Belief change: from Situation Calculus to Modal Logic

Robert Demolombe

ONERA Toulouse
2 Avenue E. Belin,
B.P. 4025
31055 Toulouse,
France

Robert.Demolombe@cert.fr

ABSTRACT. *We propose a translation into Modal Logic of the ideas that formalise belief change in the Situation Calculus. This translation is extended to the case of revision. In the conclusion is presented a set of open issues.*

KEYWORDS: *situation calculus, dynamic logic, belief change.*

1. Introduction

An interesting formalisation of belief change has been proposed in the framework of Situation Calculus [SCH 93, SHA 00, LAK 98, SCH 03]. A simpler approach has been proposed in [DEM 00]. The Situation Calculus has the great advantage to deal with classical first order logic with equality (except some limited fragments that make use of second order) [REI 01]. However, it has this original feature that some concepts, like situations and accessibility relations are integrated in the axiomatics while they only appear in the semantics in Modal Logic. Moreover, there is a large community in the field of belief change that deals with Modal Logic. That is why we think it is worth to consider how intuitive ideas expressed in Situation Calculus can be translated into Modal Logic. That was the motivation of the work presented in this paper [1].

This translation raises several technical problems. The first problem is that actions and beliefs are represented in two different heterogeneous ways in Situation Calculus. Beliefs are represented in terms of accessibility relations, as usual in Modal Logic, while actions are represented through the function symbol do that appears in the terms of the type situation. The second problem is that to go from Situation Calculus to

1. Another approach has been followed in [DEM 03] which is based on regression.

Modal Logic we have to remove situations. The third problem is that in Situation Calculus it is allowed to quantify over variables of the type action, while in Modal Logic each action has the status of an index of a modal operator, and we can not quantify over indexes.

The intuitive ideas to solve these problems are the following ones.

For the first problem we can notice [2] that a formula of the form $p(do(a, s))$ is logically equivalent to $\forall s'(s' = do(a, s) \rightarrow p(s'))$ which has the same form as $\forall s'(K(s', s) \rightarrow p(s'))$ which is used to represent a belief, and it can be abreviated by $Act(a, p, s)$, in the same way as the second formula is abreviated by $Bel(p, s)$.

For the second problem we have to distinguish situations that are quantified and situations that are represented by ground terms. In the case of ground terms we impose to all of them to refer to the same initial situation S_0. Atoms of the form $p(S_0)$ are translated by p; atoms of the form $p(do(a, S_0))$ are translated in a first step by $Act(a, p, S_0)$ and then by $Act(a, p)$ (where $Act(a, .)$ is considered here as a modal operator); in the same way $p(do(b, do(a, S_0)))$ is translated by $Act(a, Act(b, p))$. In the case of universally quantified situations, formulas of the form $\forall sp(s)$ are translated by p' where p' is obtained from $\forall sp(s)$ by removing the universal quantifier and the situation arguments in all the atoms of p. The resulting formula p' has the status of a global axiom in Fitting's terminology, and we can apply the necessitation inference rule to it.

To solve the third problem we have made a strong assumption. We assume that there is only a finite number of actions. Then, if we accept a domain closure axiom for actions, universal quantifiers can equivalently be represented by finite conjunctions. For instance, $\forall ap(a)$ can be represented by $p(\alpha_1) \wedge \ldots \wedge p(\alpha_n)$.

Independently of these technical problems we have to translate the intuitive ideas that support the formalisation of belief change in Situation Calculus. That will be presented in more details in the rest of the paper.

2. A brief introduction to Situation Calculus

We briefly recall the ideas that are relevant to the representation of belief change.

Predicates whose truth value may change when actions are performed are called fluents. They have exactly one argument of the type situation (the last argument). For instance, $p(x_1, \ldots, x_n, s)$, where s is of the type situation, represents a fluent.

