ISSN 0105-8517

Behavioural Notions for Elementary Net Systems

M. Nielsen
G. Rozenberg
P.S. Thiagarajan

DAIMI PB – 281
August 1989

COMPUTER SCIENCE DEPARTMENT
AARHUS UNIVERSITY
Ny Munkegade, Building 540
DK-8000 Aarhus C, Denmark

Abstract

We study the relationships between a number of behavioural notions that have arisen in the theory of distributed computing. In order to sharpen the understanding of these relationships we apply the chosen behavioural notions to a basic net-theoretic model of distributed systems called elementary net systems. The behavioural notions that are considered here are trace languages, non-sequential processes, unfoldings and event structures.

The relationships between these notions are brought out in the process of establishing that for each elementary net system, the trace language representation of its behaviour agrees in a strong way with the event structure representation of its behaviour.

Keywords: Net theory – Trace Languages – Non-Sequential Processes – Event Structures.

0 Introduction

Our aim here is to relate a number of behavioural notions that have evolved more or less independently of each other within the theory of distributed computing. The insights concerning the relationships between these notions are best brought out in a concrete setting. Hence we shall carry out our study by applying the selected behavioural notions to characterize the behaviour of *elementary net systems.*

Elementary net systems are a fundamental system model of net theory. This theory was initiated by Petri (1962) and it has evolved into a full-fledged theory of distributed systems and processes Brauer et al. (1987). The elementary net system model incorporates, at a primitive level, the basic features of distributed systems. An elementary net system consists of an underlying *net* which represents the *structure* of the system under study together with an *initial state.* In this setting, a net is composed out of a set of local atomic states called *conditions*, a set of local atomic transitions called *events* and a fixed neighbourhood relation between the conditions and the events. A state consists of a set of conditions that hold concurrently. The dynamics of the system is captured through a simple transition rule which specifies how the system can go from one state to another state through the occurrence of an event. Various tools have been proposed to represent the behaviour of an elementary net system.

The most primitive among these is the notion of *firing sequences.* Here the system is viewed as generating a set of strings over the events of the system. As a result, all information concerning choice and concurrency is "lost". At the other end of the spectrum, we have a *labelled event structure* denoting the behaviour of a system. In this representation we have a single poset of labelled event occurrences where information concerning the causal ordering, choice and concurrency associated with the system is clearly represented.

In between these two extremes we also have the notions of *non-sequential processes* and *traces.* A non sequential process is a labelled partially ordered set of event occurrences and condition holdings that represents a single run of the system. Here the distinction between causal ordering and concurrency is re-established (in contrast to the firing sequence approach); information concerning choice is, however, "lost". In the trace approach, a single run of the system is represented as a set of equivalent firing sequences. Here again information concerning concurrency is "recovered" through the use of a natural equivalence relation generated by the structure of the system. One then applies the tools of trace theory in a straightforward manner. As in the case of non-sequential processes, information concerning choice is lost.

Our aim here is to construct a framework in which the behavioural notions we have mentioned above can be seen to be smoothly related to each other. Indeed yet another behavioural representation called *the unfolding* will also fit into our framework. As a byproduct we can show that trace theory with its independent existence "confirms" that the labelled event structure associated with an elementary net system is the "correct" one.

The uninitiated reader is referred to Aalbersberg and Rozenberg (1986), Mazurkiewicz (1978), Nielsen et al. (1981), Rozenberg (1987) and Thiagarajan (1987) for background material. We shall survey related literature in the concluding section. Some of the results established in this paper were reported (without proofs) in the survey paper Thiagarajan (1988).

1 Elementary Net Systems

Elementary net systems, as the name suggests, are meant to be the simplest system model of net theory. They may be viewed as transition systems obeying a particular principle of change. This view of elementary net systems is explained in more detail in Thiagarajan (1988). Here, for the sake of brevity, we shall make a direct presentation.

Definition 1.1 *A* net *is a triple* $N = (S, T, F)$ *where* S *and* T *are sets and* $F \subseteq (S \times T) \cup (T \times S)$ *are such that*

(i) $S \cap T = \emptyset$

(ii) $domain(F) \cup range(F) = S \cup T$ *where*

$$domain(F) = \{x \mid \exists y.(x, y) \in F\} \text{ and}$$
$$range(F) = \{y \mid \exists x.(x, y) \in F\}.$$

□

Thus a net may be viewed as a directed bipartite graph with no isolated elements. Note that we admit the *empty* net $N_\emptyset = (\emptyset, \emptyset, \emptyset)$.

S is the set of *S-elements*, T is the set of *T-elements* and F is the *flow relation* of the net $N = (S, T, F)$. In diagrams the S-elements will be drawn as circles, the T-elements as boxes and the elements of the flow relation as directed arcs. Figure 1 is an example of a net.

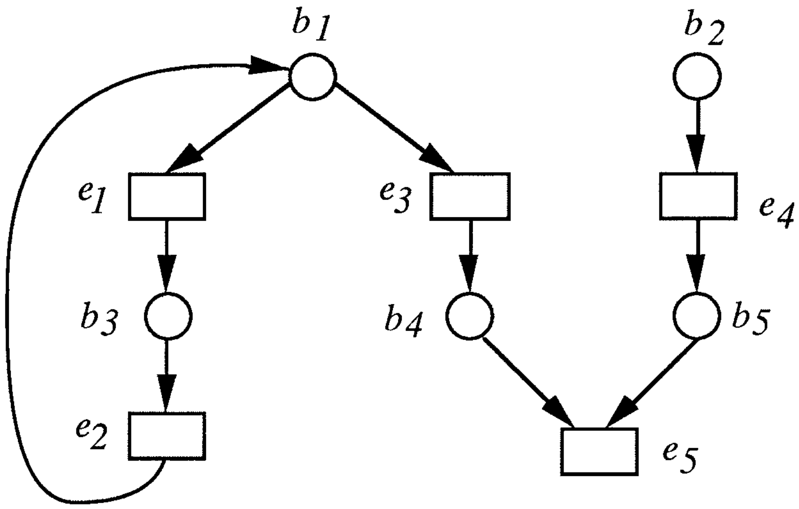

Figure 1

In this paper, the S-elements will be used to denote the (local) atomic states called *conditions* and the T-elements will be used to denote (local) atomic changes-of-states called *events.* The flow relation will model a *fixed* neighbourhood relation between the conditions and events of a system. Following usual practice, we shall represent such nets of conditions and events by triples of the form $N = (B, E, F)$.

Let $N = (B, E, F)$ be a net. Then $X_N = B \cup E$ is the set of *elements* of N. Let $x \in X_N$. Then

$$
\begin{aligned}
{}^{\bullet}x &= \{y \mid (y,x) \in F\} \quad \text{(the set of } \textit{pre-elements} \text{ of } x\text{)} \\
x^{\bullet} &= \{y \mid (x,y) \in F\} \quad \text{(the set of } \textit{post-elements} \text{ of } x\text{)}
\end{aligned}
$$

This "dot" notation is extended to subsets of X_N in the obvious way. For $e \in E$ we shall call ${}^{\bullet}e$ the set of *pre-conditions* of e and we shall call $e^{\bullet}$ the set of *post-conditions* of e.

Definition 1.2 *An* elementary net system *is a quadruple* $\mathcal{N} = (B, E, F, c_{in})$ *where*

(i) $N_{\mathcal{N}} = (B, E, F)$ *is a net called the* underlying net *of* $\mathcal{N}$.

(ii) $c_{in} \subseteq B$ *is the* initial case *of* $\mathcal{N}$. □

In diagrams the initial case will be shown by "marking" the members of c_{in}. Figure 2 is an example of an elementary net system. Through the rest of the paper we shall refer to this net system as $\mathcal{N}_2$.

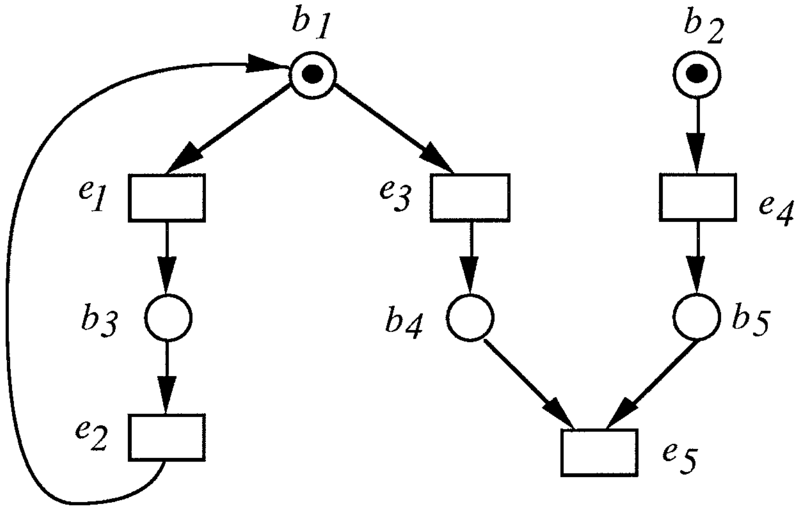

Figure 2

In this paper, we will only deal with elementary net systems. Hence we will refer to them as net systems. The dynamics of a net system are simple. A state (usually called a *case*) of the system consists of a set of conditions holding concurrently. An event can occur at a case iff all its pre-conditions and none of its post-conditions hold at the case. When an event occurs each of its pre-conditions ceases to hold and each of its post-conditions begins to hold. This simple and restrictive notion of states and changes-of-states leads to a surprisingly rich and sophisticated class of objects. Moreover, the essential features of distributed systems can be isolated and studied using net systems. First however we must formalize the dynamics of net systems.

Let $N = (B, E, F)$ be a net. Then $\longrightarrow_N \subseteq 2^B \times E \times 2^B$ is *the (elementary) transition relation generated by* N and is given by

$$\longrightarrow_N = \{(k, e, k') \mid k - k' = {}^\bullet e \wedge k' - k = e^\bullet\}$$

Definition 1.3 *Let* $\mathcal{N} = (B, E, F, c_{in})$ *be a net system.*

(i) $C_{\mathcal{N}}$, the state space of $\mathcal{N}$ *(also denoted as* $[c_{in}>$*) is the least subset of* 2^B *containing* c_{in} *such that if* $c \in C_{\mathcal{N}}$ *and* $(c, e, c') \in \longrightarrow_{N_{\mathcal{N}}}$ *then*

$c' \in C_{\mathcal{N}}$.

(ii) $TS_{\mathcal{N}} = (C_{\mathcal{N}}, E, \longrightarrow_{\mathcal{N}})$ *is* the transition system associated with $\mathcal{N}$ *where* $\longrightarrow_{\mathcal{N}}$ *is* $\longrightarrow_{N_{\mathcal{N}}}$ *restricted to* $C_{\mathcal{N}} \times E \times C_{\mathcal{N}}$.

□

For the system $\mathcal{N}_2$ shown in Figure 2, $\{\{b_1, b_2\}, \{b_1, b_5\}, \{b_4, b_2\}, \{b_4, b_5\}, \{b_3, b_2\}, \{b_3, b_5\}, \emptyset\}$ is its state space. We recall that a transition system is a triple $TS = (S, A, \rightarrow)$ where S is a set of states, A is a set of actions and $\longrightarrow \subseteq S \times A \times S$ is the (labelled) transition relation. According to the above definition there is a natural way of explaining the dynamics of a net system with the help of a transition system. We are now in a position to bring out the particular and restricted notion of change adopted in net theory.

