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Beethoven's Fifth 'Sine'-phony: the science of harmony and discord

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Can science reveal the secrets of music? This article addresses the question with a short introduction to Helmholtz's theory of musical harmony. It begins by discussing what happens when tones are played at the same time, which introduces us to the idea of beats. Next, the difference between a pure tone and a note played on an instrument is explained, which provides the insight needed to form Helmholtz's theory. This allows us to explain why a perfect fifth sounds consonant yet a diminished fifth is dissonant.

When sitting through a performance of Beethoven's Fifth Symphony, we are interpreting the complex sound wave which propagates through the air from the orchestra to our ears. We say that we are 'listening to music' and be it classical, rock or rap, music is considered a form of art. But a sound wave is something we study in physics. So just how much science is there behind the music? Can we use physics to explain why Ludwig's harmony is harmonious? And does a complete 'Theory of Music' exist which explains our enjoyment of every movement, phrase, bar and note we hear?

To start with, we need to look at the fundamental building blocks of music: these are the pure tones. If you strike a tuning fork, you will set it vibrating in simple harmonic motion. The sound that results is the simplest musical sound—a pure note all on its own (imagine an electronic 'beep' from an alarm clock). The number of vibrations the fork makes per second gives its frequency. In music this is called the *pitch* of the note. For example, any object which vibrates 261.6 times per second will sound a middle C. If it had a frequency of 392 Hz instead, then we would hear the G just above. Objects vibrating at higher frequencies sound at a higher pitch and, importantly, if the frequency of a note is *doubled*, then the pitch increases by an octave. Thus, if something vibrates at 523.3 Hz we would hear the note C an octave above middle C (this note is denoted $C^{\wedge}$, the note two octaves up $C^{\wedge\wedge}$ and so on—see figure 1).

So what happens when we hear more than one pure tone at the same time? The two sound waves combine, obeying the Principle of Superposition and the resultant wave is as in figure 2. The maximum amplitude is contained within an 'envelope' having frequency equal to the frequency difference of the two pure tones. This is the phenomenon of beating—the periodic rise and fall in loudness of the note, creating a 'fluttering' sound. Beats are important when it comes to our enjoyment of music for they can make the interval between the two pure tones sound unpleasant. The amount of irritation they cause our ear depends on how many flutters we hear per second (see figure 3). A slow rise and fall in amplitude does not annoy and is even employed in Hammond organs to create their distinctive trembling sound. A fast beating means our ears cannot distinguish the individual beats and so the effect blurs itself out in much the same way as the 25 images a second on a television set do. Provided this beating is fast enough, again we do not encounter any unpleasantness. It is when there are around 30 beats per second that the sound becomes

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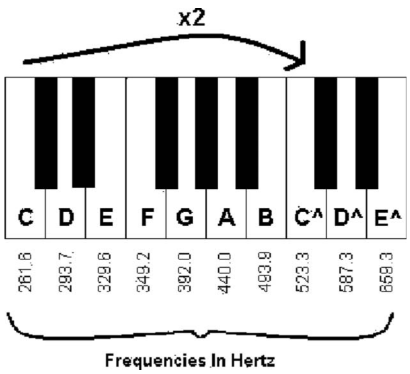


Figure 1. Notes of the keyboard with the corresponding frequency. Note that a doubling in frequency corresponds to a pitch rise of an octave.

