# Bell polynomials and generalized Laplace transforms

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**Abstract** - An extension of the Laplace transform obtained by using the Laguerre-type exponentials is first shown. Furthermore, the solution of the Blissard problem by means of the Bell polynomials, gives the possibility to associate to any numerical sequence a Laplace-type transform depending on that sequence. Computational techniques for the corresponding transform of analytic functions, involving Bell polynomials, are derived.

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## 1 Introduction

It is almost impossible to cite all the contributes of the Laplace transform:

$$\mathcal{L}(f) := \int_0^\infty \exp^{-1}(s\,t)f(t)\,dt = F(s)\,,$$

to the solution of differential equations [21, 38]. Indeed, the Fourier and the Laplace transforms are the most useful tools in Analysis and Mathematical Physics.

Actually these transforms are nothing but functions acting in function spaces, so that it is quite obvious that many others transforms can be defined similar to them.

The Bell polynomials [1] have been applied in many different fields of mathematics. In order to avoid unuseful repetitions, which would be classified as plagiarism by the modern *artificial deficiency*, we limit ourselves to recall the articles [2, 9, 16, 26].

Some generalized forms of Bell polynomials already appeared in literature (see e.g. [18, 31]). The multivariare case was also considered in [5, 25, 28, 29]. Connections with number theory have been examined in [27, 30]. See also [37].

To the author's knowledge a connection of the two before mentioned topics has not been considered in literature. This is the subject of the present article which the proposal to introduce a wide extension of the Laplace tranform by using the Blissard umbral calculus. Computational techniques in case of analytic functions are also given in the last sections. The obtained results, although formal, since they are based on an umbral approach, allow to consider infinite many other transforms which can be computed by essentially algebraic methods.

#### 2 Recalling the Bell polynomials

Considering the *n*-times differentiable functions x = g(t) and y = f(x), defined in given intervals of the real axis, the composite function  $\Phi(t) := f(g(t))$ , can be differentiated with respect to t, up to the *n*th order, by using the chain rule. We use the notations:

$$\Phi_m := D_t^m \Phi(t), \qquad f_h := D_x^h f(x)|_{x=g(t)}, \qquad g_k := D_t^k g(t).$$

Then the *n*th derivative of  $\Phi(t)$  is represented by

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n),$$

where  $Y_n$  denotes the *n*th Bell polynomial. The first few Bell polynomials are:

$$Y_{1}(f_{1}, g_{1}) = f_{1}g_{1}$$

$$Y_{2}(f_{1}, g_{1}; f_{2}, g_{2}) = f_{1}g_{2} + f_{2}g_{1}^{2}$$

$$Y_{3}(f_{1}, g_{1}; f_{2}, g_{2}; f_{3}, g_{3}) = f_{1}g_{3} + f_{2}(3g_{2}g_{1}) + f_{3}g_{1}^{3}$$

$$\cdots$$
(1)

Further examples can be found in [33], p. 49, where a recursion formula and the explicit expression given by the Faà di Bruno formula is also recalled.

A proof of the Faà di Bruno formula based on the *umbral calculus* is given in [36] and [35]. However, the Faà di Bruno is not convenient by the computational point of view,

owing the higher computational complexity with respect to the recursion. The traditional form of the Bell polynomials [11] is given by:

$$Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n) = \sum_{k=1}^n B_{n,k}(g_1, g_2, \dots, g_{n-k+1})f_k, \qquad (2)$$

where the  $B_{n,k}$  satisfy the recursion [11]:

$$B_{n,k}(g_1, g_2, \dots, g_{n-k+1}) = \sum_{h=0}^{n-k} \binom{n-1}{h} B_{n-h-1,k-1}(g_1, g_2, \dots, g_{n-h-k+1}) g_{h+1}.$$
(3)