Let $\mathcal{N} = (B, E, F, c_{in})$ be a net system, $c \in C_{\mathcal{N}}$ and $e \in E$. Then e is said to be *enabled* at c – denoted $c[e>$ – iff there exists $c' \in C_{\mathcal{N}}$ such that $(c, e, c') \in \longrightarrow_{\mathcal{N}}$. We shall often write $c \xrightarrow{e} c'$ and sometimes we shall write $c[e> c'$ in place of $(c, e, c') \in \longrightarrow_{\mathcal{N}}$.

Proposition 1.1 *Let* $\mathcal{N} = (B, E, F, c_{in})$ *be a net system* $e \in E$ *and* c, c', c_1, *etc. members of* $C_{\mathcal{N}}$. *Then the following statements hold.*

(i) $c_1 \xrightarrow{e} c_2 \wedge c_3 \xrightarrow{e} c_4 \Rightarrow$
$c_1 - c_2 = c_3 - c_4 \wedge c_2 - c_1 = c_4 - c_3$

(ii) $c[e> \Longleftrightarrow {}^\bullet e \subseteq c \wedge e^\bullet \cap c = \emptyset$

(iii) $c \xrightarrow{e} c' \wedge c \xrightarrow{e} c'' \Rightarrow c' = c''$.

□

(i) says that an event causes the same change in the system state whenever it occurs; its pre-conditions cease to hold and its post-conditions begin to hold.

(ii) says that an event is enabled at a case *if* and only if the fixed change associated with its occurrence is possible at the case. Thus no "side-conditions" are involved in the enabling of an event.

(iii) says that the transition systems associated with net systems are *deterministic*. Hence in order to connect up with other approaches to the theory of distributed systems such as CCS or CSP one must go over to *labelled* net systems. When one does so, it is possible to give an operational semantics for CCS-like processes in terms of (labelled) net systems.

Basic concepts concerning the behaviour of distributed systems such as causality, choice, concurrency, and confusion ("glitch") can now be cleanly defined – and separated from each other – with the help of net systems. The interested reader is referred to Thiagarajan (1987) for details.

We are ready to begin our study of the behaviour of elementary net systems. For the sake of convenience we fix an elementary net system $\mathcal{N} = (B, E, F, c_{in})$ and work with it throughout what follows. We shall assume that $\mathcal{N}$ is *contact-free*. In other words, we shall assume,

$$\forall c \in C_{\mathcal{N}}.\forall e \in E.[e^{\bullet} \subseteq c \Rightarrow {}^{\bullet}e \cap c = \emptyset].$$

As is well-known (see for instance Rozenberg (1987)), this does not – at least for the study of behavioural issues – involve any loss of generality.

We can now introduce the first and the most primitive of our behavioural tools. The set of *firing sequences* of $\mathcal{N}$ – denoted $FS_{\mathcal{N}}$ – is the least subset of E^* (recall that $\mathcal{N} = (B, E, F, c_{in})$) given by

(i) $\Lambda \in FS_{\mathcal{N}}$ and $c_{in}[\![\Lambda > c_{in}$

(ii) Suppose $\rho \in FS_{\mathcal{N}}$, $c_{in}[\![\rho > c$ and $c \xrightarrow{e} c'$ then $\rho e \in FS_{\mathcal{N}}$ and $c_{in}[\![\rho e > c$.

Thus $[\![\ >$ is the natural "extension" of $\longrightarrow_{\mathcal{N}}$ to $\{c_{in}\} \times E^* \times C_{\mathcal{N}}$. As may be guessed, Λ denotes the null sequence. We shall write FS instead of $FS_{\mathcal{N}}$ for convenience. For the system $\mathcal{N}_2$, some of its firing sequences are $e_1e_2e_4$, $e_4e_1e_2$ and $e_3e_4e_5$.

Firing sequences "hide" important aspects of the behaviour of a net system. To bring out this deficiancy more clearly, it will be convenient to define the notions of concurrency and conflict.

Let $e_1 \neq e_2$ and $e_1, e_2 \in E$. Let $c \in C_{\mathcal{N}}$. We say that e_1 and e_2 can occur *concurrently* at c – denoted $c[\{e_1, e_2\} >$ – iff $c[e_1 >$ and $c[e_2 >$ and $({}^\bullet e_1 \cup e_1^\bullet) \cap ({}^\bullet e_2 \cup e_2^\bullet) = \emptyset$.

Thus e_1 and e_2 can occur concurrently at c iff they can occur individually and their neighbourhoods are disjoint. For the system $\mathcal{N}_2$, at the initial case e_1 and e_4 can occur concurrently. Consequently, the firing sequences $e_1e_2e_4$ and $e_4e_1e_2$ and $e_1e_4e_2$ all represent the *same* (non-sequential) stretch of behaviour of $\mathcal{N}_2$.

The "dual" of the notion of concurrency is conflict. Then we say that e_1 and e_2 are in *conflict* at c iff $c[e_1 >$ and $c[e_2 >$ *but not* $c[\{e_1, e_2\} >$. Thus at c either e_1 may occur or e_2 may occur but not both. The choice as to whether e_1 or e_2 will occur is assumed to be resolved by the "environment" of the system. In $\mathcal{N}_2$, at the initial case e_1 and e_3 are in conflict. Hence the firing sequences $e_1e_2e_4$ and $e_3e_4e_5$ represent two conflicting (alternative) stretches of behaviour of $\mathcal{N}_2$.

It is in this sense firing sequences hide information concerning concurrency and conflict-resolution. We will now see how the theory of trace languages can be applied to extract information concerning concurrency from the firing sequences.

2 The Trace Semantics

The theory of trace languages was introduced in Mazurkiewicz (1978) to model the non-sequential behaviour of distributed programs. The basic idea is to postulate a symmetric and irreflexive independence relation over the letters of an alphabet. The elements of the alphabet set represent the actions that can be executed by a program. Two actions that are in the independence relation are supposed to occur concurrently whenever they occur "adjacent" to each other. This relation then naturally induces an equivalence relation over the language which is a sequential description of the behaviour of the program. For details we refer the reader to Aalbersberg and Rozenberg (1986). Here we shall straight away apply the notions of this formalism to net systems.

Let I denote the independence relation associated with $\mathcal{N}$ (which is actually generated by $N_{\mathcal{N}} = (B, E, F)$, the underlying net of $\mathcal{N}$),

$$I = \{(e_1, e_2) \mid e_1, e_2 \in E \wedge ({}^\bullet e_1 \cup e_1{}^\bullet) \cap ({}^\bullet e_2 \cup e_2^\bullet) = \emptyset\}$$

Since $I \subseteq E \times E$ is irreflexive and symmetric, we have a natural way of partitioning E^* using the least congruence relation generated by I via equations of the form $e_1 e_2 = e_2 e_1$, where $(e_1, e_2) \in I$. To be specific, define $\dot{\sim} \subseteq E^* \times E^*$ as,

$$\rho \,\dot{\sim}\, \rho' \overset{def}{\Longleftrightarrow} \exists \rho_1, \rho_2 \in E^* \exists (e_1, e_2) \in I. \\ \rho = \rho_1 e_1 e_2 \rho_2 \text{ and } \rho' = \rho_1 e_2 e_1 \rho_2$$

Then $\sim = (\dot{\sim})^*$ is the equivalence relation we want and for $\rho \in E^*$,

$$[\rho] \overset{def}{=} \{\rho' \mid \rho' \in E^* \text{ and } \rho' \sim \rho\}.$$

Let $\rho \in FS$. Then it is well-known that $[\rho] \subseteq FS$. (One says that FS is *consistent* with I).

Now $FS/\sim \overset{\text{def}}{=} \{[\rho] \mid \rho \in FS\}$ is the prefix-closed trace language we associate with $\mathcal{N}$. Throughout what follows we *denote* $FS/\sim$ *as* T. Thus T $(= FS/\sim)$ is a "finer" representation of the behaviour of $\mathcal{N}$ as compared to FS.

Once again it is well-known that each element of T can be (up to isomorphism) uniquely represented as a finite labelled poset of event occurrences where the labels take values in E. It turns out that information concerning choice can be recovered from T by imposing an ordering relation over T.

$\sqsubseteq \subseteq T \times T$ is given by:

$$t_1 \sqsubseteq t_2 \overset{def}{\Longleftrightarrow} \forall \rho \in t_1 \; \exists \rho' \in t_2. \; \rho \in \text{Prefix}(\rho').$$

Here Prefix (γ) denotes the set of prefixes of the string γ.

It is easy to check that $(T, \sqsubseteq)$ is a poset. Figure 3 shows an initial portion of the poset of traces associated with the net system $\mathcal{N}_2$.

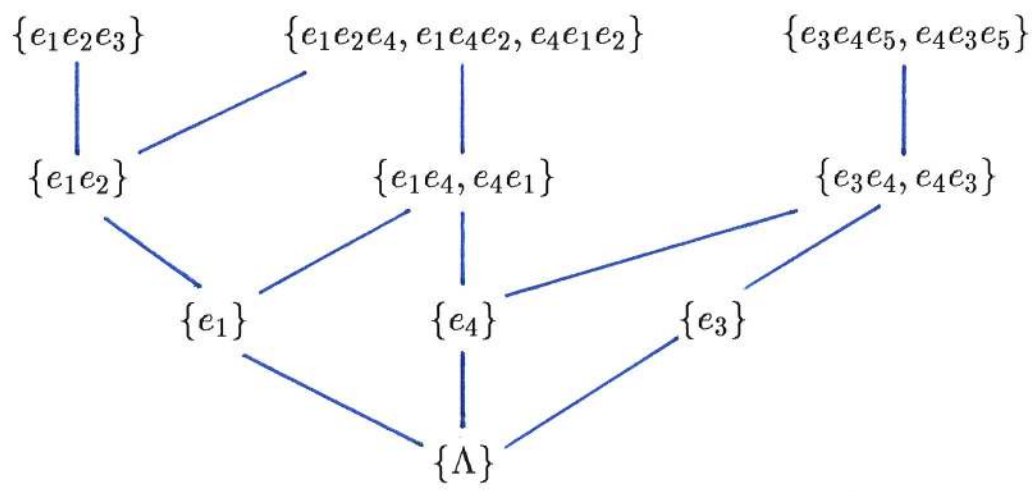

Figure 3

To "see" information concerning choice we define a "compatibility" relation over T as follows.

Let $t_1, t_2 \in T$. Then

$$t_1 \uparrow t_2 \overset{def}{\iff} \exists t \in T.\ t_1 \sqsubseteq t \text{ and } t_2 \sqsubseteq t.$$
$$t_1 \not\uparrow t_2 \overset{def}{\iff} \text{not } (t_1 \uparrow t_2)$$

If $t_1 \not\uparrow t_2$ then t_1 and t_2 represent two runs of $\mathcal{N}$ in which the individual choices that have been made to realize t_1 are not all compatible with the choices that have been made to realize t_2. In the example shown in Figure 3, $\{e_1e_2e_3\} \not\uparrow \{e_3\}$. It is easy to see that the choice of the first occurrence of e_1 in the firing sequence $e_1e_2e_3$ is opposed to the choice of e_3 in the firing sequence e_3.

3 The Finite Processes of $\mathcal{N}$

We now wish to find an alternative representation of $(T, \sqsubseteq)$. This representation will be in terms of the finite processes of $\mathcal{N}$. A process of $\mathcal{N}$ will be a *labelled net* of the form $\tilde{N} = (\tilde{B}, \tilde{E}, \tilde{F}, \tilde{\varphi})$ where $(\tilde{B}, \tilde{E}, \tilde{F})$ is a restricted kind of a net called a *causal net* and $\tilde{\varphi} : \tilde{B} \cup \tilde{E} \to B \cup E$ (recall that $\mathcal{N} = (B, E, F, c_{in})$) is the labelling function required to satisfy certain constraints. For a definition of a process along these lines, see Rozenberg (1989).

