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Behaviour of solutions of a fourth-order self-adjoint linear differential equation

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Dedicated to the memory of Jacek Szarski

Abstract. The paper deals with the equation

$$\Big(a_1(t)\big(a_2(t)(a_1(t)y')'\big)'\Big)' + a(t)y = 0, \tag{1}$$

where the coefficients $a_i(t) > 0$, $i = 1, 2$, $a(t) \geqslant 0$ are continuous on $(-\infty, \infty)$ and satisfy the conditions $\int_{t_0}^{\infty} a_i^{-1}(t)\,dt = \infty$, $i = 1, 2$. There were made various decompositions of the set of all solutions of (1), were studied the relations between these decompositions and asymptotic behaviour and also other properties of the solutions belonging to the same class.

We are dealing with the equation

$$\Big(a_1(t)\big(a_2(t)(a_1(t)y')'\big)'\Big)' + a(t)y = 0, \tag{1}$$

where $a_i(t) \in C\big((-\infty, \infty), (0, \infty)\big)$, $i = 1, 2$, $a(t) \in C\big((-\infty, \infty), [0, \infty)\big)$ and $a(t)$ does not vanish identically on any subinterval of $J = (-\infty, \infty)$. We also assume that

$$\int_{t_0}^{\infty} \frac{dt}{a_i(t)} = \infty, \quad i = 1, 2. \tag{2}$$

By a *solution* of (1) we understand a non-trivial solution. Our assumptions ensure that the solutions of (1) are either all oscillatory or all non-oscillatory ([1], [2], [3]). We use the notation

$$L_0y = y, \quad L_1y = a_1y', \quad L_2y = a_2(L_1y)' = a_2(a_1y')',$$
$$L_3y = a_1(L_2y)' = a_1\big(a_2(a_1y')'\big)', \quad L_4y = (L_3y)' = \Big(a_1\big(a_2(a_1y')'\big)'\Big)'$$

and we call L_iy the *quasi-derivative* of y of order i, $i = 0, 1, 2, 3, 4$. We say that $y = L_0y$ has *zero of order* k at t_0, if $L_iy(t_0) = 0$, $i = 0, 1, \ldots$

$\ldots, k-1, L_k y(t_0) \neq 0$. Using this notation we rewrite equation (1) in the form

$$(1') \qquad L_4 y + a(t) y = 0.$$

THEOREM 1. *Let $y(t)$ be a solution of* (1). *Then the function*

$$(3) \qquad F(y(t)) = L_0 y(t) L_3 y(t) - L_1 y(t) L_2 y(t)$$

is strictly decreasing on $(-\infty, \infty)$. If $t_0 \in (-\infty, \infty)$ is a double zero of $y(t)$ (at least), then $F(y(t)) > 0$ for $t < t_0$ and $F(y(t)) < 0$ for $t > t_0$. Thus every solution of (1) *has at most one double zero.*

Proof. We get by an easy calculation

$$F'(y(t)) = -a(t) L_0^2 y - \frac{1}{a_2(t)} L_2^2 y \leqslant 0, \qquad t \in J,$$

where $=$ does not hold in any interval. Thus $F(y(t))$ is strictly decreasing. The remaining statements in the Theorem are obvious.

Our main purpose is to investigate solutions of (1) with regard to their relation to the function $F(y(t))$; we are mainly interested in those solutions $y(t)$ for which $F(y(t)) > 0$ for $t \in J$.

We divide our considerations into two parts: I. The case where all solutions of (1) are oscillatory. II. The case where all solutions of (1) are non-oscillatory.

I. The case where all solutions of (1) are oscillatory. We start with two lemmas.

LEMMA 1. *Let $y(t)$ be a solution of* (1) *and suppose that for some $t_0 \in J$ we have*

$$(4) \qquad \begin{gathered} L_0 y(t_0) = 0, \qquad L_1 y(t_0) L_2 y(t_0) L_3 y(t_0) \neq 0, \\ \operatorname{sgn} L_1 y(t_0) = \operatorname{sgn} L_3 y(t_0) \neq \operatorname{sgn} L_2 y(t_0). \end{gathered}$$

Then for every zero s of $y(t)$ such that $s < t_0$ the following relations hold:

$$L_1 y(s) L_2 y(s) L_3 y(s) \neq 0, \qquad \operatorname{sgn} L_1 y(s) = \operatorname{sgn} L_3 y(s) \neq \operatorname{sgn} L_2 y(s).$$