The  $B_{n,k}$  functions for any k = 1, 2, ..., n are polynomials in the  $g_1, g_2, ..., g_n$  variables homogeneous of degree k and *isobaric* of weight n (i.e. they are linear combinations of monomials  $g_1^{k_1}g_2^{k_2}\cdots g_n^{k_n}$  whose weight is constantly given by  $k_1 + 2k_2 + ... + nk_n = n$ ), so that

$$B_{n,k}(\alpha\beta \, g_1, \alpha\beta^2 g_2, \dots, \alpha\beta^{n-k+1} g_{n-k+1}) = \alpha^k \beta^n B_{n,k}(g_1, g_2, \dots, g_{n-k+1}), \tag{4}$$

and

$$Y_n(f_1, \beta g_1; f_2, \beta^2 g_2; \dots; f_n, \beta^n g_n) = \beta^n Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n).$$
(5)

#### 3 The Blissard problem

John Blissard (1803-1875) published in 1861-1862 papers [6] introducing a symbolic method showing that some sequences of numbers  $\{b_k\}$  can be substituted by powers  $\{b^k\}$  so as to obtain valid formulas. The Bernoulli numbers was shown to be a first example of such a sequence. The Blissard symbolic method at present is called the *umbral calculus*, a term coined by J.J. Sylvester.

The modern version of the umbral calculus [35, 36] considers the umbral algebra, as the algebra of linear functionals on the vector space of polynomials, with the product defined by a binomial type formula. An extensive bibliogaphy of the subject can be found in [15]. The so called Blissard problem is described as follows [33].

Given the formal power series

$$e^{at} = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!},$$
(6)

associted to the sequence  $a = \{a_k\}$ , where

$$a^k := a_k, \qquad \forall k \ge 0, \qquad a_0 := 1, \tag{7}$$

the solution of the equation

$$e^{at}e^{bt} = 1 \tag{8}$$

with respect to the unknown sequence  $b = \{b_n\}$ , is given by

$$\begin{cases} b_0 := 1, \\ b_n = Y_n(-1!, a_1; 2!, a_2; -3!, a_3; \dots; (-1)^n n!, a_n), \quad (\forall n > 0), \end{cases}$$
(9)

where  $Y_n$  is the *n*th Bell polynomial [33].

#### 4 A first extension of the Laplace transform

In [12] the Laguerre-type exponentials has been defined, for every integer  $r \ge 1$ , according to the equation:

$$e_r(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{r+1}}.$$
(10)

Obviously, for r = 0, it results:  $e_0(x) \equiv \exp(x)$ .

Actually these functions are a particular case of the Le Roy functions [23], and more generally of the generalized Mittag–Leffler functions defined in [19], and deeply studied in [20].

A comparison among the functions  $[e_r(t)]^{-1}$ , (r = 1, 2),  $\exp(-t)$  and the limit value  $\lim_{r \to +\infty} [e_r(t)]^{-1} = 1/(1+t)$  shows that for every  $r \ge 1$  the functions  $[e_r(t)]^{-1} \in L^1(0, +\infty)$ , while the limit value 1/(1+t) does not satisfy this condition.

Consider the following transforms:

$$\mathcal{L}_1(f) := \int_0^\infty [e_1(s\,t)]^{-1} f(t) \, dt = \int_0^\infty \left[\sum_{k=0}^\infty \frac{(s\,t)^k}{(k!)^2}\right]^{-1} f(t) \, dt = F_1(s) \,, \tag{11}$$

and in general:

$$\mathcal{L}_{r}(f) := \int_{0}^{\infty} [e_{r}(s\,t)]^{-1} f(t) \, dt = \int_{0}^{\infty} \left[ \sum_{k=0}^{\infty} \frac{(s\,t)^{k}}{(k!)^{r}} \right]^{-1} f(t) \, dt = F_{r}(s) \,. \tag{12}$$

As in the ordinary Laplace transform, the integrals in equations (11)-(12) exist for all real numbers Re(s) > a, where the constant a, called the convergence abscissa, depends on the function f and determines the region of convergence.

Note that the increasing behaviour of the Laguerre-type exponentials, in the interval  $(0, +\infty)$  is lower with respect to the ordinary exponential, so that for any fixed f, we can choose, at least, the same convergence abscissa of the ordinary Laplace transform.

