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Behavior of the Yukawa and the Quartic Scalar Couplings in Grand Unified Theories

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Renormalization group analyses of the Yukawa and the quartic scalar couplings are made at the one-loop level in the framework of grand unified theories. The standard Weinberg-Salam model with one Higgs scalar doublet and that with two massless Higgs scalar doublets are examined numerically. The particle mass bounds, the fermion mass ratio and the behavior of the quark mixing angles are studied. In the latter model, some discrete symmetries are assumed in order to avoid flavor nonconserving neutral scalar interactions.

§ 1. Introduction

The phenomenology of the weak, electromagnetic and strong interactions seems to be described by the standard $SU(3)_c \times SU(2) \times U(1)$ gauge theory. Its extension to the grand unified theory (GUT), for example the $SU(5)$ model, explains the charge quantization.[1)]

Though there is the hierarchy puzzle,[2)] the grand unified gauge theories provide us with profound understanding of particle physics. They predict a phenomenologically favorable relation between the Weinberg angle of the neutral weak interactions and the coupling constant of the strong interactions (QCD), and in a minimal scheme the fermion mass ratio m_b/m_τ, with the help of renormalization group equations (RGE's) at the one-loop level.[3)] These agreements seem to suggest that perturbation calculations in the effective $SU(3)_c \times SU(2) \times U(1)$ gauge theory make sense, and that there is a desert up to the grand unification mass M_X from the mass of $SU(2) \times U(1)$ breaking M_W.

Of course, it is possible that there are some additional thresholds of horizontal gauge symmetries[4)] and of technicolors,[5)] etc. In such cases, the GUT predictions of the gauge coupling constants are not likely to change significantly, whereas those of the Higgs scalar structure are probably changed.

One of the useful methods to check the desert structure and the standard GUT predictions is to study the behavior of the Yukawa and the quartic scalar coupling constants using the RGE's. Discussions along this line have been made in some recent works.[6),7)] It has been made clear that there are upper bounds on the masses of fermions and a neutral Higgs boson in the model with one Higgs scalar doublet.[6)] These bounds are calculated from the values of the Yukawa

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and the quartic scalar couplings at M_W, which satisfy the following two conditions:

(1) All interactions should remain in the perturbation domain,

(2) No vacuum instability should develop,

in the whole energy range between M_W and M_X. As a perturbation constraint, we adopt in this paper that all coupling constants do not increase infinitely in calculations of the RGE's at the one-loop level. These conditions should be satisfied, if we want to compute the effects of the grand unification and its breaking. So far calculations of the RGE's are made within a simple model which contains only one Higgs doublet.

The number of Higgs scalars is one of the interesting problems in the unified gauge theory. There are some discussions for the theory to incorporate more than one Higgs doublet. First, the *CP* violation in the weak interactions can be realized in a model with more than two Higgs doublets.[8] Local or discrete horizontal symmetries usually demand some Higgs doublets.[4),9] Furthermore, Peccei and Quinn have proposed a global $U(1)$ symmetry in order to answer the strong *CP* problem.[10] In this case, two Higgs doublets are needed, and there is a pseudo-Goldstone boson, an axion.[11] Yanagida and Yoshimura have shown that two **5** scalars in the standard $SU(5)$ model can explain the baryon asymmetry of the universe by the lepto-quark interactions.[12] Recently, a supersymmetric $SU(5)$ model, or a supersymmetric $SU(3)_c \times SU(2) \times U(1)$ model, with two Higgs **5**'s is proposed to guarantee the gauge hierarchy naturally.[13),*]

In this paper, we attempt to study the behavior of the Yukawa and the quartic scalar couplings in the general cases of the standard model where N_H Higgs scalar doublets are contained. A model with one Higgs doublet and that with two massless Higgs doublets are examined numerically. In the former model, the upper bounds on fermions and a scalar masses and the fermion mass ratio m_b/m_τ are calculated with the quark mixing angles taken into account, and the evolution of the angles is examined. In the latter model, we also study them in the massless scalar theory assuming some discrete symmetries and taking account of the gauge hierarchy.

In the next section, the general framework of the RGE's in the standard $SU(3)_c \times SU(2) \times U(1)$ gauge theory is given, and the behavior of the gauge couplings in the general scheme is discussed. The numerical results of the RGE's in a model with one Higgs doublet and that with two massless Higgs doublets are given in §§ 3 and 4, respectively. The final section is devoted to conclusion.

*) In this type of models, an approach different from ours is needed, because there are many particles, scalars and Majorana spinors, as the supersymmetric partners.

§ 2. General framework

We consider the standard $SU(3)_c \times SU(2) \times U(1)$ gauge theory with N_g fermion generations and N_H Higgs doublets. Fermions obey the standard assignments. Let g_c, g and g' be the gauge coupling constants of $SU(3)_c$, $SU(2)$ and $U(1)$, respectively. The Yukawa and the quartic scalar couplings are defined as follows:

$$\mathcal{L}_Y = f^a_{ij}(\bar{\nu}_i,\ \bar{l}_i)_L \phi_a (l_j)_R + \text{h.c.}$$
$$+ h^a_{ij}(\bar{p}_i,\ \bar{n}_i)_L \phi_a (n_j)_R + \text{h.c.} + \tilde{h}^a_{ij}(\bar{p}_i,\ \bar{n}_i)_L \tilde{\phi}_a (p_j)_R + \text{h.c.}\,, \tag{1}$$

$$V(\phi) = \frac{1}{6}\lambda_{abcd}\phi_a^\dagger \phi_b \phi_c^\dagger \phi_d + (\text{lower order terms}), \tag{2}$$

where p_i, n_i, ν_i and l_i ($i=1, \cdots, N_g$) are quarks with charge 2/3 and $-1/3$ and leptons with charge 0 and -1, respectively, and ϕ_a ($a=1, \cdots, N_H$) is a complex scalar doublet,

$$\phi_a \equiv \begin{pmatrix} \phi_a^+ \\ \phi_a^0 \end{pmatrix}, \qquad \tilde{\phi}_a \equiv \begin{pmatrix} \bar{\phi}_a^0 \\ -\phi_a^- \end{pmatrix}.$$