Proof. It follows from (4) that $F(y(t_0)) = -L_1 y(t_0) L_2 y(t_0) > 0$. Therefore, $F(y(t))$ being strictly decreasing on J, we have $F(y(t)) > 0$ for $t < t_0$. Let $s < t_0$ be a zero of $y(t)$. Then we get $F(y(s)) = -L_1 y(s) L_2 y(s) > 0$, and therefore $\operatorname{sgn} L_1 y(s) \neq \operatorname{sgn} L_2 y(s)$. Let now $t_{-1} < t_0$ be the first zero of $y(t)$ to the left of t_0. From the above we know that $\operatorname{sgn} L_1 y(t_{-1}) \neq \operatorname{sgn} L_2 y(t_{-1})$. We have to prove that $\operatorname{sgn} L_3 y(t_{-1}) \neq \operatorname{sgn} L_2 y(t_{-1})$. Suppose the contrary: $\operatorname{sgn} L_3 y(t_{-1}) = \operatorname{sgn} L_2 y(t_{-1}) \neq \operatorname{sgn} L_1 y(t_{-1})$. Let $L_1 y(t_0) < 0$, $L_2 y(t_0) > 0$, $L_3 y(t_0) < 0$. Then $L_1 y(t_{-1}) > 0$, $L_2 y(t_{-1}) < 0$, $L_3 y(t_{-1}) < 0$ and $L_0 y(t) > 0$ on (t_{-1}, t_0). From (1) we see that $L_3 y(t)$ decreases on (t_{-1}, t_0) and therefore $L_3 y(t) < 0$ on $(t_{-1}, t_0]$. But then $L_2 y(t)$ also decreases on $(t_{-1}, t_0]$ and therefore $L_2 y(t) < 0$ on $(t_{-1}, t_0]$, which is a contradiction to the supposition $L_2 y(t_0) > 0$. Thus we have proved our assertion for $t_{-1} < t_0$. By induction we get the assertion for all zero $s < t_0$ of $y(t)$.

LEMMA 2. *Assume that* $t_0 < t_n$, t_0, $t_n \in J$. *Let* $y_3(t)$ *be the solution of* (1) *determined by the conditions*: $L_i y_3(t_0) = 0$, $i = 0, 1, 2$, $L_3 y_3(t_0) = 1$, *and* $u_n(t)$ *a solution of* (1) *satisfying the conditions* $L_0 u_n(t_0) = 0$, $L_0 u_n(t_n) = L_1 u_n(t_n) = 0$. *Then*: $y_3(t)$ *and* $u_n(t)$ *have no common zero in* (t_0, t_n); *between any two consecutive zeros of* $y_3(t)$ *in* $[t_0, t_n)$ *there is exactly one zero of* $u_n(t)$, *and between any two consecutive zeros of* $u_n(t)$ *in* $(t_0, t_n]$ *there is exactly one zero of* $y_3(t)$.

Proof. Denote by $y_k(t)$, $k = 0, 1, 2, 3$, the solutions of (1) satisfying the conditions

$$L_j y_k(t_0) = \delta_{jk}, \quad j, k = 0, 1, 2, 3, \tag{5}$$

δ_{jk} being the Kronecker symbol. Then the solution $u_n(t)$ has the representation

$$u_n(t) = L_1 u_n(t_0) y_1(t) + L_2 u_n(t_0) y_2(t) + L_3 u_n(t_0) y_3(t). \tag{6}$$

This means that $u_n(t)$, $y_1(t)$, $y_2(t)$, $y_3(t)$ are linearly dependent, and therefore

$$[W](u_n, y_1, y_2, y_3) = \begin{vmatrix} L_0 u_n, & L_0 y_1, & L_0 y_2, & L_0 y_3 \\ L_1 u_n, & L_1 y_1, & L_1 y_2, & L_1 y_3 \\ L_2 u_n, & L_2 y_1, & L_2 y_2, & L_2 y_3 \\ L_3 u_n, & L_3 y_1, & L_3 y_2, & L_3 y_3 \end{vmatrix} = 0 \quad \text{for all } t \in J.$$

It is easy to verify that the subdeterminants corresponding to $L_0 u_n$, $L_1 u_n$, $L_2 u_n$, $L_3 u_n$, respectively, are $L_3 y_3$, $L_2 y_3$, $L_1 y_3$, $L_0 y_3$, respectively. Therefore the above equation can be written in the form

$$L_0 u_n(t) L_3 y_3(t) - L_1 u_n(t) L_2 y_3(t) + L_2 u_n(t) L_1 y_3(t) - \\ - L_3 u_n(t) L_0 y_3(t) = 0 \tag{7}$$

for $t \in J$. Now we are able to prove that $u_n(t)$ and $y_3(t)$ have no common zero in (t_0, t_n). In fact, let $t_0 < \tau < t_n$ be a common zero of $u_n(t)$ and $y_3(t)$. Since $\tau < t_n$, from Lemma 1 we get $\operatorname{sgn} L_1 u_n(\tau) = \operatorname{sgn} L_3 u_n(\tau) \neq \operatorname{sgn} L_2 u_n(\tau)$. On the other hand, from the fact that $t_0 < \tau$ and from the initial data of $L_i y_3(t_0)$, $i = 0, 1, 2, 3$, it follows that $\operatorname{sgn} L_1 y_3(\tau) = \operatorname{sgn} L_2 y_3(\tau) = \operatorname{sgn} L_3 y_3(\tau)$. Comparing these conclusions with (7), we get a contradiction.