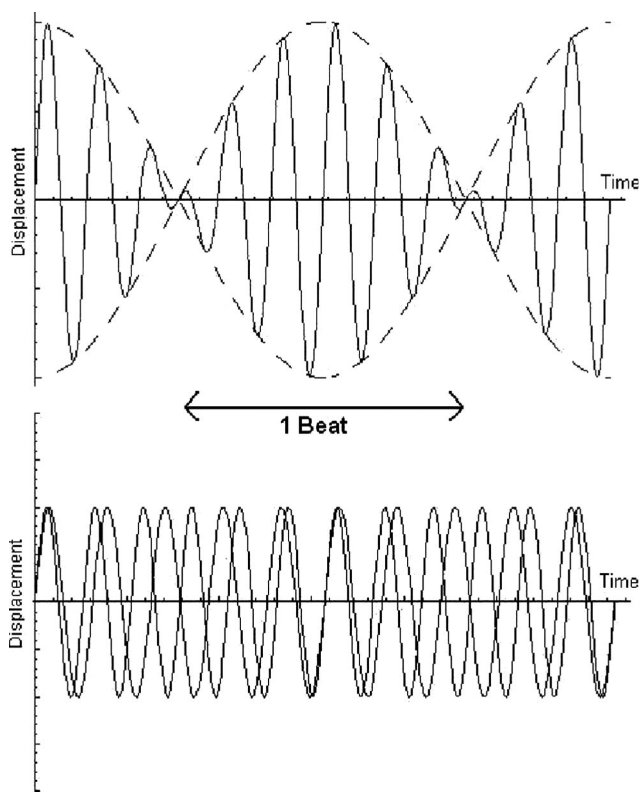


Figure 2. The red wave shows the amplitude resulting from the superposition of two pure tones with different frequencies seen below (black waves). The overall amplitude is contained within an envelope (dashed line) which creates the beating.

irksome, just as 4 images every second on a TV screen would be maddening to the eye. A difference of 30 Hz is close to the interval of a semitone and indeed, playing a C and a C# at the same time creates a jarring, dissonant

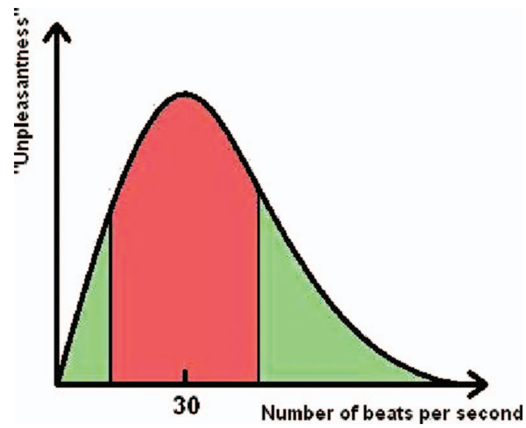
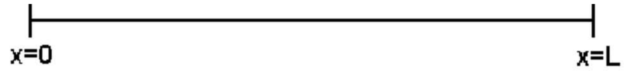


Figure 3. Around 30 beats per second creates the maximum amount of unpleasantness. The above graph represents the fall in unpleasantness at high and low beat frequencies.

sound. This idea of beats affecting the nature of the sound we hear is an important aspect of Helmholtz's theory of harmony, which is discussed later on.

To proceed further we need to observe that Beethoven did not compose any symphonies for large numbers of tuning forks. Instead he used orchestras of instruments. To exemplify what is so 'musical' about instruments we take the example of a stretched string. Figure 4 shows the result of solving the wave equation on this string. In physics we call the solutions normal modes. In music we call them harmonics and, being simple harmonic waves, they are all *pure tones*. We call the normal mode with wavelength twice that of the string's length the fundamental pure tone. This is the first harmonic of the series. The second harmonic has half this wavelength, the third one three times smaller and so on (figure 5). So if the fundamental pure tone is C (261.6 Hz), then it is also possible for this string to vibrate in such a way as to produce pure tones of frequencies which are integer multiples of 261.6 Hz. Thus, the second harmonic is C^, the third a fifth higher than this (G^) and so on.

Now, when a 'stringed' instrument is played, whether it be struck by a hammer, plucked, bowed or strummed, the string is set into a complicated motion. However, using the technique of Fourier analysis, we can break down this vibration, no matter how complicated, into a superposition of the normal mode vibrations. The amplitude of each normal mode is chosen so that when they are all superimposed, the resulting motion of the string is exactly that after the strumming, plucking etc. has been performed. Musically then, it is not only the fundamental pure tone being sounded, but the harmonics are also set into action and we hear these as well. This is the origin of the richness in sound we hear when an instrument is played. Almost always, the fundamental tone has much larger amplitude than the harmonics and the higher harmonics have rapidly



Let y be the displacement of the string, length L, and look for solutions which are waves (travelling backwards and forwards) of the form

$$y = A\exp(ikx) + B\exp(-ikx)$$

where A and B are constants. Now we impose boundary conditions at $x = 0$ and $x = L$ so that $y = 0$ there (as the string is fixed).