The Yukawa coupling constants f^a_{ij}, h^a_{ij} and $\tilde{h}^a_{ij}$ are usually denoted by $N_g \times N_g$ matrices f_a, h_a and $\tilde{h}_a$ in this paper. All elements of f_a, h_a, $\tilde{h}_a$ and λ_{abcd} are complex numbers, and the hermiticity condition for the scalar potential leads to the following relation:

$$\lambda_{abcd} = \lambda_{cdab} = \lambda^*_{badc} = \lambda^*_{dcba}\,.$$

The mass-independent renormalization group equations for these coupling constants at the one-loop level are obtained from the work of Ref. 14) as follows:

$$16\pi^2 \mu \frac{dg_c}{d\mu} = \left(-11 + \frac{4}{3}N_g\right) g_c^3\,, \tag{3·a}$$

$$16\pi^2 \mu \frac{dg}{d\mu} = \left(-\frac{22}{3} + \frac{4}{3}N_g + \frac{1}{6}N_H\right) g^3\,, \tag{3·b}$$

$$16\pi^2 \mu \frac{dg'}{d\mu} = \left(\frac{20}{9}N_g + \frac{1}{6}N_H\right) g'^3\,, \tag{3·c}$$

$$16\pi^2 \mu \frac{df_a}{d\mu} = -\left(\frac{9}{4}g^2 + \frac{15}{4}g'^2\right) f_a + \frac{1}{2} f_b f_b^\dagger f_a + f_a f_b^\dagger f_b + f_b A_{ba}\,, \tag{4·a}$$

$$16\pi^2 \mu \frac{dh_a}{d\mu} = -\left(8g_c^2 + \frac{9}{4}g^2 + \frac{5}{12}g'^2\right) h_a - 2\tilde{h}_b \tilde{h}_a^\dagger h_b$$
$$+ \frac{1}{2}(\tilde{h}_b \tilde{h}_b^\dagger + h_b h_b^\dagger) h_a + h_a h_b^\dagger h_b + h_b A_{ba}\,, \tag{4·b}$$

$$16\pi^2\mu\frac{d\tilde{h}_a}{d\mu}=-\left(8g_c{}^2+\frac{9}{4}g^2+\frac{17}{12}g'^2\right)\tilde{h}_a-2h_bh_a{}^\dagger\tilde{h}_b$$

$$+\frac{1}{2}(\tilde{h}_b\tilde{h}_b{}^\dagger+h_bh_b{}^\dagger)\tilde{h}_a+\tilde{h}_a\tilde{h}_b{}^\dagger\tilde{h}_b+\tilde{h}_bA_{ba}\,, \tag{4·c}$$

$$16\pi^2\mu\frac{d\lambda_{abcd}}{d\mu}=\frac{2}{3}(2\lambda_{abmn}\lambda_{nmcd}+\lambda_{abmn}\lambda_{cmnd}+\lambda_{amnb}\lambda_{mncd}$$

$$+\lambda_{amnd}\lambda_{cnmb}+\lambda_{amcn}\lambda_{mbnd})-3(3g^2+g'^2)\lambda_{abcd}$$

$$+\frac{9}{4}(3g^4+g'^4)\delta_{ab}\delta_{cd}+9g^2g'^2\left(\delta_{ad}\delta_{bc}-\frac{1}{2}\delta_{ab}\delta_{cd}\right)$$

$$+\lambda_{mbcd}A_{am}+\lambda_{amcd}A_{mb}+\lambda_{abmd}A_{cm}+\lambda_{abcm}A_{md}$$

$$-12H_{abcd}\,, \tag{5}$$

$$A_{ab}\equiv\mathrm{Tr}(f_a{}^\dagger f_b+3h_a{}^\dagger h_b+3\tilde{h}_a\tilde{h}_b{}^\dagger),$$

$$H_{abcd}\equiv\mathrm{Tr}(f_a{}^\dagger f_bf_c{}^\dagger f_d+3h_a{}^\dagger h_bh_c{}^\dagger h_d+3\tilde{h}_d{}^\dagger\tilde{h}_c\tilde{h}_b{}^\dagger\tilde{h}_a$$

$$+3h_bh_a{}^\dagger\tilde{h}_c\tilde{h}_d{}^\dagger+3\tilde{h}_a\tilde{h}_b{}^\dagger h_dh_c{}^\dagger-3\tilde{h}_a\tilde{h}_d{}^\dagger h_bh_c{}^\dagger-3h_dh_c{}^\dagger\tilde{h}_b\tilde{h}_a{}^\dagger).$$

These RGE's are valid for $M_W<\mu<M_X$, where M_W and M_X are the typical mass scales of the $SU(2)\times U(1)$ breaking and the grand unification, respectively. We consider M_W and M_X as the masses of W-boson and X-bosons, respectively.

Before calculating the RGE's, we should determine the low energy parameters at M_W. At energies below M_W, the effective gauge theory is described by a group $SU(3)_c\times U(1)_{em}$ (QCD and QED). The QED corrections to the gauge couplings are calculated by Marciano as[15)]

$$g^2(M_W)=\frac{4\pi\alpha(M_W)}{\sin^2\theta_W(M_W)}\,, \tag{6·a}$$

$$g'^2(M_W)=\frac{4\pi\alpha(M_W)}{1-\sin^2\theta_W(M_W)}\,, \tag{6·b}$$

$$\alpha(M_W)=\frac{1}{128.5}\,,\qquad M_W=\frac{38.5\ \mathrm{GeV}}{\sin\theta_W(M_W)}\,,$$

where $\sin^2\theta_W(M_W)$ is the observed value in the neutral current phenomena.