Here we shall define processes with the help of firing sequences. This will enable us to build up the finite processes of $\mathcal{N}$ inductively. Moreover, our method of construction will enable us to obtain the unfolding of a net system in a smooth fashion. As we will see, this method of constructing processes will be very helpful for proving the desired results. For a similar development of the process notion see Best and Devillers (1987).

For each firing sequence ρ, we will define a process $N_\rho = (B_\rho, E_\rho, F_\rho, \varphi_\rho)$. In doing so it will be convenient to keep track of the conditions that hold in $\mathcal{N}$ after the run represented by the firing sequence ρ. This set of conditions will be encoded as c_ρ.

Definition 3.1
Let $\rho \in FS$. Then $N_\rho = (B_\rho, E_\rho, F_\rho, \varphi_\rho)$ is given by:

(i) $\underline{\rho = \Lambda}$. *Then*

$N_\Lambda = (\phi, \phi, \phi, \phi)$ *and*

$c_\Lambda = \{(b, \phi) \mid b \in c_{in}\}$

recall that $\mathcal{N} = (B, E, F, c_{in})$

(ii) $\underline{\rho \neq \Lambda}$. *Let* $\rho = \rho' e$ *and assume that* $\mathcal{N}_{\rho'} = (B_{\rho'}, E_{\rho'}, F_{\rho'}, \varphi_{\rho'})$ *and* $c_{\rho'}$ *are defined. Then*

$N_\rho = (B_\rho, E_\rho, F_\rho, \varphi_\rho)$ *with*

$E_\rho = E_{\rho'} \cup \{(e, X)\}$

where $X = \{(b, D) \mid b \in^\bullet e$ *and* $(b, D) \in c_{\rho'}\}$,

$B_\rho = B_{\rho'} \cup X \cup Y$ *where* $Y = \{(b, \{(e, X)\}) \mid b \in e^\bullet\}$,

$F_\rho = F_{\rho'} \cup (X \times \{(e, X)\}) \cup (\{(e, X)\} \times Y)$, *and*

φ_ρ *is defined by:* $\forall (z, Z) \in B_\rho \cup E_\rho.\ \varphi_\rho((z, Z)) = z$.

Finally, $c_\rho = (c_{\rho'} - X) \cup Y$.

□

It will turn out that N_ρ as defined above is a labelled net. For $\rho = e_1 e_2 e_4 e_3$ in the system $\mathcal{N}_2$ we show N_ρ in Figure 4. For convenience we have displayed φ_ρ by writing the value of $\varphi_\rho(x)$ inside the graphical representation of x for each $x \in B\rho \cup E_\rho$. We will follow this convention through the rest of the paper.

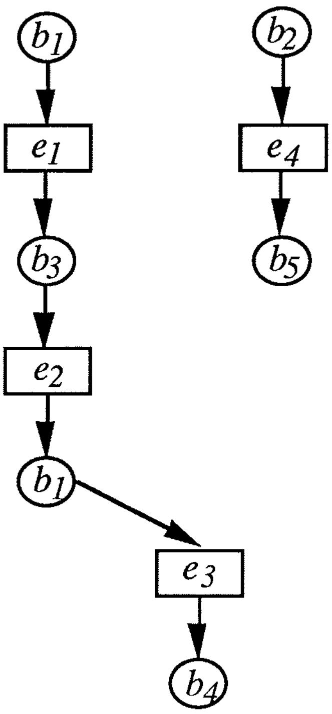

Figure 4

In order to establish a relationship between the traces of $\mathcal{N}$ and its processes it is necessary to define an ordering relation over the processes of $\mathcal{N}$.

Definition 3.2

(i) The set of finite processes of $\mathcal{N}$ is denoted as $P_\mathcal{N}$ and is given by: $P_\mathcal{N} = \{N_\rho \mid \rho \in FS\}$ where N_ρ is as given by Definition 3.1.

(ii) $\subseteq' \subseteq P_\mathcal{N} \times P_\mathcal{N}$ is defined as:

$$N_\rho = (B_\rho, E_\rho, F_\rho, \varphi_\rho) \;\; \subseteq' \;\; N_{\rho'} = (B_{\rho'}, E_{\rho'}, F_{\rho'}, \varphi_{\rho'}) \;\; \textit{iff}$$
$$B_\rho \subseteq B_{\rho'} \textit{ and } E_\rho \subseteq E_{\rho'} \textit{ and } F_\rho \subseteq F_{\rho'}.$$

□

We shall write P instead of $P_{\mathcal{N}}$.

Clearly $\subseteq'$ is a partial ordering relation. From now on we let ρ and ρ' range over FS and e range over E. We shall assume that for $\rho \in FS$, $N_\rho = (B_\rho, E_\rho, F_\rho, \varphi_\rho)$.

The set of elements c_ρ as specified in Definition 3.1 will play a crucial role in what follows. Notice that, in general, $c_\rho - B_\rho \neq \emptyset$.

It will be convenient to extend φ_ρ to $B_\rho \cup E_\rho \cup c_\rho$ as follows. By abuse of notation, this extension is also denoted as φ_ρ.

$$\forall (b, D) \in c_\rho. \ \varphi_\rho((b, D)) = b.$$

Since φ_ρ is a simple projection operation, from now on we will not display it explicitly. Our major aim in this section is to show the following:

> $(T, \sqsubseteq)$ and $(P, \subseteq')$ are isomorphic posets. In fact, $f : T \to P$ given by $f([\rho]) = N_\rho$ is an isomorphism.

Along the way we shall also show that our notion of a process "agrees" with the existing notion of a process (when restricted to the finite ones). We need a number of preliminary results.

In stating and proving these results, we will make heavy use of Definition 3.1. All the undefined terms that may crop up are to be understood with the help of Definition 3.1.

Lemma 3.3
$\varphi_\rho(c_\rho) \in C_{\mathcal{N}}$. Moreover $c_{in}[\![\rho > \varphi_\rho(c_\rho)$ (in $\mathcal{N}$).

Proof By induction on $k = |\rho|$.

<u>$k = 0$</u> Clearly $\rho = \Lambda$ and $\varphi_\Lambda(c_\Lambda) = c_{in}$ by definition.

<u>$k > 0$</u> Let $\rho = \rho' e$. Let $c_\rho, c_{\rho'}, X$ and Y be as in Definition 3.1. Set $\varphi_{\rho'}(c_{\rho'}) = c'$ and $\varphi_\rho(c_\rho) = c$.

By the induction hypothesis, $c_{in}[\rho' > c'$. We know that e is enabled at c' because $\rho' e$ is a firing sequence. Hence we must show that $c = (c' -^{\bullet} e) \cup e^{\bullet}$.

From Definition 3.1, it follows that $c_\rho = (c_{\rho'} - X) \cup Y$.

Consider $b \in c$. Then there exists $(b, D) \in c_\rho$. Suppose that $(b, D) \in Y$. Then $b \in e^{\bullet}$ by the definition of Y. Suppose that $(b, D) \in c_{\rho'} - X$. Then $b \notin^{\bullet} e$ by the definition of X. Since $(b, D) \in c_{\rho'}$, we have $b \in c'$ by the induction hypothesis. Hence $b \in c' -^{\bullet} e$. We have shown that $c \subseteq (c' -^{\bullet} e) \cup e^{\bullet}$.

Hence consider $b \in (c' -^{\bullet} e) \cup e^{\bullet}$.

If $b \in e^{\bullet}$ then clearly $(b, \{(e, X)\}) \in Y$ so that $b \in c$.
If $b \in c' -^{\bullet} e$ then there exists $(b, D) \in c_{\rho'}$ by the induction hypothesis. Moreover $(b, D) \notin X$ because $b \notin^{\bullet} e$. Hence $(b, D) \in c_{\rho'} - X$ and as a result $b \in c$. Thus $(c' -^{\bullet} e) \cup e^{\bullet} \subseteq c$.

□

Lemma 3.4
N_ρ is a (labelled) net.

Proof Follows easily from Definition 3.1 by induction on $|\rho|$.

□

The next result which is a technical one will turn out to be very useful.

Lemma 3.5

(i) $\forall (b, D) \in B_\rho \cap c_\rho \,.\, (b, D)^{\bullet} = \emptyset$ *in* N_ρ.

(ii) $\forall (y, Y) \in B_\rho \cup E_\rho \,.\, ^{\bullet}(y, Y) = Y$ *in* N_ρ.

(iii) $c_\rho - B_\rho = c_\Lambda - B_\rho$.

(iv) $|\rho| = |E_\rho|$. *In fact,* $\#_e(\rho) = |\varphi_\rho^{-1}(e)|$ *for every* $e \in E$.

Proof We will simultaneously prove all the parts of the lemma by induction on $|\rho|$. Consequently the induction hypothesis will have four parts.

<u>$k = 0$</u> Trivial.

<u>$k > 0$</u> Let $\rho = \rho' e$ and $c_{\rho'}$, X and Y be as in Definition 3.1. We will first prove that $(e, X) \notin E_{\rho'}$.

Suppose $(e, X) \in E_{\rho'}$. Then by part (ii) of the induction hypothesis, ${}^\bullet(e, X) = X$ in $N_{\rho'}$. Hence for each $(b, D) \in X$, $(e, X) \in (b, D)^\bullet$ in $N_{\rho'}$. Hence $(b, D)^\bullet \neq \emptyset$ for each $(b, D) \in X$.

Now ${}^\bullet e \neq \emptyset$ in $\mathcal{N}$ because $\mathcal{N}$ is contact-free. Hence $X \neq \emptyset$ by Lemma 3.3 (applied to ρ').

So consider $(b, D) \in X$. Then $(b, D) \in c_{\rho'}$. If $(b, D) \in c_{\rho'} - B_{\rho'}$ then clearly $(b, D)^\bullet$ is undefined in $N_{\rho'}$. If $(b, D) \in B_{\rho'}$ then $(b, D)^\bullet = \emptyset$ in $N_{\rho'}$ by part (i) of the induction hypothesis. In either case we have a contradiction. Hence $(e, X) \notin E_{\rho'}$. This shows that $E_\rho - E_{\rho'} = \{(e, X)\}$. Part (iv) of the lemma now follows from part (iv) of the induction hypothesis.

Next notice that $B_\rho = B_{\rho'} \cup X \cup Y$ and $c_\rho = (c_{\rho'} - X) \cup Y$. By the induction hypothesis, $c_{\rho'} - B_{\rho'} = c_\Lambda - B_{\rho'}$. It is now easy to show that $c_\rho - B_\rho = c_\Lambda - B_\rho$ which establishes part (iii) of the lemma.

To prove the first part of the lemma consider $(b, D) \in B_\rho \cap c_\rho$. Recall that $c_\rho = (c_{\rho'} - X) \cup Y$.

<u>**Case 1**</u> $(b, D) \in Y$. Then $D = \{(e, X)\}$.
Suppose that $(b, D)^\bullet \neq \emptyset$ in N_ρ. Then there exists an $(e_0, X_0) \in E_\rho$ such that $((b, D), (e_0, X_0)) \in F_\rho$. Recalling the definition of F_ρ in terms of $F_{\rho'}$, X, Y and e we can first rule out the possibility $((b, D), (e_0, X_0)) \in F_{\rho'}$. This is so because if this were the case then $(e_0, X_0) \in E_{\rho'}$. Two applications of part (ii) of the induction hypothesis yield $(b, D) \in {}^\bullet(e_0, X_0)$ and $(e, X) \in {}^\bullet(b, D)$ in $N_{\rho'}$. But this would lead to the known contradiction $(e, X) \in E_{\rho'}$. Hence $((b, D), (e_0, X_0)) \notin F_{\rho'}$.