At $x = 0$;

$$0 = A + B$$

$$A = -B$$

And at $x = L$

$$0 = A\exp(ikL) - A\exp(-ikL)$$

$$0 = 2i\sin(kL)$$

$$\Rightarrow kL = n\pi$$

where n is an integer ($n = 1, 2, 3\ldots$).

Hence we have standing waves of frequency $f = kv/2\pi = nv/(2L)$, where v is the speed of the wave.

Figure 4. Solving the wave equation on a fixed string results in normal modes, known in music as harmonics.

dwindling amplitudes. So, although in theory the number of harmonics can be infinite, we normally only hear up to the eleventh one layered faintly on top of the fundamental. Which of the harmonics are present and their relative amplitudes gives one of the main contributions to the *timbre* of a musical note—what makes a violin C sound different to a trumpet C. Musical instruments also emphasize certain harmonics differently through resonance effects—the cavity inside a violin will reinforce different harmonics to that of the guitar.

Some of the harmonics will be tones which 'clash' with the fundamental tone. Normally this isn't a problem since the amplitude of the high harmonics is usually very small. However, the seventh harmonic, for example, can have large enough amplitude to worry about (for a fundamental tone of C the seventh harmonic is the B flat three octaves higher). This B flat creeping into the note will create a dissonant sound. It is for this reason that piano strings are stuck around one seventh of the way down, hitting near a node of the seventh normal mode. Because this is the point at which the mode does not vibrate, it is not set into motion and the note is not heard. Conversely, it is in the interest of the instrument maker to reinforce the harmonics which create a consonant sound when heard alongside the fundamental, such as the second, third and fourth (in our example C^, G^ and C^^).

Now that we understand how an instrument is different to a tuning fork, we can build up a theory of harmony, originally devised by Helmholtz. The graph in figure 3 doesn't explain why intervals larger than a semitone sound dissonant. A diminished fifth (e.g. C and F#) sounded on two tuning forks creates so many beats per second that they cannot be distinguished by the ear and no dissonant sound is heard. However, on a keyboard things are very different: the interval sounds quite different than that of a perfect fifth (C and G), which is a consonant sound. The difference is that we are now dealing with rich, musical notes, full of harmonics and it is the *harmonics* which are clashing with each other a semitone apart, creating the beats which are so displeasing. Figure 6 shows a diminished fifth and a perfect fifth with all the harmonics which would be heard above the fundamental notes. For the diminished fifth, the second