In order to calculate the QCD correction, we use the useful formula by Georgi and Politzer,[16)]

$$\mu\frac{dg_c}{d\mu}=\beta_g=-\frac{g_c{}^3}{16\pi^2}\left[11-\frac{2}{3}\sum_{\text{quarks}}\left(\frac{1}{1+5\bar{m}_q{}^2/\mu^2}\right)\right],$$

$$\frac{\mu}{\bar{m}_q}\frac{d\bar{m}_q}{d\mu}=\gamma_m=-\frac{g_c{}^2}{2\pi^2}\left(\frac{1}{1+2\bar{m}_q{}^2/\mu^2}\right),$$

where $\bar{m}_q$ denotes the running quark mass. The on-shell mass m_q of the quark is defined by

$$\bar{m}_q(2m_q)=m_q\,. \tag{7}$$

For numerical calculations, we use the following ranges of parameters:

$$m_b=(4\sim5)\mathrm{GeV}\,,\quad m_t\gtrsim 20\ \mathrm{GeV}\,,\quad \Lambda_c=(0.1\sim0.5)\mathrm{GeV}\,,$$

where m_c, m_s, etc. are neglected, and Λ_c is defined, in this paper, by $\alpha_s\equiv g_c{}^2/4\pi$ at $\mu^2=20\ \mathrm{GeV}^2$ as

$$\alpha_s^{-1}(\mu)=\frac{11-\frac{2}{3}n_f}{2\pi}\ln\frac{\mu}{\Lambda_c}\,.\qquad (n_f=4)$$

As a result of numerical calculations, we get

$$\alpha_s(M_W)=0.115\sim0.155\,, \tag{8$\cdot$a}$$

$$m_b(M_W)=(3.1\sim4.1)\mathrm{GeV}\,. \tag{8$\cdot$b}$$

Thus, we get the fermion mass ratio at M_W as follows:

$$R(M_W)\equiv\frac{m_b(M_W)}{m_\tau(M_W)}=1.7\sim2.3\,. \tag{9}$$

Here, the QED corrections to fermion masses are neglected.

The fermion mass matrices at M_W are

$$m_l=f_a v_a\,,\qquad m_n=h_a v_a\,,\qquad m_p=\tilde{h}_a v_a\,, \tag{10}$$

where v_a ($a=1,\cdots,N_H$) is the vacuum expectation value of the neutral component $\phi_a{}^0$ of the scalar doublet. The quark mixing matrix U is defined by

$$U\equiv U_p U_n{}^\dagger\,, \tag{11}$$

where U_p and U_n are unitary matrices diagonalizing $m_p m_p{}^\dagger$ and $m_n m_n{}^\dagger$, respectively as

$$U_p m_p m_p{}^\dagger U_p{}^\dagger=(\text{a diagonal matrix}),$$

$$U_n m_n m_n{}^\dagger U_n{}^\dagger=(\text{a diagonal matrix}).$$

By removing the unphysical phase factors, we can parametrize U in terms of the quark mixing angles such as Kobayashi-Maskawa parametrization.[17] The observed mixing angles are regarded as the values at M_W in the same way as the Weinberg angle, though the observed quark masses are regarded as the values determined by Eq. (7).

The GUT condition for the gauge coupling constants are as follows:

$$g_c(M_X)=g(M_X)=\sqrt{5/3}g'(M_X).$$

From Eqs. (3), (6) and this condition, we can determine the value of M_X as

$$\ln\frac{M_X}{M_W}=\frac{48\pi}{110-N_H}\frac{1}{\alpha(M_W)}\left\{\frac{3}{8}-\sin^2\theta_W(M_W)\right\}$$

$$=\frac{16\pi}{66+N_H}\left\{\frac{3}{8\alpha(M_W)}-\frac{1}{\alpha_s(M_W)}\right\}. \tag{12}$$

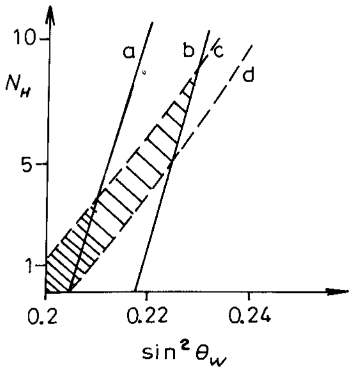

Fig. 1. Allowed region of N_H and $\sin^2\theta_W(M_W)$ from the values of M_X and $\alpha_s(M_W)$. The shaded region is allowed; (a) $M_X=10^{15}$ GeV, (b) $M_X=10^{14}$ GeV, (c) $\alpha_s(M_W)=0.155$ ($\Lambda_c=0.5$ GeV) and (d) $\alpha_s(M_W)=0.115$ ($\Lambda_c=0.1$ GeV).

The observed proton stability at present demands that $M_X>10^{14\sim15}$ GeV. We can get the allowed region of N_H and $\sin^2\theta_W$ from the values of M_X and $\alpha_s(M_W)$, Eq. (8·a), using Eq. (12) as shown in Fig. 1. The cases $N_H>8$ are not favorable, and the GUT prediction of the Weinberg angle is $\sin^2\theta_W(M_W)=0.205\sim0.225$. These values are not so strict, because we have calculated only in the one-loop approximation, and because the allowed value of M_X is not so clear now. N_g is not constrained from Eq. (12), because it is independent of N_g.

In the small Yukawa coupling limit, a constraint on N_g can be obtained from the GUT condition of the Yukawa couplings,

$$f_a(M_X)=h_a(M_X). \tag{13}$$

This is a condition of the minimal model, where only **5** scalars of $SU(5)$ (or **10** of $SO(10)$) can be coupled to fermions.