For every  $r \ge 1$ , an approximation of the transform (12) is obtained by using the truncated Laguerre-type exponential of order r, putting for a fixed integer n:

$$\int_{0}^{\infty} \left[ \sum_{k=0}^{n} \frac{(s\,t)^{k}}{(k!)^{r}} \right]^{-1} f(t) \, dt = F_{r}^{[n]}(s) \,. \tag{13}$$

Since for r = 0 the Laguerre-type exponentials give back the ordinary exponential function, then, in this case, equation (11) reduces to the ordinary Laplace transform, that is it results:

$$\mathcal{L}_0(f) = \mathcal{L}(f) := \int_0^\infty \exp^{-1}(s\,t)f(t)\,dt = F_0(s)\,.$$
(14)

#### 4.1 The inversion formula

The Laguerre-type exponentials are monotonic increasing functions in the interval  $(0, +\infty)$ , so that can be inverted in the same interval. Therefore, we can *conjecture* that the transform (12) (in particular (11)), admits the inversion formula:

$$\mathcal{L}_{r}^{-1}(f)(t) := \frac{1}{2\pi i} \lim_{\tau \to \infty} \int_{\gamma - i\tau}^{\gamma + i\tau} [e_{r}(s\,t)] F_{r}(s) \, ds \,, \tag{15}$$

where  $\gamma$  is a real number so that the contour path of integration is in the region of convergence of  $F_r(s)$ . It should also be possible to transform the contour into a closed curve, allowing the use of the residue theorem.

However, the proof of the equation (15) is not easy to carry out, since it would be necessary to introduce an extension of the Fourier transform based on Laguerre exponentials, a topic still far from being obtained.

## 5 The isomorphism $\mathcal{T}_s$ and its iterations

In previous articles (see e.g. [32]), it was shown that there exist a differential isomorphism  $\mathcal{T} := \mathcal{T}_s$ , acting into the space  $\mathcal{A} := \mathcal{A}_s$  of analytic functions of the variable s, by means of the correspondence:

$$D_s := D \equiv \frac{d}{ds} \rightarrow \hat{D}_L := D_s s D_s; \qquad s \cdot \rightarrow \hat{D}_s^{-1},$$

where

$$\hat{D}_s^{-n}F(s) := \frac{1}{(n-1)!} \int_0^s (s-\xi)^{n-1}F(\xi)d\xi.$$

The isomorphism  $\mathcal{T} := \mathcal{T}_s$  can be iterated producing a set of generalized Laguerre derivatives as follows. According to the results in [32] we put, for every integer  $m \ge 1$ ,

$$\mathcal{T}_s^{m-1}\hat{D}_L = \mathcal{T}_s^{m-1}(DsD) = DsDsD\cdots sD =: \hat{D}_{mL},$$

where the last operator contains s + 1 ordinary derivatives, denoted by  $D \equiv D_s$ . The action of  $\mathcal{T}_s$ , on powers, and consequently on all functions belonging to  $\mathcal{A} := \mathcal{A}_s$  is as follows:

$$\hat{D}_s^{-n}(1) = \frac{s^n}{n!} \; ,$$

and, by induction:

$$\mathcal{T}_s^{m-1} \hat{D}_s^{-1}(1) = \hat{D}_{\mathcal{T}_s^{m-1}}^{-1}(1) \quad \Rightarrow \quad \hat{D}_{\mathcal{T}_s^{m-1}}^{-n}(1) = \frac{s^n}{(n!)^s} \;.$$

Note that the Laguerre-type exponentials are obtained, acting with these iterated isomor-

phisms on the classical exponential, since:

$$\mathcal{T}_s^m(e^s) = \sum_{k=0}^{\infty} \frac{\mathcal{T}_s(s^k)}{(k!)^m} = \sum_{k=0}^{\infty} \frac{s^k}{(k!)^{m+1}} = e_m(s) \,.$$

It has been shown in a number of articles [3, 4, 7, 10], that new sets of special functions, namely the Laguerre-type special functions, can be introduced and some of their applications have been considered in [8, 13, 14, 24].

