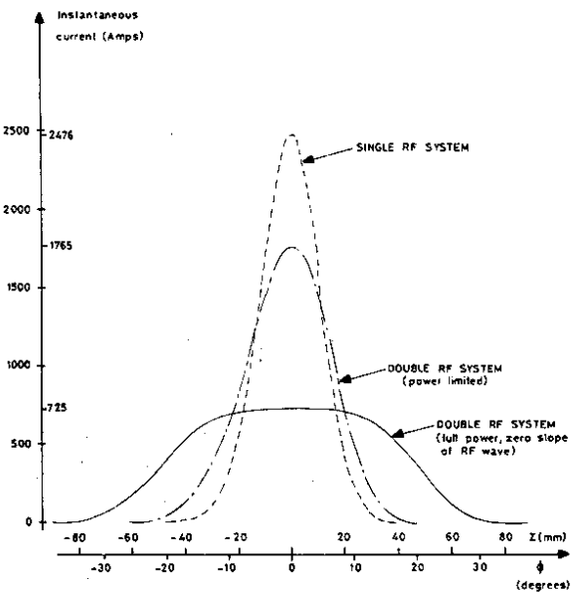

Figure 2 Computed bunch shapes →

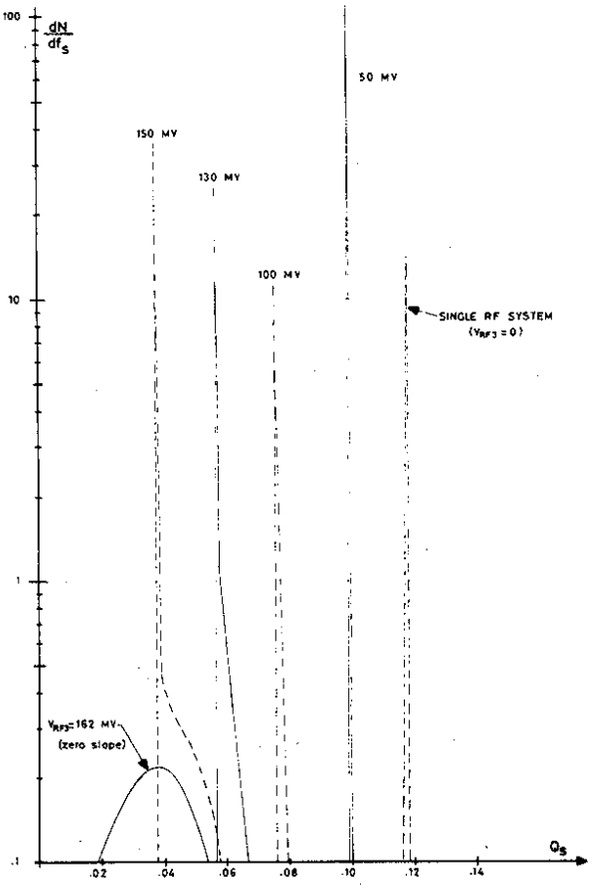

Figure 3 Computed distributions in Q_S →

4. CALCULATION OF PHASE PLANE TRAJECTORIES AS A FUNCTION OF TIME

The variations of $\dot{\phi}$, ϕ as a function of time are calculated by numerical solution of equation (2) with :

$$V_r = V_d + V_Q + V_{HM}$$

where d, Q and HM refer to the energy losses due to dipole and quadrupole magnets (synchrotron radiation), and due to higher modes in cavities and the vacuum pipe.

At each interval of time, the constant of motion (H) and the area of the trajectory A(H) associated with the $\dot{\phi}$, ϕ value are also evaluated. In this way the damping times of H, A, $\dot{\phi}$ and ϕ are calculated. The effects of variations in the frequency dispersion η is accomodated by making η as a function of $\frac{\Delta p}{p}$ in the equation for Ω_o. The frequency spectrum of the phase plane motion is also calculated in order to quantify the higher frequency components in the cases of non linear synchrotron motion.

5. APPROXIMATE ANALYTIC TREATMENT

It is useful to give approximate analytic expressions for the particle dynamics in the double RF-system which are valid for small amplitudes ($\phi \lesssim \frac{\pi}{n}$). For the bunch lengthening mode (5) the RF-wave form (1) and the potential function become

$$V(\phi) - V_r = V_o \frac{n^2-1}{6} \cos\phi_s \, \phi^3 \quad ; \quad Y^2(\phi,\phi_s) = \frac{n^2-1}{24} \cos\phi_s \, \phi^4 \, .$$

which gives for the phase plane trajectories or the Hamiltonian (3) :

$$\left(\frac{\dot{\phi}}{\Omega_o}\right)^2 + \frac{n^2-1}{12} \phi^4 = \left(\frac{\hat{\dot{\phi}}}{\Omega_o}\right)^2 = \frac{n^2-1}{12} \hat{\phi} = H$$

The phase oscillation frequency is obtained by integrating (8) :

$$\Omega_s = \Omega_o \frac{\pi}{2K(1/\sqrt{2})} \left(\frac{n^2-1}{3}\right)^{\frac{1}{4}} \left(\frac{\hat{\dot{\phi}}}{\Omega_o}\right)^{\frac{1}{2}} = \Omega_o \frac{\pi}{2K(1/\sqrt{2})} \left(\frac{n^2-1}{6}\right)^{\frac{1}{2}} \hat{\phi} \qquad (10)$$

where $K(1/\sqrt{2}) = 1.85407$ is the complete elliptic integral of modulus $1/\sqrt{2}$. The phase oscillation frequency is proportional to the amplitude $\hat{\phi}$ or to $(\hat{\dot{\phi}})^{\frac{1}{2}}$.