Let now α, β be two consecutive zeros of $y_3(t)$, $t_0 \leqslant \alpha < \beta \leqslant t_n$, and let $u_n(t) \neq 0$ on $[\alpha, \beta]$. Then an easy calculation shows that (7) can be written in the form

$$a_1(t) \left[\frac{L_2 y_3(t)}{L_0 u_n(t)} \right]' = - \frac{L_2 u_n(t) L_1 y_3(t) - L_3 u_n(t) L_0 y_3(t)}{[L_0 u_n(t)]^2},$$

whence, after dividing by $a_1(t)$,

$$\left[\frac{L_2 y_3(t)}{L_0 u_n(t)}\right]' = -\frac{L_2 u_n(t)}{[L_0 u_n]^2} y_3'(t) + \frac{[L_2 u_n(t)]'}{[L_0 u_n(t)]^2} y_3(t). \tag{8}$$

Integrating from a to β we get

$$\frac{L_2 y_3(\beta)}{L_0 u_n(\beta)} - \frac{L_2 y_3(a)}{L_0 u_n(a)} = 2\int_a^\beta \frac{y_3(t)}{u_n(t)} \left[\frac{L_0 u_n(t) L_3 u_n(t) - L_1 u_n(t) L_2 u_n(t)}{a_1(t) u_n^2(t)}\right] dt. \tag{9}$$

Since t_n is at least a double zero of $u_n(t)$, according to Lemma 1 we have $F(u_n(t)) = L_0 u_n(t) L_3 u_n(t) - L_1 u_n(t) L_2 u_n(t) > 0$ for $t < t_n$. Thus the integral on the right in (9) is not zero and its sign is $\operatorname{sgn} y_3(t) \cdot \operatorname{sgn} u_n(t)$, $t \in (a, \beta)$. Assume that $t_0 < a < \beta < t_n$. Then, by Lemma 1 and Lemma 2, we get $L_2 y_3(a) \neq 0$, $L_2 y_3(\beta) \neq 0$ and for $t \in (a, \beta)$ $\operatorname{sgn} y_3(t) = \operatorname{sgn} L_2 y_3(a) \neq \operatorname{sgn} L_2 y_3(\beta)$. Thus the left side of (9) is not zero and its sign is $-\operatorname{sgn} y_3(t) \operatorname{sgn} u_n(t)$. Therefore equality (9) yields a contradiction. If $t_0 = a < \beta < t_n$, we have to consider the interval $[t_0+\varepsilon, \beta]$, where $\varepsilon > 0$ is small.Then the equality which results from (9) on replacing a by $t_0+\varepsilon$ also yields a contradiction. We have thus proved that $u_n(t)$ has at least one zero in $(a, \beta]$. Since $y_3(t)$ and $u_n(t)$ have no common zero, we see that $u_n(t)$ has at least one zero in (a, β).

Let now γ, δ, $t_0 < \gamma < \delta \leqslant t_n$, be two consecutive zeros of $u_n(t)$. Assume that $y_3(t) \neq 0$ on $[\gamma, \delta]$. Then (7) can be written in the form

$$\left[\frac{L_2 u_n(t)}{y_3(t)}\right]' = \frac{1}{y_3^2(t)} [-L_2 y_3(t) L_1 u_n(t) + L_3 y_3(t) L_0 u_n(t)], \tag{10}$$

$t \in [\gamma, \delta]$, and integrating from γ to δ we get, similarly to (9), the equality

$$\frac{L_2 u_n(\delta)}{y_3(\delta)} - \frac{L_2 u_n(\gamma)}{y_3(\gamma)} = 2\int_\gamma^\delta \frac{u_n(t)}{a_1(t) y_3^3(t)} [L_0 y_3(t) L_3 y_3(t) - L_1 y_3(t) L_2 y_3(t)] dt. \tag{11}$$

Since t_0 is a triple zero of $y_3(t)$, according to Lemma 1 we have $F(y_3(t)) = L_0 y_3(t) L_3 y_3(t) - L_1 y_3(t) L_2 y_3(t) < 0$ for $t > t_0$. Therefore the integral on the right in (11) is not zero and its sign is $-\operatorname{sgn} y_3(t) \cdot \operatorname{sgn} u_n(t)$, $t \in (\gamma, \delta)$. Suppose that $\delta < t_n$. Then $u_n(t)$ fulfils conditions (4) at every zero point which is less than t_n. Therefore, $L_2 u_n(\gamma) \neq 0$, $L_2 u_n(\delta) \neq 0$ and $\operatorname{sgn} L_2 u_n(\gamma) \neq \operatorname{sgn} L_2 u_n(\delta) = \operatorname{sgn} u_n(t)$, $t \in (\gamma, \delta)$. This means that the right-hand side of (11) is not zero and its sign is $\operatorname{sgn} u_n(t) \operatorname{sgn} y_3(t)$, $t \in (\gamma, \delta)$. Thus equality (11) yields a contradiction. Taking into account that $y_3(t)$ and $u_n(t)$ have no common zero, we have proved that $y_3(t)$ has at least one zero in (γ, δ).

If $\delta = t_n$ and $y_3(t_n) \neq 0$, then t_n is a double zero of $u_n(t)$ and therefore $\operatorname{sgn} L_2 u_n(\delta) = \operatorname{sgn} u_n(t)$, $t \in (\gamma, \delta)$. Equality (11) gives a contradiction also in this case.