Arguments of the type situation are constants, or terms of the form $do(a, s)$, where a is of the type action and s is of the type situation. They satisfy the following axiom:

$(A1)$ $\forall a_1 \forall a_2 \forall s_1 \forall s_2 (do(a_1, s_1) = do(a_2, s_2) \rightarrow a_1 = a_2 \wedge s_1 = s_2)$

2. The formal proof can be found in [DEM 02].

This axiom forces situations to have the structure of a set of trees. In addition it is assumed that we have unique name axioms for action symbols.

A key idea to solve the frame problem in Situation Calculus is to impose to each fluent a successor state axiom. For example, we may have the successor state axiom :

$$(1)\ \forall s \forall a (p(do(a,s)) \leftrightarrow a = a_1 \vee (a = a_2 \wedge q(s)) \vee (p(s) \wedge \neg(a = a_3)))$$

This axiom intuitively says that there is no other action than a_1 and a_2 that can cause p to be true, and there is no other action than a_3 that can cause p to be false.

The general form of successor state axioms is:

$$(S_p)\ \forall s \forall a \forall \vec{x} (p(\vec{x}, do(a,s)) \leftrightarrow \Gamma_p^+(\vec{x}, a, s) \vee (p(\vec{x}, s) \wedge \neg \Gamma_p^-(\vec{x}, a, s)))$$

where $\Gamma_p^+(\vec{x}, a, s)$ and $\Gamma_p^-(\vec{x}, a, s)$ do not contain any occurence of the function do.

Beliefs are represented by formulas noted $Bel(p, s)$, whose meaning is that in the situation s it is believed that p holds. We have:

$$Bel(p,s) \overset{\text{def}}{=} \forall s'(K(s', s) \rightarrow p[s'])$$

where the predicate $K(s', s)$ plays the same role as an accessibility relation.

To define belief change we have to characterise what is the set of accessible situations after performance of an action. Two kinds of actions have to be considered. For non sensing actions the new accessible situations are the successors of the accessible ones. For sensing actions the new accessible situations are the successors of those situations that are consistent with the “sensed property”. For instance, if the sensing action α_i allows to knows whether p_i holds in the situation s, all the situations which are not consistent with the truth values of p_i in s have no successor. In addition it is assumed that sensing actions do not change the truth value of fluents. This assumption has to be encoded in the definition of successor state axioms.

The evolution of the set of accessible situations is defined by the following axiom:

$$\begin{aligned}(S_K)\ & \forall s \forall s'' \forall a (K(s'', do(a,s)) \leftrightarrow \exists s'(K(s', s) \wedge s'' = do(a, s') \wedge (\\ & (\neg(a = \alpha_1) \wedge \ldots \wedge \neg(a = \alpha_n)) \\ & \vee\ a = \alpha_1 \wedge (p_1(s) \leftrightarrow p_1(s')) \\ & \ldots \\ & \vee\ a = \alpha_n \wedge (p_n(s) \leftrightarrow p_n(s'))))\end{aligned}$$

where $\alpha_1, \ldots, \alpha_n$ is the set of all the sensing actions.

3. Translation of belief change into Modal Logic

We consider a first order Modal Logic with the following modal operators [3]:

3. As a matter of simplification we accept the Barcan formula and the converse of the Barcan formula [FIT 98].

$Act(a,p)$: whose meaning is that p holds after performance of the action a.

$Bel(p)$: whose meaning is that it is believed that p holds.

To represent the structure of the set of situations we have the axiom schemas:

$(ACT1)$ $Act(a, p \vee q) \leftrightarrow Act(a,p) \vee Act(a,q)$

$(ACT2)$ $Act(a, \neg p) \leftrightarrow \neg Act(a,p)$

These axioms are justified by the following properties of classical first order logic with equality [4]:

$\vdash \forall s \forall a(\forall s'(s' = do(a,s) \rightarrow p(s') \vee q(s')) \leftrightarrow \forall s'(s' = do(a,s) \rightarrow p(s')) \vee \forall s'(s' = do(a,s) \rightarrow q(s')))$