If all Yukawa coupling constants are negligibly small compared to the gauge coupling constants, the behavior of them are determined by the gauge couplings using Eqs. (4). In this case, the following equations can be obtained:[18]

$$h_a(\mu)=R(\mu)f_a(\mu),$$

$$16\pi^2\frac{\mu}{R}\frac{dR}{d\mu}=-8g_c{}^2+\frac{10}{3}g'^2,$$

where $R(M_W)$ is the fermion mass ratio in Eq. (9). In Fig. 2, $R(M_W)$ is shown in the N_H-$\sin^2\theta_W$ plane with $N_g=3$ and $N_g=4$. In the cases $N_g\geqq4$, there is no overlap of the allowed region.

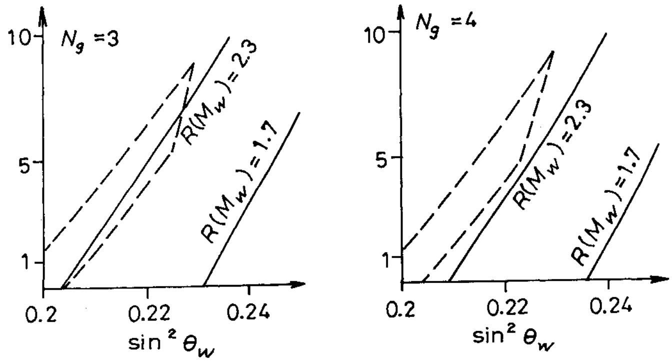

Fig. 2. Allowed region of N_H and $\sin^2\theta_W(M_W)$ in the small Yukawa coupling limit.

Now, let us discuss the Yukawa couplings in more general cases. By a suitable unitary transformation of the scalar fields $(\phi_1, \cdots, \phi_{N_H})$ to $(\phi_1', \cdots, \phi'_{N_H})$, we can always set[19]

$$\langle \phi_1'^0 \rangle = v\,, \qquad (v^2 = \sum_{a=1}^{N_H} v_a{}^2)$$

$$\langle \phi_b'^0 \rangle = 0\,. \qquad (b=2, \cdots, N_H)$$

In this choice of the scalar fields, the fermion mass matrices, Eqs. (10), are determined by f_1, h_1 and $\tilde{h}_1$. Other Yukawa coupling constants f_b, h_b and $\tilde{h}_b$ ($b=2, \cdots, N_H$) cause the flavor changing neutral scalar interactions in general.

One of natural ways to avoid these undesirable effects is to assume that all Yukawa coupling constants except for f_1, h_1 and $\tilde{h}_1$ vanish. This choice of the Yukawa couplings is apparently stable for the RGE's, Eqs. (4). In this case, the structure of the Yukawa couplings is the same as the case of $N_H=1$, which is discussed in the next section. If this is the case, and if there are no fermions whose mass is comparable with M_W, the results in Fig. 2 are reliable, that is the small Yukawa coupling limit.

The general way to avoid the flavor changing effects is to assume that all matrices f_a ($a=1, \cdots, N_H$) are simultaneously diagonalized, as well as h_a and $\tilde{h}_a$. It is not apparent whether this choice of the Yukawa couplings is stable for the RGE's or not. A simple case which is stable for renormalization is that all matrices f_a are in proportion to each other as $a_1 f_1 = a_2 f_2 = \cdots = a_{N_H} f_{N_H} = f$, where a_i ($i=1, \cdots, N_H$) is arbitrary parameter, similarly for h_a and $\tilde{h}_a$. In this case, the Yukawa couplings can be set to belong to three independent Higgs scalars by a suitable transformation of scalar fields.

Taking the GUT condition Eq. (13) into account, we can set only two of the scalar fields to couple with fermions through the Yukawa couplings. A scalar doublet ϕ_1 couples to leptons and to n_R, and another doublet ϕ_2 couples to p_R. The behavior of the Yukawa couplings of this type is discussed in § 4. In this case, the small Yukawa coupling approximation is also useful as far as all fermion masses are small as compared to M_W.

Even though the flavor changing Yukawa couplings do not vanish, large masses of the scalars make the effects small.[19] In this case, the scalar masses should be as large as several hundred GeV. This case will be briefly discussed in the final section.

§ 3. $N_H=1$

In this section, the minimal model with one Higgs doublet is studied. The Yukawa couplings and the scalar potential are defined by

$$\mathcal{L}_Y=f_{ij}(\bar{\nu}_i,\ \bar{l}_i)_L\phi(l_j)_R+\text{h.c.}$$
$$+h_{ij}(\bar{p}_i,\ \bar{n}_i)_L\phi(n_j)_R+\text{h.c.}+\tilde{h}_{ij}(\bar{p}_i,\ \bar{n}_i)_L\tilde{\phi}(p_j)_R+\text{h.c.}\,, \tag{14·a}$$

$$V(\phi)=\frac{1}{6}\lambda(\phi^\dagger\phi)^2-\mu^2\phi^\dagger\phi\,, \tag{14·b}$$

where the notations are the same as in the previous section. The mass of the physical neutral scalar and the fermion mass matrices are

$$m_H=\frac{2}{\sqrt{3}}\frac{\sqrt{\lambda}}{g}M_W\,,$$
$$m_l=\frac{\sqrt{2}}{g}fM_W\,,\qquad m_n=\frac{\sqrt{2}}{g}hM_W\,,\qquad m_p=\frac{\sqrt{2}}{g}\tilde{h}M_W\,. \tag{15}$$

First, we discuss the Yukawa couplings. Let us define the following hermitian matrices:

$$F\equiv ff^\dagger\,,\qquad H_n\equiv hh^\dagger\,,\qquad H_p\equiv\tilde{h}\tilde{h}^\dagger\,. \tag{16}$$

