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DOCTORAL THESIS

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# Benchmark Tracking Portfolio Problems with Stochastic Ordering Constraints

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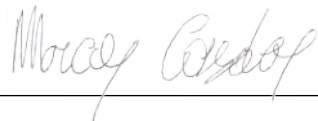


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*“ In the end, a theory is accepted not because it is confirmed by conventional empirical tests, but because researchers persuade one another that the theory is correct and relevant. ”*

Fischer Black

# *Abstract*

This work debates several approaches to solve the benchmark tracking problems and introduces different orders of stochastic dominance constraints in the decisional process. Portfolio managers usually address with the problem to compare their performance with a given benchmark. In this work, we propose different solutions for index tracking, enhanced indexation and active managing strategies. Firstly, we introduce a linear measure to deal with the passive strategy problem analyzing its impact in the index tracking formulation. This measure results to be not only theoretically suitable but also it empirically improves the solution the results. Then, proposing realistic enhanced indexation strategies, we show how to solve this problem minimizing a linear dispersion measure. Secondly, we generalize the idea to consider a functional in the tracking error problem considering the class of dilation, expected bounded risk measures and  $L^p$  compound metric. We formulate different metrics for the benchmark tracking problem and we introduce linear formulation constraints to construct portfolio which maximizes the preference of non-satiabile risk averse investors with positive skewness developing the concept of stochastic investment chain. Thirdly, active strategies are proposed to maximize the performances of portfolio managers according with different investor's preferences. Thus, we introduce linear programming portfolio selection models maximizing four performance measures and evaluate the impact of the stochastic dominance constraints in the ex-post final wealth.

**Keywords:** Benchmark tracking problem, dispersion measure of tracking error, performance measure, linear programming, stochastic dominance constraints.

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This thesis is dedicated to my loved ones.

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*Dedicated to Giuseppe, Stefania, Silvia and Paola*

# Chapter 1

## Introduction

### 1.1 Motivation

The financial world is complex and cannot be easily understood. Getting out of it becomes a demanding effort also for the best of the practitioner since financial markets represent a thick and savage jungle for most of the people who work in the financial industry. Economists have given great attention to test the concept of market efficiency and rationality. However, agents have unstable and unpredictable preferences and they do not make rational choices with them<sup>1</sup>. For this reason, the problem of choice is still a challenging issue for the financial agents or investors since “it is often said that investment management is an art, not a science” (Fabozzi, 2012).

The XXI century seems to represent a clear empirical example of the complexity of financial markets and their imperfections. Since the introduction of the financial modelling in the 1980s to price the financial derivatives (Schoutens, 2003), the level of complexity of the markets has forcefully increased, generating the so called “Century of crisis”. In fact, the first decade of XXI century will always be remembered as the

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<sup>1</sup>Financial literature assume agents with constant preferences. However, empirical evidences in the financial markets show the presence of contagious enthusiasm or worries among different kind of investors.

most dramatic economic and financial period in history. Three different crises marked these years: the Dot-com speculative Bubble in 2001-2002 (started in March 2000), the sub-prime mortgage crisis in 2007-2009 and the following Eurozone sovereign debt crisis and economic recession.

The Dot-com Bubble had already begun when people forged the term “information superhighway” in the early 1990s. The Bubble was a stock market bubble created by a period of investment and speculation in Internet firms that occurred between 1995 and 2001 and that popped with near-devastating effect in 2001, as broadly documented by (Ofek and Richardson, 2003). The year 1995 marked the beginning of a major jump in growth of Internet users who were seen by companies as potential consumers. As a result, many Internet start-ups were born in the mid and late 1990s and their IPOs emerged with ferocity and frequency, sweeping the nation up in euphoria. In 1999 there were 457 IPOs in the US market, most of which were Internet and technology related. Of these 457 IPOs, 117 doubled in price on the first day of trading. In 2001, the number of IPOs dwindled to 76 and none of them doubled on the first day of trading (Ljungqvist and Wilhelm, 2003). It is clear the presence of a significant bubble in the price of the Dot-com price characterized by high volatility. Few months later the prices suddenly slumped putting in jeopardy thousands of investors. This was only the first crisis of the XXI century.

In 2007, another financial crisis was triggered by the U.S. subprime crisis and then by the Lehman & Brothers default in September 2008 (Longstaff, 2010). These facts proved the inefficiency not only of the financial system but also of the financial modellings in their inability to price, forecast and model the complexity of the entire system<sup>2</sup>. Moreover, the assertion “too big to fail” (Sorkin, 2010) was debunked and policy makers and monetary authorities were not always willing to prevent financial institutions from defaulting. The weakness of the economic and financial theories puts in jeopardy not

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<sup>2</sup>The concept of efficiency related with financial modelling is defined as the ability to better forecast, price and measure the stylized facts observed in financial markets such as clustering of the volatility effect, heavy tails, and skewness. It is not possible to capture every feature of the financial market but it is clear that some modelizations and tools are more corrected than others.

only banks and industries but also governments ([Demyanyk and Van Hemert, 2011](#)). As a consequence, countries with high debt levels began to face more stress on their debt servicing capabilities and, hence, were penalized more. With the rescue of Greece and Ireland in 2010 and of Portugal and Greece again in 2011, it became clear that the origin of the sovereign and economic debt crisis in Europe was beyond the popular belief that it was caused by the financial sector as a result of the sub-prime bubble. The origin of the debt crisis in Greece, Italy and Ireland was the structural deficit in the government sector. Greece and Italy had a large fiscal deficit and huge public debts due to persistent imbalances, while in Ireland the crisis was mainly caused by a domestic housing boom financed by foreign borrowers who did not require a risk premium related to the probability of default ([Lane, 2011](#)).

The crises which characterized the first decade of the XXI century dictate the feebleness of the modern financial models and instruments needed to describe the financial sector and its features. The research of new concepts and diverse approaches is therefore essential to guarantee and safeguard the stability of the entire system. Moreover, financial crises are not blimps but incentives to improve the knowledge of the markets and the behavior of the agents. Crises represent an intrinsic feature of the financial world that also a perfect modelization should consider. For this reason, the research of new ways to built portfolio models is crucial to protect the investors' wealth.

To synthesize the behavior of a given market, stock indexes are efficient financial instruments to understand its trend and they allow to focus investment in a specific sector or, generally, to compare the performance of an invested portfolio with a benchmark. Analyzing the trends of two stock indexes plotted in [Figure 1.1](#), the Nasdaq 100 and the KBW Bank Index, during the period between April 1993 and October 2014, we observe how differently they behave.

Firstly, these two indexes represent the financial sectors involved in the Dot-com Bubble ([Leger and Leone, 2008](#)) and in the sub-prime mortgage crisis. The Nasdaq 100 dominates the KBW on the overall period. However, it is evident that there is no

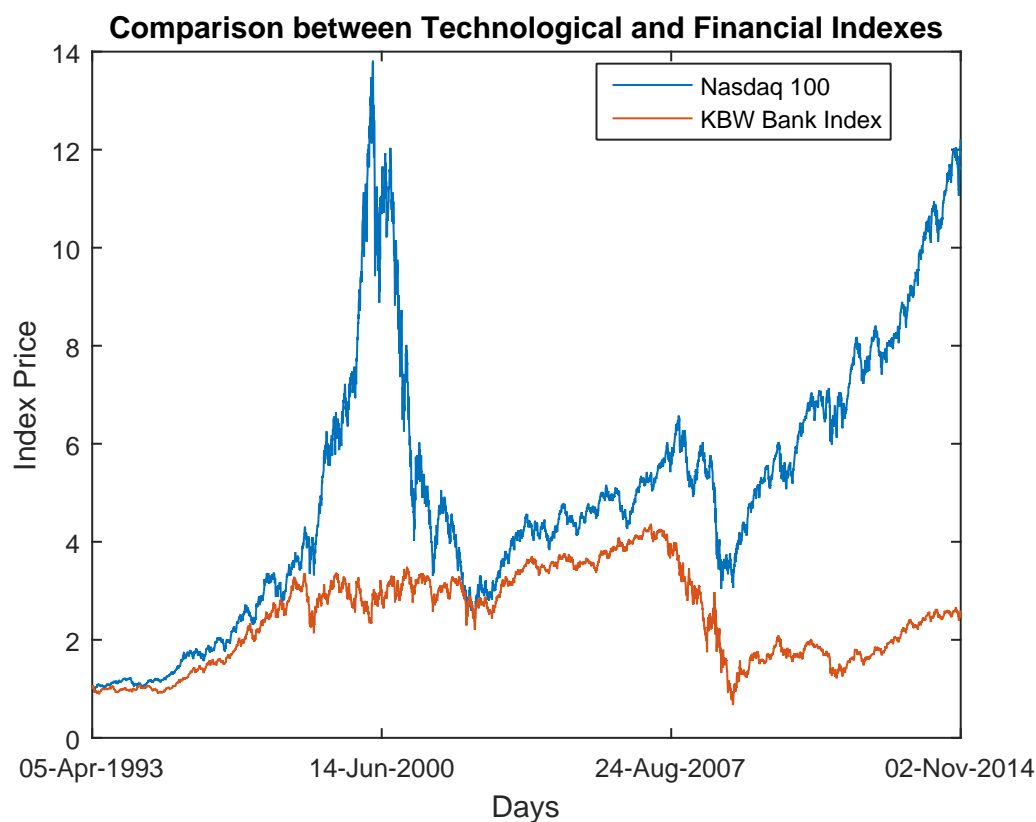


FIGURE 1.1: Comparison between Technological and Financial Indexes

correlation between them during the first financial meltdown period and also during the sub-prime crisis. In fact, at the end of the summer in 2007, their trend is inverted: it starts to fall down with different timing and momenta.

Secondly, although the speculative bubble which originated the crisis in 2000 is clearly evident analyzing the Nasdaq 100, it is not possible to observe the causes that triggered the sub-prime crisis in 2007. Focusing on the panel of the 24 American financial institutes that composes the KBW seems not to present a speculative behavior in 2000 and they are not affected and infected from the Dot-com era. However, differently from the Nasdaq 100 components, financial institutes suffer the consequences of the financial meltdown in 2007 and they had a long period of rescue to increase their market value.



In fact, the Nasdaq 100 increased forcefully after the introduction of the first FED quantitative easing in March 2009 while KBW remains stable for the following period.

This analysis stresses the importance to build portfolio models that are able to capture indexes with different behavior and number of assets. Since it is not possible to literally hold a stock index, investors and portfolio manager focus their effort to replicate or outperform the performances of these financial instruments. Nowadays, the continuous evolution in the products, rules and technologies affects the financial markets and portfolio managers are investigating different investment solutions with budget, policy and risk constraints. The benchmarking of these indexes results to be very demanding considering the behavior of assets where dimensionality and complexity represent an important variable according with the development of the financial engineering.

Figure 1.2 shows three different assets for financial and technological sectors. In the above chart, we report the three main weight components of the KBW Index (Citigroup blue line, JP Morgan red line and Wells Fargo yellow line) while the below graph illustrates three different technological stocks where two of them are in the Nasdaq 100. In the above chart of Figure 1.2 we notice the differences in the wealth paths of the three stocks. In particular, a dynamic benchmarking problem to replicate or outperform the related index should consider how they react to market movements. Whether Citi has relevant gains before the sub-prime crisis then Wells Fargo captures market opportunity in the following financial upturn and better synthesizes the behavior of the stock index. In fact, considering several combinations of the three assets it is possible to maximize investors preferences in an active or passive framework.

The below chart of Figure 1.2 presents three technological stocks. In this sector, we notice similar behavior in the wealth path of the three stocks but different amplitude in response to financial inputs. In particular, Cisco outperforms the other two stocks for the entire period but has strong shifts in the wealth path characterized from high volatility. This graph stresses also the importance of the asset picking process to select robust stocks which are not conditioned to period of financial euphoria. For this reason,

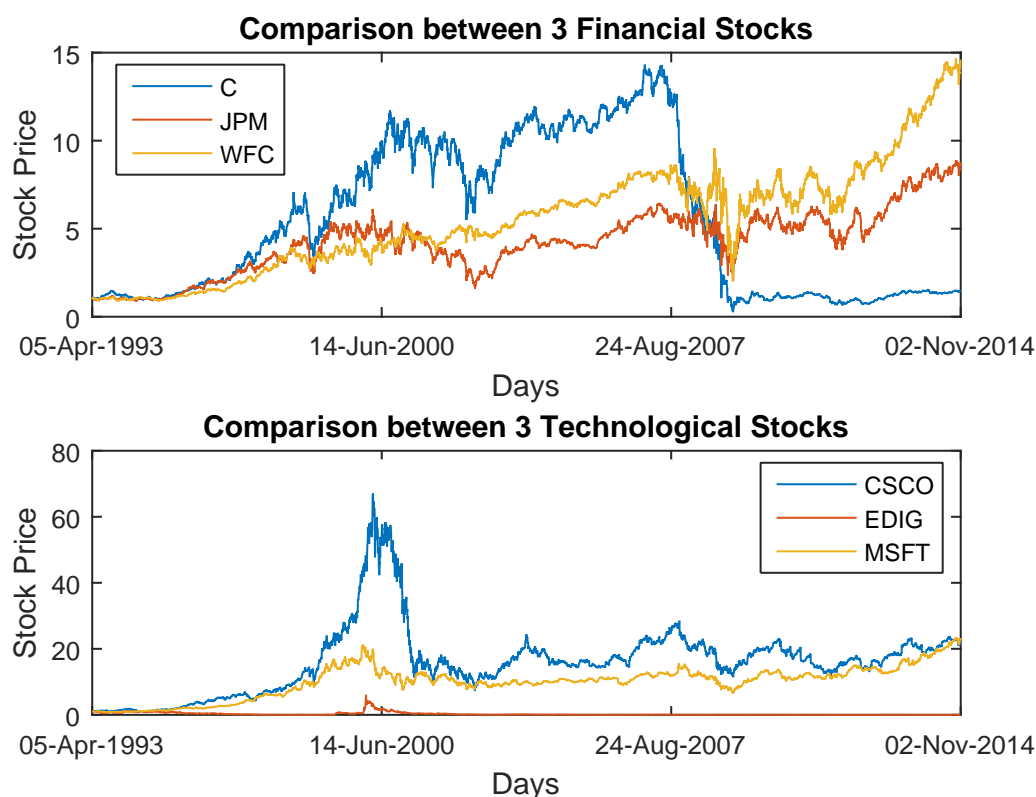


FIGURE 1.2: Comparison between 3 Financial stocks: Citigroup, JP Morgan and Wells Fargo; and 3 Technological stocks: Cisco, eDigital and Microsoft

strategies which aim to mimic or outperform benchmark returns maximizing different investors preferences should be grounded not only on the attractive or conservative returns of the stocks but also on their capability to reduce the risk with respect to the selected benchmark.

The concept of creating portfolios capable to maximize investor's utility can be obtained optimizing different measures presented in the financial literature. In particular, we observed the importance to mimic the behavior of a stock index or to replicate its returns during different phases of the financial cycle. This problem which aim to select the optimal portfolio composition in order to reduce the difference between its returns and the given benchmark ones is called benchmark tracking problem and it is part of

a more general framework and area of research known as the Modern Portfolio Theory. A brief introduction of this theory is crucial to point out the state of art and its main unsolved issues.

## 1.2 Literature survey

The Modern Portfolio Theory is a milestone in the financial literature and there have been considerable advances starting with the pathbreaking works of [Markowitz \(1952\)](#). The derivation of optimal rules for allocating wealth across risky assets in a mean-variance analysis served as the beginning of several works which aimed to define the best model to achieve relevant performances by controlling or reducing the risk. Markowitz's main idea was to propose variance as a risk measure and he introduced it in a computational model by measuring the risk of a portfolio via the covariance matrix associated with individual asset returns. This leads to a quadratic programming formulation and it was far from being the final answer to the problem of portfolio selection.

[Tobin \(1958\)](#) included the risk-free asset and showed that the set of efficient risk-return combinations was in fact a straight line, consisting in an optimal portfolio of risky and riskless assets. [Sharpe \(1963\)](#) simplified the computational burden of Markowitz's model using a single factor model. This model assumes that the return on each security is linearly related to the market index and that Tobin and Markowitz's optimal portfolio of risky assets could be formulated as the market itself. This concept leads to the development of Capital Asset Pricing Model (CAPM) ([Fama and French, 1996](#)).

Comparing the works of Markowitz and Sharpe, [Affleck-Graves and Money \(1976\)](#) pointed out possible relations between them. In particular, they observed that the results obtained with Sharpe's model became progressively better adding more indexes and diversifying the portfolio. Increasing the diversification grade, the model simulates the Markowitz one. Their study also found that Markowitz's model naturally limits the maximum weight invested in any share to about 40% considering six active shares.

This fact justify a natural diversification in the efficient portfolio. However, one of the biggest criticisms of Markowitz's model is that it does not produce portfolios that are adequately diversified.

While [Bowen \(1984\)](#) focused on the complexity of the covariance estimation in the case of small volumes of data pointing out the problem of the parameters estimation error ([DeMiguel et al., 2009](#); [Fama and French, 2004](#)) noted that portfolio managers believed that Markowitz's model lead to hold unrealistic portfolios; and they stressed the weakness in the theory of the CAPM. In fact, they argue the model failure in empirical tests implying that most applications of this model results invalid.

Recently, asymmetric risk measures have been proposed since symmetric ones do not intuitively point to risk as an undesirable result. Symmetric risk measures penalize upside deviations from the mean in the same way they penalize downside deviations. [Markowitz \(1952\)](#) also suggests maximizing expected utility instead of expected returns and compare several alternative measures of risk. [Roy \(1952\)](#) develops an equation relating portfolio variance of return to the variance of the return of the constituent securities. He advises choosing the single portfolio that maximizes  $\frac{(\mu_p - d)}{\sigma_p^2}$  where  $\mu_p$  and  $\sigma_p^2$  are the mean and the variance of the portfolio and  $d$  is a "disaster level" return the investor places a high priority of not falling below. Many authors have introduced new risk measure. While lower partial moments of  $n^{th}$  moment as a measure of risk were introduced by [Bawa \(1975\)](#) and [Bawa and Lindenberg \(1977\)](#), [Fishburn \(1977\)](#) introduced a new kind of risk measure where risk is defined by a probability-weighted function of deviations below a specific target return.

An alternative to Markowitz model is the Mean–Absolute Deviation (MAD) model, proposed by [Konno and Yamazaki \(1991\)](#) and pioneered by [Yitzhaki \(1982\)](#) that introduced and analyzed the mean–risk model using the Gini's mean difference as a risk measure. While Markowitz model assumes normality of stock returns, the MAD model does not make this assumption. The MAD model also minimizes a measure of risk, where the measure is the mean absolute deviation ([Kim et al., 2005](#); [Konno, 2011](#)). This

new measure of risk and its formulation has been broadly applied in the financial field (Zenios and Kang, 1993; Simaan, 1997; Ogryczak and Ruszczyński, 1999).

As the years go by different measures of risk have been proposed (Blume, 1971; Silvers, 1973; Merton, 1974; Fong and Vasicek, 1984; Lakonishok and Shapiro, 1986) but the acceptance of the Value at Risk (VaR) as a risk measure that focuses on the left tail of the return distribution surely represents one of the most important contribution for the financial literature in the 1990s. However, it was integrated by JP Morgan into its risk-management system since the late 1980s. In this system, they developed a service called RiskMetrics which was later spun off into a separate company called RiskMetrics Group. It is usually thought that JP Morgan invented the VaR measure. In fact, similar ideas had been used by large financial institutions in computing their exposure to market risk. The contribution of JP Morgan was that the notion of VaR was introduced to a wider audience. In the mid-1990s, the VaR measure was approved by regulators as a valid approach to calculating capital reserves needed to cover market risk (Morgan, 1996). The Basel Committee on Banking Supervision released a package of amendments to the requirements for banking institutions allowing them to use their own internal systems for risk estimation.

In fact, the main criticism to the standard deviation was its feature to be a measure of risk and not a measure of loss. In particular, it does not give any information about the possible losses of my portfolio given a level of confidence. The Value at Risk aims to solve this kind of problem and to introduce the importance of measuring risk for regulatory purposes, not only as a parameter in a model of choice. However, VaR measures the minimum loss corresponding to certain worst number of cases but does not quantify how bad these worst losses are and it presents two fundamental drawbacks: it is not sub-additive and it is not a convex function of the portfolio weights (Ortobelli et al., 2005; Rachev et al., 2008b). To overcome these efforts, Artzner et al. (1999) provide an axiomatic definition of a functional which they call a coherent risk measure. These axioms are the monotonicity, positive homogeneity, sub-additivity and invariance

and they are the fundamental structure of a new risk measure: the Conditional Value at Risk (CVaR) (Rockafellar and Uryasev, 2002, 2000), or Expected Shortfall (Acerbi and Tasche, 2002).

To conclude, we want discuss about two improvement in the vast portfolio selection framework: the introduction of the higher moment in the portfolio problem and the distribution of the portfolio returns. Starting from the seminal paper of Samuelson (1970) several works focus on the definition of portfolio selection with higher moments developing the idea that the description of the return distribution with only two parameters involve significant losses of information (Malevergne and Sornette, 2005; Konno et al., 1993; Rubinstein, 1973; Arditti, 1971).

In particular, Harvey et al. (2010) build on the Markowitz portfolio selection process by incorporating higher order moments of the assets returns. They propose the skew normal distribution as a characterization of the asset returns and using Bayesian methods they make a comparison with other optimization approaches concluding a higher expected utility of this innovative portfolio selection problem. Bayesian methods allow to create portfolio selection problems suitable for different type investor, model formulation and parameters estimation. Garlappi et al. (2007) propose a formulation that consider the uncertainty of the model and parameters building an ambiguity-averse portfolio that deliver a higher out of sample Sharpe ratio with respect to the classical ones. Finally, several paper discuss about the distribution of the portfolio returns proposing different modelization of the problem (Adcock, 2010; Li et al., 2010; Maccheroni et al., 2009; Ortobelli and Angelelli, 2009; Kole et al., 2007; Rachev et al., 2007).

The usual assumption given in several work is the Gaussianity or Normality of the asset returns. This hypothesis implies several advantages in many fields of the mathematical finance since the closed form solution of its probability density function gets it suitable in many empirical applications. In particular, Tobin (1958) shows that if asset returns are normally distributed then variance is the proper measure of risk. However, as noted by Mandelbrot (1997), Fama (1965) and recently developed by a large part

of the literature (Cont and Tankov, 2004; Carr et al., 2002; Rachev and Mittnik, 2000; Sato, 1999), the asset returns exhibit heavy tails, leptokurtosis and they are subject to volatility clustering phenomena (Rachev et al., 2011; Cont, 2001). For this reason the introduction of different statistical hypotheses have done relevant advantages in the definition of the asset returns distribution with benefits in the portfolio selection problems (Rachev et al., 2008a, 2007; Stoyanov et al., 2007; Rachev et al., 2005; Embrechts et al., 2003).

In particular, Ortobelli et al. (2010) tested the alpha stable distributional hypothesis in the stock market comparing it with the Gaussian one. They dictate that heavy tails of residuals can have a fundamental impact on the asset allocation decisions by investors and the stable Paretian model dominates the moment-based model in terms of expected utility and of the ex-post final wealth. Under the stability hypothesis, the introduction on other parameters such as the skewness and kurtosis significantly improve the description of the distributional behavior (Rachev et al., 2011). Moreover, the Student-t distribution and the Stable distribution are very used model to describe asset returns as proposed by Rachev and Mittnik (2000) and Blattberg and Gonedes (1974).

In this case, the portfolio selection problem considering higher-order could present significant improvements introducing estimators for coskewness and cokurtosis parameters as argued by Martellini and Ziemann (2010). In the financial markets, investors solve the problem to select the optimum optimizing a reward/risk performance measure (Rachev et al., 2008b; Stoyanov et al., 2007). Thus, Biglova et al. (2004) propose a different performance measure, the Rachev ratio, that maximize the utility of a not satiable nor risk averse nor risk seeker investor. This performance measure differs from the classical Sharpe ratio (Sharpe, 1994) that maximize the utility of a not satiable risk averse investor since it consider as risk and return measures the CVaR at two different percentiles of the return distribution. This work represents a starting point for others performance measure based on the return distribution as the STARR introduced by

Martin et al. (2005) and the Rachev ratio higher-moments presented by Ortobelli et al. (2009) that combine tails behavior with the introduction of skewness and kurtosis in the decisional problem.

### 1.3 Aim of the thesis

The motivation for this study was based on several avenues in the literature on portfolio selection. The pioneering work of Markowitz introduced the possibility to create portfolios based on a reward and risk measure and although sixty years of research and development have passed, it remains an unsolved puzzle in the financial literature. In fact, with the recent economic crises came the necessity of new views and approaches to improve the common financial models.

The main objective of this essay is to analyze the entire benchmark tracking problem. Facing this issue, portfolio managers want to find the optimal portfolio composition that maximizes the management style. In particular, benchmark tracking portfolio strategies could be divided into three main categories: passive, enhanced indexing and active. This essay addresses with these three problems proposing theoretical and methodological solution to maximize investors' preferences. Empirical applications involving different phases of the financial cycle during the last decade enforce the goodness of the proposed methodology. In fact, comparison with common approaches highlights the importance of innovative solutions and instruments to solve the portfolio selection problem. They are necessary considering the continuous evolution of financial engineering and researchers should provide tools to describe and modelize the complexity of the financial system.

According to Konno and Hatagi (2005), almost half the capital in the Tokyo Stock exchange is subject to passive trading strategies, while Frino et al. (2005) report that assets benchmarked against the S&P 500 exceed US\$1 trillion. Over recent years, passive portfolio management strategies have seen a remarkable renaissance. Assuming that the market cannot be beaten on the long run, these strategies aim to mimic a given market



index investing either into a replication of the benchmark, or selecting a portfolio which exhibits a behavior as similar as possible to the benchmark one. The market share of products such as exchange traded funds (ETFs) has increased significantly, and it is argued that passive portfolio management is becoming predominant. If investments are benchmarked against the index, a fund that aims to replicate this benchmark will, by definition, have a lower likelihood to severely fall below it.

The most common approach in passive portfolio management is index tracking. In this strategy, investors select portfolios that mimic the behavior of an index representing the entire market, such as the MSCI World Index, or one of its sectors. To find the optimal combination, the definition and minimization of a distance measure between tracking portfolio and benchmark index is the crucial point to efficiently manage this problem. In contrast, active portfolio management tries to generate excess returns picking stocks which are expected to outperform the market and avoiding assets that are expected to under-perform it. Both approaches have their advantages and disadvantages: active strategies rely heavily on superior predictions while passive strategies require few assumptions about future price movements. Passive strategies will also copy the benchmark's poor behavior while active strategies can react more flexibly in bear markets; etc. In the middle we find the enhanced indexing strategies that try to capture the best feature of both approaches proposing a portfolio composition that minimizes risk looking for extra-performances.

Thus, in Chapter 2 starting from the problem to mimic the performance of a financial index considering all its components or a subset only, we propose a new dispersion measure of the tracking error. This measure, called tracking error quantile regression, results to be suitable to track a given benchmark not only from a theoretical and but also from an empirical point of view. In fact, it overcomes some drawbacks of the common dispersion measures such as non-linearity and symmetry confirmed in an empirical application. The contribution made by Chapter 2 is theoretical and methodological since it describes the introduced dispersion measure based on the quantile regression with

its theoretical structure. The methodological contribution is then developed proposing a realistic LP model to solve the enhanced index problem. This problem represents a hot topic of research since every portfolio manager aims not only to track an index from above reducing the dispersion measure but also to obtain gains in the out of sample analysis. For this reason, we introduce stochastic dominance constraints in the minimization problem of the tracking error to enhance the portfolio performances. An empirical application is also developed showing the enhancement of portfolios wealth path in the out of sample analyses.

Chapter 3 generalizes the concept of dispersion measure reviewing the class of the coherent expectation bounded risk measures for the benchmark tracking problem. These measures, like the class of Gini dispersion measures, represent a useful metric to improve the decisional problem in the replication of the performances of an given index. Then, we introduce the methodology of stochastic investment chain grounded on the concept to create portfolio with stronger behavior derived from three consequent optimization steps increasing the level of stochastic dominance where the dominant portfolio become the benchmark. The contribution of this chapter is theoretical and methodological. On one hand, we analyze the linearity of these measures proposing different portfolio problems based on the dispersion component of coherent expectation bounded risk measures. In particular, since this class of measures is consistent with Rothschild–Stiglitz ordering, we could derive a tracking error problem consistent with this ordering. On the other hand, we theoretically develop linear programming formulation to solve portfolio problems with bounded third order stochastic dominance constraints. In this framework, we considering an aggressive Rachev utility function which is consistent with the preference of non-satiable nor risk seeking nor risk averse investors we develop the concept of stochastic investment chain.

Finally in Chapter 4, we deal with portfolio strategies for active management. In the Modern Portfolio Theory, the maximization of the investors' future wealth is still an relevant problem. Thus, we propose portfolio strategy which does not focus on

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the risk minimization but on the maximization of performance measures considering different ratios. The contribution of this chapter is theoretical, methodological and empirical. Since investors maximize their utility in a reward-risk sense, we implement linear portfolio optimization problems maximizing four different performance measures. In the theoretical part, we review the linear programming model of two performance measures while we develop the theoretical formulation for the Sharpe Ratio and the Mean Absolute Semideviation Ratio. Then, introducing first and second order stochastic dominance constraints we propose different portfolio selection models to strengthen the performances of invested portfolios. Finally, we empirically test the benefit to introduce stochastic dominance in portfolio problems considering its impact in the maximization of future wealth.

## Chapter 2

# Tracking Error Quantile

## Regression.

# A Dispersion Measure for the Benchmark Tracking Problem.

### 2.1 Introduction

One of the most important objective that every fund manager has to achieve is the index tracking problem. Many portfolios are managed to a benchmark or index and they are expected to replicate, its returns (e.g., an index fund), while others are supposed to be “actively managed” deviating slightly from the index in order to generate active returns. The tracking problem has been broadly described in the financial literature from different point of views. On the one hand, the research community focuses on the identification of efficient algorithms to solve the optimization problem through the development of a large diversity of heuristics and metaheuristics ([Angelelli et al., 2012](#); [Guastaroba and Speranza, 2012](#); [di Tollo and Maringer, 2009](#); [Beasley et al., 2003](#); [Gilli](#)

and K ellezi, 2002). On the other hand, several approaches have been introduced in order to describe empirical evidences, to improve the decisional problem or to propose different methodologies dealing with the index tracking problem (Krink et al., 2009; Barro and Canestrelli, 2009; Maringer and Oyewumi, 2007; Dose and Cincotti, 2005; Pope and Yadav, 1994).