From the definition of F_ρ we can now conclude that $(e_0, X_0) = (e, X)$. This implies that $(b, D) \in c_{\rho'}$. If $(b, D) \in B_{\rho'}$ then we would once again, by part (ii) of the induction hypothesis, have the contradiction $(e, X) \in E_{\rho'}$. Hence $(b, D) \notin B_{\rho'}$. But then, by part (iii) of the induction hypothesis we now have $(b, D) \in c_\Lambda - B_\rho$. We yet again have a contradiction because $D = \{(e, X)\} \neq \emptyset$ and every member of c_Λ is of the form $(b', \emptyset)$.

Case 2 $(b, D) \in c_{\rho'} - X$ and $(b, D) \notin Y$.

We know that $(b, D) \in B_\rho$. Since $B_\rho = B_{\rho'} \cup X \cup Y$ we can deduce that $(b, D) \in B_{\rho'}$. Now $(b, D)^\bullet = \emptyset$ in $N_{\rho'}$ by the induction hypothesis. Since $(b, D) \notin X$ we now have $(b, D)^\bullet = \emptyset$ in N_ρ as well by the definition of F_ρ.

We have now established the first part of the lemma.

It is now easy to establish the second part of the lemma by appealing to Definition 3.1.

□

We now wish to show that N_ρ is a causal net. Recall that a causal net is a net $N' = (B', E', F')$ such that $\forall b \in B'.\ |{}^\bullet b|, |b^\bullet| \leq 1$ and $(F')^*$ is a partial ordering relation (over $B' \cup E'$).

Lemma 3.6

N_ρ is a causal net.

Proof By induction on $k = |\rho|$.

$k = 0$ Trivial.

$k > 0$ Let $\rho = \rho' e$ and assume as before that $c_{\rho'}, X$ and Y are as in Definition 3.1.

Consider $(b, D) \in B_\rho = B_{\rho'} \cup X \cup Y$.

If $(b, D) \in Y$ then $(b, D)^\bullet = \emptyset$ in N_ρ by part (i) of the previous lemma. If $(b, D) \in X$ then clearly $(b, D)^\bullet = \{(e, X)\}$ in N_ρ by Definition 3.1. because $(b, D)^\bullet = \emptyset$ or is undefined in $N_{\rho'}$ by part (i) of the previous lemma.

Suppose $(b, D) \in B_{\rho'} - X$. Then $|(b, D)^\bullet| \leq 1$ in $N_{\rho'}$ by the induction hypothesis. Moreover ${}^\bullet(e, X) = X$ in N_ρ by part (ii) of the previous lemma. Hence $|(b, D)^\bullet| \leq 1$ for each $(b, D) \in B_\rho$.

Now suppose that $(b, D) \in Y$. Recall that $B_\rho = B_{\rho'} \cup X \cup Y$. Then $D = \{(e, X)\}$ and by part (ii) of the previous lemma, ${}^\bullet(b, D) = \{(e, X)\}$.

Next suppose that $(b, D) \in X$. If $(b, D) \notin B_{\rho'}$, then $(b, D) \in c_{\rho'} - B_{\rho'}$. This implies that $D = \emptyset$ by part (iii) of the previous lemma and ${}^\bullet(b, D) = \emptyset$ in N_ρ by part (ii) of the previous lemma.

If $(b, D) \in X \cap B_{\rho'}$ then ${}^\bullet(b, D) = D$ by part (ii) of the previous lemma and $|D| \leq 1$ by the induction hypothesis. If $(b, D) \in B_{\rho'} - X$ then $|{}^\bullet(b, D)| \leq 1$ in $N_{\rho'}$ by the induction hypothesis.

We now wish to argue that $B_{\rho'} \cap Y = \emptyset$. So consider $(b, D) \in Y$. Then $D = \{(e, X)\}$. We know from the proof of the previous lemma that $(e, X) \notin E_{\rho'}$. But $(b, D) \in B_{\rho'}$ would imply by part(ii) of the previous lemma that $(e, X) \in E_{\rho'}$. Hence $(b, D) \notin B_{\rho'}$ and thus $B_{\rho'} \cap Y = \emptyset$. Hence $|{}^\bullet(b, D)| \leq 1$ in N_ρ also.

To show that $(F_\rho)^*$ is a partial ordering relation define *depth*: $B_\rho \cup E_\rho \rightarrow \mathbf{N}_0$ as follows:

$\forall (x, X) \in B_\rho \cup E_\rho$.

$$depth\,((x, X)) = \begin{cases} 0, \text{ if } (x, X) \in c_\Lambda, \\ 1 + max\{depth((y, Y)) \mid (y, Y) \in X\}, \\ \qquad\qquad \text{otherwise} \end{cases}$$

It is easy to verify by induction on $|\rho|$ that *depth* is a well-defined map.

Suppose $(x, X) F_\rho (y, Y)$. Then clearly *depth* $((y, Y)) >$ *depth* $((x, X))$. From this it follows easily that $(F_\rho)^*$ is anti-symmetric. Clearly $(F_\rho)^*$ is reflexive and transitive.

□

We shall show in two steps that our process definition agrees with the traditional one. In doing so we shall denote $(F_\rho)^*$ by $\leq_\rho$. An *anti-chain* of a p.o. is a set of mutually unordered elements.

Lemma 3.7
Let $\hat{c} \subseteq B_\rho$ be an anti-chain in N_ρ (under the p.o. relation $\leq_\rho$). Then there exists $\rho'' \in FS$ such that $N_{\rho''} \subseteq' N_\rho$ and $\hat{c} \subseteq c_{\rho''}$.

Proof By induction on $k = |\rho|$.

<u>$k = 0$</u> Clearly $\hat{c} = \emptyset \subseteq c_\Lambda$.

<u>$k > 0$</u> Let $\rho = \rho' e$ and $c_{\rho'}, X, Y$ be as usual as given in Definition 3.1.

Recall that $B_\rho = B_{\rho'} \cup X \cup Y$. If $\hat{c} \subseteq Y$ then $\hat{c} \subseteq c_\rho$ because $c_\rho = (c_{\rho'} - X) \cup Y$. We are then done by setting $\rho'' = \rho$.

If $\hat{c} \subseteq B_{\rho'}$ then we are done thanks to the induction hypothesis. Next note that $\forall (v, V) \in X$ and $\forall (v', V') \in Y$, $(v, V) <_\rho (e, X) <_\rho (v', V')$. Hence we cannot have both $\hat{c} \cap X \neq \emptyset$ and $\hat{c} \cap Y \neq \emptyset$.

<u>Case 1</u> $\hat{c} \cap X \neq \emptyset$.

Then $\hat{c} \subseteq B_{\rho'} \cup X$. Let $\hat{c_1} = \hat{c} \cap B_{\rho'}$. Clearly $\hat{c_1}$ is an anti-chain in $N_{\rho'}$. By the induction hypothesis, there exists $\rho'' \in FS$ such that $N_{\rho''} \subseteq' N_{\rho'}$ and $\hat{c}_1 \subseteq c_{\rho''}$.

Let $\hat{c_2} = \hat{c} - \hat{c_1}$. Then $\hat{c_2} \subseteq X - B_{\rho'}$. But $X \subseteq c_{\rho'}$. Hence $\hat{c_2} \subseteq c_{\rho'} - B_{\rho'}$. By part (iii) of Lemma 3.5 we then have
$\hat{c_2} \subseteq c_\Lambda - B_{\rho'}$. Since $N_{\rho''} \subseteq' N_{\rho'}$ we know that $B_{\rho''} \subseteq B_{\rho'}$ and this implies $\hat{c_2} \subseteq c_\Lambda - B_{\rho''}$. Once again by part (iii) of Lemma 3.5, $\hat{c_2} \subseteq c_{\rho''} - B_{\rho''}$. Thus $\hat{c_2} \subseteq c_{\rho''}$ and this establishes $\hat{c} \subseteq c_{\rho''}$. Clearly $N_{\rho'} \subseteq' N_\rho$ and hence $N_{\rho''} \subseteq' N_\rho$.

<u>Case 2</u> $\hat{c} \cap Y \neq \emptyset$.

Let $\hat{c_1} = \hat{c} - Y$. We will first show that $\hat{c_1} \cup X$ is also an anti-chain in N_ρ. To see this, first note that ${}^\bullet(e, X) = X$ in N_ρ by part (ii) of Lemma 3.5. By the previous lemma, N_ρ is a causal net. Moreover $\leq_\rho = F_\rho^*$. Hence X is an anti-chain in N_ρ. $\hat{c_1}$ is an anti-chain in N_ρ because $\hat{c}$ is an anti-chain in N_ρ. Suppose that $(b_1, D_1) \in \hat{c}$, and $(b_2, D_2) \in X$ such that $(b_1, D_1) <_\rho (b_2, D_2)$ or $(b_2, D_2) <_\rho (b_1, D_1)$.

Since $\hat{c} \cap Y \neq \emptyset$, there exists $(b_3, D_3) \in \hat{c} \cap Y$. If $(b_1, D_1) <_\rho (b_2, D_2)$ then $(b_1, D_1) <_\rho (b_3, D_3)$ also because as observed earlier, $(b_2, D_2) <_\rho (e, X) <_\rho (b_3, D_3)$. This is a contradiction because $\hat{c}$ is supposed to be an anti-chain.

If $(b_2, D_2) < (b_1, D_1)$ then there exists $(b_3, D_3) \in Y$ such that $(b_3, D_3) \leq_\rho (b_1, D_1)$. This is because $(b_2, D_2)^\bullet = \{(e, X)\}$ and $(e, X)^\bullet = Y$ in N_ρ. Moreover N_ρ is a causal net and $\leq_\rho = F_\rho^*$. The case $(b_3, D_3) = (b_1, D_1)$ is ruled out because $(b_1, D_1) \in \hat{c}_1 = \hat{c} - Y$. The case $(b_3, D_3) < (b_1, D_1)$ is ruled out because $(b_3, D_3) \in Y \subseteq c_\rho$ and hence by part (i) of Lemma 3.5, $(b_3, D_3)^\bullet = \emptyset$ in N_ρ.

Thus indeed $\hat{c}_1 \cup X$ is an anti-chain in N_ρ. We now have the situation considered in the previous case. Hence there exists $\rho'' \in FS$ such that $N_{\rho''} \subseteq' N_\rho$ and $\hat{c}_1 \cup X \subseteq c_{\rho''}$.

Let $c_{in}[\![\rho'' > c''$ and $c_{in}[\![\rho' > c'$. We know that e is enabled at c'. We shall show that e is enabled at c'' also. By Lemma 3.3, $\varphi_{\rho'}(c_{\rho'}) = c'$. Hence $\varphi_{\rho'}(X) = {}^\bullet e$ by the definition of X. Since $X \subseteq c_{\rho''}$ we now have ${}^\bullet e \subseteq \varphi_{\rho''}(c_{\rho''})$. In other words, ${}^\bullet e \subseteq c''$. But then $\mathcal{N}$ is contact-free. Hence $e^\bullet \cap c'' = \emptyset$. Thus $\rho'' e$ is also a firing sequence. It is now easy to check, using Definition 3.1, that $N_{\rho'' e} \subseteq' N_\rho$. It is also easy to check that $\hat{c}_1 \cup Y \subseteq c_{\rho'' e}$. Since $\hat{c} \subseteq \hat{c}_1 \cup Y$, we are done. □

We are now prepared to compare our process definition with the "traditional" definition. Notice that we have already shown that N_ρ is a causal net for each $\rho \in FS$.

Theorem 3.8

(i) $\forall (e, X) \in E_\rho.\ \varphi_\rho({}^\bullet(e, X)) = {}^\bullet\varphi_\rho((e, X))$ *and* $\varphi_\rho((e, X)^\bullet) = (\varphi_\rho((e, X)))^\bullet$.