The quark mixing matrix U is defined by Eq. (11), where U_n and U_p are the unitary matrices diagonalizing H_n and H_p, respectively. The RGE's of the Yukawa coupling constants, Eqs. (4), lead to the following equations:

$$8\pi^2\mu\frac{dF}{d\mu}=-\left(\frac{9}{4}g^2+\frac{15}{4}g'^2\right)F+\frac{3}{2}F^2+F\,\mathrm{Tr}(F+3H_n+3H_p), \tag{17·a}$$

$$8\pi^2\mu\frac{dH_n}{d\mu}=-\left(8g_c{}^2+\frac{9}{4}g^2+\frac{5}{12}g'^2\right)H_n+\frac{3}{2}H_n{}^2$$
$$-\frac{3}{4}(H_nH_p+H_pH_n)+H_n\,\mathrm{Tr}(F+3H_n+3H_p), \tag{17·b}$$

$$8\pi^2\mu\frac{dH_p}{d\mu}=-\left(8g_c{}^2+\frac{9}{4}g^2+\frac{17}{12}g'^2\right)H_p+\frac{3}{2}H_p{}^2$$

$$-\frac{3}{4}(H_nH_p+H_pH_n)+H_p\,\mathrm{Tr}(F+3H_n+3H_p). \qquad (17\cdot\mathrm{c})$$

If H_n and H_p can be simultaneously diagonalized, all quark mixing angles vanish. In this case, U_n and U_p can be chosen to be unit matrices independent of μ. In a realistic model where the mixing angles do not vanish, U_n and U_p depend on μ. Thus, the mixing matrix U is dependent on the energy scale μ. The energy dependence of U_n and U_p can be obtained from the cross terms of H_n and H_p in Eqs. (17·b) and (17·c).

In order to illustrate the dependence, we examine the case $N_g=2$ for simplicity. The mixing matrix U in this case is defined by a mixing angle θ as

$$U\equiv\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}. \qquad (18)$$

θ is defined to be $0\leq\theta\leq\pi/2$. The evolution equation for θ can be easily calculated as follows:

$$\mu\frac{d\theta}{d\mu}=\frac{3}{32\pi^2}\cos\theta\sin\theta\left\{(H_2{}^n-H_1{}^n)\frac{H_2{}^p+H_1{}^p}{H_2{}^p-H_1{}^p}+(H_2{}^p-H_1{}^p)\frac{H_2{}^n+H_1{}^n}{H_2{}^n-H_1{}^n}\right\}, \qquad (19)$$

where $H_i{}^n$ and $H_i{}^p$ $(i=1,2)$ are eigenvalues of H_n and H_p, respectively. Unphysical phase factors in U_n and U_p are cancelled out in the calculation of Eq. (19).

In the case $H_1^{p,n}<H_2^{p,n}$, we can see from Eq. (19) that $\theta=0(\pi/2)$ is an infrared (ultraviolet) fixed point, and that the value of θ at M_X is larger than that of θ at M_W. As the differences of the eigenvalues $H_2^{p,n}-H_1^{p,n}$ become larger, θ at M_X becomes larger. The evolution equations of the eigenvalues are given by Eqs. (17). By a numerical calculation, for example, with $\theta(M_X)=\pi/4$, we obtain that the minimum of $\theta(M_W)$ is 0.12π, which is calculated in the large limit of $F_2(M_X)$, $H_2{}^n(M_X)$ and $H_2{}^p(M_X)$. In the small Yukawa-coupling limit, the mixing angle is not changed. The model with $N_g=2$ is not realistic. If, however, there are fermions of the fourth generation mixed mainly with the third generation, the mixing angle at M_X is largely different from that at M_W.

In a realistic model with $N_g=3$ which is the Kobayashi-Maskawa model, we describe here only results of numerical calculations. The quark mixing angles are defined as follows:[17]

$$U\equiv\begin{pmatrix}c_1 & s_1c_3 & s_1s_3\\ -s_1c_2 & c_1c_2c_3+s_2s_3e^{i\delta} & c_1c_2s_3-s_2c_3e^{i\delta}\\ -s_1s_2 & c_1s_2c_3-c_2s_3e^{i\delta} & c_1s_2s_3+c_2c_3e^{i\delta}\end{pmatrix}, \qquad (20)$$

where c_i and s_i ($i=1, 2, 3$) denote $\cos\theta_i$ and $\sin\theta_i$.

There are bounds on masses of the top quark and the Higgs scalar, Eqs. (15), as discussed in Ref. 6). The bounds are shown in Fig. 3(a). The calculations are made with various values of the mixing angles, and we find that the quark mixings do not change the results. Here, the values of the masses are calculated from Eqs. (5) with the coupling constants at M_W. The top quark mass is not corrected using Eq. (7) in this and the following sections. The upper bounds of the Yukawa and the quartic scalar couplings at M_W are as follows:

$$\frac{H_3{}^p(M_W)}{4\pi} \lesssim 0.16\,, \qquad \frac{\lambda(M_W)}{4\pi} \lesssim 0.25\,.$$

In Fig. 3(b), the fermion mass ratio, Eq. (9), is shown with the top quark mass. In a case of the large top quark mass, the ratio gets the effects of the large Yukawa coupling in Eqs. (17).[7] The shaded region in Fig. 3(b) corresponds to the following range of the mixing angles:

$$\theta_1(M_X) < \pi/4\,, \quad \theta_2(M_X) < \pi/4\,,$$

$$\theta_3(M_X) < \pi/4\,, \quad \delta(M_X) < \pi/4\,.$$

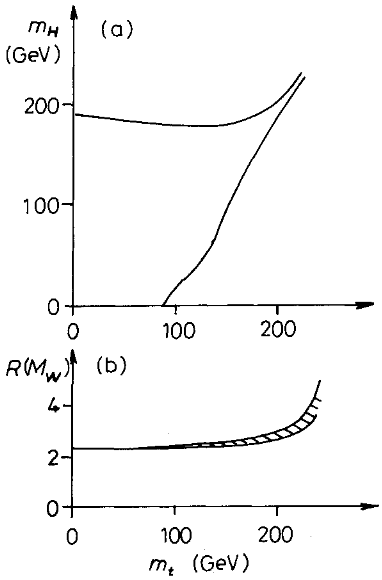

Fig. 3. (a) the bound of m_H and m_t and (b) the fermion mass ratio m_b/m_τ at M_W in the case of $N_H=1$ and $N_g=3$ with $\sin^2\theta_W(M_W)=0.205$. The shaded region in (b) corresponds to the range of the mixing angles at M_X from 0 to $\pi/4$.