$\vdash \forall s \forall a(\forall s'(s' = do(a,s) \rightarrow \neg p(s')) \leftrightarrow \neg(\forall s'(s' = do(a,s) \rightarrow p(s'))))$

To define the translation of the successor state axioms we define the function τ as follows. To make simpler the definition of the translation it is assumed that each non fluent $p(\vec{t})$ is replaced by the fluent $p(\vec{t}, s)$, and we add the related successor state axiom: $\forall s \forall a \forall \vec{t}(p(\vec{t}, do(a,s)) \leftrightarrow p(\vec{t}, s))$.

Let us first adopt the following notation:

$$\sigma_p(\vec{x}, a, s) \stackrel{\text{def}}{=} p(\vec{x}, do(a,s)) \leftrightarrow \Gamma_p^+(\vec{x}, a, s) \vee p(\vec{x}, s) \wedge \neg\Gamma_p^-(\vec{x}, a, s)$$

Then (S_p) takes the form $\forall s \forall a \forall \vec{x} \sigma_p(\vec{x}, a, s)$. We have $\tau(\forall s \forall a \forall \vec{x} \sigma_p(\vec{x}, a, s)) = \tau(\forall a \forall \vec{x} \sigma_p(\vec{x}, a, s))$ because the result of the translation is considered as a global axiom. If we have for the actions the domain closure axiom $\forall a(a = \alpha_1 \vee \ldots \vee a = \alpha_n)$, we have:

$$\tau(\forall a \forall \vec{x} \sigma_p(\vec{x}, a, s)) = \tau(\forall \vec{x} \sigma_p(\vec{x}, a_1, s)) \wedge \ldots \wedge \tau(\forall \vec{x} \sigma_p(\vec{x}, a_n, s))$$

For every α_i we have:

$$\tau(\forall \vec{x} \sigma_p(\vec{x}, \alpha_i, s)) = \forall \vec{x}(Act(\alpha_i, p(\vec{x})) \leftrightarrow \tau(\Gamma_p^+(\vec{x}, \alpha_i, s)) \vee p(\vec{x}) \wedge \neg\tau(\Gamma_p^-(\vec{x}, \alpha_i, s)))$$

Then $\tau(\Gamma_p^+)$ and $\tau(\Gamma_p^-)$ are defined in the same way as $\tau(\Gamma)$ below.

I) if $\Gamma = \neg\Gamma_1$, then $\tau(\Gamma) = \neg\tau(\Gamma_1)$

II) if $\Gamma = \Gamma_1 \vee \Gamma_2$, then $\tau(\Gamma) = \tau(\Gamma_1) \vee \tau(\Gamma_2)$

III) if $\Gamma = \exists x \Gamma_1$, then $\tau(\Gamma) = \exists x \tau(\Gamma_1)$

IV) if Γ is an atomic formula:

a) if Γ has the form $p(\vec{t}, s)$, then $\tau(\Gamma) = p(\vec{t})$

b) if Γ has the form $\alpha_i = \alpha_j$

i) if α_i and α_j are the same constant symbol, then $\tau(\Gamma) = true$

ii) if α_i et α_j are different constant symbols, then $\tau(\Gamma) = false$

4. See Theorem 1 in [DEM 02].

The result of the translation of the successor state axiom (1) of the previous section, if the set of actions is α_1, α_2, α_3 and α_4, is equivalent to the set of global axioms:

(2) $Act(\alpha_1, p) \leftrightarrow true$

(3) $Act(\alpha_2, p) \leftrightarrow q \vee p$

(4) $Act(\alpha_3, \neg p) \leftrightarrow true$

(5) $Act(\alpha_4, p) \leftrightarrow p$

The axiom (S_K) that characterises belief change in the Situation Calculus correponds to the following axiom schemas [5]:

$(ACT3)$ $Act(a, Bel(p)) \leftrightarrow Bel(Act(a, p))$ if a is not a sensing action.