(ii) If $\hat{c} \subseteq B_\rho$ *is an anti-chain in* N_ρ *then there exists* $c \in [c_{in} >$ *in* $\mathcal{N}$ *such that* $\varphi_\rho(\hat{c}) \subseteq c$.

(iii) $\forall (b, D_1), (b, D_2) \in B_\rho.\ (b, D_1) \leq_\rho (b, D_2) \vee (b, D_2) \leq_\rho (b, D_1)$.

Proof

(i) Follows easily by induction on $|\rho|$ using Lemma 3.5.

(ii) Follows easily from the previous lemma and Lemma 3.3.

(iii) Suppose that $(b, D_1), (b, D_2) \in B_\rho$ such that $\{(b, D_1), (b, D_2)\}$ is an anti-chain. By virtue of the previous lemma, it involves no loss of generality to assume that $(b, D_1), (b, D_2) \in c_\rho$. We now proceed by induction on $k = |\rho|$.

<u>$k = 0$</u> This is impossible because in this case we would have $B_\rho = \emptyset$.

<u>$k > 0$</u> Let $\rho = \rho' e$ and $c_{\rho'}$ and X and Y be as in Definition 3.1. Then $c_\rho = (c_{\rho'} - X) \cup Y$. If $(b, D_1), (b, D_2) \in Y$ then clearly $D_1 = D_2 = \{(e, X)\}$. If $(b, D_1), (b, D_2) \in c_{\rho'} - X$ then $D_1 = D_2$ by the induction hypothesis. So suppose that $(b, D_1) \in c_{\rho'} - X$ and $(b, D_2) \in Y$. Let $\varphi_{\rho'}(c_{\rho'}) = c'$. Then by Lemma 3.3, e is enabled at c' in $\mathcal{N}$. But $(b, D_2) \in Y$ implies that $b \in e^\bullet$ by the definition of Y. And $(b, D_1) \in c_{\rho'} - X$ implies that $b \in c'$ also. This is a contradiction because e is supposed to be enabled at c'.

□

We can now turn our attention to proving the main result of this section. Once again, we will first establish a number of intermediate results. These results will come in handy also in the next section .

Lemma 3.9
If $\rho \sim \rho'$ then $N_\rho = N_{\rho'}$.

Proof First suppose that $\gamma \in FS$ and $(e_1, e_2) \in I$ such that $\gamma e_1 e_2, \gamma e_2 e_1 \in FS$. We claim that $N_{\gamma e_1 e_2} = N_{\gamma e_2 e_1}$. To see this, let $E_{\gamma e_1} - E_\gamma = \{(e_1, X_1)\}$ and $(e_1, X_1)^\bullet = Y_1$ in $N_{\gamma e_1}$ and $E_{\gamma e_1 e_2} - N_{\gamma e_1} = \{(e_2, X_2)\}$ and $(e_2, X_2)^\bullet = Y_2$ in $N_{\gamma e_1 e_2}$. Since $(e_1, e_2) \in I$ it follows from Definition 3.1 and part (i) of Theorem 3.8 that $(X_1 \cup Y_1) \cap (X_2 \cup Y_2) = \emptyset$. It is now easy to verify – using yet again Definition 3.1 – that $N_{\gamma e_1 e_2} = N_{\gamma e_2 e_1}$ and also $c_{\gamma e_1 e_2} = c_{\gamma e_2 e_1}$. We can use this now to prove our result.

So now suppose that $\rho \sim \rho'$. Then there exist $\rho_1, \ldots, \rho_n \in FS$ such that $\rho = \rho_1, \rho_n = \rho'$ and for $1 \leq i < n$, $\rho_i \dot{\sim} \rho_{i+1}$. Proceeding by induction on n, the result is clearly true if $n = 1$. Hence assume that $n > 1$. Then $\rho_1 \dot{\sim} \rho_2$ and this implies the existence of $\gamma, \gamma' \in E^*$ and $(e_1, e_2) \in I$ such that $\rho_1 = \gamma e_1 e_2 \gamma'$ and $\rho_2 = \gamma e_2 e_1 \gamma'$. By the argument presented above, $N_{\gamma e_1 e_2} = N_{\gamma e_2 e_1}$ and $c_{\gamma e_1 e_2} = c_{\gamma e_2 e_1}$. Hence $N_{\rho_1} = N_{\rho_2}$. The required result now follows by the induction hypothesis.

□

Lemma 3.10
$N_\rho \subseteq' N_{\rho'} \Longleftrightarrow E_\rho \subseteq E_{\rho'}$.

Proof

$\Rightarrow$ Trivial.

$\Leftarrow$ Assume that $E_\rho \subseteq E_{\rho'}$. By part (ii) of Lemma 3.5, it suffices to show that $B_\rho \subseteq B_{\rho'}$. So consider $(b, D) \in B_\rho$. Since N_ρ is a net, ${}^\bullet(b, D) \cup (b, D)^\bullet \neq \emptyset$ in N_ρ. Suppose that $(e', X') \in^\bullet (b, D)$. Then N_ρ being a causal net, we have ${}^\bullet(b, D) = \{(e', X')\}$. Clearly $(e', X') \in E_\rho$. From Definition 3.1 it follows that $b \in (e')^\bullet$ (in $\mathcal{N}$) and $D = \{(e', X')\}$. Once again from Definition 3.1 and the fact that $E_\rho \subseteq E_{\rho'}$, it follows that $(b, D) \in B_{\rho'}$.

If $(e', X') \in (b, D)^\bullet$ in N_ρ, then $b \in {}^\bullet(e')$ (in $\mathcal{N}$) and $(b, D) \in X'$. Once again from Definition 3.1 and the fact that $E_\rho \subseteq E_{\rho'}$, it follows that $(b, D) \in B_{\rho'}$.

□

Lemma 3.11
Let $(e, X) \in E_\rho$. Then (e, X) is a maximal event in N_ρ (under $\leq_\rho$) iff there exists $\rho' e \in FS$ such that $\rho \sim \rho' e$ and $\{(e, X)\} = E_{\rho' e} - E_{\rho'}$.

Proof

$\Leftarrow$ Suppose that $\rho \sim \rho' e$ and $(e, X) \in E_{\rho' e} - E_{\rho'}$. Clearly from Definition 3.1, it follows that (e, X) is a maximal event in $N_{\rho' e}$. But by Lemma 3.9, $\rho \sim \rho' e$ implies that $N_\rho = N_{\rho' e}$. Hence (e, X) is a maximal event in N_ρ also.

$\Rightarrow$ Since $(e, X) \in E_\rho$, ρ can be expressed as $\rho = \rho_1 e \rho_2$ such that $E_{\rho_1 e} - E_{\rho_1} = \{(e, X)\}$. We now proceed by induction on $k = |\rho_2|$.

$\underline{k = 0}$ Set $\rho' = \rho_1$.

$\underline{k > 0}$ Let $\rho = \rho_1 e e' \rho_2'$. In other words $\rho_2 = e' \rho_2'$ for some $e' \in E$ and $\rho_2' \in E^*$. From the fact that (e, X) is a maximal event in N_ρ, it is easy to deduce that $(e, e') \in I$. Hence $\rho \sim \rho_1 e' e \rho_2'$. The required result now follows from the induction hypothesis and Lemma 3.9.

□

We are at last ready to prove the main result of this section.

Theorem 3.12
$(T, \sqsubseteq)$ and $(P, \subseteq')$ are isomorphic posets. In fact, $f : T \rightarrow P$ given by

$$\forall \rho \in FS. f([\rho]) = N_\rho$$

is an isomorphism.

Proof First note that if $\rho, \rho' \in FS$ such that $\rho \sim \rho'$ then $N_\rho = N_{\rho'}$ by Lemma 3.9. Hence f is well-defined.

f is obviously surjective. To verify that it is injective, assume that $\rho, \rho' \in FS$ such that $N_\rho = N_{\rho'}$. We must show that $\rho \sim \rho'$. By part (iv) of Lemma 3.5, it is clear that $|\rho| = |\rho'|$. We now proceed by induction on $k = |\rho|$.

$\underline{k = 0}$ Clearly $\rho = \rho' = \Lambda$.

$\underline{k > 0}$ Let $\rho = \rho_1 e_1$ and $\rho' = \rho'_1 e'_1$.

Case 1 $\underline{e_1 = e'_1}$.
Let $e_1 = e'_1 = e$. Furthermore, let $E_\rho - E_{\rho_1} = (e, X)$ and $E_{\rho'} - E_{\rho'_1} = (e, X')$. By Lemma 3.11 it follows that both (e, X) and (e, X') are maximal events in $N_\rho = N_{\rho'}$. Since N_ρ is a causal net and ${}^\bullet(e, X) = X$ and ${}^\bullet(e, X') = X'$, we can conclude that $X \cap X' = \emptyset$ in case $X \neq X'$. But if $X \neq X'$ and thus $X \cap X' = \emptyset$ then we would have a contradiction to part (iii) of Theorem 3.8. This is because, for each $b \in^\bullet e$, we will have some $(b, D) \in X$ and some $(b, D') \in X'$.

Moreover by virtue of the fact that (e, X) and (e, X') are maximal events, we can conclude that $X \cup X'$ is an anti-chain in N_ρ. Thus we must have $X = X'$ so that $(e, X) = (e, X')$.

This implies that $E_\rho - E_{\rho_1} = E_{\rho'} - E_{\rho'_1}$. Hence by Lemma 3.10, $N_{\rho_1} = N_{\rho'_1}$ and this in turn implies that $\rho_1 \sim \rho'_1$ by induction hypothesis. Clearly $\rho_1 e \sim \rho'_1 e$ so that $\rho \sim \rho'$.

Case 2 $\underline{e_1 \neq e'_1}$.
Let $E_\rho - E_{\rho_1} = \{(e_1, X_1)\}$ and $E_{\rho'} - E_{\rho'_1} = \{(e'_1, X'_1)\}$. By Lemma 3.11, (e'_1, X'_1) is a maximal event in $N_{\rho'}$. Since $N_\rho = N_{\rho'}$, it is a maximal event in N_ρ as well. Hence once again by Lemma 3.11, there exists $\rho'' e'_1 \in FS$ such that $\rho \sim \rho'' e'_1$ and $N_{\rho'' e'_1} = N_{\rho'}$. Since $\rho' = \rho'_1 e'_1$ we have arrived at the situation considered in case 1.

This establishes that f is a bijection.

Now suppose that $\rho, \rho' \in FS$ such that $[\rho] \sqsubseteq [\rho']$. Then there exists $\rho'' \in [\rho']$ such that ρ is a prefix of ρ''. Clearly, by Definition 3.1, $N_\rho \subseteq' N_{\rho''}$. But then $N_{\rho''} = N_{\rho'}$ by Lemma 3.9. Hence $N_\rho \subseteq' N_{\rho'}$.

Next assume that $\rho, \rho' \in FS$ are such that $N_\rho \subseteq' N_{\rho'}$. By part (iv) of Lemma 3.5, $|\rho| \leq |\rho'|$. The proof is by induction on $k = |\rho'|$.

$\underline{k = 0}$ Clearly $\rho = \rho' = \Lambda$.

$\underline{k > 0}$ Let $\rho' = \rho_1 e$. Let $E_{\rho_1 e} - E_{\rho_1} = \{(e, X)\}$. Suppose that $(e, X) \notin E_\rho$. Then $E_\rho \subseteq E_{\rho_1}$. By Lemma 3.10, this implies that $N_\rho \subseteq' N_{\rho_1}$. From the induction hypothesis, we can conclude that $[\rho] \sqsubseteq [\rho_1]$. From this we can further conclude that $[\rho] \sqsubseteq [\rho_1 e] = [\rho']$.