#### 5.1 Computation via the isomorphisms $T_s$

Acting with the isomorphism  $\mathcal{T}_s$  on both sides of equation (14), we find:

$$\int_0^\infty \mathcal{T}_s[\exp^{-1}(s\,t)]\,f(t)\,dt = \mathcal{T}_s[F_0(s)]\,,$$

that is

$$\int_0^\infty [e_1(s\,t)]^{-1} f(t)\,dt = \mathcal{T}_s[F_0(s)]\,,$$

so that, comparing this result with equation (11), we find:

$$\mathcal{T}_s[F_0(s)] = F_1(s) \,.$$

Of course this equation can be generalized starting from (12), obtaining:

$$\int_0^\infty \mathcal{T}_s[e_r(s\,t)]^{-1}f(t)\,dt = \int_0^\infty \sum_{k=0}^\infty [e_{r+1}(s\,t)]^{-1}f(t)\,dt = \mathcal{T}_sF_r(s)\,,$$

and therefore

$$\mathcal{T}_s[F_r(s)] = F_{r+1}(s) \,.$$

## 6 A more general extension of the Laplace transform

A further extension of the transforms (11)-(12) is as follows. Given the sequence  $a := \{a_k\} = (1, a_1, a_2, a_3, ...)$ , we consider the function:

$$\frac{1}{1+a_1t+a_2\frac{t^2}{2!}+a_3\frac{t^3}{3!}+\dots} \quad (t \ge 0).$$
(16)

When  $a_k = 1/(k!)^r$  the function (10) is recovered, and for r = 0 we find again  $\exp(-t)$ . Note that the functions (16) are complete monotonic functions decreasing from the initial value 1, at t = 0, and vanishing at infinity.

Therefore, according to the umbral method, we put by definition:

$$\mathcal{L}_{a}(f) := \int_{0}^{\infty} \frac{f(t)}{\sum_{k=0}^{\infty} \frac{a_{k}(s\,t)^{k}}{k!}} dt = \int_{0}^{\infty} \frac{f(t)}{\sum_{k=0}^{\infty} \frac{a^{k}(s\,t)^{k}}{k!}} dt = F_{a}(s)$$
(17)

Recalling the Blissard problem, the solution of the the umbral equation

$$\frac{1}{\sum_{k=0}^{\infty} \frac{a^k (s\,t)^k}{k!}} = \sum_{k=0}^{\infty} \frac{b^k (s\,t)^k}{k!}$$
(18)

that is

$$\exp[a(s\,t)]\exp[b(s\,t)] = 1\,,\tag{19}$$

is given by equation (9).

Therefore, the generalized Laplace transform (13) writes:

$$\mathcal{L}_{a}(f) := \int_{0}^{\infty} f(t) \left[ 1 + \sum_{k=1}^{\infty} Y_{k}(-1!, a_{1}; 2!, a_{2}; \dots; (-1)^{k} k!, a_{k}) \frac{(s t)^{k}}{k!} \right] dt = F_{a}(s) . \quad (20)$$

By using equation (2), equation (20) becomes

$$\mathcal{L}_{a}(f) = \int_{0}^{\infty} f(t) \left[ 1 + \sum_{k=1}^{\infty} \sum_{h=1}^{k} (-1)^{h} h! B_{k,h}(a_{1}, a_{2}, \dots, a_{k-h+1}) \frac{(s\,t)^{k}}{k!} \right] dt =$$

$$= \int_{0}^{\infty} f(t) \left[ 1 + \sum_{k=1}^{\infty} \sum_{h=1}^{k} (-1)^{h} h! B_{k,h}(a_{1}, a_{2}, \dots, a_{k-h+1}) \frac{(s\,t)^{k}}{k!} \right] dt = F_{a}(s).$$
(21)

It is convenient to introduce the definition

$$C_k(a) := \sum_{h=1}^k (-1)^h h! B_{k,h}(a_1, a_2, \dots, a_{k-h+1}), \quad C_0(a) := 1,$$
(22)

so that equation (20) writes

$$\mathcal{L}_{a}(f) = \int_{0}^{\infty} f(t) \sum_{k=0}^{\infty} C_{k}(a) \,\frac{(s\,t)^{k}}{k!} \,dt = F_{a}(s) \,.$$
(23)