$(ACT4)$ $p_i \rightarrow Act(\alpha_i, Bel(p_i))$ if α_i is the sensing action related to p_i.

$(ACT5)$ $\neg p_i \rightarrow Act(\alpha_i, Bel(\neg p_i))$ if α_i is the sensing action related to p_i.

$(ACT6)$ $Bel(p) \rightarrow Act(\alpha_i, Bel(p))$ if α_i is a sensing action.

If we consider the translation of these axioms in the Situation Calculus we get the following formulas:

$(act3)$ $\forall s(Bel(p, do(a_i, s)) \leftrightarrow Bel(Act(a_i, p), s))$

$(act4)$ $\forall s(p_i(s) \rightarrow Bel(p_i, do(\alpha_i, s)))$

$(act5)$ $\forall s(\neg p_i(s) \rightarrow Bel(\neg p_i, do(\alpha_i, s)))$

$(act6)$ $\forall s(Bel(p, s) \rightarrow Bel(p, do(\alpha_i, s)))$

It is proven in the Annex that the axioms $(act3)$ to $(act6)$ are logical consequences of the axiom (S_K). This guarantees that these axioms are a valid translation of (S_K). Up to now we have not proved that this translation is complete.

4. Extension to revision

The formalisation of belief change proposed by Scherl and Levesque in [SCH 93] leads to contradictions in the case of revision. To remedy this problem a new definition of beliefs has been proposed in [SHA 00]. The basic idea is to assign to each situation a plausibility level and to define belief as truth in the most plausible accessible situations [6].

In formal terms $pl(s)$ defines the plausibility level of the situation s. It is assumed that the successors of a given situation have the same plausibility level. That is we have:

---

5. In [HER 02] Herzig and Longin have independently proposed the axiom schema (SSA) which is almost the same as (ACT3).
6. For technical reasons it is said that s is more plausible than s' if $pl(s) < pl(s')$.

$(P)\ \forall s \forall a(pl(do(a,s)) = pl(s))$

For the new definition of beliefs we adopt the following notation:

$K_{max}(s', s) \stackrel{\text{def}}{=} K(s', s) \wedge \forall s''(K(s'', s) \rightarrow pl(s') \leq pl(s''))$

The predicate $K_{max}(s', s)$ characterises the most plausible accessible situations. Then we have:

$Bel(p, s) \stackrel{\text{def}}{=} \forall s'(K_{max}(s', s) \rightarrow p[s'])$

For practical reasons we think that the definition of the function pl cannot be given by extension, that is to explicitly give the $pl(s)$ value for each possible situation. Then, we propose to define it by intension, that is to give for each plausibility level l a formula p that characterises **all** the situations that have the plausibility level l. A formula which has this property is characterised by the formula denoted by $OBelp(p, l, s)$:

$OBelp(p, l, s) \stackrel{\text{def}}{=} \forall s'(K(s', s) \wedge pl(s') = l \leftrightarrow p(s'))$

Following the same idea, in Modal Logic we define the modal operator $OBelp(p, l)$ [7]. Then, the definition of plausibility levels in a given application domain is defined by the following set of assumptions:

$L = \{OBelp(p_1, l_1), \ldots, OBelp(p_n, l_n)\}$

It is assumed that the set of plausibility levels is finite, and that for each plausibility level l_i there is in L an assumption of the form $OBelp(p_i, l_i)$.

The set of all beliefs is characterised by the formula p which satisfies the property:

$(B)\ \forall l(OBelp(p, l) \wedge \forall p_i \forall l_i(OBelp(p_i, l_i) \rightarrow l \leq l_i))$

This property is represented by the modal operator $OBel(p)$. Intuitively $OBel(p)$ can be read: the set of all beliefs is defined by p. Unfortunately the property (B) is not expressed in first order modal logic because the quantifier $\forall p_i$ quantifies over the set of propositions. That means that, if we want to remain in first order modal logic, the link between $OBel(p)$ and the set (L) is not defined in the theory we consider.