In particular, Jorion (2003) introduces additional restrictions in the optimization problem reducing the higher risk of the active portfolio with respect to the index as empirically observed by Roll (1992), while Rudolf et al. (1999) illustrate the relationship between the size of bounds on permissible tactical deviations from benchmark asset class weights and their corresponding statistical tracking error measures. An interesting methodology that could be applied to solve the index tracking problem is to take advantage of the positive correlation between the price fluctuations of stocks in the same category building a stratified index portfolio Montfort et al. (2008). This portfolio is obtained dividing the index components into a large number of categories such as, for example, sectors or countries (Focardi and Fabozzi, 2004; Frino et al., 2004) and then by putting together the categories so as that each one of them is represented in the index portfolio with the same extent as in the tracked index.

These approaches are classically grounded on the minimization of a measure of dispersion of the tracking errors, i.e. the difference between the return of the replicating portfolio and the benchmark that an investor was attempting to imitate. Commonly three tracking error dispersion measures are used: 1. the mean absolute deviation (Consiglio and Zenios, 2001; Konno and Wijayanayake, 2001; Konno and Yamazaki, 1991); 2. the downside mean semideviation (Angelelli et al., 2008; Ogryczak and Ruszczyński, 1999; Kenyon et al., 1999; Harlow, 1991) which focus on the negative side of the tracking error; 3. the tracking error volatility (Corielli and Marcellino, 2006; Roll, 1992), which considers the variance of the difference between the tracked and the tracking portfolios.

The weights of the mimic portfolio can be easily determined using a least squared linear regression. Since the errors are the deviation of the index from the expected value

of the replicating portfolio, these measures mainly focus on understanding the central tendency within a data set, but they are less effective and robust at describing the behavior of data points that are distant from the line of best fit. In particular, returns distributions of the financial series are characterized by the presence of asymmetry and heavy tailness (Rachev et al., 2011; Mandelbrot, 1967; Fama, 1965) and it is interesting to investigate a methodology that addresses these features.

The contribution of this chapter is twofold. Firstly, we introduce a dispersion measure of the tracking error which captures the difference between returns of the two portfolios. This measure is suitable for this type of problem since it represents a theoretic ideal measure and empirically works better than three common dispersion measures presented in the literature. Then, the possibility to linearly formulate the index tracking problem with this measure allows to reduce the computational time and complexity of the optimization problem.

Secondly, we introduce an enhanced indexation benchmark tracking problem to guarantee extra-performances of the adopted strategy in the replication problem. For this reason, we propose a realistic model formulation with transaction costs penalty function and turnover threshold level in the minimization of the quantile regression measure of the tracking error. Then, introducing two orders of stochastic dominance constraints we enhance the performances of the invested portfolio in the out of sample analysis. The main advantage of this approach is grounded around the control of the risk source minimizing a dispersion measure of the tracking error while we try to outperform the benchmark. The proposed model is linear which allow to efficiently solve the problem also in the high dimensionality framework.

This chapter is organized as follow. In section 2.2, we introduce the classical benchmark tracking problem and we show how common measures used to solve the index tracking problem deal with it. Section 2.3 discusses the quantile regression method and we derive the related measure of dispersion for the tracking error. We theoretically formulate this measure and its properties. In section 2.4, we present the enhanced index

benchmark tracking problem with stochastic dominance constraints while in the last two section we propose an empirical application and summarize the main results.

## 2.2 Index Tracking Problem

Index tracking problems is related with a benchmark portfolio against which the performance of a managed one is compared. This comparison is based on the distribution of the active portfolio return, defined as the difference  $X - Y$ , in which  $X = r\beta$  is the random variable of the invested portfolio returns with weights represented by the vector  $\beta$  while  $Y$  denotes the benchmarks' returns. Performance and risk of the portfolio managers' strategies are based on this difference. In particular, a measure of performance of the invested portfolio relative to the benchmark is the average active return, also known as portfolio alpha, which is calculated as the difference in the sample means:

$$\hat{\alpha} = \mathbb{E}[X] - \mathbb{E}[Y] \tag{2.1}$$

Differently, a widely used risk measure of how close the portfolio returns are to the benchmark is a deviation measure of the active return, also known as tracking error (TE). The closer the tracking error is to zero, the closer the risk profile of the portfolio matches the benchmark one. These two measures are the decisional parameters in the problem of choice for portfolio managers. They compute the in sample analysis and make decision to apply to the out of sample investment period. In this case, the portfolio alpha is the expectation of the active return and the TE its standard deviation.

There exist several ways to built an index tracking portfolio since portfolio mangers have different constraints and restrictions. However, it is possible to define three main categories of strategies closely related with different levels of alpha and of the tracking error. Active portfolio strategies, that are characterized by high alphas and TEs, aim to outperform the tracked index and allow the portfolio manager to take a high risk moving

away from the real index composition, while passive strategies that are characterized by very small alphas and TEs want to replicate the index performances as close as possible (Frino and Gallagher, 2001; Sharpe, 1992). In between, there are the enhanced indexing strategies, with small to medium-sized alphas and TEs (Canakgoz and Beasley, 2009; Scowcroft and Sefton, 2003).

Thus, let  $Y$  be the return of the benchmark portfolio with realization  $y_t$  at time  $t$  (for  $t = 1, \dots, T$ ) and  $X$  be the random variable of the invested portfolio's returns such that  $x_t = \sum_{n=1}^N r_{t,n} \beta_n$  be the return of the invested portfolio, where  $r_{t,n}$  is the return of the  $n$ -th asset at time  $t$  and  $\beta$  is the vector of portfolio weight. We define a general benchmark tracking problem as follow:

$$\begin{aligned}
 & \min_{\beta} \quad \sigma(X - Y) \\
 & \text{s.t.} \quad \sum_{n=1}^N \beta_n = 1 \\
 & \quad \quad \mathbb{E}[X] - \mathbb{E}[Y] \geq K^* \\
 & \quad \quad lb \leq \beta_n \leq ub \quad \forall n = 1, \dots, N
 \end{aligned} \tag{2.2}$$

where  $\beta_n$  for  $n = 1, \dots, N$  is the portfolio optimal solution of the minimization problem and  $\sigma$  a dispersion measure generated from a given probability metric (Stoyanov et al., 2008a). The first constraints is a budget constraint while the following ones is related to institutional policy and it defines the minimum guaranteed return level  $K^*$ . Finally, the last constraint bounds the upper and lower value of the portfolio weights. In particular, it is important to define not only the maximum percentage invested in a single asset but also the admissibility of short selling position in the portfolio problem.

Generally, portfolio managers want to minimize a dispersion measure of the tracking error subject to several constraints. However, the most debatable problem is related to the cardinality constraints and the institutional rules to avoid fractional positions and a huge number of active assets to manage (Fernholz et al., 1998). On one hand, several works propose heuristics and metaheuristics approaches to reduce the number



of assets (Angelelli et al., 2012; di Tollo and Maringer, 2009; Beasley et al., 2003). On the other hand, the problem is solved through the Lagrangian method switching the discontinuity of the constraints in the optimization problem, setting the parameter lambda and introducing a penalty function to reduce the sparsity (Fastrich et al., 2014; Kopman et al., 2009; Jansen and Van Dijk, 2002). In this essay, we do not directly treat this type of problem but we propose different modelizations that reduce the number of active assets without imposing specific cardinality constraints.

### 2.2.1 Common Measure of Dispersion of Tracking Error

The tracking error is a measure of how closely a portfolio follows the index defined as its benchmark. It can be captured in different ways by a variety of dispersion measures  $\sigma(\cdot)$  of the tracking error or by a class of convex dispersion measures that satisfies an axiomatic structure and they are called deviation measures (Rockafellar et al., 2006). Let  $Y$  be the log-return of equity index with realization  $y_t, t = 1, \dots, T$  and let  $R$  be the returns of its  $N$  components being  $R = r_1, r_2, \dots, r_N$ . We define TE the tracking error  $\varepsilon_t \in \mathbb{R}$  for a single point in time  $t = 1, \dots, T$  where  $\varepsilon_t = \sum_{n=1}^N r_{t,n}\beta_n - y_t$  and  $\sigma(\varepsilon)$  a general dispersion measure. Then, the common used tracking error dispersion measures could be easily defined.

The first common dispersion measure that we review is based on the mean absolute deviation (MAD) dispersion measure (Konno and Yamazaki, 1991). Based on the Gini's measure, the MAD is used in several field of the finance (Kim et al., 2005; Consiglio and Zenios, 2001) such as in the benchmark tracking problem (Konno and Wijayanayake, 2001) for its property to do not make assumption on the returns distribution (Yitzhaki, 1982). The respective dispersion measure to solve the index tracking problem is the tracking error mean absolute deviation measure (TEMAD):

$$\text{TEMAD} \quad \sigma(\varepsilon) = \frac{1}{T} \sum_{t=1}^T |\varepsilon_t| \quad (2.3)$$

This measure takes into account the absolute variation between portfolio and benchmark returns. However, it is a symmetric measure with an equal weight for positive and negative  $\varepsilon_t$  while investors have different preferences and they show a diverse risk profile according to their aversion to negative events.

For this reason also Markowitz in its seminal papers (Markowitz, 1968, 1952) proposed to introduce in the mean-variance analysis a measure that consider only the downside risk. This measure known as downside mean semideviation (DMS) has been developed in several work (Angelelli et al., 2008; Ogryczak and Ruszczyński, 1999; Kenyon et al., 1999; Harlow, 1991) and it is defined as:

$$\text{TEDMS} \quad \sigma(\varepsilon) = \frac{1}{T} \sum_{t=1}^T |\varepsilon_t \mathbf{1}_{[\varepsilon_t < 0]}| \quad (2.4)$$

This measure is clearly asymmetric and it is suitable to capture only the downside risk and averse events but theoretically speaking in the tracking error framework this measure could leads to build portfolios with an intrinsic higher risk.

Finally, the tracking error volatility (TEV) (Corielli and Marcellino, 2006; Jorion, 2003; Roll, 1992) is the most used measure in the financial literature. It is defined as the variance of the error and a forward-looking measure which could be interpreted in terms of Value at Risk (Jorion, 2003). Moreover, a minimization of the TEV seems a sensible goal for fund sponsors or executives to evaluate an ideal active management (Roll, 1992). Then, the TEV is defined as:

$$\text{TEV} \quad \sigma(\varepsilon) = \frac{1}{T} \sum_{t=1}^T (\varepsilon_t - \bar{\varepsilon})^2 \quad (2.5)$$

where  $\bar{\varepsilon}$  is the mean of the error during the considered period. This measure is still symmetric with respect to the mean of the error and it takes into account the quadratic variation of the difference between portfolio and benchmark returns.

In the class of the quadratic measures it is preferred to the mean square error measure for two main reasons. Firstly, an ideal active management strategy would outperform the benchmark every single period by a fixed amount net of fees and expenses. This implies zero tracking error volatility. It is ideal since the fund investment could verify with complete statistical reliability that manager is adding value over an index fund alternative. Secondly, it is a valid measure to evaluate the impact of fund manager strategy during the overall time period.

In this case, we are solving a least squared regression problem and the optimal solution is based on the central tendency between the returns of tracking and tracked portfolios. Differently, it is possible to construct a regression formulation in which a linear equation relates how the quantiles of the dependent variable vary with the independent one (Koenker and Bassett, 1978). In the next section, we introduce this concept developing the relative measure of dispersion of tracking error but now we show how to solve the index tracking problem considering these common measure of dispersion of the tracking error.

## 2.2.2 Index Tracking Problem with Linear and Quadratic Dispersion Measures

The common dispersion measures of the tracking error (2.3), (2.4) and (2.5) allow to efficiently solve the benchmark tracking problem (2.2). While the tracking error volatility shows a quadratic and convex feature in the portfolio problem the other two measures could be defined as linear (Mansini et al., 2003; Speranza, 1993). This implies an efficient way to solve high dimensionality problem and to replicate index with huge number of components.

Let  $Y$  be the log-return of equity index with realization  $y_t$ ,  $t = 1, \dots, T$  and let  $R$  be the returns of its  $N$  components being  $R = r_1, r_2, \dots, r_N$ . We define  $\varepsilon_t \in \mathbb{R}$  such that  $\varepsilon_t = \sum_{n=1}^N r_{t,n}\beta_n - y_t$  tracking error (TE) and  $\sigma(\varepsilon)$  a general dispersion measure. The

benchmark tracking problem with the tracking error volatility measure (TEV) could be defined as:

$$\begin{aligned}
\min_{\beta} \quad & \sum_{t=1}^T (\varepsilon_t - \bar{\varepsilon})^2 \\
\text{s.t.} \quad & \sum_{n=1}^N \beta_n = 1 \\
& \mathbb{E}[X] - \mathbb{E}[Y] \geq K^* \\
& lb \leq \beta_n \leq ub \quad \forall n = 1, \dots, N
\end{aligned} \tag{2.6}$$

where  $\bar{\varepsilon}$  is the mean of the error during the considered period. Solving problem (2.6) we deal with a quadratic optimization with linear constraints.

Differently, the tracking error mean absolute deviation (TEMAD) (2.3) and the tracking error downside mean semideviation (TEDMS) (2.4), could be expressed in a linear formulation way. Let the following optimization problem:

$$\min_{\beta} \sum_{t=1}^T |r_t \beta - y_t| \tag{2.7}$$

that minimize the tracking error mean absolute deviation dispersion measure. We define

$$\begin{aligned}
d_t^+ &= \max \{r_t \beta - y_t, 0\} \quad \forall t = 1, \dots, T \\
d_t^- &= \max \{y_t - r_t \beta, 0\} \quad \forall t = 1, \dots, T
\end{aligned} \tag{2.8}$$

the positive and negative difference between the portfolio and benchmark returns. Then, it is possible to rewrite the optimization problem (2.7) as:

$$\begin{aligned}
\min_{\beta} \quad & \sum_{t=1}^T d_t^+ + d_t^- \\
\text{s.t.} \quad & r_t \beta - y_t = d_t^+ - d_t^- \quad \forall t = 1, \dots, T \\
& d_t^+, d_t^- \geq 0 \quad \forall t = 1, \dots, T
\end{aligned} \tag{2.9}$$

The related linear benchmark tracking problem is defined as follow:

$$\begin{aligned}
& \min_{\beta} \sum_{t=1}^T d_t^+ + d_t^- \\
& \text{s.t.} \quad r_t \beta - y_t = d_t^+ - d_t^- \quad \forall t = 1, \dots, T \\
& \quad \sum_{n=1}^N \beta_n = 1 \\
& \quad \mathbb{E}[X] - \mathbb{E}[Y] \geq K^* \\
& \quad lb \leq \beta_n \leq ub \quad \forall n = 1, \dots, N \\
& \quad d_t^+, d_t^- \geq 0 \quad \forall t = 1, \dots, T
\end{aligned} \tag{2.10}$$

Considering the asymmetric downside mean semideviation measure, we could investigate a linear formulation of the index tracking problem. In particular, for discrete random variables represented by their realizations  $x_t$  and  $y_t$  (for  $t = 1, \dots, T$ ), the downside mean semideviation is a convex piecewise linear function of the portfolio  $r_t \beta$  and the benchmark tracking problem is LP computable. Thus, we define

$$d_t^- = \min \{r_t \beta - y_t, 0\} \quad \forall t = 1, \dots, T \tag{2.11}$$

the sequence of the negative realizations given by the difference of the returns of invested and benchmark portfolios. Then, the minimization the tracking error downside mean semideviation (TEDMS) (2.4) could be defined as follow:

$$\begin{aligned}
& \min_{\beta} \sum_{t=1}^T d_t^- \\
& \text{s.t.} \quad d_t^- \leq r_t \beta - y_t \quad \forall t = 1, \dots, T \\
& \quad d_t^- \geq 0 \quad \forall t = 1, \dots, T
\end{aligned} \tag{2.12}$$

and the related LP benchmark tracking problem as:

$$\begin{aligned}
& \min_{\beta} \sum_{t=1}^T d_t^- \\
& \text{s.t.} \quad d_t^- \leq r_t \beta - y_t \quad \forall t = 1, \dots, T \\
& \quad \sum_{n=1}^N \beta_n = 1 \\
& \quad \mathbb{E}[X] - \mathbb{E}[Y] \geq K^* \\
& \quad lb \leq \beta_n \leq ub \quad \forall n = 1, \dots, N \\
& \quad d_t^- \geq 0 \quad \forall t = 1, \dots, T
\end{aligned} \tag{2.13}$$

In this benchmark tracking problem, we solve the optimization (2.13) as a LP problem.

The presented benchmark tracking problems (2.6), (2.10) and (2.13) are common methods in the financial literature to solve the problem to mimic the returns of a given benchmark. In the following section, we introduce the quantile regression method and the related dispersion measure of the index tracking.

## 2.3 The Quantile Regression

The concept of quantile regression introduced by [Koenker and Bassett \(1978\)](#) represents an interesting methodological approach to investigate and its application to the index tracking problem lead to relevant consideration. We know that any random variable  $X$  may be characterized by its right-continuous distribution function

$$F(x) = \mathbb{P}(X \leq x) \tag{2.14}$$

whereas for any  $0 < \tau < 1$ ,

$$F(\tau)^{-1} = \inf [x : F(x) \geq \tau] \quad (2.15)$$

is called the  $\tau$  *th*-quantile of  $X$ . The median,  $F^{-1}(1/2)$ , plays the central role. The quantiles arise from a simple but fundamental optimization problem. Considering a theoretic problem to find a point estimate for a random variable with distribution function  $F$  where the loss is described by the piecewise linear function

$$\rho_\tau(u) = \tau u \mathbb{1}_{[u < 0]} + (1 - \tau) u \mathbb{1}_{[u \geq 0]} \quad (2.16)$$

for some  $\tau \in (0, 1)$ . The aim is to find  $\hat{x}$  to minimize expected loss. This problem was faced by [Fox and Rubin \(1964\)](#), who studied the admissibility of the quantile estimator under the loss function minimizing

$$\mathbb{E}[\rho_\tau(X - \hat{x})] = (\tau - 1) \int_{-\infty}^{\hat{x}} (x - \hat{x}) dF(x) + \tau \int_{\hat{x}}^{+\infty} (x - \hat{x}) dF(x) \quad (2.17)$$

We could rewrite the previous formula as:

$$\begin{aligned} g(\hat{x}) &= (\tau - 1) \left[ \int_{-\infty}^{\hat{x}} x f(x) dx - \hat{x} F(\hat{x}) \right] + \\ &\quad + \tau \left[ \int_{\hat{x}}^{+\infty} x f(x) dx - \hat{x} (1 - F(\hat{x})) \right] \\ &= \tau \mathbb{E}(x) - \left[ \int_{-\infty}^{\hat{x}} x f(x) dx \right] + (1 - \tau) \hat{x} F(\hat{x}) - \tau \hat{x} (1 - F(\hat{x})) \end{aligned} \quad (2.18)$$

Thus, differentiating with respect to  $\hat{x}$ , we have:

$$\begin{aligned}
0 &= -\hat{x}f(\hat{x}) + (1 - \tau)F(\hat{x}) + (1 - \tau)\hat{x}f(\hat{x}) - \tau(1 - F(\hat{x})) + \tau\hat{x}f(\hat{x}) \\
&= (1 - \tau)F(\hat{x}) - \tau(1 - F(\hat{x})) \\
&= F(\hat{x}) - \tau F(\hat{x}) - \tau + \tau F(\hat{x}) \\
&= F(\hat{x}) - \tau \\
&= \int_{-\infty}^{\hat{x}} dF(x) - \tau \left( \int_{-\infty}^{\hat{x}} dF(x) + \int_{\hat{x}}^{+\infty} dF(x) \right) \\
&= (1 - \tau) \int_{-\infty}^{\hat{x}} dF(x) - \tau \int_{\hat{x}}^{+\infty} dF(x)
\end{aligned} \tag{2.19}$$

Since  $F$  is monotone, any element of  $x : F(x) = \tau$  minimizes the expected loss. When the solution is unique,  $\hat{x} = F^{-1}(\tau)$ ; otherwise, we have what it is called an “interval of  $\tau$ th-quantiles” from which the smallest element must be chosen to such that the empirical quantile function is left-continuous.

It is natural that an optimal point estimator for asymmetric linear loss should lead us to the quantiles, since in the symmetric case the median is the parameter that minimize the absolute loss value. Moreover, when loss is linear and asymmetric, we prefer a point estimate more likely to leave us on the flatter of the two branches of marginal loss. Thus, for example, if an underestimate is marginally three times more costly than an overestimate, we will choose  $\hat{x}$  so that  $\mathbb{P}(X \leq \hat{x})$  is three times greater than  $\mathbb{P}(X > \hat{x})$  to compensate. That is, we will choose  $\hat{x}$  to be the 75th percentile of  $F$ .

When  $F$  is replaced by the empirical distribution function

$$F_n(x) = n^{-1} \sum_{i=1}^n I(x_i \leq x) \tag{2.20}$$

We may still choose  $\hat{x}$  to minimize the expected loss:

$$\int \rho_\tau(x - \hat{x}) dF_n(x) = T^{-1} \sum_{t=1}^T \rho_\tau(x_t - \hat{x}) \tag{2.21}$$



When  $\tau$  is an integer there is some ambiguity in the solution, because we really have an interval of solutions,  $x : F_t(x) = \tau$ , but we figure out that it has weak practical consequences.

Much more important is the fact that we have expressed the problem of finding the  $\tau - th$  sample quantile, a problem that might seem inherently tied to the notion of an ordering of the sample observations, as the solution to a simple optimization problem. In fact, the problem of finding the  $\tau - th$  sample quantile may be written as

$$\min_{\xi \in \mathbb{R}} \sum_{t=1}^T \rho_{\tau}(y_t - \xi) \quad (2.22)$$

Knowing that the sample mean  $\mu$  solves the problem

$$\min_{\mu \in \mathbb{R}} \sum_{t=1}^T (y_t - \mu)^2 \quad (2.23)$$

also known as mean square error, suggest that, if we are willing to express the conditional mean of  $y$  given  $x$  as  $\mu(x) = x^T \beta$ , then  $\beta$  may be estimated by solving

$$\min_{\beta \in \mathbb{R}^p} \sum_{t=1}^T (y_t - x_t \beta)^2 \quad (2.24)$$

Similarly, since the  $\tau th$  sample quantile  $\hat{\alpha}(\tau)$  solves

$$\min_{\alpha \in \mathbb{R}} \sum_{t=1}^T \rho_{\tau}(y_t - \alpha) \quad (2.25)$$

we are led to specifying the  $\tau th$  conditional quantile function as  $Q_y(\tau|x) = x^T \beta(\tau)$ , and to consideration of  $\beta(\tau)$  solving

$$\min_{\beta \in \mathbb{R}^p} \sum_{t=1}^T \rho_{\tau}(y_t - x_t \beta) \quad (2.26)$$

where  $\rho_{\tau}(\cdot)$  is defined as  $\rho_{\tau}(\varepsilon_t) = (\tau - I(\varepsilon_t \leq 0))$  and  $\varepsilon_t = y_t - x_t \beta$  for  $t = 1, \dots, T$ .

### 2.3.1 Financial Application of the Quantile Regression

The concept of quantile regression introduced by [Koenker and Bassett \(1978\)](#) is become increasingly popular in finance ([Xiao, 2009](#); [Bassett and Chen, 2002](#)) and economics ([Fitzenberger et al., 2013](#); [Buchinsky, 1994](#)) even though it has been applied to several research fields during the last 30 years. [Chernozhukov and Umantsev \(2001\)](#) applied quantile methods to estimate the Value at Risk. [Wu and Xiao \(2002\)](#) also used quantile methods to estimate VaR and provided an example of how such approaches could be used in the context of an index fund, while [Meligkotsidou et al. \(2009\)](#) explore the impact of a number of risk factors on the entire conditional distribution of hedge fund returns. This approach provides useful insights into the distributional dependence of hedge fund returns on risk factors where the distribution of returns general deviates from normality.

[Bassett and Chen \(2002\)](#) introduce regression quantiles as a complementary tool to identify the portfolio's style signature in the time series of its returns. In particular, regression quantile extract additional information identifying the way style affects returns at places other than the expected value. The significant aspect of this approach is reflected in the estimation of the impact of style on the tails of the conditional return distribution.

Moreover, [Bassett et al. \(2004\)](#) provide an exposition of one variant of the Choquet expected utility theory as it applies to decisions under risk. They link the theory of choice under uncertainty and risk to a pessimistic decision theory that replaces the classical expected utility criterion with a Choquet expectation that accentuates the likelihood of the least favorable outcomes. By offering a general approach to portfolio allocation for pessimistic Choquet preferences, they propose a critical reexamination of the role of attitudes toward risk in this important setting. In contrast to conventional mean-variance portfolio analysis implemented by solving least squares problems, pessimistic portfolios will be constructed by solving quantile regression problems.

Engle and Manganelli (2004) propose an estimation of the Value at Risk called Conditional Autoregressive Value at Risk (CAViaR). They estimate the unknown parameters minimizing the regression quantile loss function where their evolution over the time is described by an autoregressive process. The introduction of autoregressive elements to compute the conditional VaR has been proposed by Kuester et al. (2006). Finally, Ma and Pohlman (2008) explore the full distributional impact of factors on returns of securities and find that factor effects vary substantially across quantiles of returns. Utilizing distributional information from quantile regression models, they propose two general methods for return forecasting and portfolio construction: the quantile regression alpha distribution based on assumption of the selected quantile in the forecasting process and the quantile regression portfolio distribution that incorporate the quantile information in the portfolio optimization step.

Recently, Mezali and Beasley (2013) applied the quantile regression for the index tracking proposing a double optimization problem to find a zero value of the quantile regressed intercept with a unit slope. In their formulation transaction cost, a limited number of stocks and a limited total transaction cost are included. Then, Bonaccolto et al. (2015) propose a pessimistic asset allocation model based on a performance measure derived from the quantile regression and they impose a penalty on the  $\ell_1$ -norm of the quantile regression coefficients along the line of the Least Absolute Shrinkage and Selection Operator (LASSO), introduced by Tibshirani (1996) in a standard linear regression framework.

### 2.3.2 Definition of a probability metric for the Benchmark Tracking Problem

The development of the theory of probability metrics started with the investigation of problems related to limit theorems in probability theory. The limit theorems take a very important place in probability theory, statistics, and all their applications. A well-known example by nonspecialists in the field is the celebrated Central Limit Theorem (CLT).

The central question arising is how large an error we make by adopting the approximate model. This question can be investigated by studying the distance between the limit law and the stochastic model. Generally, the theory of probability metrics studies the problem of measuring distances between random quantities (Rachev, 1991). On one hand, it provides the fundamental principles for building probability metrics while, on the other, it studies the relationships between various classes of probability metrics. The second realm of study concerns problems that require a particular metric, while the basic results can be obtained in terms of other metrics. In such cases, the metrics relationship is of primary importance.

Rachev (1991) provides more details on the methods of the theory of probability metrics and its numerous applications in both theoretical and more practical problems. Note that there are no limitations in the theory of probability metrics concerning the nature of the random quantities. This makes its methods fundamental and appealing. Actually, in the general case, it is more appropriate to refer to the random quantities as random elements. In the context of financial applications, we can study the distance between two random stocks prices, or between vectors of financial variables building portfolios, or between entire yield curves which are much more complicated objects.

One of most used approach related with the theory of the probability metrics is the problem to measure the distance between two random variable. This type of problem is known as the benchmark tracking problem and axiomatically formulated by Stoyanov et al. (2008b). Following their approach, let  $X$  be a random variable that describes the invested portfolio while  $Y$  is the benchmark, we denote by  $\mathfrak{X}$  be the space of random variables on a given probability space  $(\Omega, \mathfrak{S}, \mathbb{P})$  taking values in  $\mathbb{R}$ . By  $\mathcal{L}\mathfrak{X}_2$  we denote the space of all joint distributions  $\Pr_{X,Y}$  generated by the pairs  $X, Y \in \mathfrak{X}$ . Supposing that a mapping  $\mu(X, Y) := \mu(\Pr_{X,Y})$  is defined on  $\mathcal{L}\mathfrak{X}_2$  taking values in the extended interval  $[0, \infty]$ , it is called a *probability quasi-metric* on  $\mathfrak{X}$  if it satisfies the following two properties:

- a)  $\mu(X, Y) \geq 0$  and  $\mu(X, Y) = 0$ , if and only if  $X \sim Y$  (*Identity Property*)

b)  $\mu(X, Y) \leq \mu(X, Z) + \mu(Z, Y)$  for any  $X, Y, Z$  (*Triangle Inequality*)

In particular, we notice that the *symmetric property* does not hold in the strategy replication problem (i.e.  $\mu(X, Y) \neq \mu(Y, X)$ ) and it is suitable to differently measure positive and negative differences between the two random variables.

The following step introduced by [Stoyanov et al. \(2008b\)](#) is to define a *metric of relative deviation* and the consequence relation with the concept of *deviation metric* proposed by [Rockafellar et al. \(2006\)](#).

**Remark 2.1**

*Any quasi-metric  $\mu$  satisfying*

c)  $\mu(X + Z, Y + Z) \leq \mu(X, Y)$  for all  $X, Y, Z$ , (*Strong Regularity*)

d)  $\mu(aX, aY) = a^s \mu(X, Y)$  for all  $X, Y, a, s \geq 0$  (*Positive homogeneity of degree  $s$* )

*is said to be a (translation invariant) **metric of relative deviation** ([Stoyanov et al., 2008b](#)).*

**Remark 2.2**

*The functional  $\mu(X, Y) = D(X - Y)$  satisfies a), b), c), d) with  $s = 1$  where  $D : \mathfrak{X} \rightarrow [0, \infty]$  is a **deviation measure** such that:*

*D1.  $D(X + C) = D(X)$  for all  $X$  and constants  $C$*

*D2.  $D(0) = 0$  and  $D(\lambda X) = \lambda D(X)$  for all  $X$  and all  $\lambda > 0$*

*D3.  $D(X + Y) \leq D(X) + D(Y)$  for all  $X$  and  $Y$*

*D4.  $D(X) \geq 0$  for all  $X$ , with  $D(X) > 0$  for non-constant  $X$*

Considering the identity property a) is possible to classify different type of probability quasi-metrics and in general probability metrics considering different type of equality

held on this property. In fact, the notation  $X \sim Y$  denotes that  $X$  is equivalent to  $Y$ . The meaning of equivalence depends on the type of metrics. If the equivalence is in almost sure sense, then the metrics are called *compound*. If  $\sim$  means equality of distribution, then the metrics are called *simple*. Finally, if  $\sim$  stands for equality of some characteristics of  $X$  and  $Y$ , then the metrics are called *primary*.