The phase motion itself can be described by the Jacobian elliptic function cn(u) and the product sn(u) dn(u) which are periodic functions with the period $4K(1/\sqrt{2})$. Chosing t=0 such that ϕ has a minimum one gets

$$\dot{\phi} = \hat{\dot{\phi}} \sqrt{2} \, \mathrm{sn}\left(\frac{4K(1/\sqrt{2})}{2\pi} \Omega_s t\right) \cdot \mathrm{dn}\left(\frac{4K(1/\sqrt{2})}{2\pi} \Omega_s t\right) ; \quad \text{and} \quad \phi = \hat{\phi} \, \mathrm{cn}\left(\frac{4K(1/\sqrt{2})}{2\pi} \Omega_s t\right).$$

Since the phase motion is non-linear it contains harmonics of the basic frequency[7]

$$\dot{\phi} = \hat{\phi} \left[1.14424 \sin(\Omega_s t) + 0.15474 \sin(3\Omega_s t) + 0.01114 \sin(5\Omega_s t), \ldots\right]$$

$$\phi = \hat{\phi} \left[0.95501 \cos(\Omega_s t) + 0.04305 \cos(3\Omega_s t) + 0.00186 \cos(5\Omega_s t), \ldots\right]$$

For electrons the particle distribution in momentum is Gaussian and the normalised phase plane distribution (6) becomes :

$$\psi\left(\phi, \frac{\Delta p}{p}\right) = \frac{\sqrt[4]{8}}{\left(\Gamma(1/4)\right)^2 \sigma_\phi \, \sigma_{p/p}} \exp\left[-\frac{2\pi^2}{\left(\Gamma(1/4)\right)^4} \left(\frac{\phi}{\sigma_\phi}\right)^4 - \frac{1}{2}\left(\frac{\Delta p/p}{\sigma_{p/p}}\right)^2\right] \qquad (11)$$

where σ_ϕ is the rms bunch length measured in RF-phase angle

$$\sigma_\phi = \frac{2\sqrt{\pi}}{\Gamma(1/4)} \left(\frac{3}{n^2-1}\right)^{\frac{1}{4}} \left(\frac{h|\eta| \, \sigma_{p/p}}{Q_{so}}\right)^{\frac{1}{2}}$$

with $$Q_{so} = \frac{\Omega_o}{\omega_{rev}} = \left(\frac{h \eta e V_o \cos\phi_s}{2\pi E}\right)^{\frac{1}{2}} \quad \text{and} \quad \Gamma(1/4) = 3.6256 \, .$$

The instantaneous current (7) as a function of the longitudinal coordinate s is obtained by integrating (11) over $\Delta p/p$

$$i(s) = i_p \exp\left[-\frac{2\pi^2}{[\Gamma(1/4)]^4}\left(\frac{s}{\sigma_s}\right)^4\right]$$

where $\sigma_s = \frac{R}{h}\sigma_\phi$ is the rms bunch length in meters. The peak current i_p is related to the average current i_o of k_b bunches.

$$i_p = \frac{4\pi\sqrt{2\pi}}{\sqrt[4]{2}\,[\Gamma(1/4)]^2}\,\frac{R\,i_o}{k_b\,\sigma_s} = 2.015\,\frac{R\,i_o}{k_b\,\sigma_s}$$

The phase space area (8) of a trajectory with amplitude $\hat{\dot{\phi}}$, $\hat{\phi}$ is[7]

$$A = \frac{4\sqrt{2}}{3}\,K(1\sqrt{2})\,\hat{\phi}\,\hat{\dot{\phi}} = 3.4961\,\hat{\phi}\,\hat{\dot{\phi}} \quad .$$

Using this, the distribution (11) and the relation (10) between phase oscillation frequency and amplitude the particle distribution in Ω_s can be calculated. For convenience this frequency is normalised with :

$$q = \frac{4K(1/\sqrt{2})}{\sqrt{\pi}\,\Gamma(1/4)}\left(\frac{3}{n^2-1}\right)^{\frac{1}{2}}\frac{1}{\sigma_\phi}\,\frac{\Omega_s}{\Omega_o} = 1.154\left(\frac{3}{n^2-1}\right)^{\frac{1}{2}}\frac{1}{\sigma_\phi}\,\frac{\Omega_s}{\Omega_o}$$

to get the distribution :

$$\frac{dN}{dq} = \frac{8K(1\sqrt{2})}{\sqrt[4]{2}\,\pi\,\Gamma(1/4)}\,q^2\exp\left(-\frac{1}{2}q^4\right) = 1.095\,q^2\exp\left(-\frac{1}{2}q^4\right)$$

This distribution agrees well with the computed exact distribution shown in Fig.3. It has a maximum at q = 1 and a width (FWHM) of $q \approx 0.82$.

The fact that the Hamiltonian contains ϕ and $\dot{\phi}$ with different powers leads to a difference in the radiation damping rate for energy and phase excursion[8]. The equation of motion including the longitudinal damping rate α_ε

$$\ddot{\phi} + 2\alpha_\varepsilon\dot{\phi} + \Omega_o^2\,\frac{n^2+1}{6}\,\phi^3 = 0$$

can be solved approximately for weak damping (the Hamiltonian changes little during one oscillation) with the result

$$\hat{\phi} \propto \exp\left(-\frac{2}{3}\alpha_\varepsilon t\right) \; ; \; \hat{\dot{\phi}} \propto \exp\left(-\frac{4}{3}\alpha_\varepsilon t\right) \quad .$$

which is in good agreement with the computed results.

The energy excursion is damped faster and the phase excursion slower than for a linear RF-system. However, the phase space area $A(H) \propto \exp(-2\alpha_\varepsilon t)$ is damped with the normal rate, as expected.[9]

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