WHY STUDY SEMIGROUPS?

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1. Introduction

Before tackling the question in my title I should perhaps begin by saying what a semigroup *is*. A non-empty set S endowed with a single binary operation . is calleu a *semigroup* if, for all x , y , z in S ,

$$(xy)z = x(yz) .$$

If in addition there exists 1 in S such that, for all x in S ,

$$1x = x1 = x$$

we say that S is a *semigroup with identity* or (more usually) a *monoid.*

I shall be confining myself today to semigroups that have no additional structure. Thus, though semigroups feature quite prominently in parts of functional analysis, the *algebraic* structure of those semigroups is usually very straightforward and so they scarely rate a mention in any algebraic theory. Equally, although they are often of greater algebraic interest, I shall say nothing about topological semigroups.

Let me begin by answering a slightly different question: Who studies semigroups? Section 20 in *Mathematical Reviews* is entitled 'Groups and generalizations' and has two 'leper colonies' at the end, called 20*M Semigroups and* 20*N Other generalizations*. In the Introduction to their book *The algebraic theory of semigroups* in 1961 Clifford and Preston [2] remarked that about thirty papers on semigroups per year were currently appearing. A brief look at recent annual index volumes of *Mathematical Reviews* shows that the current figures are

1982	321
1983	310
1984	311 .

Incidentally, comparable figures for 'other generalizations' are about one third of these. So it is clear that of all generalizationsof the group concept the semigroup is the one that has attracted the most interest by far. I shall in due course hazard a guess as to why this is so.

Mathematicians are rightly a bit suspicious of theories whose only motive seems to be to generalize existing theories - and if the only motivation for semigroup theory were to examine group-theoretical results with a view to generalization, then I would have no very convincing answer to the question of my title. The test proposed by Michael Atiyah for a generalization - that it should have at least two distinct and interesting special cases - is a reasonable one provided it is not applied too dogmatically; and the semigroup concept passes the test, since from the outset semigroup theory drew its ideas partly from group theory and partly from ring theory.

For clearly every ring $(R,+,.)$ is a semigroup if we simply neglect the operation $+$. The converse is certainly not true: that is, there are semigroups $(S,.)$ with zero on which it is not possible to define an operation $+$ so as to create a ring $(S,+,.)$. The easiest way to see this is to recall the known result that a ring $(R,+,.)$ with the property that $x^2 = x$ for all x in R (a *Boolean* ring) is necessarily commutative - i.e. satisfies $xy = yx$ for all x in R . Now let $S = (A{\times}B) \cup \{0\}$, where A , B are non-empty sets and $0 \notin A \times B$, and make S into a semigroup with zero by defining

$$\begin{aligned}(a,b)0 \;=\; 0(a,b) \;&=\; 00 \;=\; 0\,,\\ (a_1,b_1)(a_2,b_2) \;&=\; (a_1,b_2)\;.\end{aligned}$$

Then S has the property that $x^2 = x$ for all x in S . On the other hand S is certainly not commutative and so cannot be made into a ring.

2. Pure mathematical reasons

Let me now turn to my question 'Why study semigroups?' I am a pure mathematician by instinct and so I begin by offering some pure mathematical reasons. I shall come later to what might be called 'applied mathematical' reasons.

My first point, not a fashionable one in these utilitarian times, is that they are *fun*, that they provide an elegant theory, with arguments that any mathematician can actively enjoy. Let me give an example. The concept of *regularity*, introduced for rings by no less a person than von Neumann, plays a much more central role in semigroup theory than it does in ring theory. We say that $(S,.)$ is *regular* if

$$(\forall\ a\ \in S)(\exists\ x\ \in S)\quad axa = a\ . \tag{1}$$

As in ring theory, *idempotent* elements - elements e such that $e^2 = e$ - are very important. From equation (1) we readily see that both ax and xa are idempotent.