### 2.3.3 Tracking Error Quantile Regression

The common measure are broadly used in the benchmark tracking but they present some intuitive drawbacks and theoretical lacks. For this reason, we consider the quantile regression method to build a dispersion measure of the tracking error suitable for the benchmark tracking problem. In quantile regression a linear equation relates how the quantile of the dependent variable vary with the independent variable. The solution of this problem does not present a closed form solution but the result are found solving a linear program. Let (2.20) be the general solution of the quantile regression, we can define the related measure of dispersion for the tracking error in a discrete case.

#### Definition 2.1

Let  $Y$  be a random variable of benchmark returns with realization  $y_t$  for  $t = 1, \dots, T$ , let  $X = r\beta$  the returns of the invested portfolio of its  $N$  components and  $\tau$  be the quantile of interest such that  $(0 \leq \tau \leq 1)$ . Let  $\varepsilon_t = \sum_{n=1}^N r_{t,n}\beta_n - y_t$  the difference between portfolio and benchmark returns at time  $t$ , we define the **tracking error quantile regression (TEQR)** at given  $\tau$  as:

$$\text{TEQR} \quad \sigma(\varepsilon|\tau) = \tau \sum_{t=1}^T \varepsilon_t \mathbb{1}_{[\varepsilon_t \geq 0]} + (1 - \tau) \sum_{t=1}^T \varepsilon_t \mathbb{1}_{[\varepsilon_t < 0]} \quad (2.27)$$

In (2.27), the first term is the sum of the positive residuals while the second term is the sum of negative residuals. The first one represents the observations that lie above the regression line and they receive a weight of  $\tau$ , while the second are the observations that lie below the regression line and they receive a weight of  $(1 - \tau)$ .

However, the concept of quantile regression is strictly connected with the Value at Risk and expected shortfall at a given confidential level  $\tau$ . It is not the objective of this dissertation to investigate this implication (Taylor, 2008; Kuester et al., 2006; Engle and Manganelli, 2004).

Portfolio managers aim to minimize a specific dispersion measure of tracking error and in the tracking error quantile regression case, when  $\tau$  increases (decreases) there will be fewer positive (negative) residuals and they will be closer to the regression line. The advantage of this approach is that when for instance  $\tau = 0.9$  (or  $\tau = 0.1$ ), the regression line fits better the extreme value of the returns distribution. In particular, low values of  $\tau$  imply a specific behavior of the portfolio manager to built a tracking portfolio that wants to reduce index outcomes greater than the mimic portfolio. It is an aggressive strategy since the quantile regression line dominates the majority of the observations.

Contrary, high values of  $\tau$  lead to a “Value at Risk” tracking portfolios and they guarantee minimum expected returns. However, both the approaches approximate the tail behavior of the benchmark returns distribution and in period of financial distress help to achieve the portfolio manager goals. Consequently, a tail selection of the quantile increase the tracking error since it exposes the portfolio to gains returns far from the central and high probable return tendency.

Fixing the value of  $\tau$  it is possible to define the tracking error quantile regression as a translation invariant metric of relative deviation and a deviation measure.

### **Proposition 2.1**

Let  $\sigma(X, Y|\tau)$  be a mapping originated by the tracking error quantile regression taking values in the extended interval  $[0, \infty]$ . Then, for a fixed  $\tau$  such that  $\tau \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ , it is a translation invariant metric of relative deviation (Stoyanov et al., 2008b) since it satisfies:

- a) *Identity Property* :=  $\sigma(X, Y|\tau) \geq 0$  and  $\sigma(X, Y|\tau) = 0$  iff  $X = Y$  a.s.
- b) *Triangle Inequality* :=  $\sigma(X, Y|\tau) \leq \sigma(X, Z|\tau) + \sigma(Z, Y|\tau) \forall X, Y, Z$

- c) *Strong Regularity*  $:= \sigma(X + Z, Y + Z|\tau) = \sigma(X, Y|\tau) \forall X, Y, Z,$
- d) *Positive homogeneity of degree  $s$*   $:= \sigma(aX, aY|\tau) = a^s \sigma(X, Y|\tau) \forall X, Y, a, s \geq 0$

**Proposition 2.2**

For a fixed  $\tau$  such that  $\tau \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ , the tracking error quantile regression is deviation measure [Rockafellar et al. \(2006\)](#) since it satisfies properties  $D1, D2, D3$  and  $D4$ .

From these definition it appears relevant the asymmetry property of the tracking error quantile regression (2.27) is linked with the value of the selected  $\tau$ . It gives a different weight to positive and negative tracking errors and it also represents an aversion risk coefficient.

As underlined by [Koenker and Bassett \(1978\)](#), the quantile regression problem does not present a close form solution as the mean square error but it is the result of a minimization problem. Let  $u$  and  $\nu$  two slack variables such that:

$$\begin{aligned} u_t &= \varepsilon_t \mathbf{1}_{[\varepsilon_t \geq 0]} \quad \forall t = 1, \dots, T \\ \nu_t &= \varepsilon_t \mathbf{1}_{[\varepsilon_t < 0]} \quad \forall i = t, \dots, T \end{aligned} \quad (2.28)$$

It is possible to express the quantile regression as a solution of the following minimization problem:

$$\min_{(\mathbf{w}, \mathbf{u}, \nu) \in \mathbb{R}^p \times \mathbb{R}_+^{2n}} \left\{ \tau \mathbf{1}^\top \mathbf{u} + (1 - \tau) \mathbf{1}^\top \nu \mid r\beta - \mathbf{u} + \nu = \mathbf{y} \right\} \quad (2.29)$$

The linearity of the formulation and its theoretical support make of the tracking error quantile regression a suitable dispersion measure to solve the benchmark tracking portfolio problem. This measure is defined fixing the value of  $\tau$ . As we discuss later, the choice of the quantile represents an interesting topic of research and the possibility to switch its value during the time allows to capture the features of the financial markets. Thus, let the log-return of equity index  $Y$  s.t.  $y_t, t = 1, \dots, T$  and of its  $N$  components being  $r_1, r_2, \dots, r_N$ . We define the slack variable  $u$  and  $\nu$  (2.28).



The linear programming benchmark tracking problem with the tracking error quantile regression dispersion measure is:

$$\begin{aligned}
\min_{\beta, u, \nu} \quad & \sum_{t=1}^T \tau u_t + (1 - \tau) \nu_t \\
\text{s.t.} \quad & r_t \beta - u_t + \nu_t = y_t \quad \forall t = 1, \dots, T \\
& \sum_{n=1}^N \beta_n = 1 \\
& \mathbb{E}[X] - \mathbb{E}[Y] \geq K^* \\
& u_t, \nu_t \geq 0 \quad \forall t = 1, \dots, T \\
& lb \leq \beta_n \leq ub \quad \forall n = 1, \dots, N
\end{aligned} \tag{2.30}$$

### 2.3.4 Reward, Risk Measure and Information Ratio

In the previous section, we introduced a dispersion measure for the benchmark tracking problem. The tracking error quantile regression is theoretically suitable to measure the distance between portfolio and benchmark returns but it depends on the assigned quantile  $\tau$ . However, investors and portfolio managers evaluate the profitability of their investment in a reward/risk sense. For this reason financial literature introduced the concept of ratio of performance to order the investor preferences for different reward and risk measures. In the context of benchmark tracking portfolio problem the performance ratio is called information ratio (IR) (Goodwin, 1998) since the selected measures are not based only on the feature of the invested portfolio but also they take into account the random variable describing a given benchmark.

In this section we firstly review the common information ratio based on the tracking error measure presented in the Section 2.2.1 and then we introduce a new information ratio based on the quantile regression discussing its properties from an empirical point of view. Finally, we propose two different strategies for portfolio managers considering static or rolling quantile decision approaches.

Generally, benchmark tracking managers evaluate the performances of their mimic portfolios considering the information ratio (IR) (i.e. the ratio between a reward measure and a dispersion measure of the tracking error) (Goodwin, 1998):

$$\text{IR} = \frac{\mu(X - Y)}{\sigma(X - Y)} \quad (2.31)$$

when  $\mu(\cdot)$  is a reward measure and  $\sigma(\cdot)$  is a dispersion measure of the tracking error. Financial literature proposes several information ratios related with a common reward measure and different risk ones. In particular, the common numerator of the (2.31) is the portfolio alpha:

$$\hat{\alpha} = \mathbb{E}[X] - \mathbb{E}[Y] \quad (2.32)$$

that identify the average difference between the return of mimic and benchmark portfolios. Then, considering the common deviation measures presented in this essay (2.3), (2.4) and (2.5) we could define three different information ratios that capture the risk of the replicating strategy in different ways.

Considering the tracking error mean absolute deviation TEMAD (2.3), it is possible to define the related information ratio mean absolute deviation as follow:

$$\text{IR}_{\text{MAD}} = \frac{\mu(X - Y)}{\sigma(X - Y)} = \frac{\hat{\alpha}}{\sigma_{\text{MAD}}(X - Y)} = \frac{\mathbb{E}[X] - \mathbb{E}[Y]}{\frac{1}{T} \sum_{t=1}^T |\varepsilon_t|} \quad (2.33)$$

Then, whether the investors are focused only on the downside risk, an efficient dispersion measure is the downside mean semideviation and the related information ratio is defined as:

$$\text{IR}_{\text{DMS}} = \frac{\mu(X - Y)}{\sigma(X - Y)} = \frac{\hat{\alpha}}{\sigma_{\text{DMS}}(X - Y)} = \frac{\mathbb{E}[X] - \mathbb{E}[Y]}{\frac{1}{T} \sum_{t=1}^T |\varepsilon_t| \mathbf{1}_{[\varepsilon_t < 0]}} \quad (2.34)$$

Finally, considering a quadratic measure to capture the distance between portfolio and benchmark returns, we define the information ratio volatility based on the measure of dispersion of the tracking error (2.5) called tracking error volatility. In this way, the

information ratio volatility is:

$$\text{IR}_V = \frac{\mu(X - Y)}{\sigma(X - Y)} = \frac{\hat{\alpha}}{\sigma_V(X - Y)} = \frac{\mathbb{E}[X] - \mathbb{E}[Y]}{\frac{1}{T} \sum_{t=1}^T (\varepsilon_t - \bar{\varepsilon})^2} \quad (2.35)$$

These measures have the advantages to be clearly defined and easy to compute. Moreover, they depend on parameters with a strong impact in the decisional problem. However, they present some important drawbacks. On one hand, they suffer the same problematics discussed in the definition of the dispersion measures. In particular, they consider a symmetric risk source such as the mean absolute deviation or they show a quadratic formulation difficult to optimize if the performance measure is the objective function of a portfolio selection model.

On the other hand, index tracking portfolio managers prefer investment with positive alphas and low dispersion measure levels. However, they have a different system of choice with respect to the magnitude of the alphas since they mimic the performance of an index where high positive alphas are related with high dispersion measures. Thus, investor with different reward-risk profiles would like to differently capture their behavior in the evaluation of an benchmark tracking strategy. Finally, it is important to mention how this information ratios are defined on  $[-\infty, +\infty]$  and when the portfolio alpha is non positive ( $\hat{\alpha} < 0$ ) the evaluation of different benchmark tracking strategies could lead to misleading results.

To cross these drawbacks, we introduce an information ratio based on the tracking error quantile regression. In fact, this measure could be easily decomposed in two parts: the sum of the positive difference between the mimic portfolio and the benchmark one and the sum of the negative side of this difference. Since the second part presents the absolute value of negative figures, we have two positive measures in the definition of the index tracking. In this framework the two parts represent a reward and a risk measure,

respectively. In particular, we could define a reward measure  $\mu$  on the tracking error as:

$$\mu(X - Y) = \mu(\varepsilon) = \frac{1}{T} \sum_{t=1}^T \varepsilon_t \mathbb{1}_{[\varepsilon_t \geq 0]} = \frac{1}{T} \sum_{t=1}^T u_t \quad (2.36)$$

This reward measure has the main feature to be defined on the positive support such that  $\mu : \mathbb{R} \rightarrow [0, +\infty]$  and as we will discuss later this property fits the possibility to compare in a reward-risk sense different benchmark tracking strategies. Then, the associate risk measure to the tracking error quantile regression is defined as the sum of the absolute value of the negative differences between portfolio and benchmark returns:

$$\sigma(X - Y) = \sigma(\varepsilon) = \frac{1}{T} \sum_{t=1}^T |\varepsilon_t| \mathbb{1}_{[\varepsilon_t < 0]} = \frac{1}{T} \sum_{t=1}^T \nu_t \quad (2.37)$$

Also in this case, we are dealing with a positive risk measure and even though it is derived from the mean downside semideviation (2.4) while the reward measure (2.36) is the mean positive semideviation.

However, benchmark tracking portfolio managers want to minimize both parts of the tracking error quantile regression with different risk aversion given by the functional  $\tau$ . On one hand, in this minimization process, they aim to obtain that  $\sum_{t=1}^T \varepsilon_t \mathbb{1}_{[\varepsilon_t \geq 0]} \geq \sum_{t=1}^T |\varepsilon_t| \mathbb{1}_{[\varepsilon_t < 0]}$  to have a replication portfolio strategy from above. On the other hand, they evaluate the goodness of their strategy considering a measure of return and a measure of risk. In this framework, portfolio managers compute the information ratio (IR) that measure the reward for unit of risk. This ratio allows also to order different investment in a reward/risk sense but it depends on how we measures the different between invested portfolio and benchmark return and how we evaluate the intrinsic risk of this difference.

Let  $\mu$  be a reward measure applied on the difference between the portfolio ( $X$ ) and benchmark ( $Y$ ) returns and  $\sigma$  a risk dispersion measure. Fixing the quantile  $\tau$  that represent the investor's risk aversion, we define the information ratio generalized

quantile regression as:

$$\text{IR}_{\text{GQR}|\tau} = \frac{\mu(X - Y)}{\sigma(X - Y)} = \frac{\tau \sum_{t=1}^T \varepsilon_t \mathbf{1}_{[\varepsilon_t \geq 0]}}{(1 - \tau) \sum_{t=1}^T |\varepsilon_t| \mathbf{1}_{[\varepsilon_t < 0]}} \quad (2.38)$$

Introducing the two slack variables  $u$  and  $\nu$  identifying the associated reward (2.36) and risk (2.37) measures, we can define a new information ratio.

### Definition 2.2

Let  $u_t = \varepsilon_t \mathbf{1}_{[\varepsilon_t \geq 0]}$  and  $\nu_t = |\varepsilon_t| \mathbf{1}_{[\varepsilon_t < 0]}$  be two slack variable representing positive and negative excess returns between tracking and tracked portfolios where  $\varepsilon_t = \sum_{n=1}^N r_{t,n} \beta_n - y_t$  for  $t = 1, \dots, T$ . We define the **information ratio generalized quantile regression** for a given  $\tau$  ( $\text{IR}_{\text{GQR}|\tau}$ ) as

$$\text{IR}_{\text{GQR}|\tau} = \frac{\tau \sum_{t=1}^T u_t}{(1 - \tau) \sum_{t=1}^T \nu_t} \quad (2.39)$$

This information ratio is a functional depending on the quantile  $\tau$ . In particular, we notice that investors with low risk aversion select high level of  $\tau$  to give a significant weight to the positive difference between tracking and tracked portfolio returns while high risk averted investor select low levels of the quantile. However, this formulation leads to focus on the different behavior of the investors and it is suitable maximizing the investor preferences in the active strategy area (Biglova et al., 2004; Sharpe, 1994; Markowitz, 1952).

To evaluate the performance of a given benchmark tracking strategy we propose a special case of the (2.39) in which  $\tau = 0.50$ . In this case, investors equally weight positive or negative performances with respect to the index.

### Definition 2.3

Let  $u_t = \varepsilon_t \mathbf{1}_{[\varepsilon_t \geq 0]}$  and  $\nu_t = |\varepsilon_t| \mathbf{1}_{[\varepsilon_t < 0]}$  be two slack variable representing positive and negative excess returns between tracking and tracked portfolios where  $\varepsilon_t = \sum_{n=1}^N r_{t,n} \beta_n -$

$y_t$  for  $t = 1, \dots, T$ . We define the *information ratio quantile regression* ( $\text{IR}_{\text{QR}}$ ) as

$$\text{IR}_{\text{QR}} = \frac{\sum_{t=1}^T u_t}{\sum_{t=1}^T \nu_t} \quad (2.40)$$

The main advantage of this information ratio is to be defined in a positive range:  $\text{IR}_{\text{GQR}} := \mathbb{R} \rightarrow [0, +\infty]$ . Since both terms of the ratio are positive it allows to compare different benchmark tracking strategies also when they have negative alphas.

Finally, we discuss two possible approaches to solve the benchmark tracking problem minimizing the tracking error quantile regression based on static or rolling strategies in the definition of the quantile. In the traditional approach, portfolio managers choose a given quantile and defining the style of portfolio strategy they keep it constant during the entire investment period. This approach becomes optimal considering policy constraints in the replication of the benchmark. In particular, a “Vale at Risk” strategy implemented selecting a fixed low quantile allows to built a mimic portfolio that is very prudential in the replication process. Differently, a high value of  $\tau$  implies a much more seeking to the risk and the possibility to achieve relevant extra-performances. Portfolio managers which follow this style have the advantages to reduce the computational time and to declare a priori which type of mimic portfolio they want to built.

Although, this approach turns out to be clear and very efficient not only from a theoretical point of view but also analyzing the empirical results, it hampers the possibility to seize some features of the financial market. For this reason a rolling strategy based on the shift of the selected quantile to solve the benchmark tracking problem seems to be optimal to capture different aspects of empirical evidences observed on the financial markets. This rolling approach consists in a decisional process executed at each portfolio optimization step that select the “best” quantile to be the more suitable to capture portfolio evolution in the following investment period.

The motivations underlying this approach are related with the opportunity to efficiently capture different phases of the financial cycle as well as the phenomena of volatility clusterings. Forecasting modeling and Black-Litterman (Meucci, 2010; Satchell and Scowcroft, 2000; Black and Litterman, 1992) views could be introduced in this formulation. In this essay, we propose a simple decisional problem to switch the quantile. This approach considers the financial trend and presents the information ratio quantile regression as a decisional tool. In particular, at each optimization step, we select the quantile that in the static approach has the higher information ratio quantile regression during a window covering the last  $d$  days.

## 2.4 Enhanced Indexing Strategy

Enhanced indexation models are related to index tracking, in the sense that they also consider the return distribution of an index as a reference. However, they aim to outperform the index by generating “excess return” (Scowcroft and Sefton, 2003) or “adding alpha” controlling the sources of risk. Enhanced indexation is a very new area of research and there is no generally accepted portfolio construction method in this field (Canakgoz and Beasley, 2009; Litterman, 2004).

In general, portfolio managers which implement an enhanced indexation strategy aim to achieve two different goals mixing the advantages of passive and active benchmark tracking portfolios. In particular, they minimize the risk maximizing possible extra-performances. In this way they try to synthesize the reduction of a dispersion measure that characterizes passive strategies while they try to outperform the benchmark as required in the active management. In this type of problem, we face the same computational issues as in index tracking such as the high dimensionality and cardinality constraints. Although the idea of enhanced indexation was formulated as early as 2000, the few enhanced indexation methods were proposed later in the research community (a review in Canakgoz and Beasley (2009)).

In the recent years the need to built portfolios that mimic a given index looking for an extra performance is getting crucial for different reasons. On one hand, portfolio managers would cover the fees and transaction costs to maximize the profits and investors' goals. Giving tracking error constraints they guarantee an extra-performance. On the other hand, the same investors would replicate the performance of an index from above and they are interested in strategies with this aim. For these reasons enhanced indexation that is arranged between passive and active strategies is a relevant topic in the recent financial literature ([Valle et al., 2015](#); [Bruni et al., 2014](#); [Chavez-Bedoya and Birge, 2014](#); [Mezali and Beasley, 2013](#); [Roman et al., 2013](#); [Lejeune, 2012](#)).

Several different approaches to the enhanced indexation problem, both exact and heuristic, have been proposed in the last decade, starting from a seminal study by [Beasley et al. \(2003\)](#). We review some of this works to give a general idea about the state of the art about this innovative area of research.

[Alexander and Dimitriu \(2005\)](#), applied a cointegration based strategy constructing two index series by adding/subtracting from the original index values a constant excess return, alpha. They seek to earn excess return by going long on the alpha plus tracking portfolio and shorting the alpha minus tracking portfolio. They used a very simple approach to decide the stocks to include in the tracking portfolios based on ranking stocks by price. [Dose and Cincotti \(2005\)](#) propose a cluster stocks method based on a distance measure between stock prices time series data. This clustering is used to decide which stocks hold in the tracking portfolio, given a priori cardinality constraint. To compute the investment in each stock they use a weighting parameter lambda ([Beasley et al., 2003](#)) as a trade-off between tracked index and excess return.

[Konno and Hatagi \(2005\)](#), used a mean absolute deviation objective function to minimize the distance between index and the tracking portfolio values. In particular, the index is scaled up by a factor alpha and normalized by the index value at the end of the time period while the mimic portfolio is normalized by the tracking portfolio value at the end of the time period. In their model transaction cost are included and the entire



problem is formulated minimizing a separable concave function with linear constraints. They performed a problem reduction test to eliminate variables. [Wu et al. \(2007\)](#), presented a double goal programming approach trying to achieve a given rate of return while they control the tracking error. In particular, it is defined in a nonlinear way, but additive, fashion. The two objective functions are the minimization of the tracking error, given by the standard deviation of the portfolio return compared to the benchmark return, and the maximization of the excess portfolio return over the benchmark.

[Canakgoz and Beasley \(2009\)](#) propose a regression based model for enhanced indexing, developing a two-stage mixed-integer linear programming approach where they respectively focused on slope-intercept and transaction cost. In the first-stage, they solve a problem achieving a regression slope as close to one as possible, subject to a constraint on the regression intercept while the minimization of transaction cost subject is central in the second-stage. [Koshizuka et al. \(2009\)](#) propose a minimization of the tracking error from an index-plus-alpha portfolio basing the selection process among the portfolios which show a high correlation with the benchmark. To solve the enhanced indexation tracking problem, they introduce a convex minimization model with linear objective function and quadratic constraints where two alternative measures of the tracking error are considered. The first one is based on the absolute deviation between the portfolio and the index-plus-alpha portfolio while the second one is the downside absolute deviation between these two quantities.

In 2011, [Meade and Beasley \(2011\)](#) investigate a momentum strategy via the maximization of a modified Sortino ratio ([Sortino and Price, 1994](#)) objective function while [Li et al. \(2011\)](#) develop a non-linear bi-objective optimization model for enhanced indexing. In a mixed integer problem where the number of units of stock are the decisional variables, they maximize the excess returns and minimize of the downside standard deviation with the introduction of an evolutionary algorithm. [Li et al. \(2011\)](#) formulate the enhanced indexing benchmark tracking problem as a bi-objective optimization model where the excess portfolio return over the benchmark is maximized, while the tracking

error, formulated by the authors as the downside standard deviation of the portfolio return from the benchmark return, is minimized. Their model includes, among other features, a cardinality constraint and buy-in threshold limits.

[Lejeune \(2012\)](#) introduce a game theoretical approach in the decisional problem of the enhanced indexing. They propose a stochastic model which aim at maximize the probability to obtain excess return of the invested portfolio with respect to the benchmark. In this formulation, they impose a threshold level ensuring that the risk source, given by the downside absolute deviation dispersion measure, does not cross it. A similar approach that consider a stochastic mixed integer nonlinear model is proposed in [Lejeune and Samatli-Paç \(2012\)](#) where asset returns and the return covariance terms are treated as random variables.

Recently, [Roman et al. \(2013\)](#) apply a second order stochastic dominance strategy ([Fábián et al., 2011](#); [Roman et al., 2006](#)) to construct a portfolio whose return distribution dominates the benchmark one. They adopt a multi-objective linear problem solved with a cutting-plane solution method presented in [Fábián et al. \(2011\)](#). Empirical analyses confirm the goodness of the proposed methodology to outperformed the benchmark. Moreover, they notice a reduction in the active asset without the introduction of cardinality constraints and the robustness of the invested portfolio which does not need to be significantly rebalanced.

Finally, [Guastaroba et al. \(2014\)](#) introduce a mixed-integer linear programming to enhance the index tracking problem maximizing the Omega ratio [Keating and Shadwick \(2002\)](#) in a linear formulation with buy-in threshold limits and cardinality constraints. In the definition of the Omega ratio they propose two different approaches where the benchmark return are defined by a fixed target or when they are random variables. [Valle et al. \(2015\)](#) discuss an extension of a three-stage approach to compute an absolute return portfolio in an enhanced indexing sense.

The contribution of this work is to propose a different methodology to solve the enhanced indexing tracking problem. Considering the tracking error quantile regression

dispersion measure we formulate a benchmark tracking problem to mimic the index's behavior enhancing its performances with stochastic dominance constraints. This methodology differently solves the problem since it neither focus on performance measures nor introduce a maximization problem related with the excess returns. Common approaches stress the need to maximize portfolio gains trying to control or limit the risk. In our approach, we still keep on reducing a dispersion measure replicating the performance of the benchmark but stochastic dominance constraints allow to force that the mimic portfolio should also dominate the tracked one.

Thus, a realistic model with transaction penalty function and buy-in threshold limits is introduced to enhanced the portfolio performance. We achieved this aim setting first order and second order stochastic dominance constraints and we discuss how different agent try to adding alpha to their portfolios. In particular, we built two kinds of enhanced indexing tracking portfolios: the first one formulate an optimization problem with first order stochastic dominance constraints whose solution is chosen by non-satiabile investors while the second one find optimal portfolio selected by non-satiabile risk averse investors.

### **2.4.1 Problem Formulation for the Enhanced Indexing**

As stress in the previous section, enhanced indexing strategies capture the best features of index tracking and active management. In fact, they reduce the risk maximizing the expected returns. Considering the classical benchmark tracking problem, portfolio managers fix the level  $K^*$  of the expected or guaranteed future returns of the tracking

portfolios in relation with the nature of their strategies.

$$\begin{aligned}
& \min_{\beta} \quad \sigma(X - Y) \\
& \text{s.t.} \quad \sum_{n=1}^N \beta_n = 1 \\
& \quad \mathbb{E}[X] \geq \mathbb{E}[Y] + K^* \\
& \quad lb \leq \beta_n \leq ub \quad \forall n = 1, \dots, N
\end{aligned} \tag{2.41}$$

Whether a zero value of  $K^*$  is set to solve the index tracking portfolio problem, increasing the required return portfolio managers switch from passive strategies to active ones. The main issue of this approach is the absence of relations between the in sample and the out of sample analyses. This aspect implies the impossibility for portfolio managers to guarantee to their investors not only given but also positive gains and consequently to pledge minima or excess returns.

To avoid this problem financial literature proposes several methodologies and portfolio formulations but, as mentioned before, they significantly diverge from the original idea of enhanced indexation strategies and sometimes they are really addressed with active portfolio management. Setting  $K^* = 0$ , we introduce an enhanced indexing benchmark tracking portfolio problem which is still minimizing a dispersion measure but also introduce two different components to make the model realistic and to obtain significant extra-performances.

Thus, applying the proposed tracking error quantile regression dispersion measure we introduce a linear penalty function in the objective formulation to reduce transaction costs and stochastic dominance constraints to enhance the performance of the invested portfolio that replicate the index from above. Moreover, a buy-in threshold level combined with transaction penalty function reduces the portfolio turnover. In the next two sections, we recall the concept of stochastic dominance and we describe the complete optimization model.

### 2.4.2 Stochastic Dominance and Benchmark Tracking

The relation of stochastic dominance is one of the fundamental concepts of the decision theory (Levy, 1992). It introduces a partial order in the space of real random variables. The first degree relation carries over to expectations of monotone utility functions, and the second degree relation to expectations of concave nondecreasing utility functions. In portfolio theory, stochastic dominance rules have been used to justify the reward-risk approaches (Mosler and Scarsini, 1991) and several behavioral finance studies have tried to characterize investors' behavior and preferences (Edwards, 1996; Friedman and Savage, 1948). However, several theoretical formulation and empirical application in the financial field has been proposed in the last twenty years (Ziemba and Vickson, 2014; Davidson and Duclos, 2013; Post and Kopa, 2013; Kopa and Tichý, 2012; Annaert et al., 2009; Ortobelli et al., 2009; Sriboonchita et al., 2009; Rachev et al., 2008b; Dentcheva and Ruszczyński, 2006; De Giorgi, 2005; Fong et al., 2005; Post and Levy, 2005; Post, 2003).

Suppose that there are two portfolios  $X$  and  $Y$ , such that all investors from a given class do not prefer  $Y$  to  $X$ . This means that the probability distributions of the two portfolios differ in a special way that, no matter the particular expression of the utility function, if an investor belongs to the given class, then  $Y$  is not preferred by that investor. In this case, we say that portfolio  $X$  dominates portfolio  $Y$  with respect to the class of investors. Generally speaking,  $X$  dominates  $Y$  with respect to the  $\alpha$  stochastic dominance order  $X \underset{\geq}{\alpha} Y$  (with  $\alpha \geq 1$ ) if and only if  $\mathbb{E}[u(Y)] \geq \mathbb{E}[u(X)]$  for all  $u$  belonging to a given class  $U_\alpha$  of utility functions (Ortobelli et al., 2009).