Of course, these angles at M_W are constrained from the present data.[20] Our aim is to see the gross feature of the mixing effects. The upper bound of the shaded region is calculated in the case $\theta_1=\theta_2=\theta_3=0$, which is consistent with the result shown in Ref. 7). In the case of large m_t, the ratio m_b/m_τ becomes larger than the value calculated in the small Yukawa coupling limit in the previous section, and the large Yukawa coupling effects are made to be mild by the quark mixings. In any case, m_t should be less than 200 GeV, and m_H should also be less than 200 GeV.

In the massless scalar theory, where the gauge hierarchy can be realized as pointed out by Weinberg,[2] the upper bound of m_t is about 80 GeV. In this case, the spontaneous breakdown of $SU(2)\times U(1)$ is caused by the Coleman-Weinberg mechanism,[21] and m_H should be less than about 10 GeV.

The behavior of the mixing angles in Eqs. (20) can also be calculated. In the

Table I. The mixing angles at M_X in the case $N_H=1$ and $N_g=3$ with an initial condition of Eqs. (21) and $\sin^2\theta_W(M_W)=0.205$.

m_t[GeV]	$s_1(M_X)$	$s_2(M_X)$	$s_3(M_X)$	$s_\delta(M_X)$
30	0.2301	0.4009	0.3007	0.0500
60	0.2304	0.4036	0.3027	0.0501
90	0.2309	0.4084	0.3062	0.0503
120	0.2317	0.4160	0.3118	0.0506
150	0.2331	0.4277	0.3205	0.0510
180	0.2354	0.4465	0.3343	0.0517
210	0.2405	0.4821	0.3604	0.0532

Kobayashi-Maskawa model, these angles at M_W are constrained from the present data.[20] We chose, as an initial condition, the following values of the mixing angles at M_W:

$$s_1(M_W)=0.23\,,\quad s_2(M_W)=0.4\,,\quad s_3(M_W)=0.3\,,\quad s_\delta(M_W)=0.05\,. \tag{21}$$

Using the current-quark-masses as an initial condition of the eigenvalues of the matrices Eqs. (16), we calculate the angles at M_X by Eqs. (17). In this calculation, we omit the GUT condition, Eq. (13). As the values of $F(\mu)$ and $H_n(\mu)$ are sufficiently small, the effect of Eq. (13) does not change the results significantly. The results are shown in Table I. The difference between the mixing angles at M_W and M_X becomes large as m_t increases. If we use another initial condition, the results will be changed, but in any case the mixing angles at M_X are a little larger than that at M_W in a model with one Higgs doublet.

§ 4. $N_H=2$

The general treatment of a model with two Higgs doublets are complicated. Though it is interesting, it does not seem to be realistic because of the flavor nonconservation in the neutral scalar interactions. We consider here the following Yukawa couplings:

$$\mathcal{L}_Y=f_{ij}(\bar{\nu}_i,\ \bar{l}_i)_L\phi_1(l_j)_R+\text{h.c.}$$
$$+h_{ij}(\bar{p}_i,\ \bar{n}_i)_L\phi_1(n_j)_R+\text{h.c.}+\tilde{h}_{ij}(\bar{p}_i,\ \bar{n}_i)_L\tilde{\phi}_2(p_j)_R+\text{h.c.} \tag{22}$$

As previously discussed, this is one of the natural ways to avoid the flavor changing effects in the neutral interaction. The fermion mass matrices are

$$m_l=\frac{\sqrt{2}}{g}\frac{v_1}{v}fM_W\,,\qquad m_n=\frac{\sqrt{2}}{g}\frac{v_1}{v}hM_W\,,$$

$$m_p=\frac{\sqrt{2}}{g}\frac{v_2}{v}\tilde{h}M_W\,.\qquad (v^2=v_1{}^2+v_2{}^2) \tag{23}$$

Let us define the matrices F, H_n and H_p by Eqs. (16). The discussions are made in the same way as the case $N_H=1$. From the RGE's, Eqs. (4), we obtain the following equations:*)

$$8\pi^2\mu\frac{dF}{d\mu}=-\left(\frac{9}{4}g^2+\frac{15}{4}g'^2\right)F+\frac{3}{2}F^2+F\,\mathrm{Tr}(F+3H_n), \tag{24·a}$$

$$8\pi^2\mu\frac{dH_n}{d\mu}=-\left(8g_c{}^2+\frac{9}{4}g^2+\frac{5}{12}g'^2\right)H_n+\frac{3}{2}H_n{}^2 \\ +\frac{1}{4}(H_nH_p+H_pH_n)+H_n\,\mathrm{Tr}(F+3H_n), \tag{24·b}$$

$$8\pi^2\mu\frac{dH_p}{d\mu}=-\left(8g_c{}^2+\frac{9}{4}g^2+\frac{17}{12}g'^2\right)H_p+\frac{3}{2}H_p{}^2 \\ +\frac{1}{4}(H_nH_p+H_pH_n)+H_p\,\mathrm{Tr}(3H_p). \tag{24·c}$$

The difference between Eqs. (17) and (24) is caused by the vertex correction and the wave function renormalization of the scalars.