The modal operators $OBelp(p, l)$ and $OBel(p)$ are not operators of a normal modal logic but operators of a classical modal logic, and their only property is the inference rule (RE) of substitutivity of equivalent formulas (see [CHE 88]).

Beside these two operators we define the operators $Belp(p, l)$ and $Bel(p)$ that respectively represent what is believed at a given level and what is believed. These two operators obey the properties of a normal modal logic. They are related to the previous ones by the following axiom schemas:

$(OB1)\ OBel(p) \rightarrow Bel(p)$

7. We adopt the same notation $OBelp$ in the Situation Calculus and in Modal Logic because we think there is no risk of misunderstanding.

$(OB2)\ \ OBelp(p, l) \rightarrow Belp(p, l)$

Now we have to define belief change in this approach.

For that purpose we only have to define the evolution of beliefs that have a given plausibility level (i.e. the binary operators $OBelp$ and $Belp$) because the set of beliefs is derived from these plausibility level dependent beliefs via the property (B).

If a is a non sensing action, since the plausibility levels are the same for the successors of a situation, according to (S_K) we have the axiom schema:

$(ACT3')\ \ Act(a, Belp(p, l)) \leftrightarrow Belp(Act(a, p), l)$

If α_i is the sensing action that allows to know whether p_i holds, from the definition of (S_K), after performance of α_i a set of beliefs represented by p is restricted to $p \wedge p_i$ if p_i holds and $p \wedge p_i$ is consistent, and it is restricted to $p \wedge \neg p_i$ if $\neg p_i$ holds and $p \wedge \neg p_i$ is consistent. Beliefs which are not consistent with the “truth value” of p_i are not “propagated” after α_i. Then, we have the following axiom schema:

$(ACT45')$ If α_i is a sensing action related to p_i:
$p_i \rightarrow (OBelp(p, l) \rightarrow Act(\alpha_i, OBelp(p \wedge p_i, l)))$ if $p \wedge p_i$ is consistent
$\neg p_i \rightarrow (OBelp(p, l) \rightarrow Act(\alpha_i, OBelp(p \wedge \neg p_i, l)))$ if $p \wedge \neg p_i$ is consistent

If a is a non sensing action and if we have $OBelp(p, l)$, the set of all the beliefs at the level l after performance of a is determined by the application of the successor state axioms. Indeed, since the successor state axioms are global axioms they are believed, and the axiom schema $(ACT3')$ shows how beliefs about the evolution of the world are equivalent to evolution of beliefs.

For instance, in the example of the previous section the formula (3) represents an instance of the successor state axiom for the action α_2:

$(3)\ \ Act(\alpha_2, p) \leftrightarrow q \vee p$

Since $Belp$ is a normal modal operator, from the global axiom (3) we can infer:

$(6)\ \ Belp(Act(\alpha_2, p) \leftrightarrow q \vee p, l)$

and

$(7)\ \ Belp(Act(\alpha_2, p), l) \leftrightarrow Belp(q \vee p, l)$

Also, from $(ACT3')$ we have:

$(8)\ \ Act(\alpha_2, Belp(p, l)) \leftrightarrow Belp(Act(\alpha_2, p), l)$

Then, we have:

$(9)\ \ Act(\alpha_2, Belp(p, l)) \leftrightarrow Belp(q \vee p, l)$

This example shows how beliefs can be propagated thanks to the successor state axioms. Then, in general we have the following axiom schema:

$(ACT7)$ If a is not a sensing action and q is a sentence that implies every formula p' such that we have: $OBelp(p,l) \rightarrow Act(a, Bel(p',l))$
then we have:

$$OBelp(p,l) \rightarrow Act(a, OBelp(q,l))$$

5. Conclusion

We have shown how the basic ideas that support belief change can be translated from Situation Calculus to Modal Logic. The result of this translation is expressed by the axiom schemas $(ACT1)$ to $(ACT6)$, and by the translation function τ that transforms the successor state axioms. A limitation of function τ definition is that it is based on a domain closure axiom for the set of actions. Also, in the domain closure axiom we have only considered actions that are represented by constant symbols. We should investigate in the future how to adapt this axiom in the case where actions are represented by general terms.