The usual first order definition of stochastic dominance (FSD) gives a partial order in the space of real random variables (Kopa and Post, 2009; Levy, 1992; Bawa, 1978). Let  $X$  and  $Y$  be r.v.s of the returns of two financial portfolios. Then, in the stochastic dominance approach, they are compared by a point-wise comparison of some performance functions constructed from their distribution functions. For a real random variable  $X$ , its first performance function is defined as the right-continuous cumulative

distribution function of  $X$ :

$$F_X(\xi) = \mathbb{P}(X \leq \xi) \quad \text{for } \xi \in \mathbb{R} \quad (2.42)$$

A random return  $X$  is said to stochastically dominate another random return  $Y$  in the first order sense, denoted  $X \underset{(1)}{\geq} Y$ , if

$$F_X(\xi) \leq F_Y(\xi) \quad \text{for } \xi \in \mathbb{R} \quad (2.43)$$

More important from the portfolio point of view is the notion of second-order dominance (SSD), which is also defined as a partial order. It is one of the most debated topic in financial portfolio selection, due to its connection to the theory of risk-averse investor behavior and tail risk minimization (De Giorgi and Post, 2008; De Giorgi, 2005; Ortobelli, 2001; Bawa, 1975).

It is equivalent to this statement: a random variable  $X$  dominates the random variable  $Y$  if  $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$  for all non-decreasing concave functions  $u(\cdot)$  for which these expected values are finite. Thus, no risk-averse decision maker will prefer a portfolio with return rate  $Y$  over a portfolio with return rate  $X$  (Ortobelli et al., 2013; De Giorgi and Post, 2008). The second performance function  $F^{(2)}$  is given by area below the distribution function  $F$ :

$$F_X^{(2)} = \int_{-\infty}^{\xi} F_X(\xi) d\xi \quad \text{for } \xi \in \mathbb{R} \quad (2.44)$$

and defines the weak relation of the second-order stochastic dominance. That is, random return  $X$  stochastically dominates  $Y$  in the second order, denoted  $X \underset{(2)}{\geq} Y$ , if

$$F_X^{(2)}(\xi) \leq F_Y^{(2)}(\xi) \quad \text{for } \xi \in \mathbb{R} \quad (2.45)$$

Changing the order of integration, the ordering  $X \underset{(2)}{\geq} Y$  is equivalent to the expected shortfall (Ortobelli et al., 2013; Ogryczak and Ruszczyński, 1999):

$$F_X^{(2)}(\xi) = \mathbb{E}[(\xi - X)_+] \text{ for } \xi \in \mathbb{R} \quad (2.46)$$

where  $(\xi - X)_+ = \max(\xi - X, 0)$ . In this case, the function  $F_X^{(2)}(\xi)$  is continuous, convex, nonnegative and non-decreasing. It is well defined for all random variables  $X$  with finite expected value.

Computational tractable and technological solvable portfolio optimization models which apply the concept of FSD or SSD were recently proposed by Ortobelli et al. (2013); Kopa and Chovanec (2008); Dentcheva and Ruszczyński (2006); Kuosmanen (2004). A common problem with index tracking models is raised by their computational difficulty due to the non-linearity of the objective function, the implementation of regulatory or trading constraints, such as the cardinality constraint which limits the number of stocks in the chosen portfolio. However, until recently, stochastic dominance was considered for its theoretical development without analyzing its implication in the benchmark tracking problem. The main reason is to ascribe to the model formulation which result intractable or at least very demanding from a computational point of view. In fact, the introduction of stochastic dominance constraints imply an increment in the complexity and in the high dimensionality of the problem since they seems to have a non-linear feature.

Here we present the methodology to linearize FSD and SSD reviewing some important work in the literature. In particular, first order stochastic dominance imply that the cumulative distribution of the dominated random variable should be greater than the dominant one (2.43). This type of formulation is presented in a non-linear form since the returns of the two portfolios have to be sorted. For this reason, Kopa (2010) and Kuosmanen (2004) propose a linear formulation of this problem through the introduction of a permutation matrix. Let  $P = \{p_{r,c}\}$  a permutation matrix with  $p_{r,c} = 0, 1$  such that  $\sum_{r=1}^T p_{r,c} = 1$  for  $c = 1, \dots, T$  and  $\sum_{c=1}^T p_{r,c} = 1$  for  $r = 1, \dots, T$ . Then portfolio

$X = x\beta$  dominates portfolio  $Y$  in a first order sense if and only if:

$$\begin{aligned}
X &\geq PY \\
\sum_{r=1}^T p_{r,c} &= 1 \quad \forall c = 1, \dots, T \\
\sum_{c=1}^T p_{r,c} &= 1 \quad \forall r = 1, \dots, T \\
p_{r,c} &\in [0, 1] \quad \forall r = 1, \dots, T; \quad \forall c = 1, \dots, T
\end{aligned} \tag{2.47}$$

Differently, several approach are presented to solve the second order stochastic dominance in a linear formulation. Let  $X = r\beta$  be the random variable of the invested portfolio in which the discrete joint distribution of the asset component  $x_t$ ,  $t = 1, \dots, T$  have the same probability and let  $Y$  be the benchmark with realization  $y_i$  (for  $i = 1, \dots, T$ ), then  $X \underset{(2)}{\geq} Y$  in the second order stochastic dominance sense if:

$$\mathbb{E}[(y_i - r_t\beta)_+] \leq \mathbb{E}[(y_i - Y)_+] \quad \forall i = 1, \dots, T \tag{2.48}$$

Then the formulation of the stochastic dominance relation 2.48 could be expressed in the following linear representation. Introducing slack variables  $s_{i,t}$  representing shortfall of  $r_t\beta$  below  $y_i$  in realization  $t$ ,  $t = 1, \dots, T$ , we can formulate the second order stochastic dominance (Ortobelli et al., 2013; Dentcheva and Ruszczyński, 2006) as:

$$\begin{aligned}
\sum_{n=1}^N r_{t,n}\beta_n + s_{i,t} &\geq y_i \quad \forall i = 1, \dots, T; \quad \forall t = 1, \dots, T \\
\sum_{t=1}^T s_{i,t} &\leq \mathbb{E}[(y_i - Y)_+] \quad \forall i = 1, \dots, T \\
s_{i,t} &\geq 0 \quad \forall i = 1, \dots, T; \quad \forall t = 1, \dots, T
\end{aligned} \tag{2.49}$$

Differently, Kopa (2010) and Kuosmanen (2004) propose another linear formulation of the second order stochastic dominance. Let us assume that the return have a discrete



joint distribution with realizations  $x_t$ ,  $t = 1, \dots, T$  having the same probability, then  $X \stackrel{(2)}{\geq} Y$  in the second order stochastic dominance sense if and only if there exists a double stochastic matrix  $Z = \{z_{r,c}\}$  with  $z_{r,c} \in [0, 1]$  such that

$$\begin{aligned}
 X &\geq ZY \\
 \sum_{r=1}^T z_{r,c} &= 1 \quad \forall c = 1, \dots, T \\
 \sum_{c=1}^T z_{r,c} &= 1 \quad \forall r = 1, \dots, T \\
 0 \leq z_{r,c} &\leq 1 \quad \forall r = 1, \dots, T; \quad \forall c = 1, \dots, T
 \end{aligned} \tag{2.50}$$

### 2.4.3 Enhanced Indexing Problem with Stochastic Dominance Constraints

A realistic formulation to solve the enhanced index benchmark tracking problem should consider the introduction of different components to make the model as real as possible. This complete formulation takes into account a linear penalty objective function and buy-in threshold level to reduce the portfolio turnover and risk management duties. Moreover, whether the introduction of stochastic dominance constraints enhances the benchmark tracking model, its formulation strongly increases the dimensionality and the computational complexity of the problem. In particular, we consider the methodologies proposed by [Kopa \(2010\)](#) and [Kuosmanen \(2004\)](#).

The enhanced index benchmark tracking problem is solved considering the minimization of a dispersion measure of the tracking error, the tracking error quantile regression (2.27), which could be formulated as linear (2.29), and the minimization of the transaction costs. To enhance the performance in the risk minimization, we introduce first and second order stochastic dominance constraints following the formulations (2.47) and (2.50). Let the log-return of equity index  $Y$  with realization  $y_t$ ,  $t = 1, \dots, T$  and of its  $N$  components being  $R = r_1, r_2, \dots, r_N$ . The tracking error  $\varepsilon_t = \sum_{n=1}^N r_{t,n} \beta_n - y_t$  is

minimized considering the tracking error quantile regression. Then, let  $tc^+$  and  $tc^-$  the transaction costs to the buying and selling portfolios  $\omega^+$  and  $\omega^-$  with a buy-in threshold level  $\theta$ . We define the enhanced indexation benchmark tracking problem with FSD constraints as:

$$\begin{aligned}
& \min_{\beta, u, \nu, p, \omega^+, \omega^-} \sum_{t=1}^T \tau u_t + (1 - \tau) \nu_t + tc^+ \omega^+ + tc^- \omega^- \\
& \text{s.t.} \quad r_t \beta - u_t + \nu_t = y_t \quad \forall t = 1, \dots, T \\
& \quad \sum_{n=1}^N \beta_n = 1 \\
& \quad \mathbb{E}[X] - \mathbb{E}[Y] \geq K^* \\
& \quad \omega_n^+ - \omega_n^- = \beta_n - \beta_n^{old} \quad \forall n = 1, \dots, N \\
& \quad \sum_n |\beta_n - \beta_n^{old}| \leq \theta \quad n = 1, \dots, N \\
& \quad X \geq PY \quad (2.51) \\
& \quad \sum_{r=1}^T p_{r,c} = 1 \quad \forall c = 1, \dots, T \\
& \quad \sum_{c=1}^T p_{r,c} = 1 \quad \forall r = 1, \dots, T \\
& \quad p_{r,c} \in \{0, 1\} \quad \forall r, c = 1, \dots, T \\
& \quad lb \leq \beta_n \leq ub \quad \forall n = 1, \dots, N \\
& \quad lb \leq \omega_n^+, \omega_n^- \leq ub \quad \forall n = 1, \dots, N \\
& \quad u_t, \nu_t \geq 0 \quad \forall t = 1, \dots, T
\end{aligned}$$

The solution of this problem is the portfolio chosen by all non-satiable investors. As treated in [Jarrow \(1986\)](#) the existence of a portfolio that stochastically dominates the index in a first order sense is equivalent to the concept of arbitrage. However, the enhanced index benchmark tracking problem (2.51) is a mixed-integer linear programming since the permutation matrix  $P$  is composed by binary variables. We notice how the dimensionality of this problem quadratically increase with the number of observation  $T$ .

The other proposed model is based on second order stochastic dominance with the introduction a double stochastic matrix  $Z$ . Thus, we define the enhanced indexation benchmark tracking problem with SSD constraints as:

$$\begin{aligned}
& \min_{\beta, u, \nu, p, \omega^+, \omega^-} \sum_{t=1}^T \tau u_t + (1 - \tau) \nu_t + tc^+ \omega^+ + tc^- \omega^- \\
& \text{s.t.} \quad r_t \beta - u_t + \nu_t = y_t \quad \forall t = 1, \dots, T \\
& \quad \sum_{n=1}^N \beta_n = 1 \\
& \quad \mathbb{E}[X] - \mathbb{E}[Y] \geq K^* \\
& \quad \omega_n^+ - \omega_n^- = \beta_n - \beta_n^{old} \quad \forall n = 1, \dots, N \\
& \quad \sum_n |\beta_n - \beta_n^{old}| \leq \theta \quad n = 1, \dots, N \\
& \quad X \geq ZY \quad (2.52) \\
& \quad \sum_{r=1}^T z_{r,c} = 1 \quad \forall c = 1, \dots, T \\
& \quad \sum_{c=1}^T z_{r,c} = 1 \quad \forall r = 1, \dots, T \\
& \quad 0 \leq z_{r,c} \leq 1 \quad \forall r, c = 1, \dots, T \\
& \quad lb \leq \beta_n \leq ub \quad \forall n = 1, \dots, N \\
& \quad lb \leq \omega_n^+, \omega_n^- \leq ub \quad \forall n = 1, \dots, N \\
& \quad u_t, \nu_t \geq 0 \quad \forall t = 1, \dots, T
\end{aligned}$$

Differently from the previous enhanced index problem with first order stochastic dominance constraints, this formulation is a linear programming and could be efficiently solved also when the computational complexity increase with the number of observations.

## 2.5 Empirical Applications

In this section, we present some empirical applications involving the tracking error quantile regression. Firstly, we present the datasets which cover a long investment period with different phases of the financial cycle and the related stock indexes are composed by different number of assets. Secondly, we solve the classical benchmark tracking problem comparing the common measures of the tracking error with the proposed dispersion one. Analyzing the behavior in sample and out of sample we also evaluate the differences between static and rolling strategies. Thirdly, we propose a realistic LP model to solve the enhanced indexation strategies with stochastic dominance constraints. In particular, we stress the advantages of our methodology to reduce the risk component obtaining extra-performances in a static or rolling framework.

### 2.5.1 Datasets Description

The empirical analysis is based on three stock indexes: Russell 1000, S&P 500 and Nasdaq 100. The first one is a very important stock index in the financial market since it considers the first 1000 U.S. public company for market capitalization. For this reason, it represents one of the most tracked index through the exchange-traded funds (ETF) and the benchmark of comparison for several portfolio strategies. The S&P 500 is an American stock market index based on the market capitalizations of 500 large companies having common stock listed on the NYSE or NASDAQ. It is one of the most commonly followed equity indices, and many consider it one of the best representations of the U.S. stock market, and a bellwether for the U.S. economy. Finally, the Nasdaq 100 is a capitalization-weighted stock market index made up of 109 equity securities issued by 100 of the largest non-financial companies listed on the NASDAQ.

The analyzed time period covers the last decade from 31st December 2002 to 31st December 2013 and we propose investment strategies with monthly recalibration (20 days) with a total number of 125 optimization steps. We generally consider an historical

moving window of 260 observations which is reduced to 120 time series data when we apply enhanced indexing strategies with first order stochastic dominance constraints. Every investment portfolio strategy starts on 12th January 2004. While the Russell 1000 presents 736 components as number of stocks during the entire period, the S&P 500 is composed by 441 assets and the Nasdaq 100 has 84 components. We set a spectrum of possible quantiles  $\tau$  in the range  $[0.01, 0.05, 0.10, 0.20, \dots, 0.90, 0.95, 0.99]$  developing the two different strategies involving the tracking error quantile regression dispersion measure: the static, fixing the referred quantile a priori or the rolling when we switch the quantile at each optimization step.

Figure 2.1 shows the normalized wealth path of the three stock indexes during the investment period and we compare our portfolio strategies with these benchmarks.

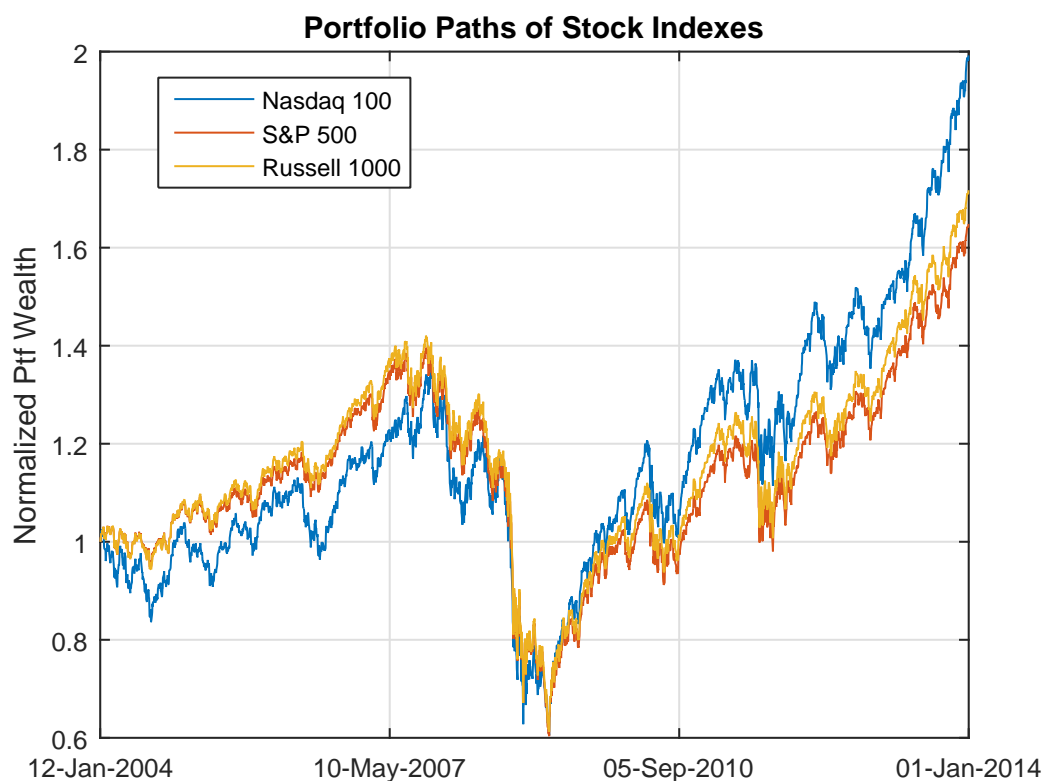


FIGURE 2.1: Portfolio Paths of Stock Indexes

On the overall period the three stock indexes have similar behavior and path. Whilst, the Russell 1000 and the S&P 500 have the same path with the first index dominating the second one, the wealth path of the Nasdaq 100 under-performs the other two before the huge sub-prime crisis inverted this trend during the following financial upturn. In particular, for the first four years the indexes steadily increase reaching a maximum level of 1.4. Then, they forcefully fall down due to the sub-prime crisis which putted in jeopardy the entire system.

After loosing more than the 84% the three stock indexes and in general the U.S. equity market rescued consequently to monetary policy of the FED and until 2011 they offer interesting gains for private and institutional investors. In 2011, the European government instability given by the Greece default rumors have consequences on the three benchmarks that loose about the 20% of their values. Finally, another period of financial growth characterized the recent years where the index has been increased more than 60% with a final wealth of about 1.7 for the Russell 1000, 1.6 for the S&P 500 and 2 for the Nasdaq 100.

For this reason the selected stock indexes represents significant benchmarks not only for their different number of components that stress the goodness of proposed portfolio selection models in the high dimensionality framework but also for the several phases of the financial cycle. In fact, we test weather the portfolio selection problems mimic, enhance or outperform the index during constant period, crisis or financial upturns in boosted markets.

### **2.5.2 Comparison Between Different Dispersion Measures in the Index Tracking Problem**

To evaluate the goodness of the proposed measure of dispersion of the tracking error, we empirically test it comparing some statistics in the in sample and out of sample analysis. In particular, we firstly solve index tracking portfolio selection problems [2.6](#), [2.10](#) and [2.13](#) with the tracking error volatility, mean absolute deviation and downside mean

semideviation as dispersion measures. Then, we solve the portfolio problem 2.30 for the tracking error quantile regression. In the Table 2.1, 2.2 and 2.3, we report 5 different statistics for the in sample and out of sample analyses. In particular, we evaluate common and innovative strategies considering the portfolio alpha  $\alpha$ , the tracking error mean absolute deviation, the tracking error downside mean semideviation, the tracking error volatility and the average number of active assets.

Generally, we notice how the in sample analysis is not related with the out of sample and it is difficult for portfolio managers take decision based on the in sample information. Identifying the best criteria of selection, it is not possible to fix the in sample quantile and obtain the best out of sample statistics. Focusing on the alpha of the portfolio we could see how in the left side of the Table 2.1 increasing the level of the quantile we obtain higher values of portfolio alpha according to the aim to build a VaR tracking portfolio or one that wants to have better performances. Whether Table 2.1 and 2.2 produce very low values of alpha considering a tracking portfolio with more than 60 active assets, the replication of the Nasdaq 100 (Table 2.3) is obtained with a smaller portfolio and the results show an increment in the extra-performances of the invested portfolios with respect to the benchmark.

In a deviation framework, we observe how the tracking error quantile reduce this risk source with respect to the common measures of dispersion. It is possible to find a quantile with lower measure of dispersion than the common ones but since there are no relations between the in sample and the out of sample analysis it is not possible to select an efficient quantile in a priori window. Moreover, the dispersion is strictly related with the portfolio performances. A tracking portfolio with an high alpha has also an higher dispersion. However, the number of active assets represent a significant parameter to evaluate the degree of diversification but it is also a negative parameter considering transaction and managing costs. Finally, since it is not possible to select a priori best quantile, the introduction of rolling strategy allow to have a unique solution to this problem.

TABLE 2.1: Index Tracking Strategy Statistical Analysis, Russell 1000

Russell 1000	In Sample				Out of Sample				
	$\alpha$	TEMAD	TEDMS	TEV #	$\alpha$	TEMAD	TEDMS	TEV #	
Str. min TEQR									
tau 0.01	1.63	2.89	1.61	3.60	59	35.53	18.09	50.61	61
tau 0.05	1.68	2.47	1.60	3.05	62	35.33	18.13	50.18	62
tau 0.10	1.84	3.13	1.51	3.91	49	35.03	18.17	50.08	61
tau 0.20	1.65	2.85	1.63	3.52	59	35.08	18.08	50.76	62
tau 0.30	1.68	2.89	1.61	3.58	60	34.97	18.12	50.29	62
tau 0.40	1.69	2.98	1.60	3.65	57	34.47	17.87	48.65	61
tau 0.50	1.62	2.73	1.65	3.38	63	35.04	18.27	50.93	62
tau 0.60	1.64	2.95	1.62	3.62	59	34.00	18.02	47.45	61
tau 0.70	1.83	2.67	1.57	3.38	58	35.39	17.9	48.84	60
tau 0.80	1.73	2.74	1.63	3.38	62	36.3	18.14	50.79	59
tau 0.90	1.76	2.82	1.65	3.51	60	36.24	18.28	51.08	59
tau 0.95	1.90	2.68	1.64	3.36	58	36.53	18.43	51.19	57
tau 0.99	2.11	2.59	1.53	3.22	58	36.54	18.19	51.81	57
Str. min TEMAD	-0.31	2.63	1.47	3.34	61	34.41	17.80	50.46	62
Str. min TEDMS	0.69	3.88	1.37	4.76	42	42.55	21.90	60.61	43
Str. min TEV	-0.19	2.68	1.50	3.15	63	34.38	17.75	49.63	62



TABLE 2.2: Index Tracking Strategy Statistical Analysis, S&amp;P 500

S&P 500	In Sample				Out of Sample					
	$\alpha$	TEMAD	TEDMS	TEV #	$\alpha$	TEMAD	TEDMS	TEV #		
Str. min TEQR										
tau 0.01	1.67	1.90	1.07	2.43	72	-0.07	28.49	14.28	38.97	68
tau 0.05	1.80	1.73	1.06	2.39	72	-0.18	28.16	14.40	38.53	68
tau 0.10	1.82	1.98	1.07	2.64	70	-0.01	28.38	14.28	38.82	67
tau 0.20	1.88	1.96	1.02	2.68	68	0.14	28.68	14.39	39.55	67
tau 0.30	1.82	2.03	1.04	2.71	69	-0.03	28.43	14.20	39.61	68
tau 0.40	1.81	2.03	1.03	2.69	69	-0.02	28.45	14.37	39.66	67
tau 0.50	1.84	2.03	1.02	2.74	64	0.10	28.48	14.25	39.77	67
tau 0.60	1.85	2.04	1.06	2.63	64	-0.18	28.31	14.25	40.09	67
tau 0.70	1.86	1.94	1.03	2.58	68	0.06	28.14	14.24	39.52	67
tau 0.80	1.86	2.03	1.04	2.67	68	0.05	28.11	14.49	39.19	66
tau 0.90	1.82	2.00	0.99	2.60	68	-0.13	28.66	14.24	39.74	66
tau 0.95	2.01	2.23	0.97	2.89	61	-0.02	28.78	14.38	40.08	66
tau 0.99	2.16	2.16	0.97	2.77	62	-0.18	29.28	14.59	41.02	65
Str. min TEMAD	-0.23	1.91	1.09	2.39	71	-0.54	28.84	14.69	39.40	66
Str. min TEDMS	0.86	2.35	0.75	2.96	55	-0.29	35.22	17.76	48.55	48
Str. min TEV	0.10	1.96	0.91	2.36	64	-0.65	31.54	16.09	49.34	63

TABLE 2.3: Index Tracking Strategy Statistical Analysis, Nasdaq 100

Nasdaq 100	In Sample				Out of Sample				
	$\alpha$	TEMAD	TEDMS	TEV #	$\alpha$	TEMAD	TEDMS	TEV #	
Str. min TEQR									
tau 0.01	3.86	4.84	2.38	6.75	37	62.44	28.78	86.79	36
tau 0.05	4.20	5.01	2.37	7.17	39	60.28	28.85	83.47	37
tau 0.10	4.30	5.65	2.35	8.17	37	59.00	28.63	82.01	37
tau 0.20	4.43	6.15	2.23	9.00	32	57.84	29.05	80.25	38
tau 0.30	4.40	6.34	2.28	9.60	32	56.98	29.42	79.04	38
tau 0.40	4.43	6.26	2.29	9.46	32	56.55	28.83	78.19	38
tau 0.50	4.35	6.09	2.17	9.27	29	56.28	28.64	77.59	38
tau 0.60	4.35	5.70	2.34	8.87	33	55.75	28.73	76.50	38
tau 0.70	4.42	5.56	2.02	8.83	36	55.07	28.85	75.19	39
tau 0.80	4.55	5.49	2.00	8.60	37	54.93	28.73	74.86	38
tau 0.90	4.77	5.30	2.08	7.96	37	54.28	29.01	74.26	39
tau 0.95	5.20	5.81	2.16	8.51	35	54.76	29.46	74.22	39
tau 0.99	5.63	6.31	1.96	9.21	33	55.97	28.87	76.11	39
Str. min TEMAD	-0.06	3.35	1.70	4.49	49	45.35	20.71	60.80	51
Str. min TEDMS	1.32	3.72	1.20	5.06	45	48.83	22.50	65.17	47
Str. min TEV	-0.03	3.40	1.66	4.13	54	44.42	20.07	59.86	53

In a rolling framework, we switch the quantile at each optimization step according with the maximum information ratio 2.40 in the last month. Figures 2.2, 2.3 and 2.4 illustrate the wealth path of the common tracking error strategies and the rolling one in the out of sample analyses. In these Figures we report the wealth paths of the minimization of the tracking error mean absolute deviation (blue line) solving the portfolio problem 2.10, the tracking error downside mean semideviation (red line) solving the portfolio model 2.13 and the quadratic formulation 2.6 for the tracking error volatility (yellow line). The violet line represents the wealth path of the rolling strategy which switches the quantile at each optimization step according with the higher in sample information ratio quantile regression 2.40 during the last month and we compare all these strategies with the given benchmark (green line).

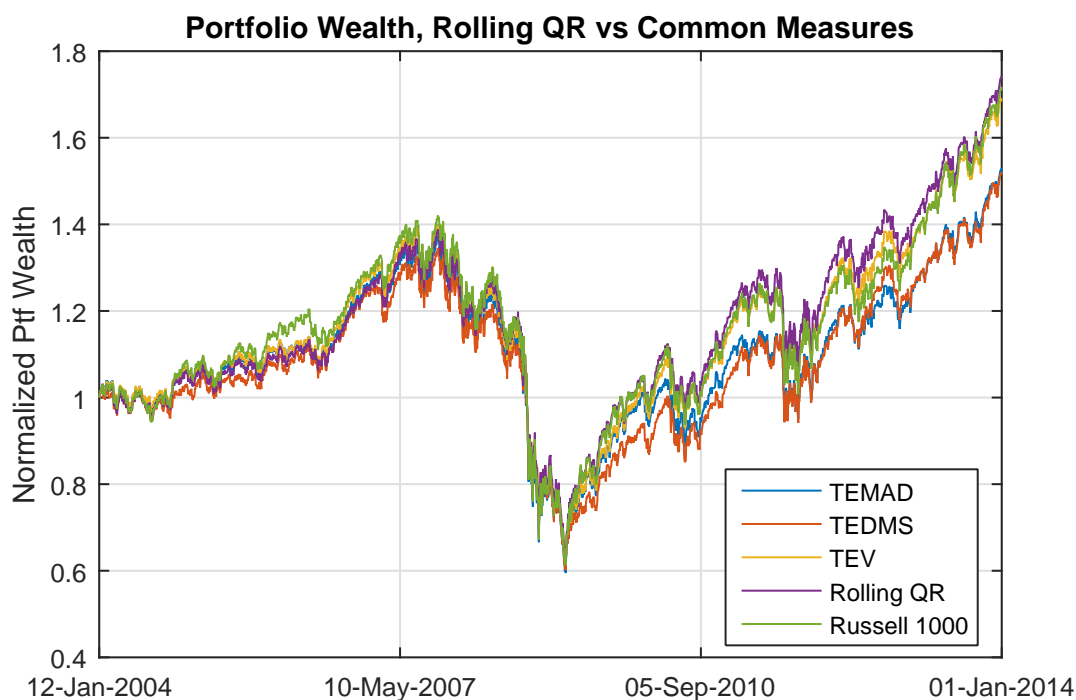


FIGURE 2.2: Out of Sample Portfolio Wealth of Index Tracking Rolling Strategy, Russell 1000

Figure 2.2 shows the wealth paths regarding the index tracking of the Russell 1000.

We notice that during the first period of the strategy the replication portfolios mimic the Russell 1000 from below with a relevant difference in 2006. Then, there are an adjustment of the portfolios during the sub-prime crisis. In this case, the proposed formulation is suitable to capture period of financial instability. The financial upturn in 2009, marks two principal features in the benchmark tracking strategies.

Firstly, the minimization of the tracking error mean absolute deviation and tracking error downside mean semideviation leads to a strong replication of the benchmark from below. This implies inefficient tracking process since the difference between the two mimic and the benchmark portfolios steadily increase. Secondly, while the tracking error volatility approach matches the portfolio and index returns, the rolling strategy involving the tracking error quantile regression efficiently deal with the benchmark tracking problem since it is the only one who shows the matching from a above. Table 2.4 reports the statistics of the out of sample analysis. In particular, we present also the information ratios related with the portfolio alpha and the different dispersion measure. It is clear the importance of the introduced information ratio quantile regression to improve the decisional problem to order in a reward risk sense benchmark tracking strategies with negative  $\alpha$ .