Next, note that a semigroup $(S,.)$ is regular if and only if

$$(\forall\ a\ \in S)(\exists\ a' \in S)\quad aa'a = a\ ,\quad a'aa' = a'\ . \tag{2}$$

It is clear that $(2) \Rightarrow (1)$. To show that $(1) \Rightarrow (2)$ simply take $a' = xax$; then from (1) we have

$$aa'a = a\ , \qquad a'aa' = a'\ .$$

The element a' is usually called an *inverse* of a , but it should be noted that this is a weaker concept of inverse than the one used in group theory: for example in the four element semigroup with Cayley table

	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

it is easy to check that every element is an inverse of every other element.

Theorem 1. *The following coniditons on a regular semigroup S are equivalent:*

(1) *Idempotents commute;*

(2) *Inverses are unique* .

Proof. (1) ⟹ (2) . Suppose that idempotents commute. Let a' , a^* be inverses of a . Then

$$\begin{aligned} a' &= a'aa' = a'aa^*aa' = \underline{a'aa^*a}a^*ac' && \text{by } (2) \\ &= a^*aa'\underline{aa^*aa'} = a^*aa'aa'aa^* && \text{by commuting idempotents} \\ &= a^*aa^* = a^* && \text{by } (2)\ . \end{aligned}$$

(2) ⟹ (1) . Let e , f be idempotents and let x be the unique inverse of ef :

$$efxef = ef\ , \quad xefx = x\ .$$

Then fxe is idempotent, since

$$(fxe)^2 = f(xefx)e = fxe\ ;$$

and ef is an inverse of fxe :

$$\begin{aligned} (fxe)(ef)(fxe) &= f(xefx)e = fxe\ , \\ (ef)(fxe)(ef) &= efxef = ef\ . \end{aligned}$$

But an idempotent i is its own unique inverse $(iii = i\ ,\ iii = i)$ and so $ef = fxe$, an idempotent. Similarly fe is idempotent.

The unique inverse of ef is thus ef itself. On the other hand fe is an inverse of ef , since

$$(ef)(fe)(ef) = (ef)^2 = ef\ , \quad (fe)(ef)(fe) = (fe)^2 = fe\ .$$

It follows that $ef = fe$, as required.

That argument goes back to the early 1950s, to some fundamental work by Vagner [16 , 17] and Preston [12 , 13 , 14] . A regular semigroup satisfying either one (and hence both) of the conditions in Theorem 1 is called an *inverse* semigroup. A very impressive theory has been created for such semigroups, as is evidenced by the publication in 1984 of a 674-page monograph by Petrich [11] devoted entirely to semigroups of this kind.

But do semigroups of this kind occur 'in nature'? Let me give a not very well known example due to Schein [15] and McAlister [10] . Let G be a group and let $K(G)$ be the set of all right cosets of G . This includes G itself and also the cosets of the subgroup 1 , which are effectively the elements of G . Define an operation $\star$ on $K(G)$ by

$$Ha \star Kb = (H \vee aKa^{-1})ab .$$

This is a natural definition: it is not hard to check that $Ha \star Kb$ is the smallest coset containing the product $HaKb$. [Certainly

$$HaKb = (HaKa^{-1})ab \subseteq (H \vee aKa^{-1})ab .$$

Conversely, suppose that $HaKb \subseteq Pc$ $(\in K(G))$. Then in particular $ab \in Pc$ and so $Pc = Pab$. Now

$$Hab \subseteq HaKb \subseteq Pab \text{ and so } H \subseteq P ;$$

also

$$(aKa^{-1})ab = aKb \subseteq HaKb \subseteq Pab$$

and so $aKa^{-1} \subseteq P$. Thus $H \vee aKa^{-1} \subseteq P$ and so

$$(H \vee aKa^{-1})ab \subset Pab = Pc .]$$

It is a routine matter to check that $\star$ is an associative operation and that $(a^{-1}Ha)a^{-1}$ is an inverse of Ha in the semigroup $(K(G) , \star)$. Now suppose that Ha is idempotent:

$$Ha = Ha \star Ha = (H \vee aHa^{-1})a^2 .$$

Then in particular $a^2 = 1a^2$ Ha ; i.e. $a^2 = ha$ for some h in H .

Hence $a = h \in H$ and so $Ha = H$. In fact the idempotents of $(K(G), *)$ are precisely the *subgroups* of G. For any two subgroups H, K of G

$$H * K = H \vee K = K * H .$$

Thus idempotents commute and so $(K(G), *)$ is an inverse semigroup.