Note that the ordinary Laplace transform corresponds to the sequence: a = (1, 1, ..., 1, ...), that is  $a_k \equiv 1, \forall k \ge 1$ . Therefore, it results:

$$\mathcal{L}_{(1,1,\dots,1,\dots)}(f) \equiv \mathcal{L}(f) \,. \tag{24}$$

In this case, we find:

$$B_{k,h}(1,1,\ldots,1) = S(k,h),$$

where S(k, h) are the Stirling numbers of the second kind [11]. Then

$$C_k(1, 1, ..., 1) = \sum_{h=1}^k (-1)^h h! S(k, h), \quad C_0(a) := 1.$$

Recalling the known identity

$$\sum_{h=1}^{k} (-1)^{k-h} h! S(k,h) = 1,$$

we find

$$C_k(1, 1, \dots, 1) = (-1)^k, \quad \forall k \ge 0,$$

so that the ordinary expression of the Laplace transform is recovered.

A number of sums defining the coefficients  $C_k(a)$  corresponding to different sequences  $a = \{a_k\}$  can be found in [30], however, in what follows, we will assume the fundamental hypothesis that preserves the property of the ordinary Laplace transform:

**HP.** For every fixed s in the region of convergence, the power series  $\sum_{k=0}^{\infty} C_k(a) (st)^k / k!$ , in equation (23) has an exponential decay to zero when  $t \to \infty$ .

In this framework, another possibility is to assume  $a_k = k!$ . In this case equation (16) becomes:

$$\frac{1}{1+t+t^2+t^3+\dots} \quad (t \ge 0).$$
(25)

The truncation of the geometric series at the denominator in equation (25) produces graphs corresponding to the sequences

$$(1, 1, 0, 0, 0, \dots), (1, 1, 1, 0, 0, 0, \dots), (1, 1, 1, 1, 0, 0, 0, \dots).$$

The decreasing character of the corresponding graphs increases as the number of units increase.

**Remark 1.** Increasing the values of the sequence  $\{a_k\}$  in equation (16), the corresponding graphs exibilit a more fast decreasing character. Then the transforms corresponding to the relevant truncations can be limiteded to a small interval of the type [0, L], as the values of the function (16) become negligible outside this interval.

### 7 Properties

From the definition (23) the following properties are derived:

#### • Linearity

$$\mathcal{L}_a(Af_1 + Bf_2) = A \mathcal{L}_a(f_1) + B \mathcal{L}_a(f_2).$$
<sup>(26)</sup>

#### • Homothetic property

Putting:  $x a := (xa_1, x^2a_2, \ldots, x^na_n, \ldots)$ , by using the isobaric property (5) of the Bell polynomials, it results:

$$\mathcal{L}_{xa}(f) = F_a(xs) \,. \tag{27}$$

This can be interpreted as an homothety between the space of the a parametes and that of the variable s.

• Scaling property

$$\mathcal{L}_a(f(d\,t)) = \frac{1}{d} F_a\left(\frac{s}{d}\right) \,. \tag{28}$$

**Proof.** From equation (23) we find:

$$F_a\left(\frac{s}{b}\right) = \int_0^\infty f(t) \sum_{k=0}^\infty C_k(a) \left(\frac{s}{b}\right)^k \frac{t^k}{k!} dt \,,$$

changing variable, putting t = b x, it results

$$F_a\left(\frac{s}{b}\right) = b \int_0^\infty f(b\,x) \sum_{k=0}^\infty C_k(a) \,\frac{(s\,x)^k}{k!} \, dx = \mathcal{L}_a(f(b\,x)) \,,$$

that is equation (28), up to the change of name of the variable t.

• Action on the derivative

$$\mathcal{L}_{a}(f') = \int_{0}^{\infty} f'(t) \sum_{k=0}^{\infty} C_{k}(a) \frac{(s t)^{k}}{k!} dt =$$

$$= -s \int_{0}^{\infty} f(t) \sum_{k=0}^{\infty} C_{k+1}(a) \frac{(s t)^{k}}{k!} dt - f(0) .$$
(29)

**Proof.** It is sufficient to integrate by parts and to use the above HP.

## 8 Computational techniques

According to the above definitions, it is possible to prove the theorems

**Theorem 1.** Let f(t) be an analytic function on the real axis. Using the Taylor expansion

of the function f(t), centered at the origin:

$$f(t) = \sum_{k=0}^{\infty} c_k \frac{t^k}{k!}, \qquad (30)$$

equation (23) writes:

$$\mathcal{L}_{a}(f) = \int_{0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} c_{n-k} C_{k}(a) \frac{(s t)^{k}}{k!} dt = F_{a}(s).$$
(31)