Changing the benchmark and focusing on the replication of the S&P 500 we observe a different behavior of the wealth paths of the strategies. In particular, the minimization of the tracking error mean absolute deviation and the downside mean semideviation significantly improves and they mimic efficiently the benchmark index during the overall period. In contrast, we notice how the minimization of the tracking error volatility under-performs the benchmark after the sub-prime crisis. Table 2.6 shows the benefits of rolling approach with respect to the common dispersion measures of the tracking error. In particular, comparing this table with the results of the Russell 1000 we observe a reduction of the dispersion keeping similar reward value.

Analyzing the index tracking strategies to mimic the Russell 1000 and the S&P 500, we dictate the dominance of the proposed tracking error quantile regression dispersion

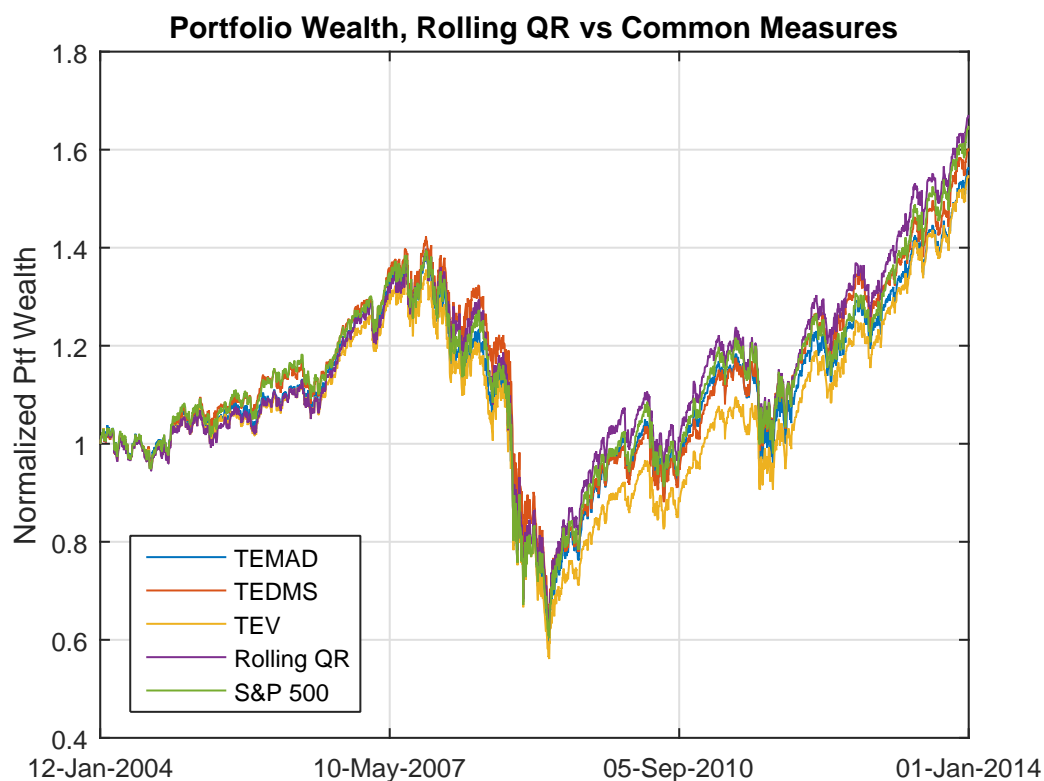


FIGURE 2.3: Out of Sample Portfolio Wealth of Index Tracking Rolling Strategy, S&P 500

measure to replicate the benchmarks. Developing a rolling strategy which proposes to switch the quantile according to the greatest information ratio quantile regression evaluated in a monthly in sample period we obtain an index tracking portfolio with positive alpha a low dispersion.

Finally, Figure 2.4 illustrates the wealth paths of the comparison between the three common tracking error measures and the quantile rolling strategy. We notice that all the index tracking strategies outperform the benchmark for the overall period with a similar path. In particular, they show extra gains before the sub-prime crisis and in the following financial upturn they steadily increase since the Nasdaq 100 is not strongly affected from the European sovereign debt crisis. Although the mimic portfolios have

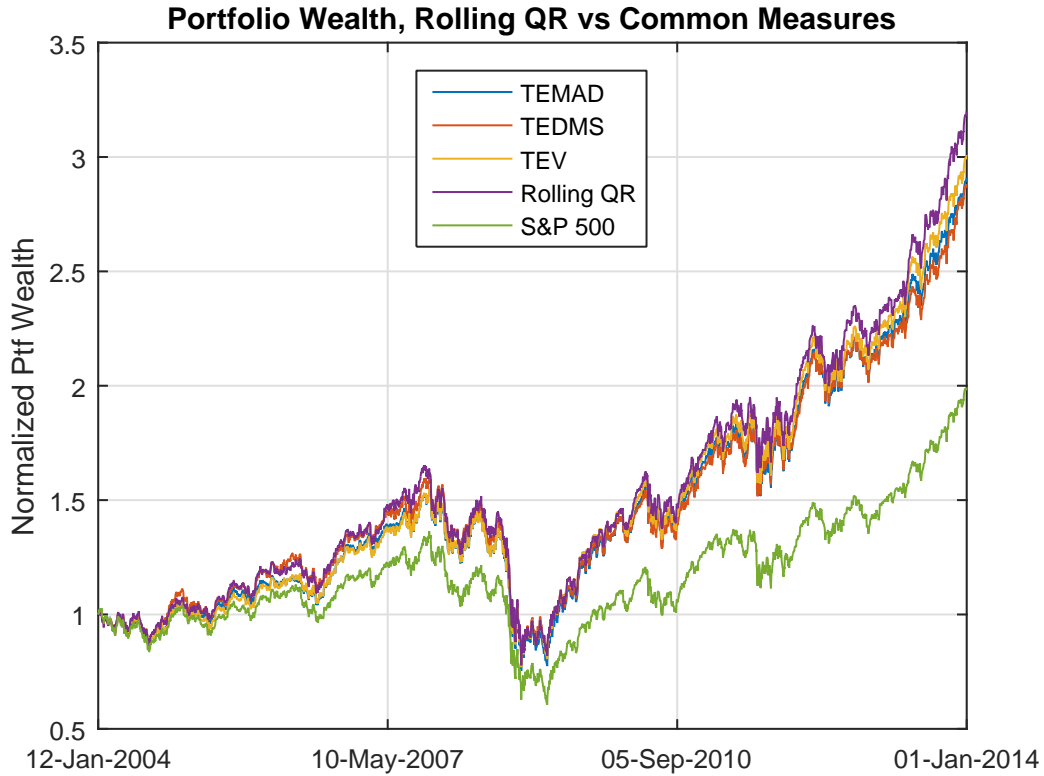


FIGURE 2.4: Out of Sample Portfolio Wealth of Index Tracking Rolling Strategy, Nasdaq 100

similar path the rolling quantile regression dominates the others in terms of wealth with a final value greater than 3.3.

### 2.5.3 A Realistic Formulation for the Enhanced Indexation Problem with Stochastic Dominance Constraints

The comparison between the tracking error quantile regression and the common dispersion measures to mimic the performance of a benchmark highlights the theoretical and empirical impact of this measure in the benchmark tracking problem. Thus, we propose two realistic models to solve the enhanced indexation benchmark tracking problem introducing first and second order stochastic dominance constraints. We solve the

optimization problems (2.51) and (2.52) with  $tc^+ = tc^- = 15\text{bps}$  and the turnover constraints  $\theta = 50\%$ . It means the impossibility to roll more than 50% of the invested portfolio at each optimization step. We remark that for the problem (2.51) we consider a rolling time series of 130 historical observations since the problem is still demanding considering the presence of  $130 \times 130 = 16900$  integer random variables.

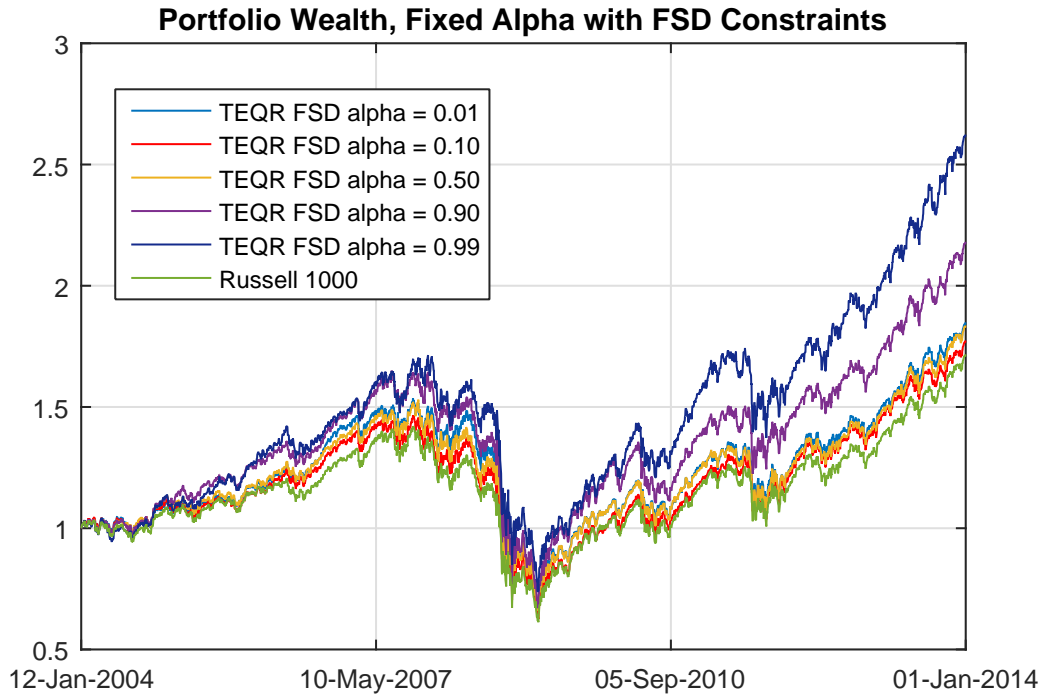


FIGURE 2.5: Out of Sample Portfolio Wealth of Enhanced Indexation Static Strategies, Russell 1000

In this approach we compute the static analysis fixing the spectrum of  $\tau$  and keeping constant the selected one for the entire period. Then, we propose the rolling strategy introducing the information ratio quantile regression (2.40) as decisional variable and we report the results of the wealth paths. Figure 2.5 reports the wealth paths of five static strategies to solve the enhanced indexation problem with first order stochastic dominance constraints. We notice that increasing the level of the quantile value we obtain higher returns and consequent dispersion. This pattern is not totally confirmed in the

analysis of the enhanced indexation strategies with second order stochastic dominance constraints where the benchmark is the Nasdaq 100. In this case extreme values of the quantile lead to have the best portfolios in terms of final wealth as reported in Table 2.8. Similar behavior is showed in the Figure 2.6 where we report the out of sample wealth path of the enhanced indexation static strategies. We observe how the introduction of these portfolio constraints allows to have a better performance than the tracked index. These observations stress the idea to introduce also a rolling strategy in the enhanced indexation framework to have a unified approach and to take advantages from the better enhance in a given period.

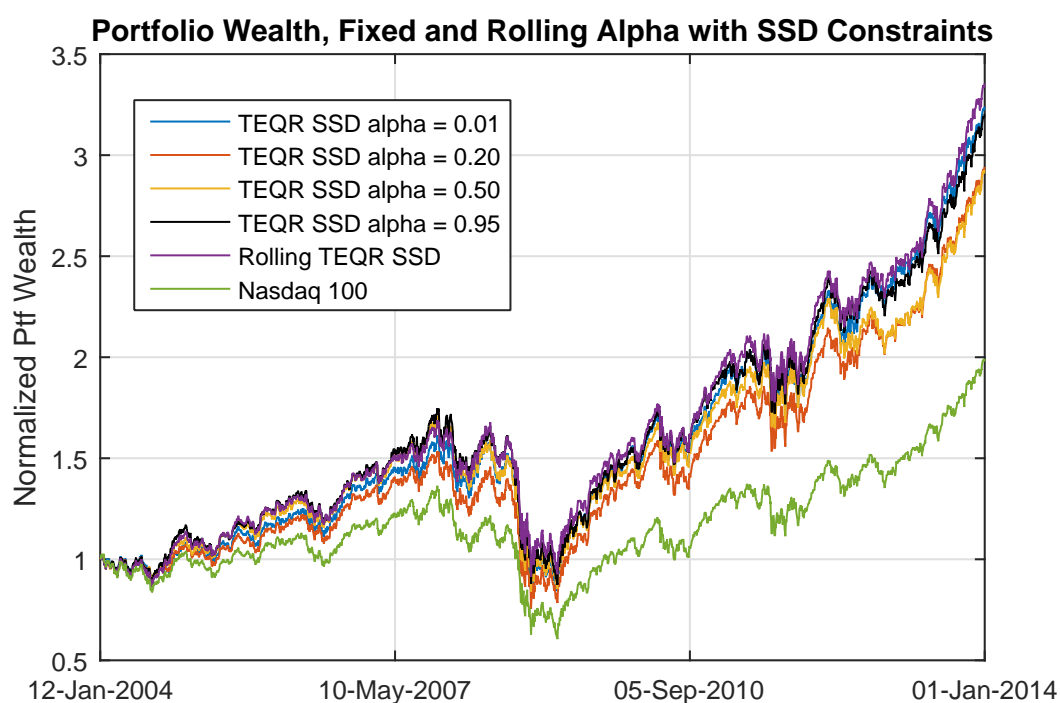


FIGURE 2.6: Out of Sample Portfolio Wealth of Enhanced Indexation Static Strategies, Nasdaq 100

Thus, Figures 2.7 and 2.8 show the wealth paths of the enhanced indexing rolling strategies with first and second order stochastic dominance constraints with Russell 1000 and S&P 500 as benchmarks. Analyzing the results, we deduce the importance of



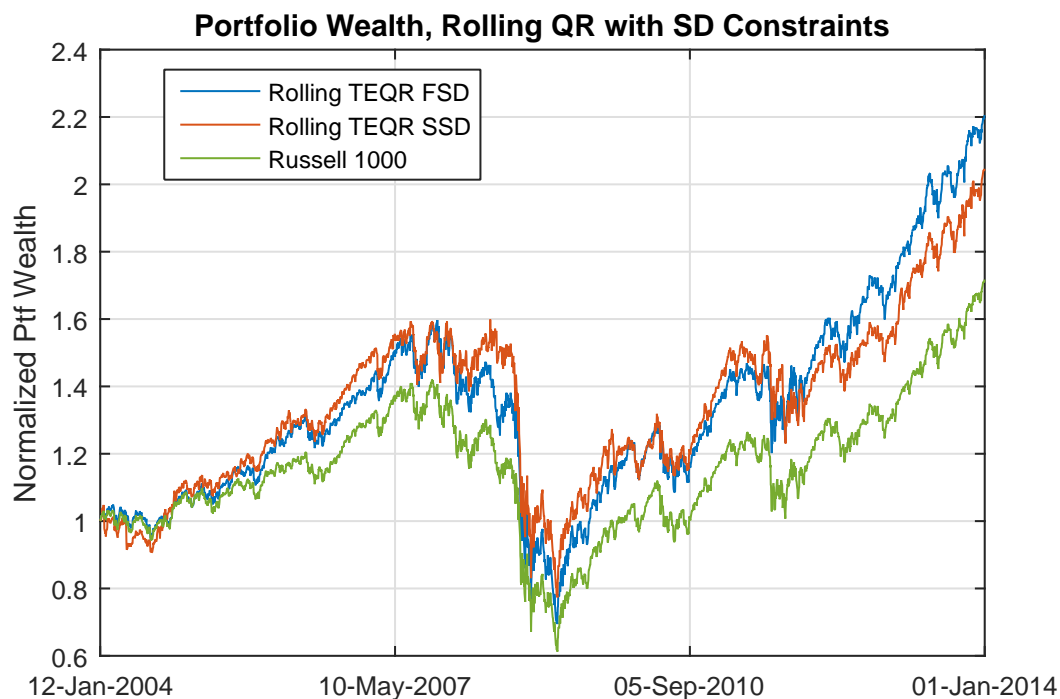


FIGURE 2.7: Out of Sample Portfolio Wealth of Enhanced Indexation Rolling Strategy, Russell 1000

stochastic dominance constraints in the portfolio selection problem. The wealth paths of the enhanced index portfolio with first and second order stochastic dominance outperform the Russell 1000 on the overall period. Only during the first period when the market is constant the portfolio with second order stochastic dominance constraints could not show extra-performances. Then, during the first investment period the enhanced indexation strategies outperform the benchmark of about 20% while at the end of 2013 the extra-performances are about 30% and 50% for FSD and SSD portfolios. Considering the S&P 500 as a benchmark, we obtain the same patterns with a lower level of wealth for the entire period. In fact, the final value of the enhanced indexing portfolio with first order stochastic dominance is about 2.1 and for second order is about 1.8.

Interesting analyses are developed in the Tables 2.5 and 2.7 where we report the

statistical values of the rolling strategy in the tracking error problem and in the enhanced ones. The main feature is not only the increment of the portfolio return but also the small increment in the dispersion measure produced by the rolling enhanced indexation strategy with first order stochastic dominance constraints. In fact, this type of approach results to be very interesting from portfolio managers since the increment of the risk is very controlled with respect to the significant extra-performances in the portfolio final wealth.

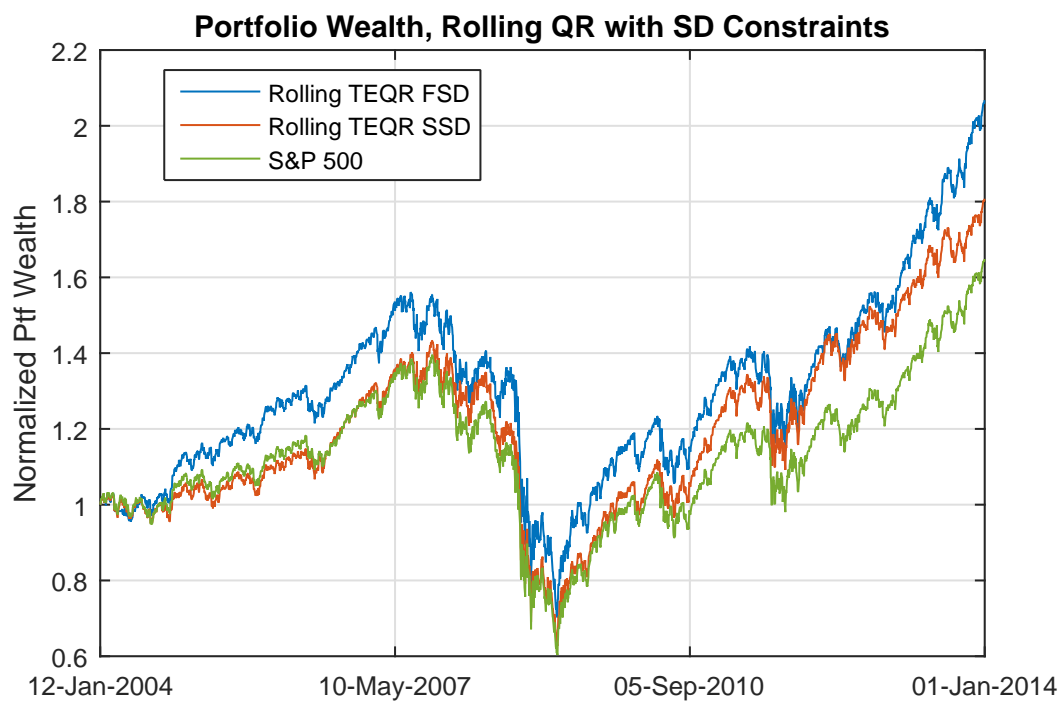


FIGURE 2.8: Out of Sample Portfolio Wealth of Enhanced Indexation Rolling Strategy, S&P 500

TABLE 2.4: Index Tracking Rolling Strategy Out of Sample Statistical Analysis, Russell 1000

Russell 1000		Out of Sample								
Rolling tau	TEQR	TEMAD	TEDMS	TEV	IR <sub>MAD</sub>	IR <sub>DMS</sub>	IR <sub>V</sub>	IR <sub>QR</sub>	#	
Str. max	IR <sub>QR</sub>	0.15	35.59	17.72	50.33	0.44	0.87	0.31	1.009	60
Str. min	TEMAD	-1.19	34.41	17.8	48.47	-3.47	-6.71	-2.46	0.933	62
Str. min	TEDMS	-1.26	42.55	21.9	60.62	-2.95	-5.74	-2.07	0.943	43
Str. min	TEV	-0.16	34.38	17.75	49.63	-0.47	-0.90	-0.32	0.937	62

TABLE 2.5: Enhanced Indexation Rolling Strategy Out of Sample Statistical Analysis, Russell 1000

Russell 1000		Out of Sample								
Rolling tau	TEQR	TEMAD	TEDMS	TEV	IR <sub>MAD</sub>	IR <sub>DMS</sub>	IR <sub>V</sub>	IR <sub>QR</sub>	#	
Str. max	IR <sub>QR</sub>	0.15	35.59	17.72	50.33	0.44	0.87	0.31	1.009	60
Str. max	IR <sub>QR</sub> FSD	2.59	43.23	20.32	65.26	6.00	12.76	3.97	1.128	61
Str. max	IR <sub>QR</sub> SSD	1.82	69.18	33.68	129.94	2.63	5.39	1.40	1.054	49

TABLE 2.6: Index Tracking Rolling Strategy Out of Sample Statistical Analysis, S&amp;P 500

<b>S&amp;P 500</b>		<b>Out of Sample</b>								
<b>Rolling tau</b>	<b>TEQR</b>	<b>TEMAD</b>	<b>TEDMS</b>	<b>TEV</b>	<b>IR<sub>MAD</sub></b>	<b>IR<sub>DMS</sub></b>	<b>IR<sub>V</sub></b>	<b>IR<sub>QR</sub></b>	<b>#</b>	
<b>Str. max</b>	<b>IR<sub>QR</sub></b>	0.13	28.68	14.22	39.30	0.45	0.91	0.33	1.017	67
<b>Str. min</b>	<b>TEMAD</b>	-0.54	28.84	14.69	39.40	-1.88	-3.68	-1.37	0.963	66
<b>Str. min</b>	<b>TEDMS</b>	-0.29	35.22	17.76	48.55	-0.83	-1.64	-0.60	0.984	48
<b>Str. min</b>	<b>TEV</b>	-0.65	31.54	16.09	49.34	-2.06	-4.04	-1.32	0.960	63

TABLE 2.7: Enhanced Indexation Rolling Strategy Out of Sample Statistical Analysis, S&amp;P 500

<b>S&amp;P 500</b>		<b>Out of Sample</b>								
<b>Str. rolling tau</b>	<b>TEQR</b>	<b>TEMAD</b>	<b>TEDMS</b>	<b>TEV</b>	<b>IR<sub>MAD</sub></b>	<b>IR<sub>DMS</sub></b>	<b>IR<sub>V</sub></b>	<b>IR<sub>QR</sub></b>	<b>#</b>	
<b>max</b>	<b>IR<sub>QR</sub></b>	0.13	28.68	14.22	39.30	0.45	0.91	0.33	1.017	67
<b>max</b>	<b>IR<sub>QR</sub> FSD</b>	2.35	36.59	17.12	55.16	6.41	13.71	4.26	1.137	62
<b>max</b>	<b>IR<sub>QR</sub> SSD</b>	0.95	30.55	14.8	41.93	3.12	6.45	2.28	1.064	62



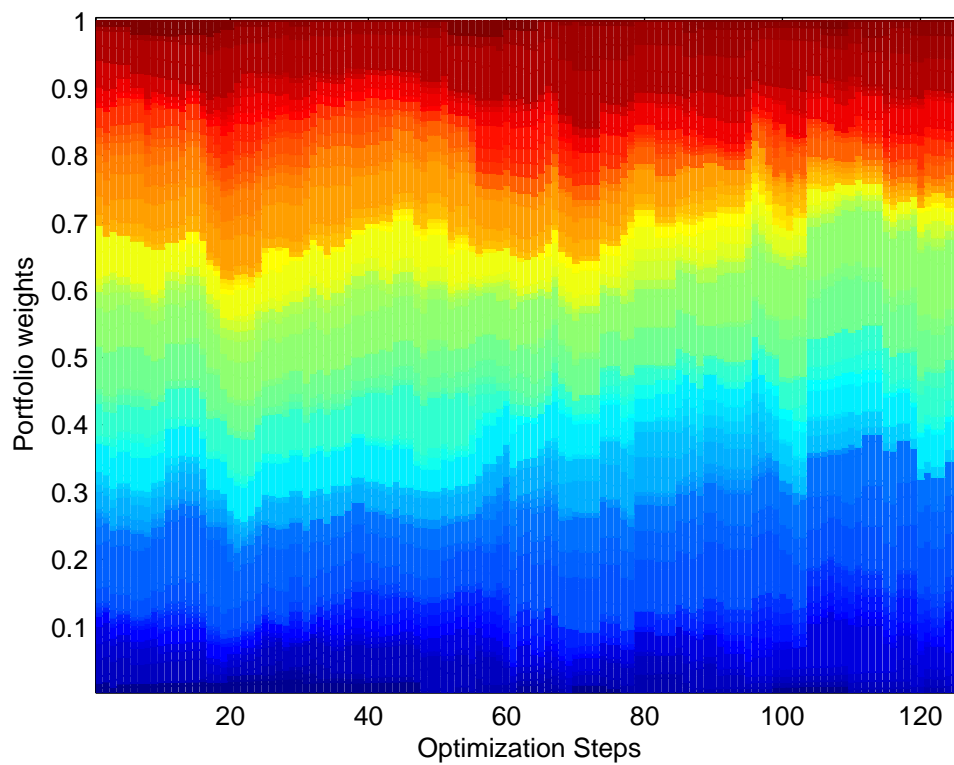


FIGURE 2.9: Enhanced Indexation SSD Rolling Strategy Portfolio Weights, Russell 1000

Finally, in Figure 2.9 we report the different weights of the portfolio composition in the enhanced indexation strategy with second order stochastic dominance constraints. One advantage of the introduction of stochastic dominance constraints in the decisional problem is related with the stability of portfolio weight. However, for management duties this approach implies not only lower transaction costs but also a risk management simplified process to evaluate and estimate possible different sources of risk in the investment process.

## 2.6 Final Remarks

In this chapter, we introduce a dispersion measure for the index tracking portfolio problem. This type of problem aims to replicate the performance of a given stock index as benchmark. The tracking error quantile regression allows to better mimic the behavior of the index during different phases of the financial cycle. For a theoretical point of view this measure belongs to the class of relative deviation metrics and it satisfy a general structure of properties. Analyzing the results of an empirical application, a rolling strategy based on the switching of the quantile during at each optimization step produce a unique and interesting approach to solve this problem. Then, we propose a realistic enhanced indexation problem minimizing the tracking error quantile regression with first and second order stochastic dominance constraints. The introduction of this formulation aims to enhance the performances of the invested portfolio minimizing the dispersion. Empirical applications on Nasdaq 100, S&P 500 and Russell 1000 stock indexes confirm the goodness of the proposed approach.

## Chapter 3

# Dispersion Measures for the Benchmark Tracking Portfolio Problem and Third Order Stochastic Dominance Constraints.

### 3.1 Introduction

General deviation measures are introduced and studied for their potential applications to risk management in areas like portfolio optimization and engineering. Such measures include standard deviation as a special case but need not be symmetric with respect to ups and downs. The main component presented in a random variable is its uncertainty that is most commonly measured considering the standard deviation or other indicators, such as mean absolute deviation. In many situations, however, there is interest in treating the extent to which a random variable falls short of its expected value differently



from the extent to which it exceeds its expected value. This suggests to focus on the concept of general deviation measures as a bigger class of the common ones but they do not need to be symmetric between positive and negative shift with respect to a average value. In particular, this concept find several application in the financial sector. The asymmetry is one of the main properties in the benchmark tracking portfolio problem for several motivations. Of course, risk analysis can go far beyond portfolios, and advances in the subject can be beneficial in other areas of management and engineering.

In this chapter, we theoretically develop the two main contributions of the previous section. In particular, we classify the introduced measure of dispersion of the tracking error in a more general framework considering the class of the coherent expectation bounded risk measures. Linking the concept of risk measure with respect to a given orderings, we derive other linear measures suitable to solve the benchmark tracking problem. Then, we theoretically generalize the introduction of third order stochastic dominance in the portfolio problem introducing a linear formulation for its constraints. In this way, we obtain portfolio chosen by non-satiable risk averse investors with positive skewness. This concept allow to build portfolio which dominates in the third order sense an optimal portfolio with respect to the second order maximizing a performance measure that is consistent with none of the previous orders. Empirically speaking, this contribution is focused on the concept of stochastic investment chain which are grounded on a three step portfolio optimization problem with different orders of stochastic dominance.

This chapter is organized as follow. In section 3.2, starting from the definition of dispersion measure, we consider different classes of risk measure for benchmark tracking problem. In section 3.3, we introduce the benchmark tracking problem considering a linear dispersion measure derived from the class of expectation bounded risk measure and from the  $L^p$  metric. Section 3.4 debates the linearization of an aggressive utility function and of the third order bounded stochastic dominance constraints. Then, we propose a stochastic investment chain methodology to apply the third order bounded stochastic constraints in the portfolio optimization model. Finally, in the last section,

we summarize the results. Theoretical contributions are introduced as Proposition and Theorem while Corollary and Remarks are used to report works from other author which would be useful to completely understand the contribution of this chapter.

## 3.2 Coherent Expectation Bounded Risk Measures

In financial applications, many researchers have already delved into particular deviations other than standard deviation, in one aspect or another. [Markowitz \(1952\)](#) suggested the use of a downside form of standard deviation. Possible advantages of mean absolute deviation and its downside version, most notably in relation to linear programming computations of optimal portfolios, have been explored in [Mansini et al. \(2003\)](#), [Konno and Shirakawa \(1994\)](#), [Feinstein and Thapa \(1993\)](#), [Speranza \(1993\)](#). Here, we link dilation with expectation bounded risk measures and we propose linear formulation for the benchmark tracking portfolio problem basing on the axiomatic approach of [Rockafellar et al. \(2006\)](#) and [De Giorgi \(2005\)](#).