So assume that, $(e, X) \in E_\rho$. (e, X) is a maximal event in $N_{\rho'}$ by Lemma 3.11. Since $N_\rho \subseteq' N_{\rho'}$, it follows that (e, X) is a maximal event in N_ρ as well. Once again by Lemma 3.11, we can find $\rho'' e \in FS$ such that $\rho \sim \rho'' e$ and $E_{\rho'' e} - E_{\rho''} = \{(e, X)\}$. Since $\rho \sim \rho'' e$ we must have $N_\rho = N_{\rho'' e}$. Since $E_\rho \subseteq E_{\rho'}$ we can now conclude that $E_{\rho''} \subseteq E_{\rho_1}$ (recall that $\rho' = \rho_1 e$ and that $E_{\rho'} - E_{\rho_1} = \{(e, X)\}$). This implies that $N_{\rho''} \subseteq' N_{\rho_1}$ and by the induction hypothesis we then have $[\rho''] \sqsubseteq [\rho_1]$. Finally this lets us to conclude that $[\rho'' e] \sqsubseteq [\rho_1 e]$ and since $\rho \sim \rho'' e$ and $\rho' = \rho_1 e$, we are done.

□

4 The Labelled Event Structure of $\mathcal{N}$

Our method of constructing the finite processes of $\mathcal{N}$ leads to a simple definition of the unfolding of $\mathcal{N}$.

Definition 4.1 *The unfolding of $\mathcal{N}$ – denoted as $UF_{\mathcal{N}}$ – is the quadruple $UF_{\mathcal{N}} = (\hat{B}, \hat{E}, \hat{F}, \hat{\varphi})$, where (keeping in mind that $N_\rho = (B_\rho, E_\rho, E_\rho)$ for each $\rho \in FS$ as specified in Definition 3.1.)*

(i) $\hat{B} = \bigcup_{\rho \in FS} B_\rho$,

(ii) $\hat{E} = \bigcup_{\rho \in FS} E_\rho$,

(iii) $\hat{F} = \bigcup_{\rho \in FS} F_\rho$, *and*

(iv) $\forall (x, X) \in \hat{B} \cup \hat{E}.\ \hat{\varphi}(x, X)) = x$.

□

As an example, part of the unfolding of $\mathcal{N}_2$ is shown in Figure 5.

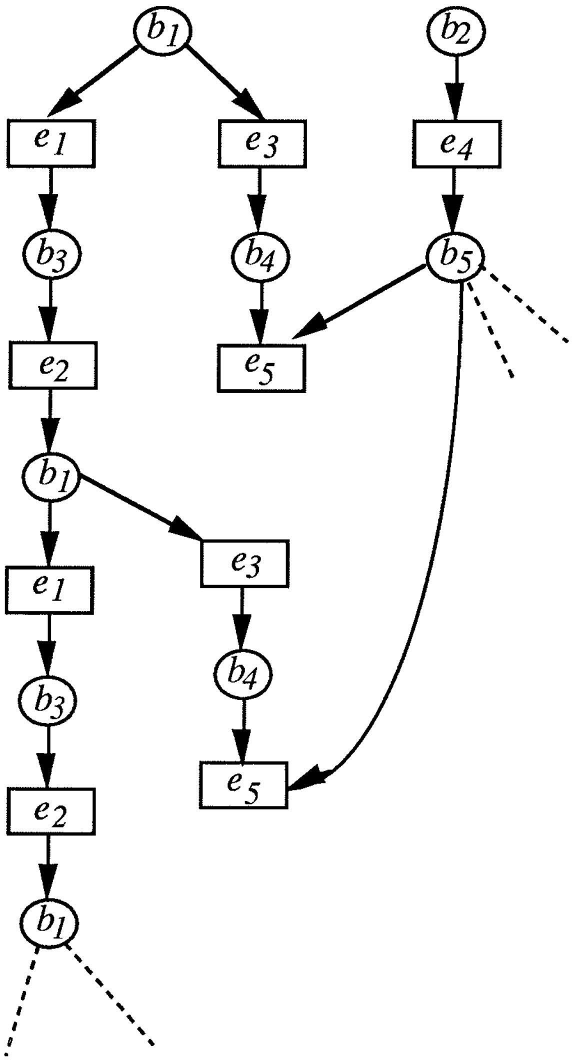

Figure 5

$\hat{N} = (\hat{B}, \hat{E}, \hat{F})$ is called the *underlying net* of $UF_{\mathcal{N}}$. It is easy to check that $\hat{N}$ is indeed a net. $\hat{\varphi} : \hat{B} \cup \hat{E} \to B \cup E$ so that $UF_{\mathcal{N}}$ is a labelled net. Our first task is to show that $\hat{N}$ is an occurrence net in the sense of Nielsen et al. (1981). Before doing so it will be convenient to adopt some notations concerning posets.

Let $PO = (X, \leq)$ be a poset and $Y \subseteq X$.
Then $\uparrow Y = \{x \in X \mid \exists y \in Y.y \leq x\}$. $\downarrow Y = \{x \in X \mid \exists y \in Y.\ x \leq y\}$. In case $Y = \{y\}$ is a singleton we shall write $\uparrow y$ ($\downarrow y$) instead of $\uparrow \{y\}$ ($\downarrow \{y\}$). For $Y \subseteq X$ we say that Y is *compatible* (bounded) – and this is denoted by $Y \uparrow$ – in case there exists $x \in$ X such that $y \leq x$ for every

$y \in Y$. If $Y = \{y_1, y_2\}$ then we shall write $y_1 \uparrow y_2$ instead of $\{y_1, y_2\} \uparrow$. $y_1 \not\uparrow y_2$ will denote the negation of $y_1 \uparrow y_2$. Y is said to be *pair-wise compatible* in case $y_1 \uparrow y_2$ for every $y_1, y_2 \in Y$.

Recall that an *occurrence net* is a net $N' = (B', E', F')$ such that

(i) $\forall b' \in B'. |{}^\bullet b'| \leq 1$,

(ii) $(F')^*$ is a partial ordering relation, and

(iii) $\forall e_1, e_2 \in E'. [e_1 \neq e_2 \wedge {}^\bullet e_1 \cap {}^\bullet e_2 \neq \emptyset \Rightarrow \uparrow e_1 \cap \uparrow e_2 = \emptyset]$.

Here $\uparrow e_1$ and $\uparrow e_2$ are assumed to be defined w.r.t. the partial ordering relation $\leq' = (F')^*$.

Through the rest of this section we shall assume that $\hat{N} = (\hat{B}, \hat{E}, \hat{F})$ is the underlying net of the unfolding of $\mathcal{N}$ as specified in Definition 3.1. We set $\hat{X} = \hat{B} \cup \hat{E}$ and $\hat{\leq} = (\hat{F})^*$. For each $\rho \in FS$ we shall assume $N_\rho = (B_\rho, E_\rho, F_\rho)$ as specified in Definition 3.1. We set $X_\rho = B_\rho \cup E_\rho$ and $\leq_\rho = (F_\rho)^*$. We shall show in two steps that $\hat{N}$ is an occurrence net .

Lemma 4.2 *Let* $(x, X), (y, Y) \in \hat{X}$.

(i) $(x, X)\hat{F}(y, Y)$ *iff* $(x, X) \in Y$,

(ii) $(x, X)\hat{\leq}(y, Y)$ *iff* $\forall \rho \in FS. [(y, Y) \in X_\rho \Rightarrow (x, X) \in X_\rho \wedge (x, X) \leq_\rho (y, Y)]$.

Proof By the definition of $\hat{F}$ we know that $(x, X)\hat{F}(y, Y)$ iff there exists $\rho \in FS$ such that $(x, X)F_\rho(y, Y)$. By part (ii) of Lemma 3.5 we then have that $(x, X)F_\rho(y, Y)$ iff $(x, X) \in Y$.

To prove the second part note that $(x, X)\hat{\leq}(y, Y)$ iff there exist $(x_1, X_1), (x_2, X_2), \ldots, (x_n, X_n) \in \hat{X}$ $(n \geq 1)$ such that $(x, X) = (x_1, X_1)$, $(x_n, X_n) = (y, Y)$ and for $1 \leq i < n$, $(x_i, X_i)\hat{F}(x_{i+1}, X_{i+1})$. We now do induction on n.

<u>$n = 1$</u> There is nothing to prove.

$\underline{n > 1}$ By the first part of the lemma, $(x_1, X_1)\hat{F}(x_2, X_2)$ iff $(x_1, X_1) \in X_2$. But $(x_1, X_1) \in X_2$ iff $\forall \rho \in FS.\ (x_2, X_2) \in X_\rho$ implies that $(x_1, X_1)F_\rho(x_2, X_2)$ which in turn implies that $(x_1, X_1) \in X_\rho$ as well. This follows once again from part (ii) of Lemma 3.5. The required result now follows from the induction hypothesis.

□

Theorem 4.3
$\hat{N} = (\hat{B}, \hat{E}, \hat{F})$ is an occurrence net.

Proof Let $(b, D) \in \hat{B}$. Suppose that $(e_1, X_1)\hat{F}(b, D)$ and $(e_2, X_2)\hat{F}(b, D)$. Then by the first part of the previous lemma, $(e_1, X_1), (e_2, X_2) \in D$. By the definition of $\hat{B}$ we know that, for some $\rho \in FS, (b, D) \in B_\rho$. By part (ii) of Lemma 3.5, ${}^\bullet(b, D) = D$ in N_ρ. But then $|D| \leq 1$ because N_ρ is a causal net. Hence $(e_1, X_1) = (e_2, X_2)$.

Clearly $\hat{\leq}$ is reflexive and transitive. So assume that $(x, X), (y, Y) \in \hat{X}$ such that $(x, X)\hat{\leq}(y, Y)$ and $(y, Y)\hat{\leq}(x, X)$.

Let $\rho \in FS$ be such that $(y, Y) \in X_\rho$. Then by part (ii) of Lemma 3.2, $(x, X) \in X_\rho$ and $(x, X) \leq_\rho (y, Y)$. Since $(x, X) \in X_\rho$, $(y, Y)\hat{\leq}(x, X)$ would imply once again by the second part of the previous lemma that $(y, Y) \leq_\rho (x, X)$. Hence $(x, X) = (y, Y)$, because N_ρ is a causal net, and so $\leq_\rho$ is anti-symmetric.

Now suppose that $(b, D) \in \hat{B}$ and $(e_1, X_1), (e_2, X_2) \in \hat{E}$ are such that $(e_1, X_1) \neq (e_2, X_2)$ and $(b, D)\hat{F}(e_1, X_1)$ and $(b, D)\hat{F}(e_2, X_2)$. We must prove that $\uparrow (e_1, X_1) \cap \uparrow (e_2, X_2) = \emptyset$. Suppose $(y, Y) \in \uparrow (e_1, X_1) \cap \uparrow (e_2, X_2)$. Let $\rho \in FS$ such that $(y, Y) \in X_\rho$.

Then by part (ii) of the previous lemma, we have $(b, D), (e_1, X_1), (e_2, X_2) \in X_\rho$. By part (i) of the previous lemma and part (ii) of Lemma 3.5 we would then have $(b, D)F_\rho(e_1, X_1)$ and $(b, D)F_\rho(e_2, X_2)$. This is a contradiction because N_ρ is a causal net. □

We can now give the event structure semantics of $\mathcal{N}$. First we note that the definition of an occurrence net allows one to specify a conflict relation between the elements of an occurrence net in a natural way. Instead of giving the general definition, we shall straightaway specify the conflict relation for the occurrence net $\hat{N} = (\hat{B}, \hat{E}, \hat{F})$.