If $\delta = t_n$ and $y_3(t_n) = 0$ then, as we will see in Remark 1, t_n is a triple zero of $u_n(t)$. In this case

$$\lim \frac{L_2 y_3(t)}{y_3^2(t)} u_n(t) = 0 \quad \text{as } t \to t_n^-$$

and relations (10) and (11) are valid. Since

$$\operatorname{sgn} \lim_{t \to t_n^-} \frac{L_2 u_n(t)}{y_3(t)} = \operatorname{sgn} \frac{L_3 u_n(t_n)}{y_3'(t_n)} = \operatorname{sgn} u_n(t) y_3(t), \qquad t \in (\gamma, \delta),$$

equality (11) again leads to a contradiction. In view of the fact that $u_n(t)$ and $y_3(t)$ have no common zero in (t_0, t_n), summarizing, we have proved that between any two consecutive zeros γ, δ of $u_n(t)$, $t_0 < \gamma < \delta \leqslant t_n$, there is at least one zero of $y_3(t)$. This ends the proof of Lemma 2.

Remark 1. It follows from (7) that t_n is a zero of $y_3(t)$ if and only if t_n is a triple zero of $u_n(t)$.

DEFINITION 1. Denote by E the set of all solutions of (1) having the property: $u(t) \in E$ iff $F(u(t)) > 0$ for all $t \in J$.

DEFINITION 2. Denote by U the set of all solutions of (1) having the property: $u(t) \in U$ iff at every zero ϱ of $u(t)$

$$\begin{gathered} L_1 u(\varrho) L_2 u(\varrho) L_3 u(\varrho) \neq 0, \\ \operatorname{sgn} L_1 u(\varrho) = \operatorname{sgn} L_3 u(\varrho) \neq \operatorname{sgn} L_2 u(\varrho). \end{gathered} \tag{12}$$

THEOREM 2. *The set U is non-empty. There are at least two elements $u(t), v(t) \in U$ linearly independent.*

Proof. Let $t_0 \in J$ and $y_k(t)$, $k = 0, 1, 2, 3$, be solutions of (1) satisfying (5). Let $\{t_n\}_{n=1}^{\infty}$, $t_0 < t_n$, be a sequence such that $\lim t_n = \infty$ as $n \to \infty$. Denote by $u_n(t)$ the solution of (1) satisfying the conditions

$$u_n(t_0) = 0, \quad L_0 u_n(t_n) = L_1 u_n(t_n) = 0, \tag{13}$$

$$L_1^2 u_n(t_0) + L_2^2 u_n(t_0) + L_3^2 u_n(t_0) = 1. \tag{14}$$

From (14) it follows that the sequences $\{L_i u_n(t_0)\}_{n=1}^{\infty}$, $i = 1, 2, 3$, are bounded. Therefore one can find a sequence $\{n_j\}_{j=1}^{\infty} \subset \{n\}_{n=1}^{\infty}$ such that $\{L_i u_{n_j}(t_0)\}_{j=1}^{\infty}$, $i = 1, 2, 3$, converges to c_i, $i = 1, 2, 3$, and $c_1^2 + c_2^2 + c_3^2 = 1$. Using for $u_{n_j}(t)$ the representation (6), i.e. writing

$$u_{n_j}(t) = L_1 u_{n_j}(t_0) y_1(t) + L_2 u_{n_j}(t_0) y_2(t) + L_3 u_{n_j}(t_0) y_3(t), \tag{15}$$

we see that $\{u_{n_j}(t)\}$ converges uniformly on every compact subset of J to a solution $u(t)$:

$$u(t) = c_1y_1(t)+c_2y_2(t)+c_3y_3(t), \tag{16}$$

which is not trivial. For $u_{n_j}(t)$ we have, by Lemma 1, $F(u_{n_j}(t)) > 0$ for $t < t_{n_j}$. In view of the continuity of F and the convergence of $\{u_{n_j}(t)\}$ to $u(t)$ we get $\lim_{j\to\infty} F(u_{n_j}(t)) \geqslant 0$ for all $t \in J$. Then from the strict monotonicity of $F(u(t))$ it follows that $F(u(t)) > 0$ for all $t \in J$ and therefore every zero of $u(t)$ is simple.

Let ϱ be a zero of $u(t)$. Then from $F(u(t)) > 0$ we have $-L_1u(\varrho)\times \times L_2u(\varrho) > 0$ or $\operatorname{sgn} L_1u(\varrho) \neq \operatorname{sgn} L_2u(\varrho)$, $L_1u(\varrho) \neq 0$, $L_2u(\varrho) \neq 0$. Suppose that, say, $L_1u(\varrho) > 0$, $L_2u(\varrho) < 0$ and that $L_3u(\varrho) \leqslant 0$. Let τ be the first zero of $u(t)$ to the right of ϱ. Such a zero exists because $u(t)$ is oscillatory. Then $L_0u(t) > 0$ on (ϱ, τ), $L_1u(\tau) < 0$ (τ is a simple zero of $u(t)$) and from (1) we get that $L_3u(t)$ is non-increasing on (ϱ, τ). Thus $L_3u(t) \leqslant 0$ on $(\varrho, \tau]$ and $L_2u(t)$ is non-increasing on $(\varrho, \tau]$. Therefore $L_2u(t) < 0$ on $[\varrho, \tau]$. But $F(u(\tau)) = -L_1u(\tau)L_2u(\tau) > 0$ and therefore $L_2u(\tau)$ must be positive. This is a contradiction, which proves that $L_3u(\varrho) > 0$. We have thus proved that $u(t) \in U$.