### 3.2.1 From Deviation Measure to Expectation Bounded Risk Measure

In the previous chapter, we review the concept of deviation measure and we show how the tracking error quantile regression dispersion measure belongs to this family. However, deviation measures are designed for applications to problems involving risk, they are not “risk measures” in the sense proposed by [Artzner et al. \(1999\)](#). The connection between deviation measures and risk measures is close, but a crucial distinction must be appreciated clearly. Instead of measuring the uncertainty in  $X$ , in the sense of non-constancy, a risk measure evaluates the “overall seriousness of possible losses” associated with  $X$ , where a loss is an outcome below 0, in contrast to a gain, which is an outcome above 0. In applying a risk measure, this orientation is crucial; if the concern is over the extent to which a given random variable  $X$  might have outcomes  $X(\omega)$  that drop below a threshold  $C$ , one needs to replace  $X$  by  $X - C$ .

Remembering that coherent risk measure is defined as a functional  $\mu : \mathcal{L}^2 \rightarrow [-\infty, +\infty]$  satisfying the following properties:

- A1) Translation Invariance :=  $\mu(X + C) = \mu(X) - C$  for all  $X$  and constant  $C$ ,
- A2) Positive Homogeneity :=  $\mu(0) = 0$  and  $\mu(\lambda X) = \lambda\mu(X)$  for all  $X$  and all  $\lambda$ ,
- A3) Subadditivity :=  $\mu(X + X') \leq \mu(X) + \mu(X')$  for all  $X$  and  $X'$ ,
- A4) Monotonicity :=  $\mu(X) \leq \mu(X')$  for all  $X \geq X'$ ,

The idea to link coherence and deviation risk measures could be find introducing the concept of expectation bounded risk measure. They are defined as any functional  $\mu : \mathcal{L}^2 \rightarrow [-\infty, +\infty]$  satisfying the axioms A1, A2, A3 (not necessary A4) and

- A5)  $\mu(X) \geq E[-X]$  for all non-constant  $X$ .

According to the analysis of [Rockafellar et al. \(2006\)](#), when all the axioms A1-A5 are satisfied, we speak of *coherent expectation bounded risk measure*. Moreover, there is a one-to-one relation between deviation measures and expectation bounded risk measures such that if  $D : \mathfrak{X} \rightarrow [0, \infty]$  is a deviation measure satisfying axioms D1 – D4 and  $\mu : \mathcal{L}^2 \rightarrow [-\infty, +\infty]$  is a expectation bounded risk measure, then:

- 1)  $D(X) = \mu(X - EX)$
- 2)  $\mu(X) = \mathbb{E}[-X] + D(X)$

Thus, not every coherent measures are expectation bounded for the axioms A5 while it is interesting to investigate the relation between deviation and expectation bounded risk measures. In particular, an expectation bounded risk measure is never a deviation measure since the translation invariance property does not hold while it is possible to obtain a relation between  $\mu$  and  $D$  based on the coherency. In fact,  $\mu$  is coherent if and only if  $D$  is lower range dominated satisfying the following property:

D5)  $D(X) \leq \mathbb{E}[X] - \inf X$  for all  $X$

This formulation allows to define a class of coherent risk measure defined through a dispersion measure and an expectation on the referred random variable. This set is called expectation bounded risk measure and it is the based to formulate several linear risk measure suitable for the benchmark tracking problem.

### 3.2.2 CVaR and Coherent Gini Type Measures

Considering the set coherent expectation bounded risk measure, we bring out two important cases: the Conditional Value at Risk (CVaR) (Mansini et al., 2007; Rockafellar and Uryasev, 2002; Pflug, 2000; Artzner et al., 1999) and the class of Gini measures (Shalit and Yitzhaki, 1984). Let  $X$  and  $Y$  be two random variables. We remember that  $X$  dominates  $Y$  with respect to  $\alpha$  inverse stochastic dominance order (Muliere and Scarsini, 1989)  $X \succeq_{-\alpha} Y$  (with  $\alpha > 1$ ) if and only if for every  $p \in [0, 1]$ ,

$$\begin{aligned} F_X^{(-\alpha)}(p) &= \frac{1}{\Gamma(\alpha)} \int_o^p (p-u)^{\alpha-1} dF_X^{-1}(u) \geq F_Y^{(-\alpha)}(p), & \alpha > 1 \\ F_X^{(-1)}(p) &\geq F_Y^{(-1)}(p), & \alpha = 1 \end{aligned} \tag{3.1}$$

where  $F_X^{-1}(0) = \lim_{p \rightarrow 0} F_X^{-1}(p)$  and  $F_X^{-1}(p) = \inf \{x : \Pr(X \leq x) = F_X(x) \geq p\} \forall p \in (0, 1]$  is the left inverse of the cumulative distribution function  $F_X$ . In this case,  $-F_X^{(-\alpha)}(p)$  is the risk measure associated with this risk ordering. In the risk management literature, the opposite of the  $p$ -quantile  $F_X^{-1}(p)$  of  $X$  is referred to as Value at Risk (VaR) (Pflug, 2000; Jorion, 1996) of  $X$ , i.e.  $VaR_p(X) = -F_X^{-1}(p)$ . VaR refers to the maximum loss among the best  $1-p$  percentage cases that could occur for a given horizon. In particular, when  $\alpha = 2$ , we obtain  $F_X^{(-2)}(p) = L_X(p) = \int_0^p F_X^{-1}(t) dt$  the absolute Lorenz curve of stock  $X$  with respect to its distribution function  $F_X$ . The absolute concentration curve  $L_X(p)$  valued at  $p$  shows the mean return accumulated up to the lowest  $p$  percentage of the distribution. Both measures and  $L_X(p)$  have important financial and economic

interpretations and are widely used in the recent risk literature. In particular, the negative absolute Lorenz curve divided by probability  $p$  is a coherent risk measure in the sense of Artzner et al. (1999) that is called conditional value-at-risk (CVaR), or expected shortfall (Pflug, 2000), and is expressed as

$$CVaR_p(X) = -\frac{1}{p}L_X(p) = \inf_u \left\{ u + \frac{1}{p}\mathbb{E}[(-X - u)_+] \right\} \quad (3.2)$$

where the optimal value  $u$  is  $VaR_p(X) = -F_X^{-1}(p)$ . As a consequence of Rockafellar et al. (2006) definition we obtain the following corollary:

**Corollary 3.1**

*For any  $p \in (0, 1)$ , the functional*

$$D(X) = CVaR_p(X - \mathbb{E}[X]) \quad (3.3)$$

*is a continuous, lower range dominated deviation measure and it correspond to the following coherent expectation bounded risk measure:*

$$\mu(X) = CVaR_p(X) \quad (3.4)$$

As proved by Pflug (2000) the minimization of the CVaR for a fixed mean is obtained solving a LP problem.

**3.2.3 Gini Tail Measures Associated with a Dilation Order**

In many portfolio selection problems some concentration measures have been used to measure the variability in choices. Starting from the linearization of the CVaR, other coherent risk measures using specific functions for the Lorenz curve can be easily obtained. In particular, we observe that some classic Gini-type (GT) measures are coherent measures. By definition, for every  $v > 1$  such that  $\alpha = v + 1$  and for every  $\beta \in (0, 1)$  we

have that:

$$\begin{aligned}
 GT_{(\beta,v)}(X) &= -\Gamma(v+1) \frac{1}{\beta^v} F_X^{(-(v+1))}(\beta) \\
 &= -(v-1)v \frac{1}{\beta^v} \int_0^\beta (\beta-v)^{v-2} L_X(u) du
 \end{aligned} \tag{3.5}$$

is consistent with  $\succeq_{-(v+1)}$  order, where  $\Gamma(v+1) = \Gamma(\alpha) = \int_0^{+\infty} z^{\alpha-1} e^{-z} dz$ . Then, using the coherency of CVaR, we remark that:

**Remark 3.1**

For every  $v \geq 1$  and for every  $\beta \in (0, 1)$  the measure  $GT_{(\beta,v)}(X) = -\Gamma(v+1)F_X^{(-(v+1))}(\beta)/\beta^v$  is a linearizable coherent risk measure associated with the  $(v+1)$  inverse stochastic dominance order (Ortobelli et al., 2013). The measure  $GT_{(\beta,v)}(X)$  generalizes the CVaR that we get when  $v = 1$ .

Other classic example of concentration measure is Gini's mean difference (GMD) and its extensions related to the fundamental work of Gini (Shalit and Yitzhaki, 1984). Gini's mean difference is twice the area between the absolute Lorenz curve and the line of safe asset joining the origin with the mean located on the right boundary vertical. In addition to GMD, we consider the extended Gini's mean difference (Ortobelli et al., 2013; Shalit and Yitzhaki, 2010; Yitzhaki, 1983) that takes into account the degree of risk aversion as reflected by the parameter  $v$ . This index can also be derived from the Lorenz curve as follows:

$$\begin{aligned}
 \Gamma_X(v) &= \mathbb{E}[X] - v(v-1) \int_0^1 (1-u)^{v-2} L_X(u) du \\
 &= v \operatorname{cov}(X, (1 - F_X(X))^{v-1})
 \end{aligned} \tag{3.6}$$

From this definition, it follows that  $\Gamma_X(v) - \mathbb{E}[X] = -\Gamma(v+1)F_X^{(-(v+1))}$  characterizes the previous Gini orderings.

**Remark 3.2**

The extended Gini's mean difference is a measure of spread associated with the expected bounded coherent risk measure  $\Gamma_X(v) - \mathbb{E}[X]$  for every  $v > 1$ .

Several application of the Gini's mean difference in the portfolio theory has been developed in [Ortobelli et al. \(2013\)](#) and [Shalit and Yitzhaki \(1984\)](#). One of the most interesting is the use of Gini measures to the extended the Gini “tail” measures (for a given  $\beta$ ) that are associated with a dilation order ([Fagioli et al., 1999](#)):

$$\Gamma_{X,\beta}(1) = \mathbb{E}[X] - \frac{1}{\beta} \left( \int_0^\beta F_X^{-1}(u) du \right) \quad (3.7)$$

These measures can also be extended using  $v > 1$  and the tail measure:

$$\begin{aligned} \Gamma_{X,\beta}(v) &= \mathbb{E}[X] - \frac{v}{\beta^v} \left( \int_0^\beta (\beta - u)^{v-1} F_X^{-1}(u) du \right) \\ &= \mathbb{E}[X] - \frac{v(v-1)}{\beta^v} \int_0^\beta (\beta - u)^{v-2} L_X(u) du \end{aligned} \quad (3.8)$$

for some  $\beta \in [0, 1]$ .

**Remark 3.3**

For every  $v > 1$  the Gini “tail” measure associated with a dilation order  $\Gamma_{X,\beta}(v) = \mathbb{E}[X] - \Gamma(v+1)F_X^{(-v+1)}(\beta)/\beta^v$  is the deviation measure associated with the expectation bounded coherent risk measure  $\Gamma_{X,\beta}(v) - \mathbb{E}[X]$ .

As mentioned above, we would like to consider a linear risk measure to be suitable in the proposed formulation for the benchmark tracking problem. Then, the following propositions hold:

**Proposition 3.1**

The quantile regression measure is a coherent expectation bounded risk measure associated with the Gini “tail” measure when  $v = 1$  for less than a multiplicative factor.

**Proof.** Considering the (3.8) when  $v = 1$ , we obtain:

$$\begin{aligned} \Gamma_{X,\beta}(v) &= \mathbb{E}[X] - \frac{1}{\beta} \int_0^\beta F_X^{-1}(u) du \\ \Rightarrow \beta \Gamma_{X,\beta}(v) &= \beta \mathbb{E}[X] - \int_0^\beta F_X^{-1}(u) du \end{aligned} \quad (3.9)$$

Since  $\mathbb{E}[X] = \int_0^\beta F_X^{-1}(u)du + \int_\beta^1 F_X^{-1}(u)du$ . Then, we obtain:

$$\begin{aligned} \beta \Gamma_{X,\beta}(v) &= \beta \int_0^\beta F_X^{-1}(u)du + \beta \int_\beta^1 F_X^{-1}(u)du - \int_0^\beta F_X^{-1}(u)du \\ &= (\beta - 1) \int_0^\beta F_X^{-1}(u)du + \beta \int_\beta^1 F_X^{-1}(u)du \end{aligned} \quad (3.10)$$

Then, with the following proposition, we could extend the linearity concept also for  $v \geq 2$ . □

**Remark 3.4**

*For  $v \geq 2$  the Gini “tail” measure is a linear coherent expectation bounded risk measure associated with a dilation order.*

In this case, we will discuss the linear benchmark tracking problem for  $v = 2$  in the Section 3.4 when we show how to use the Gini “tail” measure  $\Gamma_{X,\beta}(2)$  for a spectrum of  $\beta \in (0, 1)$ .

**Remark 3.5**

*For every  $v \geq 1$  and for every  $\beta \in (0, 1)$  the Gini tail measure  $\Gamma_{X,\beta}(v) = \mathbb{E}[X] - \Gamma(v + 1)F_X^{-(v+1)}(\beta)/\beta^v$  is consistent with Rothschild–Stiglitz order. Moreover, if  $\Gamma_{X,\beta}(v) = \Gamma_{Y,\beta}(v)$  for any  $\beta \in [0, 1]$  then  $F_X = F_Y$  (Ortobelli et al., 2013).*

Finally, it is possible to link the previous formulation with the portfolio theory. According to Rockafellar et al. (2006), we can express the following proposition

**Proposition 3.2**

*For any expected bounded risk measures consistent with the Rothschild–Stiglitz ordering, we could derive a tracking error problem consistent with Rothschild–Stiglitz.*

**3.2.4 The  $L^p$  Compound Metric**

Finally, let us to introduce the class of concentration  $L^p$ -metrics. Considering the class of compound metrics we propose a concentration measure which could be applied to the



benchmark tracking portfolio problem. Following [Ortobelli et al. \(2013\)](#), we say that  $X$  is preferred to  $Y$  with respect to the  $\mu$ -compound distance from  $Z$  (namely,  $X \succeq_{\mu Z} Y$ ) if and only if there exists a probability functional  $\rho : \Lambda \times Z \times B \rightarrow \mathbb{R}$  dependent on  $\mu$  such that for any  $t \in B$  and  $X, Y \in \Lambda$ ,  $\rho_X(t) \leq \rho_Y(t)$ . In this case, the equality  $\rho_X = \rho_Y$  implies a distributional equality  $F_g(X, Z) = F_g(Y, Z)$  for compound distances  $\mu$  and a distance equality  $g(F_X, F_Z) = g(F_Y, F_Z)$  for simple distances  $\mu$ , where  $g(x, z)$  is a distance in  $\mathbb{R}$ . We call  $\rho_X$  the (tail) tracking-error measure (functional) associated with the  $\mu$  tracking error ordering.

Consequently, in benchmark tracking strategies we minimize the tracking-error functional  $\rho_X$  associated with the  $\mu$ -tracking-error ordering. In essence, probability metrics can be used as tracking-error measures. In solving the portfolio problem with a probability distance, we intend to “approach” the benchmark and change the perspective for different types of probability distances. Hence, if the goal is only to control the variability of an investor’s portfolio or to limit its possible losses, mimicking the uncertainty or the losses of the benchmark can be done with a primary probability distance. When the objective for an investor’s portfolio is to mimic the entire benchmark, a simple or compound probability distance should be used. In addition to its role of measuring tracking errors, a compound distance can be used as a measure of variability. If we apply any compound distance  $\mu(X, Y)$  to  $X$  and  $Y = X_1$  that are *i.i.d.*, we obtain:

$$\mu(X, X_1) = 0 \quad \text{iff} \quad \mathbb{P}(X = X_1) = 1 \quad \text{iff} \quad X \text{ is a constant almost surely.}$$

For this reason, we refer to  $\mu(X, X_1) = \mu_I(X)$  as a concentration measure derived by the compound distance  $\mu$ . Similarly, if we apply any compound distance  $\mu(X, Y)$  to  $X$  and  $Y = \mathbb{E}[X]$  (either  $Y = M(X)$ , i.e. the median or a percentile of  $X$ , if the first moment is not finite), we get:

$$\mu(X, \mathbb{E}[X]) = 0 \quad \text{iff} \quad \mathbb{P}(X = \mathbb{E}[X]) = 1 \quad \text{iff} \quad X \text{ is a constant almost surely.}$$

Hence,  $\mu(X, \mathbb{E}[X]) = \mu_{\mathbb{E}[X]}(X)$  can be referred to as a dispersion measure derived by the compound distance  $\mu$ . Then, for each probability compound metric we can generate a probability compound distance  $\mu_H(X, Y) = H(\mu(X, Y))$  with parameter  $K_H$ .

Considering the  $L^p$  average compound metric, for every  $p \geq 0$  we recall the  $L^p$ -metrics:  $\mu_p(X, Y) = \mathbb{E}[|X - Y|^{\min(1, 1/p)}]$ ; the associated concentration measures are  $\mu_{I,p}(X) = \mathbb{E}[|X - X_1|^{\min(1, 1/p)}]$ , where  $X_1$  is an *i.i.d.* copy of  $X$ ; and the associated dispersion measures are the central moments  $\mu_{\mathbb{E}[X],p}(X) = \mathbb{E}[|X - \mathbb{E}[X]|^{\min(1, 1/p)}]$ . The dispersion and concentration measures  $\mu_{\mathbb{E}[X],p}(X)$  and  $\mu_{I,p}(X)$  are variability measures consistent with the  $(p + 1)$  Rothschild–Stiglitz order for any  $p \geq 1$ . We can consider for  $L^p$  metrics the tracking-error measures

$$\begin{aligned} \rho_{X,p}(t) &= (\mu_p(X \mathbf{1}_{|X-Z| \geq t}, Z \mathbf{1}_{|X-Z| \geq t}))^{\min(1, 1/p)} - t^p \mathbb{P}(|X - Z| \geq t) \\ &= \mathbb{E}[ (|X - Z|^p - t^p)_+ ] \end{aligned} \quad (3.11)$$

for any  $t \in [0, +\infty)$  associated with a  $\mu_p$  tracking error ordering. Moreover,  $\rho_{X,p} = \rho_{Y,p}$  implies that  $F_{|X-Z|} = F_{|Y-Z|}$ .

### 3.3 Different Metrics for the Benchmark Tracking Problem

After the description of two different measure for the tracking error, in this section, we propose the two linear problem for the Gini “tail” measure and the concentration measure associate with the  $L^p$  average compound metric. Considering the Gini “tail” measure  $\Gamma_{X,\beta}(v)$  (3.8) we notice that for  $v = 1$ , we obtain the quantile regression dispersion measure while here, we propose a benchmark tracking portfolio with  $v = 2$ . Thus, let the log-return of equity index be a random variable  $Y$  with discrete realization  $y_t$  for  $t = 1, \dots, T$  and let  $X = r\beta$  the random variable of the invested portfolio with realization  $r_t\beta, t = 1, \dots, T$  where  $\beta_n$  is a vector represented the portfolio weights  $n = 1, \dots, N$  (total number of index components) and  $r$  an  $T \times N$  matrix of stocks returns. We define

the random variable  $Z$  such that  $Z = X - Y$ . Then, considering the Gini measure  $\Gamma_{Z,\gamma}(v)$  of ordering 2 such that:

$$\Gamma_{Z,\gamma}(2) = \mathbb{E}[Z] - \frac{2}{\gamma^2} \int_0^\gamma L_Z(u) du \quad (3.12)$$

where  $L_X(u)$  is the Lorenz curve. The general formulation of the benchmark tracking problem is:

$$\begin{aligned} \min_Z \quad & \Gamma_{Z,\gamma}(2) \\ \text{s.t.} \quad & \sum_{n=0}^N \beta_n = 1 \\ & \mathbb{E}[X] - \mathbb{E}[Y] \geq K^* \\ & lb \leq \beta_n \leq ub \quad \forall n = 1, \dots, N \end{aligned} \quad (3.13)$$

Then following the integral rule derived from fractional integral theory define the benchmark tracking LP as:

$$\begin{aligned} \min_{\beta, b, \nu} \quad & \frac{1}{T} \sum_{t=1}^T (r_t \beta - y_t) - \frac{2T}{s^2} \sum_{i=1}^s \left( \frac{i}{T} b_i + \frac{1}{T} \sum_{t=1}^T \nu_{t,i} \right) \\ \text{s.t.} \quad & \nu_{t,i} \geq -r_t \beta + y_t - b_i \quad \forall i = 1, \dots, s; \forall t = 1, \dots, T \\ & \sum_{n=0}^N \beta_n = 1 \\ & \mathbb{E}[X] - \mathbb{E}[Y] \geq K^* \\ & lb \leq \beta_n \leq ub \quad \forall n = 1, \dots, N \\ & \nu_{t,i} \geq 0 \quad \forall i = 1, \dots, s; \forall t = 1, \dots, T \\ & b_i \in \mathbb{R} \quad \forall i = 1, \dots, s \end{aligned} \quad (3.14)$$

where  $s = \gamma T$  for every  $\gamma \in [0, 1]$ . We notice that for problem (3.14) we could define a spectrum of the possible values of  $\gamma$  and, as in the quantile regression dispersion measure, solve a static or rolling index tracking strategy. Also for the benchmark tracking problem

with the Gini measure is possible to define the related LP enhanced indexation strategy with stochastic dominance constraints.

Finally, we propose a LP benchmark tracking problem for the concentration measure derived from the  $L^p$ -metric (3.11). Let  $X$  and  $Y$  be two random variables representing the portfolio and the benchmark with realization  $r_t\beta$  and  $y_t$  for  $t = 1, \dots, T$ . Then, all investors who choose the portfolios that are consistent with the  $\mu_p$  tracking error ordering, where  $p = 1$ , solve the following LP benchmark tracking problem for some given  $q > 0$ :

$$\begin{aligned}
 \min_{\beta, u, \nu} \quad & \frac{1}{T} \sum_{t=1}^T u_t \\
 \text{s.t.} \quad & \nu_t \geq r_t\beta - y_t \quad \forall t = 1, \dots, T \\
 & \nu_t \geq y_t - r_t\beta \quad \forall t = 1, \dots, T \\
 & u_t \geq \nu_t - q \quad \forall t = 1, \dots, T \\
 & \sum_{n=0}^N \beta_n = 1 \\
 & \mathbb{E}[X] - \mathbb{E}[Y] \geq K^* \\
 & lb \leq \beta_n \leq ub \quad \forall n = 1, \dots, N \\
 & u_t \geq 0; \nu_t \in \mathbb{R} \quad \forall t = 1, \dots, T
 \end{aligned} \tag{3.15}$$

where  $K^*$  is the extra-performance of the invested portfolio with respect to the benchmark. Also in this case the previous problem (3.15) could be solved for different values of  $q > 0$  and it is possible to define an enhanced indexation benchmark tracking problem introducing different stochastic dominance constraints.

### 3.3.1 $L^P$ Average Compound Metrics with Stochastic Dominance Constraints

To evaluate the importance of the introduction of stochastic dominance constraints in the benchmarking portfolio problem we propose three realistic models which takes into

account a penalty function with transaction costs and a turnover threshold. For this reason, it is possible to evaluate in a static and rolling framework the goodness of  $L^P$  metric and the impact of the stochastic dominance constraints in the out of sample wealth path.

Let  $X = r\beta$  a random variable of the portfolio returns and  $Y$  the random variable of the benchmark returns with realizations  $r_t\beta$  and  $y_t$  for  $t = 1, \dots, T$ . Let  $\beta \in \mathbb{S} = \{\beta \in \mathbb{R}^n \mid \sum_{n=1}^N \beta_n = 1\}$  and  $q$  be a parameter such that  $q \in \mathbb{Q} = \{\delta, \max_{\beta}(\max_t[r_t\beta, y_t,])\}$  with  $\delta$  small enough. Let  $tc^+$  and  $tc^-$  the transaction costs to buy and sell new securities and  $\beta^{old}$  the composition of the invested portfolio before the optimization step. Then, the optimal portfolio composition which solve the benchmark tracking problem is obtained fixing the value of  $q$  and solving the following linear programming problem:

$$\begin{aligned}
 \min_{\beta, u, \nu, \omega^+, \omega^-} \quad & \frac{1}{T} \sum_{t=1}^T u_t + tc^+ \omega^+ + tc^- \omega^- \\
 \text{s.t.} \quad & \nu_t \geq r_t\beta - y_t & \forall t = 1, \dots, T \\
 & \nu_t \geq y_t - r_t\beta & \forall t = 1, \dots, T \\
 & u_t \geq \nu_t - q & \forall t = 1, \dots, T \\
 & \sum_{n=0}^N \beta_n = 1 & (3.16) \\
 & \mathbb{E}[X] - \mathbb{E}[Y] \geq K^* \\
 & \omega_n^+ - \omega_n^- = \beta_n - \beta_n^{old} & \forall n = 1, \dots, N \\
 & \sum_n |\beta_n - \beta_n^{old}| \leq \theta & n = 1, \dots, N \\
 & lb \leq \beta_n \leq ub & \forall n = 1, \dots, N \\
 & u_t \geq 0; \nu_t \in \mathbb{R} & \forall t = 1, \dots, T
 \end{aligned}$$

where  $u, \nu$  are two variable to linearize the associate benchmark tracking measure of the  $L^P$  metric and  $\omega^+, \omega^-$  two slack variables of the portfolio changes.

Then, we introduce first and second order stochastic dominance constraints in the linear formulation proposed in [Kopa \(2010\)](#) and [Kuosmanen \(2004\)](#) through the permutation matrix  $P$  for the first order and the double stochastic matrix  $Z$  for the second one. Let  $P = \{p_{r,c}\}$  a permutation matrix with  $p_{r,c} \in \{0, 1\}$  s.t.  $\sum_{r=1}^T p_{r,c} = 1$  for  $c = 1, \dots, T$  and  $\sum_{c=1}^T p_{r,c} = 1$  for  $r = 1, \dots, T$ , we could define the following enhanced index mixed-integer linear problem with FSD constraints:

$$\begin{aligned}
 & \min_{\beta, u, \nu, \omega^+, \omega^-, p} \quad \frac{1}{T} \sum_{t=1}^T u_t + tc^+ \omega^+ + tc^- \omega^- \\
 \text{s.t.} \quad & \nu_t \geq r_t \beta - y_y & \forall t = 1, \dots, T \\
 & \nu_t \geq y_y - r_t \beta & \forall t = 1, \dots, T \\
 & u_t \geq \nu_t - q & \forall t = 1, \dots, T \\
 & \sum_{n=0}^N \beta_n = 1 \\
 & \mathbb{E}[X] - \mathbb{E}[Y] \geq K^* \\
 & \omega_n^+ - \omega_n^- = \beta_n - \beta_n^{old} & \forall n = 1, \dots, N \\
 & \sum_n |\beta_n - \beta_n^{old}| \leq \theta & n = 1, \dots, N \tag{3.17} \\
 & X \geq PY \\
 & \sum_{r=1}^T p_{r,c} = 1 & \forall c = 1, \dots, T \\
 & \sum_{c=1}^T p_{r,c} = 1 & \forall r = 1, \dots, T \\
 & p_{r,c} \in \{0, 1\} & \forall r, c = 1, \dots, T \\
 & lb \leq \beta_n \leq ub & \forall n = 1, \dots, N \\
 & lb \leq \omega_n^+, \omega_n^- \leq ub & \forall n = 1, \dots, N \\
 & u_t \geq 0; \nu_t \in \mathbb{R} & \forall t = 1, \dots, T
 \end{aligned}$$

where  $K^*$  is the extra-performance of the portfolio with respect to the benchmark.

This kind of problem is mixed-integer since the introduction of the permutation matrix require binary variable for the first order stochastic dominance conditions.

Then, let  $Z = \{z_{r,c}\}$  a double stochastic matrix with  $z_{r,c} \in [0, 1]$  s.t.  $\sum_{r=1}^T z_{r,c} = 1$  for  $c = 1, \dots, T$  and  $\sum_{c=1}^T z_{r,c} = 1$  for  $r = 1, \dots, T$ , we could define the following enhanced index linear problem with SSD constraints:

$$\begin{aligned}
 & \min_{\beta, u, \nu, \omega^+, \omega^-, z} \quad \frac{1}{T} \sum_{t=1}^T u_t + tc^+ \omega^+ + tc^- \omega^- \\
 \text{s.t.} \quad & \nu_t \geq r_t \beta - y_y & \forall t = 1, \dots, T \\
 & \nu_t \geq y_y - r_t \beta & \forall t = 1, \dots, T \\
 & u_t \geq \nu_t - q & \forall t = 1, \dots, T \\
 & \sum_{n=0}^N \beta_n = 1 \\
 & \mathbb{E}[X] - \mathbb{E}[Y] \geq K^* \\
 & \omega_n^+ - \omega_n^- = \beta_n - \beta_n^{old} & \forall n = 1, \dots, N \\
 & \sum_n |\beta_n - \beta_n^{old}| \leq \theta & n = 1, \dots, N \tag{3.18} \\
 & X \geq ZY \\
 & \sum_{r=1}^T z_{r,c} = 1 & \forall c = 1, \dots, T \\
 & \sum_{c=1}^T z_{r,c} = 1 & \forall r = 1, \dots, T \\
 & 0 \leq z_{r,c} \leq 1 & \forall r, c = 1, \dots, T \\
 & lb \leq \beta_n \leq ub & \forall n = 1, \dots, N \\
 & lb \leq \omega_n^+, \omega_n^- \leq ub & \forall n = 1, \dots, N \\
 & u_t \geq 0; \nu_t \in \mathbb{R} & \forall t = 1, \dots, T
 \end{aligned}$$

The condition of second order stochastic dominance are weaker than the first order and the range of the variables of the double stochastic matrix is the interval  $[0, 1]$ . It

is important to stress the dramatical increment in the number of variables given by the introduction of this two strong sets of constraints.

### 3.4 Stochastic Investment Chain

After the dissertation of linear expectation bounded risk measures, a relevant problem in the benchmark tracking is linked with the introduction of stochastic dominance constraints. The previous chapter shows the impact of these constraints in portfolio selection problem to enhance performances and increment gains. For this reason, we try to investigate a methodology to obtain strengthen portfolio considering different orders of stochastic dominance in a linear programming framework. We propose a three steps portfolio optimization model where we increase the order of stochastic dominance maximizing suitable utility functions where, at the next level order, the dominant portfolio becomes the dominated one. This chain has the consequence to increase the robustness of the invested portfolio obtaining higher wealth in the out of sample analysis or to improve the performance ratios.