The *conflict relation associated with* $\hat{N}$, denoted by $\hat{\#}$, is the least subset of $\hat{X} \times \hat{X}$ given by:

- $\forall \hat{e_1}, \hat{e_2} \in \hat{E}.\ [\hat{e_1} \neq \hat{e_2} \wedge {}^\bullet\hat{e_1} \cap {}^\bullet\ \hat{e_2} \neq \emptyset\ (in\ \hat{N}) \Rightarrow \hat{e_1}\hat{\#}\hat{e_2}]$,
- $\forall \hat{x}, \hat{y}, \hat{z} \in \hat{X}.\ [\hat{x}\hat{\#}\hat{y}\hat{\leq}\hat{z} \Rightarrow x\hat{\#}\hat{z}.]$

Definition 4.4 *The* labelled event structure *of* $\mathcal{N}$ *– denoted* $ES_{\mathcal{N}}$ *– is the quadruple* $ES_{\mathcal{N}} = (\hat{E}, \leq, \#, \varphi)$ *where (recall that* $UF_{\mathcal{N}} = (\hat{B}, \hat{E}, \hat{F}, \hat{\varphi})$*)*

(i) $\leq$ *is* $\hat{\leq}$ *(*$= (\hat{F})^*$*) restricted to* $\hat{E} \times \hat{E}$*,*

(ii) $\#$ *is* $\hat{\#}$ *(the conflict relation associated with* $\hat{N}$*) restricted to* $\hat{E} \times \hat{E}$*, and*

(iii) φ *is* $\hat{\varphi}$ *restricted to* $\hat{E}$*.*

□

$(\hat{E}, \leq, \#)$ is called the *underlying event structure* of $ES_{\mathcal{N}}$. By abuse of notation we shall denote this triple also as $ES_{\mathcal{N}}$.

Recall that an event structure is a triple $ES = (E', \leq', \#')$ where

(i) E' is a set of events,

(ii) $\leq' \subseteq E' \times E'$ is a partial ordering relation called the *causality relation* of ES,

(iii) $\#' \subseteq E' \times E'$ is an irreflexive and symmetric relation called the *conflict relation* of ES, and

(iv) $\#'$ is required to be "inherited" via $\leq'$ in the sense that

$$\forall e_1, e_2, e_3 \in E'[e_1 \#' e_2 \leq' e_3 \Rightarrow e_1 \#' e_3].$$

From the fact that $\hat{N}$ is an occurrence net it is easy to deduce that $ES_{\mathcal{N}}$ is indeed a (labelled) event structure. An initial portion of $ES_{\mathcal{N}_2}$is shown in Figure 6. The sguiggly lines represent the "minimal" elements of the conflict relation. The remaining elements of the conflict relation are precisely those that can be deduced using the axiom that conflict is inherited via the causality relation.

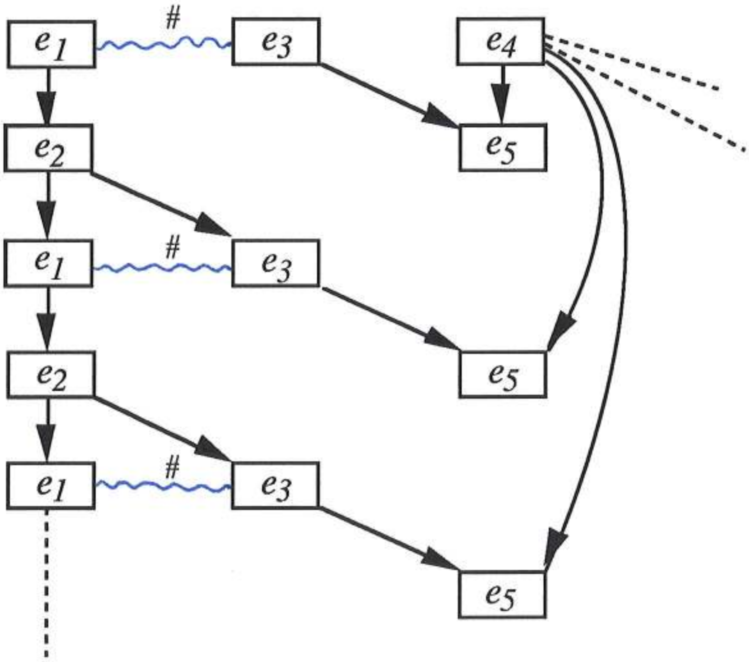

Figure 6

The states of an event structure are called *configurations*. A configuration represents a state of affairs that has been reached after the occurrence of a set of events.

For an event to occur, all the events that lie in its "past" (as specified by the causality relation of the relation) must have occurred. No two events which are in conflict can both have occurred in a state of affairs represented by a configuration. These considerations underly the definition of configurations.

Definition 4.5
Let $ES = (E', \leq', \#')$ be an event structure.

(i) Let $d \subseteq E'$. Then d is a *configuration* of ES iff $d = \downarrow d$ (left-closed) and $\#' \cap (d \times d) = \emptyset$ (conflict-free),

(ii) C_{ES} is the set of configurations of ES, and

(iii) C_{ES}^{fin} is the set of *finite* configurations (i.e. each member of C_{ES}^{fin} is a finite set) of ES.

□

Let $d_1 \subseteq d_2$ with $d_1, d_2 \in C_{ES}$. Then the state d_1 is "earlier" than d_2. In other words set inclusion is the natural causality relation over configurations.

For $ES_{\mathcal{N}}$, we let $\hat{C}_{\mathcal{N}}$ denote the set of *finite configurations* of $ES_{\mathcal{N}}$, the event structure associated with $\mathcal{N}$. We can at last establish the main result of this paper; the trace semantics of $\mathcal{N}$ as represented by $(T, \sqsubseteq)$ and the event structure semantics of $\mathcal{N}$ as represented by $ES_{\mathcal{N}} = (\hat{E}, \leq, \#)$ "agree" with each other.

Theorem 4.6
$(P, \subseteq')$, and $(\hat{C}_{\mathcal{N}}, \subseteq)$ are isomorphic posets. In fact $g : P \rightarrow \hat{C}_{ES}$ given by:

$$\forall \rho \in FS.\ \ g(N_\rho = (B_\rho, E_\rho, F_\rho)) = E_\rho$$

is an isomorphism.

Proof We shall first show that g is well-defined. Let $\rho \in FS$. Then we must show that E_ρ is left-closed and conflict-free in $ES_{\mathcal{N}} = (\hat{E}, \leq, \#)$. Suppose that $(e, X) \in E_\rho$ and $(e', X') \in \hat{E}$ such that $(e', X') \leq (e, X)$. Then $(e', X') \hat{\leq} (e, X)$ in $\hat{N}$. By part (ii) of Lemma 4.2, we then have $(e', X') \in E_\rho$ as well. Hence E_ρ is left-closed.

Suppose that $(e_1, X_1), (e_2, X_2) \in E_\rho$ such that $(e_1, X_1) \# (e_2, X_2)$. Then $(e_1, X_1) \hat{\#} (e_2, X_2)$ in $\hat{N}$. By definition of $\hat{\#}$ it follows that there exist $(e_1', X_1'), (e_2', X_2') \in \hat{E}$ and $(b, D) \in \hat{B}$ such that the following condition are satisfied:

(i) $(e_1', X_1') \hat{\leq} (e_1, X_1)$ and $(e_2', X_2') \hat{\leq} (e_2, X_2)$,

(ii) $(e_1', X_1') \neq (e_2', X_1')$, and

(iii) $(b, D) \hat{F} (e_1', X_1')$ and $(b, D) \hat{F} (e_2', X_2')$.

Since $(e_1, X_1), (e_2, X_2) \in E_\rho$, it follows once again from part (ii) of Lemma 3.2 that $(b, D)F_\rho(e_1', X_1')$ and $(b, D)F_\rho(e_2', X_2')$. This is a contradiction because N_ρ is a causal net. Thus g is well-defined.

g is clearly 1-1. We must argue that g is onto. So consider $d \in \hat{C}_{\mathcal{N}}$. The proof is by induction on $k = |d|$.

$\underline{k = 0}$ Then $g(N_\lambda) = \emptyset$.

$\underline{k > 0}$ Let (e, X) be a maximal element in d under $\leq$.
Let $d' = d - \{(e, X)\}$. By the induction hypothesis, there exists $\rho' \in FS$ such that $g(N_{\rho'}) = d'$. In otherwords, $E_{\rho'} = d'$.

Claim $X \subseteq c_{\rho'}$ (recall Definition 3.1).

Proof of claim. Let $(b, D) \in X$. If $D = \emptyset$ then $(b, \emptyset) \in c_\Lambda$. Hence if $(b, \emptyset) \notin c_{\rho'}$, then there exists $(e', X') \in E_{\rho'}$ such that $(b, \emptyset) \in X'$. This follows from Definition 3.1. But this would imply, by part (i) of Lemma 4.2, that $(b, \emptyset)\hat{F}(e', X')$ and $(b, \emptyset)\hat{F}(e, X)$. Clearly $(e, X) \neq (e', X')$ because $d' = E_{\rho'} = d - \{(e, X)\}$. Hence $(e', X')\hat{\#}(e, X)$ which in turn implies that $(e', X')\#(e, X)$. This is a contradiction because d, by hypothesis, is conflict-free.

If $D \neq \emptyset$ then $|D| = 1$, because $\hat{N}$ is an occurrence net. Let $D = \{(e', X')\}$, then $(e', X')\hat{F}(b, D)\hat{F}(e, X)$. Since d is left-closed, this implies that $(e', X') \in E_{\rho'}$. This in turn implies that $(b, D) \in B_{\rho'}$. Clearly $\{(b, D)\}$ is an anti-chain in $N_{\rho'}$. Hence by Lemma 3.7, there exists $\rho'' \in FS$ such that $N_{\rho''} \sqsubseteq' N_{\rho'}$ and $(b, D) \in c_{\rho''}$. From Theorem 3.12, we know that $[\rho''] \sqsubseteq [\rho']$. Hence without loss of generality we can assume, using Theorem 3.12 once again, that ρ'' is a prefix of ρ'. If we start from $N_{\rho''}$ and follow the construction of $N_{\rho'}$ then according to Definition 3.1, $(b, D) \notin c_{\rho'}$ just in case there exists $(e'', X'') \in E_{\rho'} - E_{\rho''}$ such that $(b, D) \in X''$. As before, this would imply that $(e'', X'')\#(e, X)$ which contradicts the fact that d is conflict-free.

Thus $(b, D) \in c_{\rho'}$ and consequently $X \subseteq c_{\rho'}$. Let $c_{in}[\![\rho' > c'$ in $\mathcal{N}$. Then from part (i) of Theorem 3.12, Lemma 3.3, and the definitions of the various labelling functions it follows that ${}^\bullet e \subseteq c'$. Since $\mathcal{N}$ is contact-free this implies that e is enabled at c'. In other words, $\rho' e$ is a firing sequence of $\mathcal{N}$. It is now routine to verify that $E_{\rho' e} = d$.

Let $\rho, \rho' \in FS$. Then according to Lemma 3.10, $N_\rho \subseteq' N_{\rho'}$ iff $E_\rho \subseteq E_{\rho'}$. This completes the proof.

□

Corollary 4.7 *$(T, \sqsubseteq)$ and $(\hat{C}_{\mathcal{N}}, \subseteq)$ are isomorphic posets.*

Proof Follows at once from Theorem 3.12 and Theorem 4.6.

□

We have related $(T, \sqsubseteq)$ to $(\hat{C}_{\mathcal{N}}, \subseteq)$ rather than to $(\hat{E}, \leq, \#)$ for technical convenience. It turns out that $(\hat{E}, \leq, \#')$ and $(\hat{C}_{\mathcal{N}}, \subseteq)$ are in some sense "equivalent"representations, one can smoothly go back and forth between these two structures.