As discussed in the previous section, the evolution of the quark mixing angles can be obtained from Eqs. (24). In a toy model with $N_g=2$, where the mixing angle θ is defined by Eq. (18), we find

$$\mu\frac{d\theta}{d\mu}=\frac{-1}{32\pi^2}\cos\theta\sin\theta\left\{(H_2{}^n-H_1{}^n)\frac{H_2{}^p+H_1{}^p}{H_2{}^p-H_1{}^p}+(H_2{}^p-H_1{}^p)\frac{H_2{}^n+H_1{}^n}{H_2{}^n-H_1{}^n}\right\}. \tag{25}$$

The sign of the r.h.s. is opposite to that of Eq. (19). Thus, $\theta=0(\pi/2)$ is an ultraviolet (infrared) fixed point in contrast to the case $N_H=1$, and the angle at M_X is smaller than that at M_W.

In a realistic model with $N_g=3$, where the angles are defined by Eq. (20), the results of the numerical calculations are listed in Table II. The calculations are made with an initial condition of Eqs. (21) at M_W. We have assumed that $v_1=v_2$ to obtain the eigenvalues of F, H_n and H_p at M_W from Eqs. (23), and the GUT

Table II. The mixing angles at M_X in the case $N_H=2$ and $N_g=3$ with an initial condition of Eqs. (21) and $\sin^2\theta_W(M_W)=0.210$.

m_t[GeV]	$s_1(M_X)$	$s_2(M_X)$	$s_3(M_X)$	$s_\delta(M_X)$
30	0.2299	0.3994	0.2996	0.0500
60	0.2297	0.3975	0.2982	0.0499
90	0.2294	0.3939	0.2955	0.0498
120	0.2287	0.3873	0.2906	0.0496
150	0.2274	0.3726	0.2796	0.0491

*) In the Appendix of Ref. 22), the RGE's in the same model can be found, but those of the Yukawa couplings are incorrect.

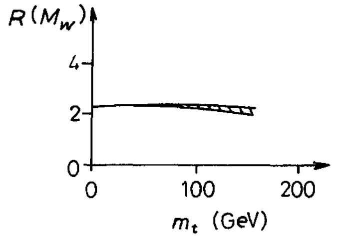

Fig. 4. The ratio m_b/m_τ at M_W in the case $N_H=2$ and $N_g=3$ with $\sin^2\theta_W(M_W)=0.210$. The shaded region corresponds to the range of the mixing angles at M_X from 0 to $\pi/4$.

condition, Eq. (13), is omitted.

The fermion mass ratio m_b/m_τ in this model with $N_g=3$ is shown in Fig. 4 where the shaded region corresponds to the values of the mixing angles at M_X in a range from 0 to $\pi/4$. The lower bound of the shaded region is calculated in the case $\theta_1=\theta_2=\theta_3=0$. The top quark masses in Fig. 4 are calculated from Eqs. (23) with an assumption $v_1=v_2$.

In the case of large $H_3{}^p$, the ratio becomes smaller than the value calculated in the small Yukawa coupling limit. Thus in this case, the results in Fig. 2 are recovered by the large Yukawa coupling effects. The model with $N_g=4$ may be also favorable in the cases $N_H \geqq 2$.

Now, let us discuss the scalar potential. In order to guarantee the Yukawa couplings to be the type of Eq. (22), we impose symmetries under the following discrete transformation:[22]

$$[\phi_1 \to -\phi_1,\ (l_i)_R \to -(l_i)_R,\ (n_i)_R \to -(n_i)_R,\ \text{others unchanged}] \qquad (26\cdot\text{a})$$

and/or

$$[\phi_2 \to -\phi_2,\ (p_i)_R \to -(p_i)_R,\ \text{others unchanged}]. \qquad (26\cdot\text{b})$$

We consider here the massless scalar theory in order to realize the gauge hierarchy[2] and not to increase the number of parameters. The general potential term of two massless Higgs doublets invariant under (26) is

$$\begin{aligned} V(\phi)=&\frac{1}{6}\lambda_1(\phi_1{}^\dagger\phi_1)^2+\frac{1}{6}\lambda_2(\phi_2{}^\dagger\phi_2)^2+\frac{1}{3}\lambda_3(\phi_1{}^\dagger\phi_1)(\phi_2{}^\dagger\phi_2)\\ &+\frac{1}{3}\lambda_4(\phi_1{}^\dagger\phi_2)(\phi_2{}^\dagger\phi_1)+\frac{1}{6}\lambda_5[(\phi_1{}^\dagger\phi_2)^2+(\phi_2{}^\dagger\phi_1)^2], \end{aligned} \qquad (27)$$

where all λ's are real. The effective potential of this model at the one-loop level is studied in Ref. 22). The RGE's of the λ's at the one-loop level can be obtained from Eq. (5) and be found in the appendix of Ref. 22).

The vacuum stability condition is as follows:

$$\begin{aligned} &\lambda_1 \geq 0\,, \qquad \lambda_2 \geq 0\,, \qquad \lambda_C \equiv \sqrt{\lambda_1\lambda_2}+\lambda_3 \geq 0\,,\\ &\lambda_B \equiv \lambda_C+\lambda_4 \geq 0\,, \qquad \lambda_A \equiv \lambda_B+\lambda_5 \geq 0\,. \end{aligned} \qquad (28)$$

If, and only if, $\lambda_A=0$, $SU(2)\times U(1)$ is spontaneously broken to $U(1)_{\text{em}}$ by the Coleman-Weinberg mechanism.[21] The ratio of the vacuum expectation values of $\phi_1{}^0$ and $\phi_2{}^0$ is real, and is given by

$$v_1{}^2/v_2{}^2=\sqrt{\lambda_2/\lambda_1}\ .$$

The relation $\lambda_1=\lambda_2$ is not stable for RGE's because of the fermion loop contribution. The top quark mass in Fig. 4 and Table II, which is calculated with an assumption $v_1=v_2$, should be understood to be written as an example to see the gross feature.