To achieve this aim, we build a theoretical structure with sufficient conditions to express the third order bounded stochastic dominance constraints in the portfolio formulation which represent the preferences of all non-satiable risk averse investors with positive skewness. Financial literature develops several works with the concept of third-order stochastic dominance ([Post et al., 2014](#); [Le Breton and Peluso, 2009](#); [Schmid, 2005](#); [Gotoh and Konno, 2000](#); [Tehrani, 1980](#); [Bawa, 1978](#); [Whitmore, 1970](#)) but there is not a unified framework to express in a linear formulation the possibility to introduce in the portfolio choice the preference of these type of investors. To introduce third order stochastic dominance in the investment chain, the key point reflects the choice of utility function which should not be consistent with second order stochastic dominance. In this case, if we build an optimal portfolio in a second order stochastic sense maximizing a function consistent with this order, then it is not possible to find a dominant portfolio



introducing an higher order and keeping the same utility function. For this reason, we consider the Rachev utility function (Rachev et al., 2008a) which is not consistent with first and second order stochastic dominance.

In the following part of this section, we firstly develop the linear programming formulation problem to maximize the Rachev utility function. Secondly, we introduce the third order stochastic dominance condition for the portfolio selection problem developing their linear formulation to efficiently solve the portfolio problem. Finally, we present the decisional steps of the stochastic investment chain to build portfolios that dominate a dominant one to higher of different orders.

### 3.4.1 The Rachev Utility Function

Financial agents and investors behave differently in their approach to the financial markets. For years several studies propose utility functions to describe their behavior and the preferences trying to draw a complete picture of the entire universe. In this section, we develop linear formulation problems to maximize one kind of investors' preferences represented by the the Rachev utility functions (Rachev et al., 2008a). It reflects the behavior of non-satiable nor risk averse nor risk seeking investors. In particular, maximizing this kind of preference, we solve the following portfolio selection problem.

Let  $X = r\beta$  be a random variable of the portfolio returns and let  $\alpha_1$  and  $\alpha_2$  be two confidential levels. The Rachev utility is defined as the different between the Conditional Value at Risk of the two sides of the returns distribution. Thus, let  $a, b \in \mathbb{R}^+$  be two positive coefficients, a non-satiable nor risk averse nor risk seeking investor choose the solution of the following portfolio problem:

$$\begin{aligned}
 & \max_{\beta} \quad b CVaR_{\alpha_1}(-X) - a CVaR_{\alpha_2}(X) \\
 & \text{s.t.} \quad \sum_{n=1}^N \beta_n = 1 \\
 & \quad \quad 0 \leq \beta_n \leq 1 \quad \quad \quad \forall n = 1, \dots, N
 \end{aligned} \tag{3.19}$$

Since the Rachev utility function could be rewritten as  $CVaR_{\alpha_1}(-X) - \frac{a}{b} CVaR_{\alpha_2}(X)$  and following [Stoyanov et al. \(2007\)](#), the portfolio selection problem 3.19 could be defined in a mixed-integer linear programming way. As discussed in [Stoyanov et al. \(2007\)](#), it is necessary to introduce integer variable to bound the reward side of the utility function. For this reason, the introduction of variable  $B \geq |r\beta|$ .

**Proposition 3.4**

The solution of the maximization of the Rachev utility function (3.19) with linear constraints is defined with the following mixed-integer linear programming portfolio selection problem:

$$\begin{aligned}
 & \underset{\beta, g, \lambda, \gamma, d}{max} && \frac{1}{\lceil \alpha_1 T \rceil} \sum_{t=1}^T g_t - \frac{a}{b} \left( \gamma + \frac{1}{\alpha_2 T} \sum_{t=1}^T d_t \right) \\
 & s.t. && \sum_{n=1}^N \beta_n = 1 \\
 & && g_t \leq B \lambda_t && \forall t = 1, \dots, T \\
 & && g_t \geq r_t \beta - B(1 - \lambda_t) && \forall t = 1, \dots, T \\
 & && g_t \leq r_t \beta + B(1 - \lambda_t) && \forall t = 1, \dots, T \\
 & && -r_t \beta - \gamma \leq d_t && \forall t = 1, \dots, T \\
 & && lb \leq \beta_n \leq ub && \forall n = 1, \dots, N \\
 & && \lambda_t \in \{0, 1\}, \gamma \in \mathbb{R} && \forall t = 1, \dots, T \\
 & && g_t \geq 0, d_t \geq 0 && \forall t = 1, \dots, T
 \end{aligned} \tag{3.20}$$

where  $\lambda$  is the binary variable.

**Proof**

Since the the combination between two convex functions is still convex, we could follow [Stoyanov et al. \(2007\)](#) to linearize the problem 3.19 with linear constraints. In particular, we notice that if the second terms could be linearized following [Rockafellar and Uryasev \(2002\)](#), the linearization of the first term leads to an unbounded problem and the objective function draws to infinite. For this reason the problem could be defined as mixed-integer

introducing a binary set of variables  $\lambda_t$  and an artificial upper bound  $B$  which is big enough to bound the possible maximum return of the created portfolio. The introduction of the binary variables imply to consider two cases:

1. Suppose that  $\lambda_t = 0$ , then:

$$\begin{aligned} g_t &\leq 0 \\ g_t &\geq r_t\beta - B \\ g_t &\leq r_t\beta + B \\ g_t &\geq 0 \end{aligned}$$

Since  $B$  is a very large number, then  $g_t = 0$ .

2. Suppose that  $\lambda_t = 1$ , then:

$$\begin{aligned} g_t &\leq B \\ g_t &\geq r_t\beta \\ g_t &\leq r_t\beta \\ g_t &\geq 0 \end{aligned}$$

From this system of equation  $g_t$  should be positive and bounded from  $B$  with a unique solution such that  $g_t = r_t\beta$ .

□

### 3.4.2 Linear Formulation for Third Order Stochastic Dominance Constraints

The introduction of third order stochastic dominance relates on the possibility to build portfolio which dominates a benchmark which is optimal in the second order stochastic sense. Considering the maximization of an aggressive function such as the Rachev utility it is possible to construct portfolio with a suitable behavior for the investors. To

express the concept of the third order stochastic dominance constraints in a linear formulation, we propose an approach based on three main works presented in the literature (Dentcheva and Ruszczyński, 2003; Hanoch and Levy, 1970, 1969).

In particular, whether Dentcheva and Ruszczyński (2003) propose a linear formulation for the second order stochastic dominance while Hanoch and Levy (1970) introduce some condition to satisfy second order stochastic dominance which can be extended to other higher order. For this reason, we remember the main proposition introduced in Dentcheva and Ruszczyński (2003).

**Remark 3.6**

Assuming to consider two random variables  $X$  and  $Y$  such that  $Y$  has a discrete distribution with realization  $y_i$  and cumulative density function  $F_Y(y_t) = u_t$  for  $t = 1, \dots, T$ . Then,  $X \underset{(2)}{\succeq} Y$  with if and only if:

$$\mathbb{E} [(y_i - X)_+] \leq \mathbb{E} [(y_i - Y)_+] \tag{3.21}$$

where  $(y_i - X)_+ = \max[0, y_i - X]$  (Dentcheva and Ruszczyński, 2003).

Then, they build a portfolio optimization problem to guarantee linear second order stochastic dominance constraints introducing a slack variable representing the shortfall of  $X$  below  $y_t$ . Thus, it is possible to derive a necessary and sufficient condition for the existence of stochastic dominance relation between two random variables.

**Proposition 3.8**

Let  $X$  and  $Y$  be two random variables uniformly distributed (i.e.  $\mathbb{P}(X = r_t\beta) = \frac{1}{T} = \mathbb{P}(Y = y_t)$ ) with discrete realizations  $y_t$  and  $x_t\beta$  for  $t = 1, \dots, T$  where  $\beta$  is the portfolio weights. If  $\beta \in \mathbb{S}_n = \left\{ \beta \in R^n \mid \sum_{n=1}^N \beta_n = 1; \beta = 1 \right\}$ , such that  $\min_t r_t\beta \geq y_1$  then  $X \underset{(\alpha)}{\not\succeq} Y, \forall \alpha \geq 1$ , where  $y_1$  is the first observation of the cumulative random variable  $F_Y$ .

**Proof**

Since  $X \succeq_{\frac{1}{\alpha}} Y$ ,  $\forall \alpha \geq 1$  implies that  $F_X(y_1) \leq F_Y(y_1) = \frac{1}{T}$  then  $\min_t r_t \beta \geq y_1$ . □

Following [Hanoch and Levy \(1969\)](#) we find a necessary condition to guarantee the existence of a stochastic dominance relation such that  $X \succeq_{(\alpha+1)} Y$  but not to the previous order:  $X \not\prec_{(\alpha)} Y$ . In particular, we focus our analysis on the third order stochastic dominance constraints such that  $X \succeq_{\frac{3}{3}} Y$  and  $X \not\prec_{\frac{2}{2}} Y$ . Then, it is possible to define the following conditions that guarantee the previous relation.

**Theorem 3.1**

Let  $X$  and  $Y$  be two random variables with finite number of discrete realizations  $x_t$  and  $y_t$  for  $t = 1, \dots, T$ . Whether exists a  $\bar{t}$  such that  $\bar{t} \leq T$  and

$$\begin{aligned} F_X^{(2)}(t) &\leq F_Y^{(2)}(t) \quad \forall t \leq \bar{t} \\ F_X^{(2)}(t) &\geq F_Y^{(2)}(t) \quad \forall t > \bar{t} \\ \mathbb{E}[X] &> \mathbb{E}[Y] \end{aligned} \tag{3.22}$$

then,  $X \succeq_{\frac{3}{3}} Y$  and  $X \not\prec_{\frac{2}{2}} Y$ .

**Proof**

Recall that  $X \succeq_{\frac{3}{3}} Y$  iff  $F_X^{(3)}(u) = \int_{-\infty}^u F_X^{(2)}(t) dt \leq F_Y^{(3)}(u) = \int_{-\infty}^u F_Y^{(2)}(t) dt$ . Then,  $\lim_{t \rightarrow +\infty} \int_{-\infty}^u (F_Y^{(2)}(t) - F_X^{(2)}(t)) dt > 0$  because  $F_X^{(3)}(t) = \frac{1}{2} \mathbb{E}[(t - X)_+^2]$  and we obtain that (1)  $F_Y^{(3)}(t) - F_X^{(3)}(t) = \frac{1}{2}(t^2(F_Y(t) - F_X(t)) + \mathbb{E}(Y^2 I_{[Y \leq t]}) - \mathbb{E}(X^2 I_{[X \leq t]}) + 2t(E(X) - E(Y)))$ .

We notice that, for  $t$  large this difference is always positive since  $\mathbb{E}(X) > \mathbb{E}(Y)$  and  $X$  and  $Y$  are bounded random variables.

Therefore:

- $\forall u \leq \bar{t}$  the integral  $\int_{-\infty}^u (F_X^{(2)}(t) - F_Y^{(2)}(t)) dt \leq 0$ , while
- $\forall u > \bar{t}$ ,  $\int_{-\infty}^u (F_X^{(2)}(t) - F_Y^{(2)}(t)) dt = \int_{-\infty}^{\bar{t}} (F_X^{(2)}(t) - F_Y^{(2)}(t)) dt + \int_{\bar{t}}^u (F_X^{(2)}(t) - F_Y^{(2)}(t)) dt \leq 0$

Since the first part is less than zero, the second part is greater than zero by hypothesis but the second part never overcomes the first one for the presence of (1).  $\square$

This theorem can be used even in linear portfolio selection framework when instead of classic third stochastic dominance we consider the third bounded stochastic dominance order. As a matter of fact, Fishburn (1980) has shown that for any order greater than the second one, the classic stochastic dominance order implies the bounded one but the contrary is not always true.

**Proposition 3.9**

Assuming to consider two random variables  $X$  and  $Y$  that have discrete distributions with realization  $r_t\beta$  and  $y_t$  for  $t = 1, \dots, T$  which belong to the joint support  $\mathbb{U} = \{u_{(1)} \leq \dots \leq u_{(j)} \leq \dots \leq u_{(J)}\}$  where  $J \leq 2T$ . Then,  $X \succeq_{(3),b} Y$  (i.e.,  $X$  dominates  $Y$  in the bounded third order stochastic dominance sense) if there exists a  $\bar{t} \in \mathbb{U}$  such that:

$$\begin{aligned} F_X^{(2)}(u_j) &\leq F_Y^{(2)}(u_j) \quad \forall u_j \in \mathbb{U} \leq \bar{t} \\ F_X^{(2)}(u_j) &\geq F_Y^{(2)}(u_j) \quad \forall u_j \in \mathbb{U} > \bar{t} \\ \mathbb{E}[X] &> \mathbb{E}[Y] \end{aligned} \tag{3.23}$$

**. Proof**

Since  $\forall j, u_j \in \mathbb{U}$  and for any random return  $X$  the function  $F_X^{(2)}(u_j)$  is convex. Then, for the first inequality of (3.23) such that  $u_j \leq \bar{t}$  we have that:

$$F_X^{(2)}(u_j) \leq \lambda F_X^{(2)}(u_{(1)}) + (1 - \lambda) F_X^{(2)}(u_{(J)}) \leq \lambda F_Y^{(2)}(u_{(1)}) + (1 - \lambda) F_Y^{(2)}(u_{(J)}) = F_Y^{(2)}(u_j),$$

where  $\lambda = \frac{u_{(J)} - u_j}{u_{(J)} - u_{(1)}}$ . The last inequality follows from the linearity of  $F_Y^{(2)}(u_j)$  in  $\mathbb{U}$ .

Differently if  $u_j \geq \bar{t}$ , we have that:

$$F_X^{(2)}(u_j) \geq \lambda F_X^{(2)}(u_{(1)}) + (1 - \lambda) F_X^{(2)}(u_{(J)}) \geq \lambda F_Y^{(2)}(u_{(1)}) + (1 - \lambda) F_Y^{(2)}(u_{(J)}) = F_Y^{(2)}(u_j),$$

where  $\lambda = \frac{u_{(J)} - u_j}{u_{(J)} - u_{(1)}}$ .  $\square$

We generally know that we cannot consider the ui of the support of all portfolio problem but we have a good approx when the element of the number of the support are very large.

### 3.4.3 Two Investment Strategies with Different Stochastic Orders

The concept of stochastic investment chain allows investors and portfolio manager to create aggressive strategies improving the reward-risk combination. For this reason, we propose an investment strategy which is based on a three steps portfolio optimization process when we introduce a new stochastic order to dominate previous dominant portfolio.

Let  $A, B$  and  $C$  be three steps which the related optimal portfolios (i.e.  $X^A = r\beta^A$ ) and let  $Y$  be a random variable of the index benchmark returns. Considering the Rachev utility function we follow the following portfolio selection rule to build the stochastic investment chain:

- A)  $X^A$  FSD  $Y$ ;
- B)  $X^B$  SSD  $X^A$ ;
- C)  $X^C$  TSD  $X^B$ .

At each investment step, we evaluate the risk and the reward of the strategy to consider the advantages and disadvantages of this approach. In particular, we solve the following three portfolio selection problems. Let  $X^A$  and  $Y$  be two random variables with discrete realization  $r_t\beta^A$  and  $y_t$  for  $t = 1, \dots, T$  which belong to the joint support  $\mathbb{U} = \{u_{(1)} \leq \dots \leq u_{(j)} \leq \dots \leq u_{(J)}\}$  where  $J \leq 2T$ . Let  $X^A$  be the return of the invested portfolio and  $Y$  the benchmark ones. The optimal portfolio composition maximizing the Rachev utility function at the first step  $A$  is the solution of the following mixed-integer

linear portfolio problem:

$$\begin{aligned}
 & \max_{\beta^A} \quad b \text{CVaR}_{\alpha_1}(-X^A) - a \text{CVaR}_{\alpha_2}(X^A) \\
 & \text{s.t.} \quad X^A \underset{(1)}{\succeq} Y \\
 & \quad \mathbb{E}[X^A] - \mathbb{E}[Y] \geq K^* \\
 & \quad \sum_{n=1}^N \beta_n^A = 1 \\
 & \quad 0 \leq \beta_n^A \leq 1 \quad \forall n = 1, \dots, N
 \end{aligned} \tag{3.24}$$

where  $X^A \underset{(1)}{\succeq} Y$  means that  $X^A$  FSD  $Y$ . Considering the linearization problem (3.20) of the Rachev utility function and the mixed-integer feature of the first order stochastic dominance constraints (Kopa, 2010) we get a mixed-integer linear problem.

Then, the next step aims to find a portfolio that outperforms  $X^A$  in a second order stochastic dominance sense solving the following linear portfolio selection problem. Let  $X^B$  and  $X^A$  be two random variables with discrete realization  $r_t \beta^B$  and  $r_t \beta^A$  for  $t = 1, \dots, T$  which belong to the joint support  $\mathbb{U} = \{u_{(1)} \leq \dots \leq u_{(j)} \leq \dots \leq u_{(J)}\}$  where  $J \leq 2T$ . Let  $X^A$  be the return of the invested portfolio and  $X^A$  the optimal portfolio obtained from the problem (3.24), we solve:

$$\begin{aligned}
 & \max_{\beta^B} \quad b \text{CVaR}_{\alpha_1}(-X^B) - a \text{CVaR}_{\alpha_2}(X^B) \\
 & \text{s.t.} \quad X^B \underset{(2)}{\succeq} X^A \\
 & \quad \mathbb{E}[X^B] - \mathbb{E}[X^A] \geq K^* \\
 & \quad \sum_{n=1}^N \beta_n^B = 1 \\
 & \quad 0 \leq \beta_n^B \leq 1 \quad \forall n = 1, \dots, N
 \end{aligned} \tag{3.25}$$

where  $X^B \underset{(1)}{\succeq} X^A$  means that  $X^B$  SSD  $X^A$ . In this case, we introduce the linear definition of the second order stochastic dominance constraints through the double stochastic



matrix (Kuosmanen, 2004). We notice how the problem (3.25) is solved in a linear programming way.

Finally, we solve the step  $C$  of the stochastic investment chain. Let  $X^C$  and  $X^B$  be two random variables with discrete realization  $r_t\beta^C$  and  $r_t\beta^B$  for  $t = 1, \dots, T$  which belong to the joint support  $\mathbb{U} = \{u_{(1)} \leq \dots \leq u_{(j)} \leq \dots \leq u_{(J)}\}$  where  $J \leq 2T$ . We solve the following linear problem with third order bounded stochastic dominance constraints:

$$\begin{aligned}
 & \max_{\beta^C} \quad b CVaR_{\alpha_1}(-X^C) - a CVaR_{\alpha_2}(X^C) \\
 \text{s.t.} \quad & X^C \underset{(3),b}{\succeq} X^B \\
 & \mathbb{E}[X^C] - \mathbb{E}[X^B] \geq K^* \\
 & \sum_{n=1}^N \beta_n^C = 1 \\
 & 0 \leq \beta_n^C \leq 1 \quad \forall n = 1, \dots, N
 \end{aligned} \tag{3.26}$$

where the solution of the linear programming problem (3.26) is the portfolio  $X^C = r\beta^C$  which dominated the benchmark represented by the optimal portfolio  $X^B = r\beta^B$  in the second order sense. In particular, following the Proposition 3.9 we introduce the linear formulation of the third order bounded stochastic dominance constraints in the portfolio problem.

### 3.5 Empirical Applications

In this section, we propose two empirical application to the benchmarking problem. Considering the Russell 1000 stock index as the selected benchmark, we firstly address with the problem to build portfolio which mimic the performances of a stock index and then we introduce first and second order stochastic dominance constraints to evaluate the enhanced strategies in a static and rolling framework. Secondly, we propose an empirical application of the stochastic investment chain proposed in Section 3.4 when

we maximize the Rachev utility function. At each optimization step, we solve (A), (B) and (C) phases of the chain and we evaluate the out of sample wealth path of the three portfolios.

### 3.5.1 Benchmark Tracking Error Problem with $L^P$ Measure

The analyzed time period covers the last decade from 31st December 2002 to 31st December 2013 and we propose investment strategies with monthly recalibration (20 days) with a total number of 125 optimization steps. We generally consider an historical moving window of 260 observations which is reduced to 120 time series data when we apply enhanced indexing strategies with first order stochastic dominance constraints. Every investment portfolio strategy starts on 12th January 2004. The Russell 1000 presents 736 components as number of stocks during the entire period.

We set a spectrum of 13 possible  $q \in \mathbb{R}$  in the range of the join support between portfolio and benchmark and we consider the information ratio quantile regression 2.40 as decisional rule to switch from the static to the rolling approach. Figure 3.1 shows the out of sample wealth path of the rolling strategies obtained minimizing the dispersion measure derived from the  $L^P$  metric. The aim of portfolio manager is to obtain portfolio as much as possible closed to the benchmark. The blue line represents the rolling strategy obtained minimizing the realistic index tracking portfolio problem (3.16) while the yellow and red line represents the wealth path of the enhanced indexation problem with second and first order stochastic dominance constraints. In these cases we solve the portfolio problem (3.18) and (3.17).

Analyzing the behavior of the three benchmarking strategies is clear the impact of stochastic dominance constraints in the optimization problem. Firstly, the dispersion measure derived from the  $L^P$  metric results to be a useful tool to track the benchmark performances since the blue lines mimic very well the Russell 1000. Secondly, whether second order stochastic dominance constraints produces a relative weak impact in the

final wealth with gain more than 10% on the overall period, the contribution of the FSD strategy is very impressive.

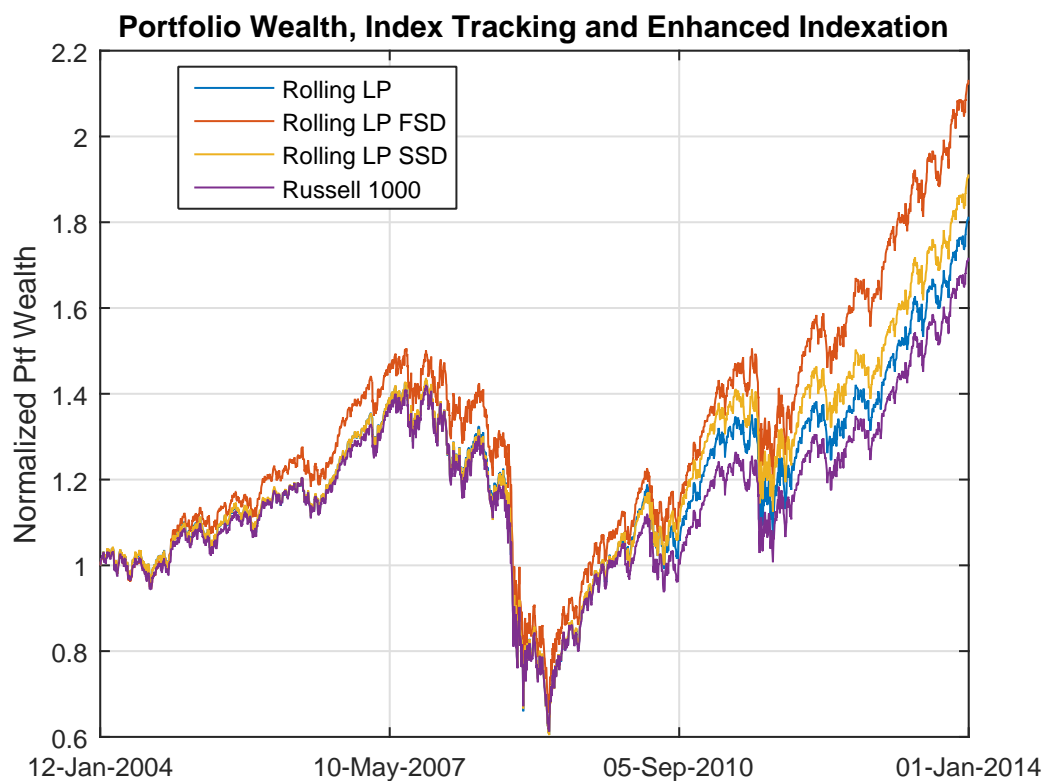


FIGURE 3.1: Out of Sample Portfolio Wealth of Index Tracking and Enhanced Indexation Rolling Strategy LP Metrics, Russell 1000

In fact, solving the enhanced indexation problem with first order stochastic dominance constraints we obtain portfolios which outperforms the index and the previous one for the entire investment period. In this case we have a peak in the period before the sub-prime crisis with gains about 50% and a value of the final wealth of about 2.1 times the initial one.

### 3.5.2 Three Stochastic Order Steps Maximizing the Rachev Utility Function

In this section, we analyze the results of empirical applications of the stochastic investment chain problem. In particular, considering the Russell 1000 as initial benchmark and for an investment period of 10 years, we compare three portfolio strategies which correspond to the three steps of the stochastic investment chain. The analyzed time period covers the last decade from 31st December 2002 to 31st December 2013 and we propose investment strategies with monthly recalibration (20 days) with a total number of 125 optimization steps. Setting an historical moving window of 120 daily observations we solve the portfolio problems (3.24), (3.25) and (3.26) at each optimization step. Every investment portfolio strategy starts on 12th January 2004 and the Russell 1000 presents 736 components as number of stocks during the entire period.

Figure 3.2 shows the wealth path of the three portfolios  $X^A$ ,  $X^B$  and  $X^C$  during the overall period. We notice the high different behavior between them. In particular, starting from the maximization of the Rachev utility with first order stochastic dominance (blue line) we notice that the wealth increase forcefully after the sub-prime and the Greece debt crises in 2008 and 2011. However, the wealth reach a final value more than 3 times the initial one. Differently, the solution of the step (B), obtained maximizing the Rachev utility with the second order stochastic constraints where the benchmark is the previous optimal portfolio, shows a wealth path relative smooth for the entire period. In fact, comparing these two strategies we notice how the red line which represent the portfolio  $X^B$  is much more conservative in the risk exposition and additive shift during the overall period. Finally, the introduction of third order stochastic dominance constraints has the opposite behavior in the out of sample portfolio wealth. The yellow line represents the entire path and it shows how the portfolio is strongly exposed to upward and downward shifts. Before sub-prime crisis in 2007 the value of the portfolio was more than 3 times the initial one and after losing all the previous gains in few months the

wealth restart to increase forcefully during the following financial upturn with a final figure of about 5.3.

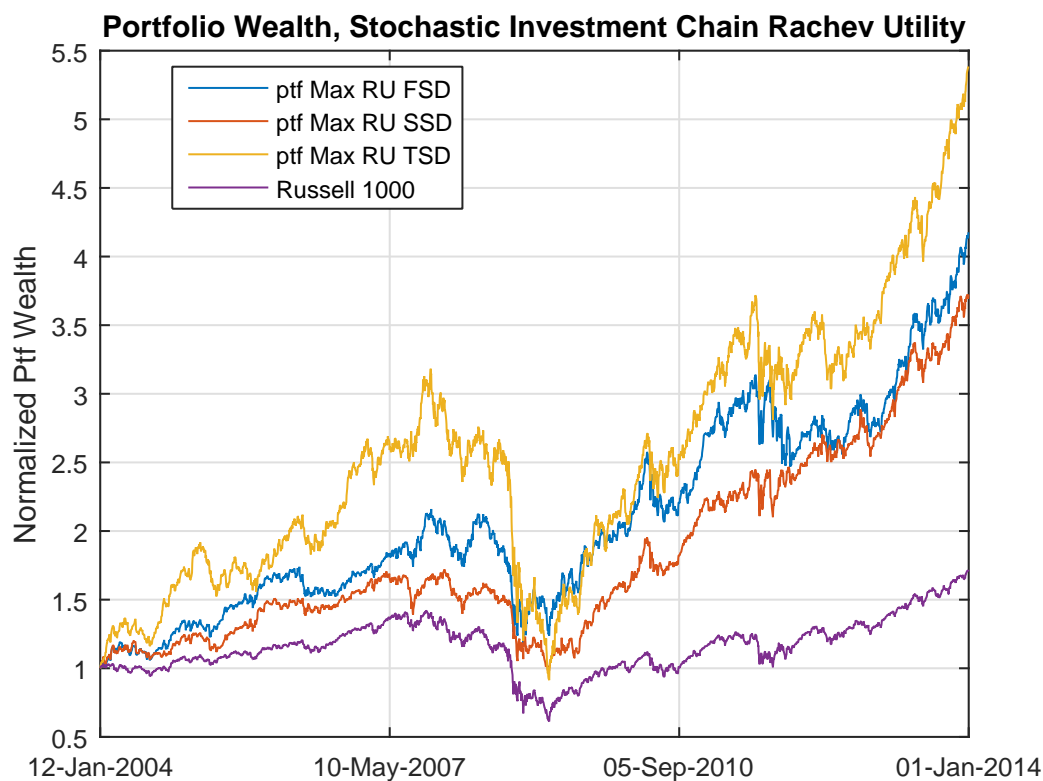


FIGURE 3.2: Out of Sample Portfolio Wealth of Stochastic Investment Chain Maximizing Rachev Utility Function, Russell 1000

### 3.6 Final Remarks

In this Chapter, we extended the concept of deviation measure to the class of the risk measure. For this reason, we show how a deviation measure could be defined as an expectation bounded risk measure and we illustrate some applications. In particular, introducing the class of Gini measure we prove how the quantile regression results to be a special case of this very important class in the financial literature and we propose

two linear portfolio selection models based on a different ordering of the Gini measure and a concentration measure derived from the  $p$ -average compound metrics. Then, we introduce a theoretical construction to extend the linearity constraints to the third order bounded stochastic dominance. This approach has the aim to introduce stochastic dominance chain and build portfolios that dominate an optimal one to an higher order.

## Chapter 4

# Linear Programming Active Management Strategy. The Maximization of Performance Measures.

### 4.1 Introduction

The active benchmark tracking portfolio problem is a investment strategy which aims to exceed the performance of a selected target benchmark and it is sometimes referred to as active portfolio management ([Sharpe, 1994](#)). It is well known that many professional investors achieve this benchmarking strategy: for instance, bond funds try to beat the Barclays Bond Index, commodity funds seek to beat the Goldman Sachs Commodity Index while several mutual funds take the Standard and Poors (S&P) 500 Index as their benchmark.

However, a famous study by [DeMiguel et al. \(2009\)](#) questions the efficiency of active strategies in the mean-variance optimization relative to naive diversification, i.e., relative to a strategy that places a weight of  $1/N$  on each of the  $N$  assets under consideration. The authors of the study implement 14 variants of the standard mean-variance model for a number of datasets and find that there is no single model that consistently delivers a Sharpe ratio or a CEQ return that is higher than that of the  $1/N$  portfolio.