Then, we have proposed an extension of this translation to the case where we deal with plausibility levels in order to solve the problem of revision. In that case the result of the translation is represented by the set of axiom schemas: $(ACT1)$, $(ACT2)$, $(ACT3')$, $(ACT45')$and $(ACT7)$. However, there are some important issues that deserve further work. One of them is to find a formal representation of the fact that in (L) we assume that we have a complete set of assumptions. Another one is that the link between the modal operator $OBel(p)$ and $OBelp(p,l)$ requires to quantify over the propositions p_i and over the modal operator indexes l_i. Finally, in the axiom schema $(ACT7)$ the link between the propositions p and q is not defined inside the theory. For all these reasons, this proposal should be considered as a preliminary step for further works.

Acknowledgement. I would like to thank Andreas Herzig and Ivan Varzinczak for their help in writing this paper.

6. References

[CHE 88] CHELLAS B. F., *Modal Logic: An introduction*, Cambridge University Press, 1988.

[DEM 00] DEMOLOMBE R., POZOS-PARRA M. P., "A simple and tractable extension of Situation Calculus to Epistemic Logic", RAS Z. W., OHSUGA S., Eds., *Proc. of 12th International Symposium ISMIS 2000*, Springer. LNAI 1932, 2000.

[DEM 02] DEMOLOMBE R., "Correspondance entre théorie de l'interaction et calcul des situations", report num. 3/07165/DTIM, 2002, ONERA Toulouse.

[DEM 03] DEMOLOMBE R., HERZIG A., VARZINCZAK I., "Regression in Modal Logic", C. ARECES AND P. BLACKBURN, Ed., *Methods for Modalities*, 2003.

[FIT 98] FITTING M., MENDELSOHN R. L., *First-Order Modal Logic*, Kluwer, 1998.

[HER 02] HERZIG H., LONGIN D., "Sensing and revision in a modal logic of belief and action", VAN HARMELEN F., Ed., *Proceedings of ECAI02*, IOS Press, 2002.

[LAK 98] LAKEMEYER G., LEVESQUE H., "AOL: a logic of acting, sensing, knowing and only knowing", *Proc. of the 6th Int. Conf. on Principles of Knowledge Representation and Reasoning*, 1998, p. 316-327.

[REI 01] REITER R., *Knowledge in Action: Logical Foundations for Specifying and Implementing Dynamical Systems*, MIT Press, 2001.

[SCH 93] SCHERL R., LEVESQUE H., "The Frame Problem and Knowledge Producing Actions", *Proc. of the National Conference of Artificial Intelligence*, AAAI Press, 1993.

[SCH 03] SCHERL R., LEVESQUE H., "Knowledge, action and the frame problem", *Artificial Intelligence*, vol. 144, 2003, p. 1–39.

[SHA 00] SHAPIRO S., PAGNUCO M., LESPÉRANCE Y., LEVESQUE H., "Iterated belief change in the Situation Calculus", *Proc. of the 7th Conference on Principles on Knowledge Representation and Reasoning (KR2000)*, Morgan Kaufman Publishers, 2000.