The *normal* subgroups N of G are such that, for all Ha in $K(G)$,

$$N * Ha = (N \vee H)a = (NH)a .$$
$$Ha * N = (H \vee aNa^{-1})a = (H \vee N)a = (NH)a ,$$

and so are *central* idempotents in $(K(G), *)$. Conversely, if N is a central idempotent then for all a in G

$$Na = N * 1a = 1a * N = (1 \vee aNa^{-1})a = aN$$

and so N is normal.

That is a slightly quaint example, but I mention it because I have the feeling that it has not yet been adequately exploited. The main reason that semigroups turn up in mathematics is that one is very often interested in self-maps of a set of one kind or another, and whenever f, g, h are such maps it is automatically the case that

$$(f \circ g) \circ h = f \circ (g \circ h) .$$

If the maps are bijections then the appropriate abstract idea is that of a group; if not then inevitably we must consider a semigroup. It is this connection with maps (arising from the associative axiom) that is the strongest reason why semigroups are more important both theoretically and in applications than the various non-associative generalizations of groups.

There is another pure mathematical reason for being interested in semigroups. It is possible to take a very general standpoint in algebra and to discuss a so-called Ω-algebra A having a family $\Omega = \{\omega_i : i \in I\}$ of operations, where $\omega_i : A^{n_i} \rightarrow A$ is an n_i-ary operation. [For example, in a group one can take $I = \{1,2\}$, $n_1 = 2$, $n_2 = 1$, with

$$[\omega_1(a_1,a_2) = a_1a_2 , \quad \omega_2(a_1) = a_1^{-1} .]$$

If $\phi : A \rightarrow B$ a is map between Ω-algebras then we say that ϕ is a *morphism* if (for all i in I and for all $a_1 , \ldots , a_{r_i}$ in A)

$$\phi(\omega_i(a_1 , \ldots , a_{n_i})) = \omega_i(\phi(a_1) , \ldots , \phi(a_{n_i})) .$$

If we regard ϕ as applying to A^{ni} in an obvious way then we can express this property succinctly as a commuting condition

$$\phi \circ \omega_i = \omega_i \circ \phi . \tag{3}$$

A *congruence* on A is an equivalence relation $\sim$ with the property that (for all i)

$$a_1 \sim a_1' , \ldots , a_{n_i} \sim a_{n_i}' \Rightarrow \omega_i(a_1 , \ldots , a_{n_i}) \sim \omega_i(a_1' , \ldots , a_{n_i}') . \tag{4}$$

Consider the quotient set $A/\sim$, whose elements are equivalence classes

$$[a] = \{x \in A : x \sim a\} .$$

The congruence property means that $A/\sim$ inherits the Ω-algebra structure from A : we simply define

$$\omega_i([a_1] , \ldots , [a_{n_i}]) = [\omega_i(a_1 , \ldots , a_{n_i})] \tag{5}$$

and the compatibility condition (4) ensures that the definition makes sense. There is a natural map $\natural : A \rightarrow A/\sim$ defined by

$$\natural(a) = [a] \qquad (a \in A)$$

and the definition (5) can be interpreted as saying that

$$\omega_i \circ \natural = \natural \circ \omega_i ;$$

hence, comparing with (3), we see that $\natural$ is a morphism.

Now suppose that ϕ is a morphism from A *onto* B; we say that B is a *morphic image* of A. Define $\sim$ on A by the rule that

$$a \sim a' \quad \text{if and only if} \quad \phi(a) = \phi(a') .$$

It is easy to verify that $\sim$ is a congruence. The *first isomorphism theorem* for Ω-algebras is then as follows:

Theorem 2. *Let* A, B *be* Ω*-algebras, let* $\phi : A \to B$ *be a morphism with* $im\ \phi = B$, *and let* $\sim$ *be the congruence on* A *defined by* (6). *Then there is an isomorphism* $\alpha : A/\sim \to B$ *such that the diagram*

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "\natural"'] & B \\
A/\sim \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

commutes.