In the construction of the solution $u(t)$ we choose the point t_0 arbitrarily. We showed that t_0 is a simple zero of $u(t)$ ($u(t)$ has only simple zeros). If now we choose $t_0' \neq t_0$ such that $u(t_0') \neq 0$ and construct the solution $v(t)$ in the same way as it was done with $u(t)$, we get that $v(t) \in U$ and $v(t_0') = 0$ and therefore $u(t)$ and $v(t)$ are linearly independent.

THEOREM 3. $E = U$.

Proof. Let $u(t) \in U$. Then (12) holds at every point ϱ which is a zero of $u(t)$ and therefore $F(u(\varrho)) > 0$ at those points. Now, $F(u(t))$ being strictly decreasing and $u(t)$ oscillatory, this implies that $F(u(t)) > 0$. Thus $u(t) \in E$.

Let $u(t) \in E$. Then $F(u(t)) > 0$ for all $t \in J$. Let ϱ be a zero of $u(t)$. Then $F(u(\varrho)) = -L_1u(\varrho)L_2u(\varrho) > 0$. Thus at every point which is a zero of $u(t)$ we have: $L_1u(\varrho) \neq 0$, $L_2u(\varrho) \neq 0$, $\operatorname{sgn} L_1u(\varrho) \neq \operatorname{sgn} L_2u(\varrho)$. Assume that for a certain zero point τ of $u(t)$ we have: $L_1u(\tau)\cdot L_2u(\tau) \neq 0$, $\operatorname{sgn} L_1u(\tau) \neq \operatorname{sgn} L_2u(\tau)$, $\operatorname{sgn} L_3u(\tau) = \operatorname{sgn} L_2u(\tau)$. Then, if $\xi > \tau$ is the next zero of $u(t)$ to the right of τ, we get, using the same reasoning as in the proof of Theorem 2, $L_1u(\xi)\cdot L_2u(\xi)\cdot L_3u(\xi) \neq 0$, $\operatorname{sgn} L_1u(\xi) = \operatorname{sgn} L_2u(\xi) = \operatorname{sgn} L_3u(\xi)$. Therefore, $F(u(\xi)) = -L_1u(\xi)\cdot L_2u(\xi) < 0$, which is a contradiction. Thus, at any zero point τ of $u(t)$ the following relations hold: $L_1u(\tau)\cdot L_2u(\tau)\cdot L_3u(\tau) \neq 0$, $\operatorname{sgn} L_1u(\tau) = \operatorname{sgn} L_3u(\tau) \neq \operatorname{sgn} L_2u(\tau)$ and $u(t) \in U$.

We can inspect the structure of the set U from another point of view using the classification of solutions due to U. Elias [1].

Let $S(c_0, c_1, \ldots, c_n)$ denote the number of sign changes in the sequence $c_0, c_1, \ldots, c_n$, where all $c_i \neq 0$. Let $y(t)$ be a solution of the equation

$$(*) \qquad L_n y + a(t) y = 0,$$

where $L_n y = \Big(a_{n-1}(t)\Big(a_{n-2}(t)\Big(\ldots\big(a_1(t)(a_0(t)y)'\big)'\ldots\Big)'\Big)'\Big)'$ and $a_i(t)$, $i = 0, 1, \ldots$ $\ldots, n-1$, $a(t)$ are continuous in $(-\infty, \infty)$, $a_i(t) > 0$, $i = 0, 1, \ldots, n-1$, $a(t) > 0$ or $a(t) < 0$. Write

$$S(y, t^-) = \lim_{\xi \to t^-} S\big(L_0 y(\xi), L_1 y(\xi), \ldots, L_n y(\xi)\big).$$

$$S(y, t^+) = \lim_{\xi \to t^+} S\big(L_0 y(\xi), -L_1 y(\xi), \ldots, (-1)^n L_n y(\xi)\big).$$

PROPOSITION 1 [1]. *For every non-trivial solution $y(t)$ of $(*)$ there exists a t_0 such that $S(y, t^+)$ and $S(y, t^-)$ are constants for $t > t_0$. If $S(y, t^+) = k$ on (t_0, ∞) then $S(y, t^-) = n-k$ on (t_0, ∞) and $(-1)^{n-k} a(t) < 0$. On (t_0, ∞) $L_0 y$, $L_1 y, \ldots, L_{n-1} y$ may have only simple zeros.*

We will denote $\lim S(y, t^+)$ as $t \to \infty$ by $S(y)$. Following Proposition 1 it is possible to sort the solutions of $(*)$ into classes $S_k = \{y \mid S(y, t^+) = k$ for sufficiently large values of $t\}$ indexed by integers k, $0 \leqslant k \leqslant n$, such that $(-1)^{n-k} a(t) < 0$.

PROPOSITION 2 [1]. *The set of non-trivial solutions of $(*)$ is the union of the disjoint, non-empty sets S_k, $0 \leqslant k \leqslant n$, such that $(-1)^{n-k} a(t) < 0$. Each one of the above sets consists either of oscillatory or of non-oscillatory solutions only.*

According to Propositions 1 and 2 we will have in the case of equation (1) only two sets, S_1 and S_3.

THEOREM 4. $U = S_1$.