Annex

THEOREM 1. — *Let ϕ be a formula of classical first order logic with equality in which occurs the term t.We note $\phi[t]$ the formula ϕ in which a given occurence of t has been fixed. We note $\phi[x]$ the formula obtained from $\phi[t]$ by replacing this occurence of t by x. We have:*

$$\models \phi[t] \leftrightarrow \forall x(x = t \rightarrow \phi[x])$$

PROOF. — See the proof of Theorem 1 in [DEM 02]. ■

THEOREM 2. — *The translation in the Situation Calculus of the axiom schema: $(ACT3)$ $Act(a, Bel(p)) \leftrightarrow Bel(Act(a, p))$ if a is not a sensing action, is a logical consequence of the axiom (S_K).*

PROOF. — The translation of $(ACT3)$ in the Situation Calculus gives:

(1) $\forall s(Bel(p, do(a_i, s)) \leftrightarrow Bel(Act(a_i, p), s))$

We adopt the following notations :

(2) $Bel(p, do(a_i, s))$

(3) $Bel(Act(a_i, p), s)$

We are going to prove that (2) is equivalent to (3).

>From the definitions we have:

$(2) \Leftrightarrow (4)$ $\forall s''(K(s'', do(a_i, s)) \rightarrow p(s''))$

$(3) \Leftrightarrow (5)$ $\forall s'(K(s', s) \rightarrow Act(a_i, p, s'))$

$(5) \Leftrightarrow (6)$ $\forall s'(K(s', s) \rightarrow \forall s''(s'' = do(a_i, s') \rightarrow p(s'')))$

Since a_i is not a sensing action, from (S_K) we have:

(7) $\forall s \forall s''(K(s'', do(a_i, s)) \leftrightarrow \exists s'(K(s', s) \wedge s'' = do(a_i, s')))$

Then, from (7) we have:

(4) $\Leftrightarrow$ (8) $\forall s''(\exists s'(K(s', s) \wedge s'' = do(a_i, s')) \rightarrow p(s''))$

(8) $\Leftrightarrow$ (9) $\forall s'(K(s', s) \rightarrow \forall s''(s'' = do(a_i, s') \rightarrow p(s'')))$

Therefore we have (6) $\Leftrightarrow$ (9) and (2) $\Leftrightarrow$ (3).

■

THEOREM 3. — *The translation in the Situation Calculus of the axiom schema:* $(ACT4)$ $p_i \rightarrow Act(\alpha_i, Bel(p_i))$ *if* α_i *is the sensing action related to* p_i*, is a logical consequence of the axiom* (S_K).

PROOF. — The translation of $(ACT4)$ in the Situation Calculus gives:

(1) $\forall s((p_i(s) \rightarrow Bel(p_i, do(\alpha_i, s)))$

We adopt the following notation:

(2) $p_i(s) \rightarrow Bel(p_i, do(\alpha_i, s))$

>From the definitions we have:

(2) $\Leftrightarrow$ (3) $p_i(s) \rightarrow \forall s''(K(s'', do(\alpha_i, s)) \rightarrow p_i(s''))$

>From (S_K), for the sensing action α_i we have:

(4) $\forall s''(K(s'', do(\alpha_i, s)) \leftrightarrow \exists s'(K(s', s) \wedge s'' = do(\alpha_i, s') \wedge (p_i(s) \leftrightarrow p_i(s'))))$

>From (4) we have:

(3) $\Leftrightarrow$ (5) $p_i(s) \rightarrow \forall s''(\exists s'(K(s', s) \wedge s'' = do(\alpha_i, s') \wedge (p_i(s) \leftrightarrow p_i(s'))) \rightarrow p_i(s''))$

(5) $\Leftrightarrow$ (6) $\forall s'(p_i(s) \rightarrow (K(s', s) \wedge (p_i(s) \leftrightarrow p_i(s')) \rightarrow \forall s''(s'' = do(\alpha_i, s') \rightarrow p_i(s''))))$

>From the Theorem 1 we have:

(6) $\Leftrightarrow$ (7) $\forall s'(p_i(s) \wedge K(s', s) \wedge (p_i(s) \leftrightarrow p_i(s')) \rightarrow p_i(do(\alpha_i, s')))$

Sensing actions do not change the truth values of the fluent. Then, for each fluent p we have $\forall s'(p(do(\alpha_i, s')) \leftrightarrow p(s'))$. Therefore, for any formula p_i we have:

(8) $\forall s'(p_i(do(\alpha_i, s')) \leftrightarrow p_i(s'))$

>From (8) we have:

(7) $\Leftrightarrow$ (9) $\forall s'(p_i(s) \wedge K(s', s) \wedge (p_i(s) \leftrightarrow p_i(s')) \rightarrow p_i(s'))$

(9) $\Leftrightarrow$ (10) $\forall s'(K(s', s) \rightarrow (p_i(s) \wedge (p_i(s) \leftrightarrow p_i(s')) \rightarrow p_i(s')))$

We can easily prove that (11) is a theorem of classical first order logic.