This is, in one form or another, one the cornerstones of abstract algebra. It says in effect that an Ω-algebra A carries its morphic images 'within itself' and that to reveal them we need only consider the quotients of A by its various congruences.

The result applies to groups, of course, but it is not usually stated in quite this way. This is because for a group A there is a one-one correspondence between congruences $\sim$ and normal subgroups N given by

$$N = \{a \in A : a \sim 1\} , \quad \text{or}$$

$$a \sim b \quad \text{if and only if} \quad ab^{-1} \in N .$$

The quotient $A/\sim$ is always denoted by A/N. Similarly, for a ring A there is a one-one correspondence between congruences $\sim$ and two-sided ideals I given by

$$I = \{a \in A : a \sim 0\}, \quad \text{or}$$

$$a \sim b \quad \text{if and only if} \quad a - b \in I,$$

and the quotient $A/\sim$ is always written A/I .

In semigroups no such device is available and we must study congruences as such. So semigroups consititute the simplest, most manageable and most natural class of algebras to which the methods of universal algebra must be applied.

3. Applied Mathematical reasons

Let me now turn to less exalted reasons for studying semigroups. One of the striking aspects of semigroup conferences these days is that many of the participants, between a third and a half, at a guess, come from departments of computer science. The reason is that semigroups have found significant applications in the theory of automata, languages and codes.

If A is a non-empty set (an *alphabet*, as we often want to call it) then the set of all finite words

$$w = a_1a_2 \dots a_n$$

in the alphabet A is a semigroup if we define multiplication by juxtaposition. Denote the *length* of w by $|w|$. If we include the empty word 1 (with $|1| = 0$) then we obtain a monoid, which we denote by A^* . This is the *free monoid* generated by A . The set of non-empty words in A^* is usually denoted by A^+ . A subset of A^* is called a *language* .

Now let Q be a finite non-empty set and suppose that we have a map $f : Q \times A \rightarrow Q$; we normally write $f(q,a)$ simply as qa and think of A as 'acting' on Q . The function f can be extended to $Q \times A^*$ by defining (inductively)

$$q1 = q(q \in Q)$$

$$q(wa) = (qw)a \qquad (q \in Q, \ w \in A^*, \ a \in A) .$$

We say that $\mathsf{A} = (Q,f)$ is an A^**-automation*. (In the terminology of

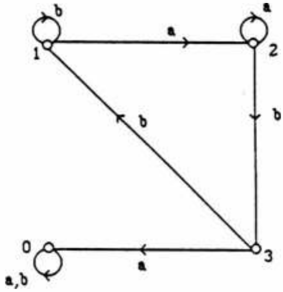
Eilenberg [4] this is a complete deterministic automaton.) We may think of it as a very rudimentary machine whose *states* (the elements of Q) can be altered by various *input* (the elements of A) .

Suppose now that among the elements of Q there is an element i which we call the *initial* state and that there is a subset T of Q called the set of *terminal* states. Let

$$L = \{w \in A^* ; \quad iw \in T\} .$$

Then we say that L is the *language recognized by the automaton* A . A language L is called *recognizable* if there exists an automaton A recognizing L .

Example. We can picture an automaton via its *state graph*. If $A = \{a\ b\}$, $Q = \{0,1,2,3\}$, $0a = 0b = 0$, $1a = 2a = 2$, $3a = 0$, $1b = 1$, $2b = 3$, $3b = 1$, then we can draw the picture



Let $i = 1$, $T = \{1,2,3\}$. It is easy to see that 0 is a 'sink' state from which no escape is possible, and that

$$1aba = 2aba = 0 .$$

In fact L , the language recognized by this automaton, consists of all words in A^* not containing aba as a segment.