Proof. Let $y(t) \in U$ and suppose that $y(t_0) = 0$, $L_1 y(t_0) > 0$, where t_0 is large enough. Then there exists $\varepsilon > 0$ such that $L_1 y(t) L_2 y(t) L_3 y(t) \neq 0$ for $t \in (t_0, t_0 + \varepsilon)$. In view of (12) we have $\operatorname{sgn} L_1 y(t_0) = \operatorname{sgn} L_3 y(t_0) \neq \operatorname{sgn} L_2 y(t_0)$. Therefore $L_0 y(t) > 0$, $L_1 y(t) > 0$, $L_3 y(t) > 0$, $L_2 y(t) < 0$, $L_4 y(t) < 0$ for every $t \in (t_0, t_0 + \varepsilon)$. This means that $S(y) = S\big(L_0 y(t), -L_1 y(t), L_2 y(t), -L_3 y(t), L_4 y(t)\big) = 1$. Thus $y \in S_1$.

Let now $y \in S_1$. This means that there is t_0 such that $S(y, t^+) = 1$ for $t \in (t_0, \infty)$. The solution $y(t)$ can have only one multiple zero point. Therefore there exists x_0 such that $y(t)$ has only simple zeros in (x_0, ∞). Let $T > \max\{t_0, x_0\}$ and let $\varrho > T$ be a zero of $y(t)$. We may suppose that $L_1 y(\varrho) > 0$ without loss of generality. Then there exists $\varepsilon_1 > 0$ such that for $t \in (\varrho, \varrho + \varepsilon_1)$ we have $L_i y(t) \neq 0$, $i = 0, 1, 2, 3, 4$, and $L_0 y(t) > 0$, $L_1 y(t) > 0$. Thus in the sequence $\{L_0 y(t), -L_1 y(t), L_2 y(t), -L_3 y(t), L_4 y(t)\}$ we have certainly one change of sign on the first place. This means that there is no change of sign on others places, because $y(t) \in S_1$. Consequently $L_2y(t) < 0$, $L_3y(t) > 0$, $L_4y(t) < 0$. It follows that condition (12) is satisfied at ϱ, and since ϱ is an arbitrary zero of $y(t)$ greater than T, we infer that $y(t) \in U$.

THEOREM 5. *Let* $u(t) \in U$. *Then*

$$\int^{\infty} a(t)u^2(t)dt < \infty, \quad \int^{\infty} a_2^{-1}(t)L_2^2u(t)dt < \infty. \tag{17}$$

Proof. Using the properties of $F(u(t))$ we have

$$0 < F(u(t)) = u(t)L_3u(t) - L_1u(t)L_2u(t)$$
$$= F(u(\tau)) - \int_{\tau}^{t} [a(s)u^2(s) + a_2^{-1}(s)L_2^2u(s)]ds \quad \text{for } t \geqslant \tau.$$

Since $\lim F(u(t)) = L \geqslant 0$ as $t \to \infty$, inequalities (17) follow.

COROLLARY 1. *Let* $a(t) \geqslant m > 0$, $a_2(t) \leqslant M_2$. *Then*

$$\int^{\infty} y^2(t)dt < \infty, \quad \int^{\infty} L_2^2y(t)dt < \infty \tag{18}$$

for all $y \in U$.

In what follows we will need

LEMMA 3 ([4], Lemma 6). *Suppose that a function* $f(t)$ *has a bounded derivative on* $[t_0, \infty)$ *and that* $\int_{t_0}^{\infty} f^2(t)dt < \infty$. *Then* $\lim f(t) = 0$ *as* $t \to \infty$.

THEOREM 6. *Let* $0 < m \leqslant a(t) \leqslant M$, $0 < m_1 \leqslant a_1(t) \leqslant M_1$, $a_2(t) \leqslant M_2$ *for* $t \geqslant t_0$, *where* m, M, m_1, M_1, M_2 *are constants. Then*

$$\int^{\infty} a_1^{-1}(t)L_3^2y(t)dt < \infty, \quad \int^{\infty} L_3^2y(t)dt < \infty, \quad \int^{\infty} L_2^2y(t)dt < \infty, \tag{19}$$
$$|L_3y(t)| \text{ is bounded and } \lim L_2y(t) = 0 \text{ as } t \to \infty$$

for every $y(t) \in U$.

Proof. Multiply (1) by L_2y, where $y(t) \in U$. We get $-L_2y(t)L_3'y(t) = a(t)y(t)L_2y(t)$, which can be rewritten as

$$-(L_2y(t)L_3y(t))' + a_1^{-1}(t)L_3^2y(t) = a(t)y(t)L_2y(t).$$

Integrating this equality from a to t, where a is a zero point of $L_3y(t)$ greater than t_0, we get

$$-L_2y(t)L_3y(t) + \int_a^t a_1^{-1}(s)L_3^2y(s)ds = \int_a^t a(s)y(s)L_2y(s)ds.$$

Then using the inequality

$$-(y^2(t) + L_2^2y(t)) \leqslant 2y(t)L_2y(t) \leqslant (y^2(t) + L_2^2y(t)),$$

we obtain

$$-\int_a^t \big(a(s)y^2(s)+ML_2^2y(s)\big)ds$$

$$\leqslant -2L_2y(t)\,L_3y(t)+\int_a^t a_1^{-1}(s)L_3^2y(s)ds$$

$$\leqslant \int_a^t \big(a(s)y^2(s)+a(s)L_2^2y(s)\big)ds \leqslant \int_a^t \big(a(s)y^2(s)+ML_2^2y(s)\big)ds.$$