**Proof.** - In fact, from equations (23)-(30), by using the Cauchy product of power series, we find

$$\mathcal{L}_{a}(f) = \int_{0}^{\infty} \sum_{k=0}^{\infty} c_{k} \frac{t^{k}}{k!} \sum_{k=0}^{\infty} C_{k}(a) \frac{(st)^{k}}{k!} dt = \int_{0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} c_{n-k} C_{k}(a) \frac{(st)^{k}}{k!} dt,$$

that is the result.

**Theorem 2.** Let f(t) be a function expressed by the Laurent expansion:

$$f(t) = \sum_{k=0}^{\infty} c_k \frac{t^{-k}}{k!} \,, \tag{32}$$

then, equation (23) writes:

$$\mathcal{L}_{a}(f) = \int_{0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} c_{n-k} C_{k}(a) \frac{s^{k} t^{-n+2k}}{n!} dt = F_{a}(s).$$
(33)

**Proof.** - In fact, considering the Cauchy product:

$$\sum_{k=0}^{\infty} c_k x^{-k} \sum_{k=0}^{\infty} a_k x^k = \sum_{n=0}^{\infty} \sum_{k=0}^n c_{n-k} x^{-n+k} a_k x^k = \sum_{n=0}^{\infty} \sum_{k=0}^n a_k c_{n-k} x^{-n+2k} .$$
(34)

from equations (23)-(32), we find

$$\mathcal{L}_{a}(f) = \int_{0}^{\infty} \sum_{k=0}^{\infty} c_{k} \frac{t^{-k}}{k!} \sum_{k=0}^{\infty} C_{k}(a) \frac{(s\,t)^{k}}{k!} dt = \int_{0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{c_{n-k}}{(n-k)!} \frac{C_{k}(a)}{k!} s^{k} t^{-n+2k} dt,$$

that is the result.

### 9 A general isomorphism $\mathcal{T}_s(a)$

The results of Section 5 suggests the possibility to introduce a more general isomorphisms. The isomorphism  $\mathcal{T}_s$ , defined in Section 5 is determined by the sequence  $a^{[1]} := (1, 1/2!, 1/3!, \ldots)$ . Now, given a sequence of nonvanishing real numbers  $a := (a_1, a_2, a_3, \ldots)$ ,  $(a_k \neq 0, \forall k)$ , we can define a correspondence acting into the space  $\mathcal{A} := \mathcal{A}_s$  of analytic functions of the *s* variable by means of the position:

$$\mathcal{T}_s(a)s^n = a_n s^n$$

In particular, the isomorphism  $\mathcal{T}_s$  is recovered, since it results:  $\mathcal{T}_s(a^{[1]}) \equiv \mathcal{T}_s$ . Even if this isomorphism is not derived form a differential operator, it is still possible to apply it to the generalized Laplace transform (23), obtaining the equation:

$$\mathcal{T}_s(a)[F_a(s)] = \int_0^\infty f(t) \sum_{k=0}^\infty C_k(a) a_k (s t)^k dt.$$

### 10 Conclusion

It has been shown that, by exploiting Laguerre-type exponentials, it is possible to introduce generalized forms of the Laplace transform that it is supposed to be applied in the treatment of differential equations that use the Laguerre derivative instead of the ordinary one. For these transforms it was also possible to deduce the transformed functions by means of a differential isomorphism studied in previous articles.

The particular form of Laguerre exponentials also suggested a wider extension of the Laplace transform, which is associated with a sequence of numbers denoted by the umbral symbol *a*. This extension, through the solution of the Blissard problem, which uses Bell's polynomials in a natural way, has made it possible to formally define a whole class of transforms, each of which is associated with a fixed sequence. Some fundamental calculation rules have been demonstrated for all the generalized transformations considered. Numerous problems remain open, first of all the existence and the analytical proof of the inverse transformation, which should be based on the extension of the Fourier transform to the Laguerrian case, with all the problems related to the study of a completely new Fourier type analysis.

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