The physical scalars in this model are as follows: A massive charged scalar $\chi^{\pm}$, two massive neutral scalars χ_1 and χ_2, and a scalon S which is a pseudo-Goldstone boson associated with the spontaneous breakdown of the scale invariance. Their masses are represented as follows:[22]

$$m_{\chi^\pm}=\frac{2}{3}\frac{\sqrt{\lambda_C}}{g}M_W\ , \tag{29·a}$$

$$m_{\chi_1}=\frac{2}{\sqrt{3}}\frac{\sqrt{\sqrt{\lambda_1\lambda_2}}}{g}M_W\ , \tag{29·b}$$

$$m_{\chi_2}=\frac{2}{\sqrt{3}}\frac{\sqrt{\lambda_B}}{g}M_W\ , \tag{29·c}$$

$$m_S{}^2=\frac{g^2}{32\pi^2M_W{}^2}\{2m_{\chi^\pm}^4+m_{\chi_1}^4+m_{\chi_2}^4+6M_W{}^4+3M_Z{}^4-4\sum_i(m_{l_i}^4+3m_{n_i}^4+3m_{p_i}^4)\}. \tag{29·d}$$

Of course, these masses are given at M_W with a condition $\lambda_A=0$.

If $\lambda_5=0$, which is stable for the RGE's, Eqs. (22) and (27) are invariant under separate phase transformations of ϕ_1 and ϕ_2. This is the $U(1)_{PQ}$ symmetry,[10] and χ_2 is an axion.[11]

The relation $\lambda_A(M_W)=0$ should be satisfied, and all λ's should satisfy the vacuum stability condition (28) in the whole energy regions from M_W to M_X. The parameters λ's should be chosen at M_X just like that, using the RGE's. In this way the gauge hierarchy can be realized. This is not fine tuning, if the masslessness of the scalars are guaranteed by some mechanism.

By numerical calculations in that way, we can find the upper bounds of the scalar masses, Eqs. (29). The results are as follows:

$$m_{\chi^\pm}\lesssim 240\text{ GeV}\ ,\qquad m_{\chi_1},\ m_{\chi_2}\lesssim 200\text{ GeV}\ ,\qquad m_S\lesssim 40\text{ GeV}\ .$$

The upper bounds of the λ's are $O(1)$. The scalon mass can be as heavy as 40 GeV in this model in contrast to the case $N_H=1$.

The Yukawa coupling constants are bounded themselves from Eqs. (24) as in the case $N_H=1$. But, in the case of the large Yukawa couplings, the vacuum stability condition and $\lambda_A(M_W)=0$ cannot be satisfied. In the massless scalar theory, further constraint on the Yukawa coupling constants is obtained. The upper bound of the top quark mass in the model with two massless Higgs doublets is

$$m_t \lesssim 120 \text{ GeV} .$$

§ 5. Conclusion

We have studied the RGE's of the Yukawa and the quartic scalar couplings, and examined mainly the behavior of the Yukawa coupling constants in the cases $N_H=1$ and $N_H=2$, Eqs. (14·a) and (22), respectively. These are applicable to the general cases of N_H with an assumption of the flavor conservation in the neutral scalar interactions at the tree level.

It is possible to suppress the flavor changing effects by giving large masses to the scalars. As discussed in Ref. 19), the scalars not concerned with the Higgs mechanism may have arbitrary large bare mass terms. In this case, however, it is difficult to explain naturally the gauge hierarchy.

In the massless scalar theory, the values of the quartic scalar coupling constants at M_W in the general cases of N_H have upper bounds of $O(1)$ as examined, for example, in the previous two sections. Thus, after the spontaneous breakdown of $SU(2)\times U(1)$ by the Coleman-Weinberg mechanism, the upper bound of the scalar masses is about a few hundred GeV. On the other hand, the Yukawa coupling constants related to the flavor changing effects may have the values of $O(1)$. We should suppress these couplings to be sufficiently small compared to the gauge couplings.

After all, one of the natural ways to avoid flavor changing neutral interactions is to choose the Yukawa couplings of the type Eq. (14·a) or Eq. (22). Whether these are the cases or not, the gross feature of the results in §§ 3 and 4 are likely unchanged, because the additional Yukawa coupling constants cannot be large. In the model with $N_g=3$, only H_3^p can be large, which depends on the top quark mass.

The main difference between the results of the cases $N_H=1$ and $N_H=2$ is in the quark mixing angles, Tables I and II. The mixing angles at M_X can be observed in the interactions caused by the X-boson exchange, for example the proton decay. Among them, θ_1 may be observed easily in comparison with others. In the case $N_H=1$, $\theta_1(M_X)$ is a little larger than $\theta_1(M_W)$. On the other hand, in the case $N_H=2$, $\theta_1(M_X)$ is smaller than $\theta_1(M_W)$. If m_t is smaller than the bound calculated in the massless scalar theory, the difference is very small, about 1% at most. If m_t is sufficiently large, or if there are fermions of the fourth generation with masses comparable to M_W, the difference may be observed in the proton decays. The observation of $\theta_1(M_X)$ being different from $\theta_1(M_W)$ is an evidence of the large Yukawa couplings. If this is the case, the scalar mass terms seem to be needed in both cases of $N_H=1$ and $N_H=2$. We have studied the massless scalar theory in § 4, but the results of the Yukawa couplings are independent of the scalar mass terms.

We have studied the standard GUT prediction. If evidences being at variance with our results are found, the standard GUT should be modified.

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