From $\int_a^\infty a_2^{-1}(s)L_2^2y(s)ds<\infty$ and $a_2^{-1}(t)\geqslant M^{-2}$ it follows $\int_a^\infty L_2^2y(s)ds<\infty$. Using also the fact that $\int_a^\infty a(s)y^2(s)ds<\infty$, and letting t pass through the zeros of $L_3y(t)$ to infinity, we get from the above inequality $\int_a^\infty a_1^{-1}(s)\times$ $\times L_3^2y(s)ds<\infty$; and since $M_1^{-1}\leqslant a_1^{-1}(t)$, $t\geqslant t_0$, we have $\int_a^\infty L_3^2y(t)dt<\infty$.

Now multiply (1) by $L_3y(t)$. We obtain

$$-L_3y(t)L_3'y(t) = a(t)y(t)L_3y(t).$$

Integration over $[a,t]$, where $a<t$ is a zero of $L_3y(t)$, yields

$$-L_3^2y(t) = 2\int_a^t a(s)y(s)L_3y(s)ds$$

and similarly as above we get

$$-\int_a^\infty \big(a(s)y^2(s)+ML_3^2y(s)\big)ds \leqslant -\int_a^\infty \big(a(s)y^2(s)+a(s)L_3^2y(s)\big)ds \leqslant -L_3^2y(t)$$

$$\leqslant \int_a^\infty \big(a(s)y^2(s)+a(s)L_3^2y(s)\big)ds$$

$$\leqslant \int_a^\infty \big(a(s)y^2(s)+ML_3^2y(s)\big)ds.$$

Thus $|L_3y(t)|$ is bounded and this, together with the assumption $0<m_1$ $\leqslant a_1(t)$, implies the boundedness of $|L_2'y(t)|$ on (t_0,∞).

An application of Lemma 3 gives $\lim L_2y(t)=0$ as $t\to\infty$.

THEOREM 7. *Let the assumptions of Theorem 6 be fulfilled and suppose that* $0<m_2\leqslant a_2(t)$ *for* $t>t_0$. *Then* (19) *holds evidently and, moreover,*

(20) $|L_1y(t)|$ *is bounded*, $\lim y(t)=0$ *as* $t\to\infty$,

$\lim F\big(y(t)\big)=0$ *as* $t\to\infty$, $\quad \lim L_1y(t)=0$, *and*

$\lim L_3y(t)=0$ *as* $t\to\infty$, $\quad$ *for all* $y(t)\in U$.

Proof. Let $y(t) \in U$. The function $F(y(t))$ being strictly decreasing on $(-\infty, \infty)$, we have $L_1y(t)L_2y(t) < y(t)L_3y(t)$. Then

$$\left(L_1^2y(t)\right)' < 2a_2^{-1}(t)y(t)L_3y(t) \leqslant a_2^{-1}(t)\left(y^2(t)+L_3^2y(t)\right) \leqslant m_2^{-1}\left(y^2(t)+L_3^2y(t)\right).$$

Integration of this inequality gives

$$L_1^2y(t)-L_1^2y(t_1) \leqslant m_2^{-1}\Big(\int_{t_1}^{t} y^2(s)\,ds+\int_{t_1}^{t} L_3^2y(s)\,ds\Big)$$
$$\leqslant m_2^{-1}\Big(\int_{t_1}^{\infty} y^2(s)\,ds+\int_{t_1}^{\infty} L_3^2y(s)\,ds\Big) < \infty.$$

Thus $L_1^2y(t)$ is bounded on (t_1, ∞). Taking into account the boundedness of $a_1(t)$ and applying Lemma 3, we get $\lim y(t) = 0$ as $t \to \infty$.

From this fact and from Theorem 6 it follows immediately that $\lim F(y(t)) = 0$ as $t \to \infty$.

Further, we have to prove that $\lim L_1y(t) = 0$ as $t \to \infty$. Since $L_2y(t)$ is bounded and $a_2(t)$ is also bounded, we infer that $L_1'y(t)$ is bounded. An easy calculation gives

$$a_1^{-1}(t)L_1^2y(t) = \left(y(t)L_1y(t)\right)'-a_2^{-1}(t)y(t)L_2y(t).$$

Integrating this equality and respecting the facts already known, we obtain

$$M_1^{-1}\int_a^t L_1^2y(s)\,ds \leqslant \int_a^t a_1^{-1}(s)L_1^2y(s)\,ds \leqslant |y(t)L_1y(t)-y(t_0)L_1y(t_0)|+$$
$$+M_2^{-1}\int_a^t \left(y^2(s)+L_2^2y(s)\right)ds \leqslant K_1 \quad \text{for } t \geqslant a.$$

Thus $\int_a^\infty L_1^2y(t)\,dt < \infty$. An application of Lemma 3 gives $\lim L_1y(t) = 0$ as $t \to \infty$.