To bring this out we need to introduce some additional notions concerning posets. Let $PO = (X, \leq)$ be a poset. Then for $Y \subseteq X, \sqcup Y$ will denote the l.u.b. of Y in PO if it exists. $p \in X$ is called a *prime element* iff for every $Y \subseteq X$, s.t. $\sqcup Y$ exists $p \leq \sqcup Y$ implies that $p \leq y$ for some $y \in Y$. Let PR denote the set of prime elements of PO. Then PO is *prime algebraic* iff $\forall x \in X,\ x = \sqcup\{p \mid p \in PR \text{ and } p \leq x\}$. Next we need the notions of coherence and finite coherence. $PO = (X, \subseteq)$ is said to be *coherent* iff every pair-wise compatible subset $Y \subseteq X$ has a l.u.b. in PO. PO is said to be *finitely coherent* iff every *finite* pair-wise compatible subset $Y \subseteq X$ has a l.u.b. in PO. Finally, $PO = (X, \leq)$ is said to be *finitary* iff $\forall x \in X.\ \downarrow x$ is a finite set. The event structure $ES = (E', \leq', \#')$ is finitary iff $(E', \leq')$ is finitary. Clearly, $ES_{\mathcal{N}}$ is a finitary event structure, and $(\hat{C}_{\mathcal{N}}, \subseteq)$ is a finitary poset.

Theorem 4.8
Let $ES = (E', \leq, \#')$ be a finitary event structure. Let C_{ES}^{fin} denote the set of finite configurations of ES. Then $PO_{ES} = (C_{ES}^{fin}, \subseteq)$ satisfies the following properties:

(i) PO_{ES} is finitary,

(ii) PO_{ES} *is prime algebraic with* $\{\downarrow e \mid e \in E'\}$ *as its set of prime elements, and*

(iii) PO_{ES} *is finitely coherent.*

Proof The proof can be easily extracted from Nielsen et al. (1981).

□

Now let $PO = (X, \leq)$ be a poset that satisfies the three properties stated in Theorem 4.8. Let PR denote the set of prime elements of PO. Then $ES_{PO} = (PR, \leq', \#')$ is given by:

(i) $\leq'$ is $\leq$ restricted to $PR \times PR$, and

(ii) $\forall p_1, p_2 \in PR.\ p_1 \#' p_2$ iff $p_1 \not\uparrow p_2$ in PO.

Then it is easy to prove that ES_{PO} is a finitary event structure. What is more surprising and pleasant is the following.

Theorem 4.9
Let $PO = (X, \leq)$ *be a finitary, prime algebraic and finitely coherent poset. Let* $ES_{PO} = (PR, \subseteq', \#')$ *be defined as above. Then* PO *and* $(C^{fin}_{ES_{PO}}, \subseteq)$ *are isomorphic posets. In fact* $h : X \to PR$ *given by:*

$$\forall x \in X.\ \ h(x) = \{p \in PR \mid p \leq x\}$$

is an isomorphim.

□

Finally, suppose we are given a finitary event structure

$$ES = (E', \leq', \#')$$

with associated poset of finite configurations

$$PO_{ES} = (C^{fin}_{ES}, \subseteq).$$

We may associate an event structure $ES'' = (E'', \leq'', \#'')$ with PO_{ES} as outlined above. This is because, by Theorem 4.8, PO_{ES} enjoys the required properties. It is once again routine to prove that ES and ES'' are isomorphic event structures in the obvious sense.

Thus we are justified in claiming that $ES_{\mathcal{N}}$ and $(\hat{C}_{\mathcal{N}}, \subseteq)$ are "equivalent" representations.

5 Discussion

In this paper we have formalized a number of notions of the behaviour of elementary net systems. In particular, starting with the simple notion of firing sequences we have derived the three notions of behaviour called traces, processes and event structures. We have proved strong formal relationships in terms of isomorphisms between the associated structures (of traces, processes and finite configurations of the event structures).

In between we have managed to include the notion of unfoldings as a stepping stone to the definition of the event structure behaviour. As a mater of fact we could have defined the event structure behaviour of an elementary net system directly in terms of its processes: The events being the union of the events of the processes; one event being causally dependent on another iff it is so in every process in which they both occur; two events being in conflict iff they do not both appear in any process. However we decided to include the unfolding here to illustrate how well it fits into our framework. Once the notion of processes is worked out as we have done here the unfolding "falls out" through the simple device of "gluing" together the processes.

It is clear that our results are related to and depend upon the well-known results relating occurrence nets, event structures and prime algebraic domains Nielsen et al. (1981). It is also known that in a fairly general setting traces and event structures can be related to each other Bednarczyk (1988), Kiehn (1988), Shields (1989) and Rozoy and Thiagarajan (1987). However the questions addressed in this paper are of a different nature. Here we have considered the relationship between the *independent* applications of these models to characterize the behaviour of a given class of systems; in our case, elementary net systems which are the basic system model of net theory.

The reader familiar with the various behavioural models (processes, traces, event structures) will have no trouble in understanding our main results. However he/she might be taken aback by the technical complexity involved in proving these results. Of course it is entirely possible that a much neater derivation of the results has been missed by us. It might also be the case that some marginal advantage might have been gained by permitting isolated elements in our nets. However, we feel that the difficulties encountered in proving our results have to do with the fact that – as already pointed out – the various behavioural notions have to be related to each other in the context of studying the behaviour of *fixed* and *restricted* classes of systems. For example, in the literature on non-sequential processes Best and Fernandez (1988) one will find that Lemma 3.6 and Theorem 3.8 together constitute *definition* of the notion of a process for elementary net systems. Here we have had to prove them to be consequences of our more basic definitions. Indeed deriving these two results constitutes the bulk of the technical labour involved.

The informed reader might also be puzzled by the fact that our results are formulated in terms of finite objects only. In particular, the event structures are represented in terms of the poset of *finite* configurations whereas the standard definition includes all configurations. Given the fact that our event structures are necessarily finitary it turns out that the representation in terms of finite configurations is adequate for our purposes as detailed in Theorems 4.8 and 4.9. More, we are forced to consider only finite configurations and finite processes since it is not clear how the theory of trace languages extends to infinitary strings. Fortunately this commitment to dealing with only finite objects involves no permanent loss of information, concerning infinite behaviours. For instance the prime algebraic domain of *all* configurations of a finitary event structure can be easily obtained upto isomorphism by the standard ideal completion of our chosen poset of finite configurations. This remark applies as well to the poset of finite processes . For instance, to obtain a generalization of Corollary 4.7 we would only have to consider the ideal completion of the poset of traces. As yet another example, the notion of a computation advocated in Mazurkiewicz (1989), Reisig (1984) and

Stark (1987) to eliminate certain fairness notions is simply defined as a maximal ideal of $(T, \sqsubseteq)$ and hence $(\hat{C}_{ES}, \subseteq)$.

We take these observations as an indication that in many of the applications, it is sufficient for our behavioural notions to cater for finite objects (finite firing sequences, finite traces, finite processes, finite configurations) only.

We now wish to point out that our work can be viewed in a broader context. We have established two ways of associating a prime algebraic coherent poset with an elementary net system; one via the processes and one via the traces. In other words we have two maps – say f and g – from the class of elementary net systems to the class of prime algebraic coherent posets. Our main result is that for each $\mathcal{N}$, $f(\mathcal{N})$ and $g(\mathcal{N})$ are isomorphic to each other.

It is well known that elementary net systems (viewed as safe Petri nets) and prime algebraic coherent posets can be equipped with "behaviour preserving" morphisms to yield the categories $\mathcal{EN}$ and $\mathcal{PPO}$ respectively (see Winskel (1987)). It turns out that the maps f and g we have been considering can be smoothly lifted to become a pair of functors from $\mathcal{EN}$ to $\mathcal{PPO}$. In this case our main result generalizes to the existence of a natural isomorphism between these two functors in the sense of MacLane (1971).

Going further down this road the informed reader may have noticed that our notion of unfolding is different from the one presented in Winskel (1987). The difference arises mainly because we do not allow isolated elements in the underlying nets of elementary net systems. As a consequence, the nice categorical characterization of the unfolding in Winskel (1987) does not work in our case. However with a slightly different notion of morphisms between elementary net systems we can construct a new category $\mathcal{EN}'$ of elementary net systems. *Now* we obtain a similar characterization of the unfolding, namely the existence of a special morphism from $UF_{\mathcal{N}}$ (clearly, the unfolding of $\mathcal{N}$ is also an object in $\mathcal{EN}'$) to $\mathcal{N}$ in $\mathcal{EN}'$ which is co-free over $\mathcal{N}$. It so happens that this new notion of net morphisms between elementary net systems (and in fact, between safe Petri nets) possesses some pleasing properties. For instance, the *empty* elementary net system is both the initial and final object in $\mathcal{EN}'$. This might have some positive impact on categorical studies in net theory.

Acknowledgements

This work has been part of joint work of ESPRIT Basic Research Actions CEDISYS and DAEMON from which support is acknowledged.

The third author acknowledges support from the Dutch National Concurrency Project REX sponsored by NFI.

The authors also wish to thank two anonymous referees for valuable comments.

6 References

(Note: LNCS is an abbreviation for Lecture Notes in Computer Science, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo.)

Aalbersberg IJ J, Rozenberg G (1986) Theory of Traces. Technical Report no. 86-16, Institute of Applied Mathematics and Computer Science, University of Leiden, Leiden, The Netherlands

Bednarczyk M (1988) Categories of Asynchronous Systems. Ph.D. Thesis, Computer Science Department, University of Sussex, Brighton, G.B.

Best E, Devillers R (1987) Sequential and Concurrent Behaviour in Petri Net Theory. TCS 55:87-136

Best E, Fernandez C (1988) Non-Sequential Processes, A Petri Net View. EATCS Monographs on Theoretical Computer Science 13. Springer-Verlag

Brauer W, Reisig W, Rozenberg G (eds.) (1987) Advances in Petri Nets 1986 part I. LNCS 254

Brauer W, Reisig W, Rozenberg G (eds.) (1987) Advances in Petri Nets 1986 part II. LNCS 255

Kiehn A (1988) On the Interrelation Between Synchronized and Non-Synchronized Behaviour of Petri Nets. Journal of Information Processing and Cybernetics EIK 24 1/2:3-18

Mac Lane S (1971) Categories for the Working Mathematician. Springer-Verlag

Mazurkiewicz A (1978) Concurrent Program Schemes and their Interpretation. DAIMI Report PB-78 Computer Science Department, Aarhus University, Aarhus, Denmark

Mazurkiewicz A (1989) Basic Notions of Trace Theory. LNCS 354:285-363

Nielsen M, Plotkin G, Winskel G (1981) Petri Nets, Event Structures and Domains: Part I TCS 13:85-108

Petri C A (1962) Kommunikation mit Automaten. Inst für Instrumentelle Mathematik, Schriften des IIM Nr 2

Reisig W (1984) Partial Order Semantics versus Interleaving Semantics for CSP-like Languages and its Impact on Fairness. LNCS 172:403-413

Rozenberg G (1987) Behaviour of Elementary Net Systems. LNCS 254:60-94

Rozoy B, Thiagarajan P S (1987) Event Structures and Trace Monoids. Technical Report 87-47, Laboratoire d'Informatique Théorique et Programmation University of Paris VII Paris, France

Shields M N (1989) Behavioural Presentations. LNCS 354:673-689

Stark E W (1987) Concurrent Transition Systems and Semantics of Process Networks. 14th ACM Conference on Principles of Programming Languages 199-210

Thiagarajan P S (1987) Elementary Net Systems. LNCS 254:26-59

Thiagarajan P S (1988) Some Behavioural Aspects of Net Theory. LNCS 317:630-653

Winskel G (1987) Event Structures. LNCS 255:325-392