(11) $\forall s \forall s'((p_i(s) \wedge (p_i(s) \leftrightarrow p_i(s'))) \rightarrow p_i(s'))$

Therefore, from (11) we have:

$(10) \Leftrightarrow (12)\ \ \forall s'(K(s',s) \rightarrow true)$

Since (12) is a tautology and (2) is equivalent to (12) we have proved that (2) is a tautology. Then, we have (1).

■

THEOREM 4. — *The translation in the Situation Calculus of the axiom schema:* $(ACT5)\ \ \neg p_i \rightarrow Act(\alpha_i, Bel(\neg p_i))$ *if* α_i *is the sensing action related to* p_i*, is a logical consequence of the axiom* (S_K)*.*

PROOF. — The proof is very close to the proof of the Theorem 3. ■

THEOREM 5. — *The translation in the Situation Calculus of the axiom schema:* $(ACT6)\ \ Bel(p) \rightarrow Act(\alpha_i, Bel(p))$ *if* α_i *is a sensing action, is a logical consequence of the axiom* (S_K)*.*

PROOF. — The translation of (ACT6) in the Situation Calculus gives:

$(1)\ \ \forall s(Bel(p,s) \rightarrow Bel(p, do(\alpha_i, s)))$

>From the definitions we have:

$(1) \Leftrightarrow (2)\ \ \forall s(\forall s'(K(s',s) \rightarrow p(s')) \rightarrow \forall s''(K(s'', do(\alpha_i, s'')) \rightarrow p(s'')))$

We adopt the following notations:

$(3)\ \ \forall s'(K(s',s) \rightarrow p(s'))$

$(4)\ \ \forall s''(K(s'', do(\alpha_i, s'')) \rightarrow p(s''))$

>From the axiom (S_K) for the sensing action α_i we have:

$(5)\ \ \forall s \forall s''(K(s'', do(\alpha_i, s)) \leftrightarrow \exists s'(K(s',s) \wedge s'' = do(\alpha_i, s') \wedge (p_i(s) \leftrightarrow p_i(s'))))$

>From (5) we have:

$(4) \Leftrightarrow (6)\ \ \forall s''(\exists s'(K(s',s) \wedge s'' = do(\alpha_i, s') \wedge (p_i(s) \leftrightarrow p_i(s'))) \rightarrow p(s''))$

$(6) \Leftrightarrow (7)\ \ \forall s'(K(s',s) \wedge (p_i(s) \leftrightarrow p_i(s')) \rightarrow \forall s''(s'' = do(\alpha_i, s') \rightarrow p(s'')))$

>From the Theorem 1 we have:

$(7) \Leftrightarrow (8)\ \ \forall s'(K(s',s) \wedge (p_i(s) \leftrightarrow p_i(s')) \rightarrow p(do(\alpha_i, s')))$

Since sensing actions do not change the truth values of the fluents, it is the same for any formula. Then, we have:

$(9)\ \ \forall s'(p(do(\alpha_i, s')) \leftrightarrow p(s'))$

>From (9) we have:

$(8) \Leftrightarrow (10)\ \ \forall s'(K(s',s) \wedge (p_i(s) \leftrightarrow p_i(s')) \rightarrow p(s'))$

Therefore we have (3) that implies (10), and (3) implies (4) because (4) is equivalent to (10). This proves that we have (2) and (1). ■