Now, we are going to prove that also $\lim L_3y(t) = 0$ as $t \to \infty$. In fact, $|L_3'y(t)| = |-a(t)y(t)| \leqslant M|y(t)| < K_2$. Thus $L_3'y(t)$ is bounded. Then in view of the fact that $\int^\infty L_3^2y(t)\,dt < \infty$, Lemma 3 implies $\lim L_3y(t) = 0$ as $t \to \infty$.

II. The case where all solutions of (1) are non-oscillatory. In this case (see [5]) the quasi-derivatives $L_iy(t)$, $i = 0, 1, 2, 3$, of the solution $y(t)$ of (1) are monotone on some interval $[T_y, \infty)$ and therefore the limits $\lim_{t\to\infty} L_iy(t)$, $i = 0, 1, 2, 3$, exist (finite or infinite). The set of all solutions of (1) can be divided into 4 disjoint classes: V_0, V_1, V_2, and V_3 in the following way.

DEFINITION 3. A solution $y(t)$ of (1) belongs to the class V_k, $k \in (0, 1, 2, 3)$, if $\lim L_k y(t)$ as $t \to \infty$ is finite and $|\lim L_i y(t)| = \infty$ for $i < k$.

LEMMA 4 (see Lemma 2, [5]). *If* $y(t) \in V_k$, *then*

(a) $(-1)^{i+1} y(t) L_i y(t) > 0$, $i = k+1, \ldots, n-1$, *for* $t \geq t_1 \geq T_y$;

(b) $\lim L_i y(t) = 0$ *as* $t \to \infty$, $i = k+1, \ldots, n-1$;

(c) $\lim L_i y(t) = \infty \operatorname{sgn} y(t)$ *as* $t \to \infty$, $i = 0, 1, \ldots, k-1$.

THEOREM 8. $U = E = V_0 \cup V_1$ *and*

$$(21)\quad \int^{\infty} a(t) y^2(t)\, dt < \infty, \qquad \int^{\infty} a_2^{-1}(t) L_2^2 y(t)\, dt < \infty \quad \textit{for all } y(t) \in E.$$

Proof. Let $y(t) \in V_0 \cup V_1$. Then, in view of (a) and (c) of Lemma 4, we have $\operatorname{sgn} y(t) = \operatorname{sgn} L_3 y(t)$, $\operatorname{sgn} L_1 y(t) \neq \operatorname{sgn} L_2 y(t)$ for $t \geq t_1$. Therefore $F(y(t)) > 0$ for $t \geq t_1$ and thus $y(t) \in E$. The same reasoning as in the proof of Theorem 5 leads to inequalities (21).

Now, let $y(t) \in E$. Then $F(y(t)) > 0$ and therefore (21) holds. Suppose that $y(t) \in V_2$ and $y(t) > 0$ for $t \geq t_1$. Then $L_3 y(t) > 0$ for $t \geq t_1$ and so $L_2 y(t)$ increases. If we have $L_2 y(t) < 0$ for $t \geq t_1$, then $L_1 y(t)$ decreases for $t \geq t_1$ and must be positive to avoid contradiction with the assumption that $y(t) > 0$. But then $\lim L_1 y(t)$ is finite and therefore $y(t) \in V_1 \cup V_0$, which contradicts the assumption that $y(t) \in V_2$. Thus $L_2 y(t)$ must be positive for $t \geq t_2 \geq t_1$ and increasing. Therefore $L_2 y(t) > L_2 y(t_2) > 0$ for $t > t_2$. From this and the second part of (21) we get $L_2^2 y(t_2) \int^{\infty} a_2^{-1}(t)\, dt < \infty$, which contradicts (2). We get the same contradiction if we suppose that $y(t) \in V_2$, $y(t) < 0$ for $t < t_2$. Summarizing, we have proved that if $y(t) \in E$ then $y(t) \notin V_2$.

Let now $y(t) \in E$ and $y(t) \in V_3$. Suppose that $y(t) > 0$ for $t \geq t_1$. Then $L_3 y(t) > 0$ and decreases for $t \geq t_1$; according to (c) of Lemma 4 we have $\lim L_2 y(t) = \infty$ as $t \to \infty$. This, jointly with the second part of (21), gives a contradiction of the same kind as above. We get the same contradiction if we assume that $y(t) \in E$, $y(t) \in V_3$ and $y(t) < 0$ for $t \geq t_1$. Thus we have shown that $y(t) \in E$ implies $y(t) \notin V_3$.

THEOREM 9. $U = E = V_0 \cup V_1 = S_1$.

Proof. Let $y(t) \in V_0 \cup V_1$. Then from (a) and (c) of Lemma 4 and from equation (1) we get $\operatorname{sgn} y(t) = \operatorname{sgn} L_1 y(t) = \operatorname{sgn} L_3 y(t) \neq \operatorname{sgn} L_2 y(t) = \operatorname{sgn} L_4 y(t)$ for t large enough and therefore $S(y) = 1$.

Let $y(t) \in V_2 \cup V_3$. Then from (a) and (c) of Lemma 4 and from equation (1) we have $\operatorname{sgn} y(t) = \operatorname{sgn} L_1 y(t) = \operatorname{sgn} L_2 y(t) = \operatorname{sgn} L_3 y(t) \neq \operatorname{sgn} L_4 y(t)$. Therefore $S(y) = 3$.

This paper generalizes the results of paper [4].

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