If L_1 , $L_2 \subseteq A^*$ then $L_1 . L_2$ is defined as $\{w_1w_2 : w_1 \in L_1 , w_2 \in L_2\}$. If $L \subseteq A^*$ then $\langle L \rangle$ denotes the submonoid of A^* generated by L . Let F be the set of all finite subsets of A^* . Then the set

Rat A^* of *rational* subsets of A^* is the set of subsets of A^* obtained from F by means of the operations of $\cup$ (finite union), . and $<\ >$. This leads to an important characterization of recognizable subsets of A^* :

Theorem 3. (Kleene [7]). *A language L is recognizable if and only if it is rational.*

Another characterization of recognizable languages is more algebraic in character. If L is any subset of A^* then the *syntactic congruence* $\sim$ of L is the relation $\sim$ on A^* defined by

$$w_1 \sim w_2 \quad \text{iff } [uw_1\, v \in L \text{ iff } uw_2\, v \in L] \ . \tag{7}$$

(I have not gone into the mathematical theory of grammar at all, but one can perhaps dimly see that this is saying that w_1 and w_2 are mutually interchangeable in the set of 'meaningful' sentences that constitutes L . Thus (very roughly speaking) if u = (the), v = (sat on the mat) then cat $\sim$ dog but cat $\not\sim$ black . So 'cat' and 'dog' are syntactically equivalent but 'cat' and 'black' are not.) It is easy to verify that $\sim$ is a congruence on A^* . Then $A^*/\sim$ is called the *syntactic monoid* of L and is usually denoted by $M(L)$.

Theorem 4. *L is recognizable if and only if its syntactic monoid $M(L)$ is finite.*

This is by no means as deep as the Kleene theorem. Indeed one way round it is virtually obvious. If $M(L)$ is finite we can define an action of A on $M(L)$ by

$$f(\overline{w},a) \;=\; \overline{w}\,a \;=\; \overline{wa} \quad (\overline{w} \in M(L) \;=\; A^*/\sim)$$

making $\mathrm{A} = (M(L)\ ,f)$ into an A^*-automation. If we then take $\overline{1}$ as initial state and $\overline{L} = \{\overline{w} : w \in L\}$ as the set of terminal states then A recognizes z $(\in A^*)$ if and only if $\overline{1}z \in \overline{L}$, i.e. if and only if $\overline{z} \in \overline{L}$, i.e. if and only if there exists w in A^* such that $w \in L$ and $w \sim z$. But then $1w1 \in L$ and hence $z = 1z1 \in L$ by (7) . We conclude that the automation $\mathrm{A} = (M(L)\ ,f)$ recognizes L .

I mention this proof (or rather half-proof) because it emphasised the very close links between automata and monoids. A further connection is provided by the theory of *codes*.

It is not the case that every submonoid of a free monoid is free. For example, in $\{a,b\}^*$ consider the submonoid $M = M^+ \cup \{1\}$, where $M^+ = \{a^i : i \geq 2\}$. The *base* of M (i.e. the set $M^+ \setminus (M^+)^2$ of indecomposable elements of M) is $\{a^2,a^3\}$, but M is certainly not freely generated by $\{a^2,a^3\}$. We say that a subset C of A^* is a *code* if it is the base of a *free* submonoid of M. For example,

$$C_1 = \{a^2, aba, ab^2, b\}, \qquad C_2 = \{a^4, b, ba^2, ab, aba^2\}$$

are codes. This means, for example, that any word in four letters - say $x_3x_1x_4x_2x_3$ - can be encoded unambiguously (in C_1, say) as

$$ab^2a^2baba^2b^2 .$$

C_1 is in fact an example of a *prefix* code: any word in $<C_i> = C_1^*$ can be decoded without hesitation by reading from the left. This is because $C_1 \cap C_1A^+ = \emptyset$. By contrast if one tries to decode

$$aba^6ba^2$$

using C_2 the answer (unique since C_2 is a code) is $x_5x_1x_3$, but one might first have tried x_4x_1 ? or $x_5x_1x_2$? before reaching the correct solution.

That the theory of codes is intimately bound up with the theory of monoids is illustrated by.

Theorem 5. *Let M be a submonoid of a free monoid A^* and let C be its base. Then C is a prefix code if and only if M satisfies*

$$u, \ ux \in M \Longrightarrow x \in M .$$

(M is called left unitary.)

This has been a very sketchy introduction to a rich and rapidly growing area. Those whose appetites have been whetted should turn to Eilenberg [4,5] , Lallement [8] , Berstel and Perrin [1] , and Lothaire [9] .

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