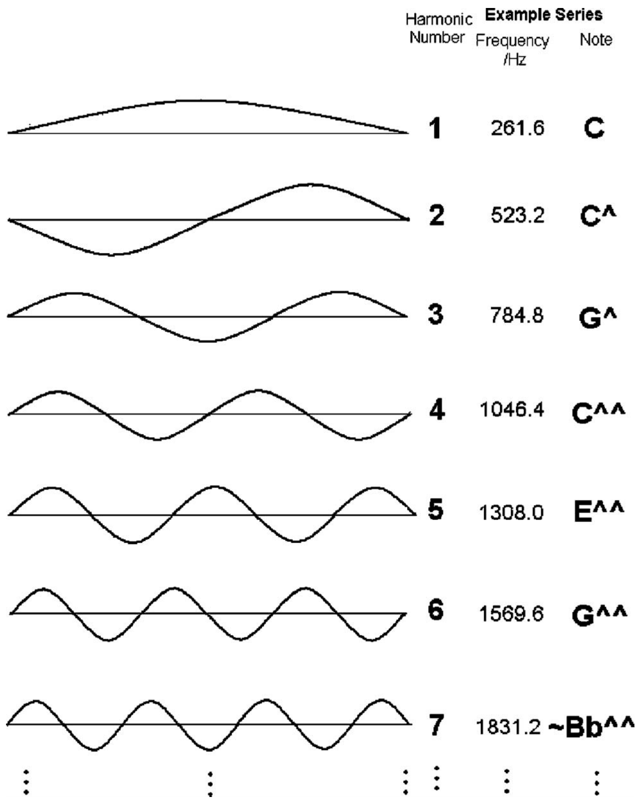


Figure 5. The displacement of the vibrating string is shown for the first seven harmonics, along with corresponding frequencies for a fundamental note of C.

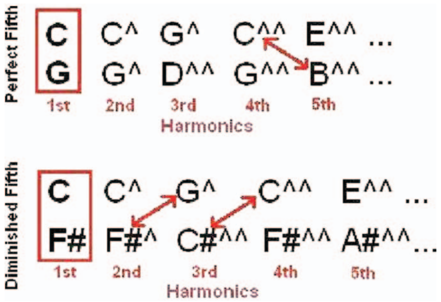


Figure 6. We hear the harmonics as well as the two fundamental notes (boxed in red) when an interval is played on an instrument. For a perfect fifth, the first harmonics to be around 30 beats apart are the 4th and 5th ones which are very quiet. Therefore, it is a consonant interval. For the diminished fifth, louder harmonics beat with each other creating a dissonant sound.

and third harmonics are clashing a semitone apart, creating those 30 beats per second and thus producing an uncomfortable sound. In contrast, it is only when we get to the fifth harmonic of the perfect fifth that the 30 beats are formed. Since the fifth harmonic is much quieter, the beats are almost inaudible and the lack of them forms the consonant sound.

By looking for unpleasant beating in the harmonics as well as the fundamental tone of a note, we can analyse any number of notes played at the same time and discover how discordant or harmonious the sound will be. This is obviously an important method which can be used to examine whole pieces of music—if we wanted to, we could do this for the entire score of Beethoven's Fifth and would mostly find it to consist of consonant chords.

We are now closer to answering the questions posed at the start of this article. Firstly, it is clear that the science of music is an extensive and complex subject. Here, we have focused only on the physical characteristics of the 'musical' sound wave. Many other effects have not been mentioned: for example the modification of the sound wave by the physical dimensions of the concert hall. This area of acoustics is fascinating and its study has enabled buildings such as the Sydney Opera House to be constructed.

What of our all encompassing 'Theory of Music'? We have seen how any piece of music can be built out of pure tones—sine waves of all frequencies combine to create the final sound wave which reaches our ears. Beethoven really did compose a 'Sine'-phony. And Helmholtz's theory explains the pleasantness of harmony, so the scientist knows why the symphony does not consist entirely of diminished fifths. When Beethoven chose his chords, for the most part he spared his listeners the discomfort of 'beating' notes. But this is the limit of our scientific theory. Why a particular sequence of notes one after the other creates a beautiful melody or why varying the length of them makes for a foot-stomping rhythm is something which, for now, requires an artistic mind.

So what is the difference between the sound wave of a composed melody and a sound wave of randomly played notes? The understanding of why only the former is musical must lie in the understanding of the biological mechanisms which begin in the workings of the ear and finish in the brain. We all have an intuitive 'feel' for the musical melody, but the physical distinction is extremely subtle. Clearly something quite profound is at work. It is humbling that although science can delve deep into the heart of an atom and explore the furthest reaches of the universe, it is at a loss when trying to describe more than the very basics of musical enjoyment. Maybe this is an area science will never be able to conquer. The secrets of the finer aspects of music could stay forever locked away in the mind of the musician.

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Tom Melia is a 4th year undergraduate studying physics at Exeter College, Oxford. This article was originally a talk he prepared as part of a second year project—it was a natural subject choice for, being a musician, he has always been interested in the science behind music. He hopes to continue his studies next year.