The main reason of this result is attributed to the presence of estimation error and constraints on portfolio holdings with significant implication in presence of huge assets with short available time series data. [Pástor \(2000\)](#) and [Pástor and Stambaugh \(2000\)](#) use Bayesian methods to address the issue of parameter uncertainty. [Ledoit and Wolf \(2003\)](#) develop an optimal shrinkage methodology for covariance matrix estimation and find that it improves the out-of-sample performance of mean-variance optimization methods. [Jagannathan and Ma \(2003\)](#) consider ad hoc short-sale constraints and position limits and show that these restrictions are a form of shrinkage that improves portfolio performance by reducing the ex post effect of estimation error. [Kan and Zhou \(2007\)](#) use an innovative approach to develop a three-fund asset allocation strategy that optimally diversifies across both factor and estimation risk.

Thus, the problem to identify the “best” composition to beat a given benchmark or market portfolio is the main topic of this chapter since a decisional process to identify the selection criteria is still an open question in the financial literature. Different approaches to address with the active management strategy involve decisional problems based on historical observations or scenario simulations. In this essay, we consider historical observations as decisional variables while strategies involving the creation of future scenarios are discussed in the [Appendix A](#). There, we present a well-known methodology based on an ARMA-GARCH process to obtain future realization of the log-returns of the asset’s price when the innovation is chosen considering three distributional hypotheses: Gaussian, Student and Stable Paretian ([Rachev and Mitnik, 2000](#)).



The aim of this chapter is to solve the benchmark tracking problem implementing active strategies to manage portfolio which outperforms the benchmark index. In this framework, we face the high dimensionality problem looking for efficient solutions to maximize the utility of different investors' preferences. The main contribution of this chapter is the development of linear formulation portfolio optimization problems maximizing four different performance measures. Then, introducing first and second order linear stochastic dominance constraints, we evaluate their impact in the out of sample wealth path of the invested portfolios. In particular, a linear programming formulation of the optimization problem significantly reduces the computational complexity of the active strategies and efficiently solves the problem to maximize a performance measure when the number of asset is lower than the available historical observations.

This chapter is organized as follow. In Section 4.2, we review three main performance measures presented in the financial literature and we introduce a new one based on the mean absolute semi-deviation. Section 4.3 discusses the linear formulation approach to solve the active management while Section 4.4 describe the linear and mixed-integer linear programming with stochastic dominance enhancement. An empirical application is proposed in Section 4.5 and we summarize the main results in the final Section 4.6.

## 4.2 Performance Measures and Different Investors' Profiles

In the active strategy framework, the goal of portfolio managers is to maximize their future or final wealth considering different reward/risk investors' profiles. In particular, maximizing future investors wealth, we generally use a reward/risk portfolio selection model applied either to historical series or to simulated scenario models (see, among others, [Rachev et al. \(2008a\)](#) and [Biglova et al. \(2004\)](#)). Let  $Y$  be the random variable representing the return of a given benchmark with realization  $y_t$  at time  $t$  for  $t = 1, \dots, T$  composed by  $N$  assets with returns  $R = [r_1, \dots, r_N]'$ . Thus, the vector of the returns

of an invested portfolio is defined by a random variable  $X$  such that  $X = r\beta$  with realization  $x_t = \sum_{n=1}^N r_{n,t}\beta_n$  and the tracking error is a random variable  $Z$  such that  $Z = X - Y$ . To maximize the performances of a portfolio in the reward/risk framework, we provide the maximum expected reward  $\mu$  per unit of risk  $\rho$ . This optimal portfolio is commonly called the market portfolio and it can be obtained with several possible reward/risk performance ratios (Cogneau and Hübner, 2009a,b) defined as:

$$G(Z) = \frac{\mu(Z)}{\rho(Z)} \quad (4.1)$$

Recall that a performance ratio must be isotonic with investors' preferences, i.e. if  $Z$  is preferable to  $V$ , then  $G(Z) \geq G(V)$  (Rachev et al., 2008a). Although the financial literature agrees that investors are non-satiable, there is no common vision about their risk aversion. Investors' choices should be isotonic with non-satiable investors' preferences, i.e. if  $Z \geq V$  then  $G(Z) \geq G(V)$ , and several behavioral finance studies suggest that most investors are neither risk averse nor risk seeking (Cogneau and Hübner, 2009a,b; Rachev et al., 2008a). Here, we review and present four measure of performances given by the ratios between a reward and a risk measure.

### 4.2.1 The Sharpe Ratio

The Sharpe ratio is a commonly used measure of portfolio performance. However, because it is based on the mean-variance theory, it is valid only for either normally distributed returns or quadratic preferences. In other words, the Sharpe ratio is a meaningful measure of portfolio performance when the risk can be adequately measured by standard deviation. When return distributions are non-normal, the Sharpe ratio can lead to misleading conclusions and unsatisfactory paradoxes, see, for example, (Ortobelli et al., 2005; Bernardo and Ledoit, 2000; Hodges, 1998).

According to the Markowitz mean-variance analysis, [Sharpe \(1994\)](#) suggests that investors should maximize what is now referred to as the Sharpe Ratio (SR) given by:

$$\text{SR}(Z) = \frac{\mathbb{E}[Z]}{\text{STD}(Z)} \quad (4.2)$$

where the numerator is the expected value and the denominator represent the standard deviation of excess returns. Maximizing the Sharpe Ratio, we obtain a market portfolio which should be optimal for non-satiable risk-averse investors, but that is not dominated in the sense of second-order stochastic dominance. This performance measure is fully compatible with elliptically distributed returns, but it leads to incorrect investment decisions when the returns distribution presents heavy tails or skewness.

#### 4.2.2 Rachev Ratio

The Rachev Ratio ([Biglova et al., 2004](#)) is based on tail measures and it is isotonic with the preferences of non-satiable investors that are neither risk averse nor risk seekers. The Rachev Ratio (RR) is the ratio between the average of earnings and the mean of losses; that

$$\text{RR}(Z, \alpha_1, \alpha_2) = \frac{\text{CVaR}_{\alpha_2}(-Z)}{\text{CVaR}_{\alpha_1}(Z)} \quad (4.3)$$

where the Conditional Value-at-Risk (CVaR), is a coherent risk measures ([Rockafellar and Uryasev, 2002](#); [Artzner et al., 1999](#)) defined as:

$$\text{CVaR}_{\alpha}(Z) = \frac{1}{\alpha} \int_0^{\alpha} \text{VaR}_q(Z) dq \quad (4.4)$$

and

$$\text{VaR}_q(Z) = -F_Z^{-1}(q) = -\inf\{z | \mathbb{P}(Z \leq z) > q\} \quad (4.5)$$

is the Value-at-Risk (VaR) of the random return  $Z$ . If we assume a continuous distribution for the probability law of  $Z$ , then  $\text{CVaR}_{\alpha}(Z) = -\mathbb{E}[Z | Z \leq \text{VaR}_{\alpha}(Z)]$  and, therefore CVaR can be interpreted as the average loss beyond VaR. Typically, we use historical

observations to estimate the portfolio return and a risk measures. A consistent estimator of  $\text{CVaR}_\alpha(Z)$  is given by:

$$\text{CVaR}_\alpha(Z) = \frac{1}{[\alpha T]} \sum_t^{[\alpha T]} Z_{t:T} \quad (4.6)$$

where  $T$  is the number of historical observations of  $Z$ ,  $[\alpha T]$  is the integer part of  $\alpha T$  and  $Z_{t:T}$  is the  $t$ -th observation of  $Z$  ordered in increasing values. Similarly an approximation of  $\text{VaR}_q(Z)$  is simply given by  $-Z_{[qT]:T}$ .

### 4.2.3 The STARR

In 2005, [Martin et al. \(2005\)](#) introduce a different reward risk measure: the Stable Tail Adjusted Return Ratio ( $\text{STARR}_\alpha$ ). This measure is a generalization of the Sharpe Ratio but it allows to overcome the drawbacks of the standard deviation as a risk measure ([Artzner et al., 1999](#)). In particular, STARR focus on the downside risk and it is not a symmetric and unstable measure of risk when returns present heavy-tailed distribution. Thus, let a random variable  $Z$  be the difference between two random variables representing the portfolio and benchmark returns, the STARR at the confidence level  $\alpha$  is expressed as:

$$\text{STARR}(Z, \alpha) = \frac{\mathbb{E}[Z]}{\text{CVaR}_\alpha(Z)} \quad (4.7)$$

The STARR differently from the Sharpe Ratio considers a coherent risk measure and not a deviation one as risk sources.

### 4.2.4 Mean Absolute Semideviation Ratio

Finally, we introduce a performance measure splitting the two components of the quantile regression dispersion measure. This ratio is based on the idea to divide positive and negative difference between the returns of the invested and benchmark portfolios and evaluate their mean in the absolute sense. Thus, we introduce reward and risk measures

applied to the tracking error  $Z = X - Y$  where  $X$  is the invested portfolio and  $Y$  is the benchmark return.

**Definition 4.1**

Let  $Z$  be a random variable with realization at time  $t$  equal to  $\varepsilon_t$  which represent the difference between the returns of the invested portfolio and the benchmark (i.e.  $\varepsilon_t = r_t\beta - y_t$ ) and let  $u_t = (r_t\beta - y_t)\mathbb{1}_{[r_t\beta \geq y_t]}$  and  $\nu_t = |(y_t - r_t\beta)\mathbb{1}_{[r_t\beta < y_t]}|$  be two positive variables representing the two sides of the excess returns between investing and benchmark portfolios. We define the **Mean Absolute Semideviation Ratio** (MASDR) as:

$$\text{MASDR}(Z) = \frac{\mathbb{E}[Z\mathbb{1}_{[Z \geq 0]}]}{\mathbb{E}[Z\mathbb{1}_{[Z < 0]}]} = \frac{\frac{1}{T} \sum_{t=1}^T u_t}{\frac{1}{T} \sum_{t=1}^T \nu_t} = \frac{\mathbb{E}[\max(r_t\beta - y_t, 0)]}{\mathbb{E}[\max(y_t - r_t\beta, 0)]} \quad (4.8)$$

The main advantage of this ratio is the positive support when it is defined and it allows to compare investment also when the alpha of the portfolio is negative. Moreover, as developed in the following section the maximization of the Mean Absolute Semideviation Ratio (4.8) could be efficiently solved as a linear programming portfolio problem.

We notice that this ratio is the a special case of the Farinelli-Tibiletti Ratio (FTR) (Farinelli et al., 2008):

$$\text{FTR}(Z, p, q) = \frac{\mathbb{E}[Z^p \mathbb{1}_{[Z \geq 0]}]^{1/p}}{\mathbb{E}[Z^q \mathbb{1}_{[Z < 0]}]^{1/q}} \quad (4.9)$$

when  $p = q = 1$ . In this case, as treated in Stoyanov et al. (2007) we deal with a non-quasi concave reward-risk ratio. In the general formulation with  $p \geq q > 1$  it is not possible to define a linear programming formulation of the portfolio problem (4.10) while we show how linearize this special case in the next section.

### 4.3 LP Problem for Active Strategies

In the Modern Portfolio Theory, the problem of choice to maximize the performance of an investor with different preferences is still an open question in the financial literature. In recent time, one of the main issue is to deal with portfolio problem characterized by high dimensionality and numerous assets. In particular, the first concept relates with the situation when the number of assets are greater than the historical observations (Kondor et al., 2007; Papp et al., 2005). Considering these type of problems is essential to have an easily solvable structure of the portfolio model and the class of LP problem results to be suitable to achieve this aim. Linear programming implies the possibility to find an optimal solution reducing the computational time. Dealing with non-linear optimization problem with huge number of assets, we have to develop our research in the reduction of the dimensionality, in the linearization of the objective function and in the definition of linear constraints to efficiently find the optimal portfolio composition.

In this section, we address both of these problems. Here we discuss the reduction of the number of assets while in the next subsections we develop linear formulation to solve the maximization problem of the four performance measures presented before.

In the financial applications, several problems need a preliminary reduction when the number of assets is still numerous. This concept is strictly linked with the ordering problem (Ortobelli et al., 2013, 2009; Ortobelli and Shalit, 2008; Ortobelli et al., 2008) which identify different criteria to select the “best” and reject the “worst” assets according to investor’s preference. For instance, whether a portfolio manager bases its decision in a mean-variance framework the Sharpe Ratio results to be an optimal criteria to select a subset of assets with high expected value and low standard deviation.

The main technique for a preliminary reduction leads to the introduction of pre-selection steps before the portfolio optimization. Pre-selection consist to order the asset considering a criteria which reflect the investor preferences and active only the assets that satisfy the given condition. Following this way, we could reduce the number of

portfolio's inputs. In this essay, we pre-select the "best" assets ordering with respect to the performance measure considered in the optimization problem. This performance measure is calculated on the historical observation of the rolling window and then, we select the first  $d$  assets such that  $d \leq N$  with higher value. In fact, they have the best reward for unit of risk and they would be suitable to outperform the index also in the next investment period.

### 4.3.1 Active Strategies Maximizing a Performance Measure

The second problem to address is related with the linearization of the four performance measures presented before solving the following portfolio selection problem. Let  $X = r\beta$  be the return of the invested portfolio with realization  $x_t = r_t\beta$  at time  $t$  where  $r_t$  is the row of the asset returns and let  $Y$  be the returns of the benchmark with realization  $y_t$ . We define the common portfolio selection problem as follow:

$$\begin{aligned} \max_{\beta} \quad & \frac{\mu(X - Y)}{\rho(X - Y)} \\ \text{s.t.} \quad & \sum_{n=1}^N \beta_n = 1 \\ & lb \leq \beta_n \leq ub \quad \forall n = 1, \dots, N \end{aligned} \tag{4.10}$$

where  $\beta_n$  for  $n = 1, \dots, N$  is the portfolio weight vector and optimal solution of the minimization problem,  $lb$  the lower bound and  $ub$  the upper bound as maximum amount invested in a given asset. In particular, fixing the value of the upper bound it is possible to implicitly define the number of minimum active assets in the portfolio selection problem.

We could notice that the objective function is non linear since in the ratio the variable  $\beta$  appears both to the numerator and to the denominator. For this reason analyzing the nature of different reward and risk measure, we linearize these objective functions following the theoretical structure in [Stoyanov et al. \(2007\)](#). In particular, we

review two active management portfolio strategies linearizing the STARR (4.3) and the Rachev Ratio (4.7) following the approach proposed in [Stoyanov et al. \(2007\)](#) proposing an empirical application. Then, we develop a linear formulation for the Sharpe Ratio (4.2) and the Mean Absolute Semideviation Ratio (4.8). The following two Remarks are important to linearly develop the Sharpe Ratio and the Mean Absolute Semideviation Ratio.

**Remark 4.1**

If  $\mu : \mathbb{D}_1 \subset \mathbb{R}^n \rightarrow \mathbb{R}^{++}$  is a concave function and  $\rho : \mathbb{D}_2 \subset \mathbb{R}^n \rightarrow \mathbb{R}^{++}$  is a convex function then

1. the ratio  $\mu/\rho : \mathbb{D}_1 \cap \mathbb{D}_2 \rightarrow \mathbb{R}^{++}$  is quasi-concave;
2. the ratio  $\rho/\mu : \mathbb{D}_1 \cap \mathbb{D}_2 \rightarrow \mathbb{R}^{++}$  is quasi-convex;
3. the following relationship holds:  $\arg \max_x \frac{\mu(x)}{\rho(x)} = \arg \min_x \frac{\rho(x)}{\mu(x)}$

as proved in [Stoyanov et al. \(2007\)](#).

**Remark 4.2**

Suppose that  $\mu(\cdot)$  and  $\rho(\cdot)$  are functional satisfying the following properties: The reward measure  $\mu$  is assumed to be a positive functional on the space of real-valued random variables that is:

1. positive homogeneous:  $\mu(tX) = t\mu(X)$ ,  $t > 0$
2. concave:  $\mu(\alpha X_1 + (1 - \alpha)X_2) \geq \alpha\mu(X_1) + (1 - \alpha)\mu(X_2)$ ,  $\alpha \in [0, 1]$

The risk-measure is a positive functional on the space of real-valued random variables which is assumed to be:

1. positive homogeneous:  $\rho(tX) = t\rho(X)$ ,  $t > 0$
2. sub-additive:  $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$



Then, the reward function  $\mu(x_t\beta - y_t) : X \cap \mathbb{R}^n \rightarrow \mathbb{R}^{++}$  is concave and the risk function  $\rho(x_t\beta - y_t) : X \cap \mathbb{R}^n \rightarrow \mathbb{R}^{++}$  is convex, provided that the domain  $X$  is a convex set and the problem (4.10) could be linearized as developed in [Stoyanov et al. \(2007\)](#).

#### 4.3.1.1 Portfolio with maximum STARR

Applying the linearization technique developed in [Stoyanov et al. \(2007\)](#) and [Rockafellar and Uryasev \(2002\)](#), we could reformulate the problem (4.10) where the objective function is represented by the  $\text{STARR}_\alpha(X)$  (4.7). Let  $\beta = \frac{w}{g}$  and  $X^* = rw$  we obtain the following LP portfolio selection problem:

$$\begin{aligned}
\min_{w, g, d, \gamma} \quad & \gamma + \frac{1}{\alpha T} \sum_{t=1}^T d_t \\
\text{s.t.} \quad & \mathbb{E}[X^*] - g\mathbb{E}[Y] = 1 \\
& -r_t w + g y_t - \gamma \leq d_t \quad \forall t = 1, \dots, T \\
& \sum_{n=1}^N w_n = g \\
& g lb \leq w_n \leq g ub \quad \forall n = 1, \dots, N \\
& g \geq 0, d_t \geq 0 \quad \forall t = 1, \dots, T
\end{aligned} \tag{4.11}$$

where  $\alpha$  is the confident level. This problem is linear and introducing  $T + 2$  variables it could be efficiently solved. In particular, the dimensionality does not increase so much with the length of the historical observations. However, the choice of the confidential level is crucial to obtain feasible solution.

#### 4.3.1.2 Mixed-Integer linear programming to maximize the Rachev Ratio

Differently, there are reward-risk ratios suggested in literature that are not in the class of the quasi-concave functions because both the numerator and the denominator are convex. Such are for instance the Farinelli-Tibiletti Ratio ([Farinelli et al., 2008](#)) and

the Generalized Rachev Ratio (Biglova et al., 2004). However, Stoyanov et al. (2007) propose a mixed-integer linear programming (MILP) formulation for the Rachev Ratio introducing binary variables  $\lambda_t$ ,  $\forall t = 1, \dots, T$  and the threshold  $B$  such that  $B \geq |x_t w|$ ,  $\forall t = 1, \dots, T$ .

In this case, setting an extremely high value of  $B$  we could solve the following MILP problem which maximizes the  $RR_{\alpha_1, \alpha_2}(X)$ :

$$\begin{aligned}
& \min_{w, g, f, d, \lambda, \gamma} && - \frac{1}{\lceil \alpha_2 T \rceil} \sum_{t=1}^T f_t \\
\text{s.t.} &&& f_t \leq B \lambda_t && \forall t = 1, \dots, T \\
&&& f_t \geq r_t w - B(1 - \lambda_t) && \forall t = 1, \dots, T \\
&&& f_t \leq r_t w + B(1 - \lambda_t) && \forall t = 1, \dots, T \\
&&& \sum_{t=1}^T \lambda_t = \lceil \alpha_2 T \rceil \\
&&& \gamma + \frac{1}{\lceil \alpha_1 T \rceil} \sum_{t=1}^T d_t \leq 1 && (4.12) \\
&&& -r_t w - \gamma \leq d_k && \forall t = 1, \dots, T \\
&&& \sum_{n=1}^N w_n = g \\
&&& g \text{ lb} \leq w_n \leq g \text{ ub} && \forall n = 1, \dots, N \\
&&& g \geq 0, d_t \geq 0 && \forall t = 1, \dots, T \\
&&& f_t \geq 0, \lambda_t \in [0, 1] && \forall t = 1, \dots, T
\end{aligned}$$

when the real portfolio weights are obtained dividing the vector  $w$  for the scalar  $g$  such that  $\beta = \frac{w}{g}$  and  $\lceil \alpha T \rceil$  is the ceiling integer number of  $\alpha T$ . The feasibility of this problem is strictly connected with the length of historical observation having an impact on the number of binary variable in the MILP problem. The number of real variables is  $2T + 2$  while the mixed-integer are  $T$  for a total number of  $3T + 2$  variables. For this reason it is not possible to solve the problem with long historical time series data

but an interesting analysis that we do not discuss in this essay could be based on the introduction of weekly data to cover a long historical time period with small number of observation. For a practical point of view this approach is adopted for several class of the asset management like as the pension fund when the availability of the data has a weekly frequency.

#### **4.3.1.3 Maximization of the Sharpe Ratio**

In this section, we propose a linear formulation to maximize the common portfolio problem (4.10) when the measure of performance is the Sharpe Ratio (4.2). It is well-known that this type of problem could be solved in a quadratic form minimizing the risk or maximizing the return with a set of linear constraints (Stoyanov et al., 2007). In this essay, we propose a linear approximation of the portfolio variance (Ortobelli et al., 2013) to introduce stochastic dominance constraints in the optimization problem. Since in the enhanced indexation strategies we highlight the importance of this set of stochastic dominance constraints to strengthen the investor future wealth, we propose a linearization technique based on an integral rule derived from fractional integral theory (Ortobelli et al., 2013; Fishburn, 1980, 1976) which allows to efficiently solve the problem.

#### **Proposition 4.1**

*The maximization of the Sharpe Ratio should be solve in a linear programming formulation considering that the reward measure satisfies the positive homogeneity property and it is concave while the risk measure is positive homogeneous and sub-addictive.*

Considering the fractional integral theory, we maximize the Sharpe Ratio solving the following problem:

$$\begin{aligned}
& \min_{w,v,u,g} \sum_{i=1}^{M-1} \frac{1}{T} \sum_{k=1}^T v_{k,i} + u_{k,i} \\
& \text{s.t.} \quad \mathbb{E}[X^*] - g\mathbb{E}[Y] = 1 \\
& \quad \sum_{n=1}^N w_n = g \\
& \quad v_{k,i} \geq c + \frac{i}{M} (\mathbb{E}[X^*] - g\mathbb{E}[Y] - c) - r_k w + g y_k \quad \forall k = 1, \dots, T, i = 1, \dots, M \\
& \quad u_{k,i} \geq c + \frac{i}{M} (-\mathbb{E}[X^*] - g\mathbb{E}[Y] - c) + r_k w - g y_k \quad \forall k = 1, \dots, T, i = 1, \dots, M \\
& \quad g lb \leq w_n \leq g ub \quad \forall n = 1, \dots, N \\
& \quad g \geq 0, u_{k,i} \geq 0, v_{k,i} \geq 0 \quad \forall k = 1, \dots, T, i = 1, \dots, M
\end{aligned} \tag{4.13}$$

where  $M$  is a large integer and  $c = -\max(|\min_{\beta} \min_t (r_t \beta - y_t)|, |\max_{\beta} \max_t (r_t \beta - y_t)|)$ . Thus, the optimal portfolio weight  $\beta_n = \frac{w_n}{g}$  for  $n = 1, \dots, n$ .

In the previous portfolio optimization problem, we number of variable increase to  $2T \times M + 1$  where  $T$  is the fixed number of historical observations and it represent the length of the past window while  $M$  is a decisional input representing the discretization of the fractional integration. Thus, the main issue is to define the trade-off between the approximation and the computational complexity of the portfolio problem.

#### 4.3.1.4 LP problem to maximize the Mean Absolute Semideviation Ratio

Finally, we propose a linear programming portfolio model which maximizes the Mean Absolute Semideviation Ratio (4.8). In this formulation we have a reward and a risk measure defined as the mean of the positive and negative deviation. Thus, we have to define the properties of these two measures.

#### Proposition 4.2

Let  $\mu(Z) = \mathbb{E}[Z \mathbf{1}_{[Z \geq 0]}]$  be a reward measure which satisfies the positive homogeneity

and the concavity properties and let  $\rho(Z) = \mathbb{E} [|Z|\mathbf{1}_{[Z<0]}]$  a risk measure that satisfies the positive homogeneity and the sub-additivity properties. Then, the reward functional  $\mu(Z)$  is concave and the risk functional  $\rho(Z)$  is convex.

To maximize the Mean Absolute Semideviation Ratio we solve the optimization problem (4.10) with the given performance measure as objective function.

**Proposition 4.3**

The general performance measure optimization problem (4.10) is equivalent to the following linear programming problem:

$$\begin{aligned}
\min_{w,d,g} \quad & \sum_{t=1}^T d_t \\
s.t. \quad & \mathbb{E}[X^*] - g\mathbb{E}[Y] = 1 \\
& \sum_{n=1}^N w_n = g \\
& d_t \geq gy_t - r_t w \quad \forall t = 1, \dots, T \\
& g lb \leq w_n \leq g ub \quad \forall n = 1, \dots, N \\
& g \geq 0, d_t \geq 0 \quad \forall t = 1, \dots, T
\end{aligned} \tag{4.14}$$

where the optimal portfolio composition  $\beta = w/g$ .

**Proof.** We want to maximize the following performance measure:

$$\max_{\beta} \frac{\mathbb{E} [(X - Y)\mathbf{1}_{[X-Y \geq 0]}]}{\mathbb{E} [|X - Y|\mathbf{1}_{[X-Y < 0]}]} = \max_{\beta} \frac{\mathbb{E} [\max(X - Y, 0)]}{\mathbb{E} [\max(Y - X, 0)]} =$$

Then, reformulating the numerator (Guastaroba et al., 2014), we obtain:

$$= \max_{\beta} \frac{\mathbb{E} [X - Y] - \mathbb{E} [\max(Y - X, 0)]}{\mathbb{E} [\max(Y - X, 0)]} = \max_{\beta} \frac{\mathbb{E} [X - Y]}{\mathbb{E} [\max(Y - X, 0)]} - 1$$

Since the numerator is a concave function and the denominator is convex for Remark 4.1, we have

$$\max_{\beta} \frac{\mathbb{E}[X - Y]}{\mathbb{E}[\max(Y - X\beta, 0)]} - 1 = \min_{\beta} \frac{\mathbb{E}[\max(Y - X, 0)]}{\mathbb{E}[X - Y]}$$

Substituting  $g = \mu^{-1}(r_t\beta - y_t)$  and assuming the positive homogeneity we could rewrite the previous minimization problem as follow:

$$\begin{aligned} \min_{w, g} \quad & \mathbb{E}[\max(gY - X^*, 0)] \\ \text{s.t.} \quad & \mathbb{E}[X^*] - g\mathbb{E}[Y] = 1 \\ & \sum_{n=1}^N w_n = g \\ & g \text{ lb} \leq w_n \leq g \text{ ub} \quad \forall n = 1, \dots, N \\ & g \geq 0 \quad \forall t = 1, \dots, T \end{aligned} \tag{4.15}$$

where  $w = \beta g$ . Then, introducing a slack variable  $d = \max(gY - X^*, 0)$  we obtain the portfolio problem 4.14.  $\square$

We notice that, the linear programming portfolio problem 4.14 which maximizes the Mean Absolute Semideviation Ratio has a relative small increment in the number of variables equal to  $T + 1$ .

## 4.4 Active Management of Stochastic Dominance Constraints

In chapter 2, we show how the introduction of stochastic dominance constraints leads to obtain better results in terms of final wealth building portfolios which dominates the benchmark during the overall period. For this reason, in this section we introduce the linear formulation of the first and second orders stochastic dominance constraints to increase the wealth of the invested portfolio keeping the optimization problem linear or mixed-integer linear programming. For this reason, starting from the optimization

problems 4.11 and 4.12, we introduce the double stochastic matrix  $Z = z_{r,c}$  such that  $\sum_{r=1}^T z_{r,c} = 1$  for  $c = 1, \dots, T$ ,  $\sum_{c=1}^T z_{r,c} = 1$  for  $r = 1, \dots, T$  and  $z_{r,c} \in [0, 1]$  to include the second order stochastic dominance constraints as suggested in Kopa (2010) and Kuosmanen (2004) while the consider the permutation matrix  $P = p_{r,c}$  such that  $\sum_{r=1}^T p_{r,c} = 1$  for  $c = 1, \dots, T$ ,  $\sum_{c=1}^T p_{r,c} = 1$  for  $r = 1, \dots, T$  and  $p_{r,c} \in \{0, 1\}$  to consider the first order stochastic dominance constraints. Then, we introduce the linear and mixed-integer linear programming to maximize the four performance measure in the portfolio optimization problem (4.10) with first and second order stochastic dominance constraints.

#### 4.4.1 Maximize the STARR with FSD and SSD constraints

The next two portfolio problems maximize the STARR with first and second order stochastic dominance constraints, respectively. In particular, the portfolio which maximize the STARR at a confidential level  $\alpha$  is obtain solving the following MILP problem:

$$\begin{aligned}
& \min_{w,g,d,\gamma,p^*} \quad \gamma + \frac{1}{\alpha T} \sum_{t=1}^T d_t \\
& \text{s.t.} \quad \mathbb{E}[X^*] - g\mathbb{E}[Y] = 1 \\
& \quad \quad -r_t w + g y_t - \gamma \leq d_t \quad \forall t = 1, \dots, T \\
& \quad \quad \sum_{n=1}^N w_n = g \\
& \quad \quad X^* \geq P^* Y \\
& \quad \quad \sum_{r=1}^T p_{r,c}^* = g \quad \forall c = 1, \dots, T \\
& \quad \quad \sum_{c=1}^T p_{r,c}^* = g \quad \forall r = 1, \dots, T \\
& \quad \quad g \in \mathbb{N}, p_{r,c}^* \in \mathbb{N} \quad \forall r, c = 1, \dots, T \\
& \quad \quad 0 \leq p_{r,c}^* \leq g \quad \forall r, c = 1, \dots, T \\
& \quad \quad g \text{ lb} \leq w_n \leq g \text{ ub} \quad \forall n = 1, \dots, N \\
& \quad \quad d_t \geq 0 \quad \forall t = 1, \dots, T
\end{aligned} \tag{4.16}$$

where  $P^*$  is a modified permutation matrix with integer values and the optimal portfolio composition  $\beta = w/g$ . In fact, according with [Kopa \(2010\)](#) and [Kuosmanen \(2004\)](#),  $X$  dominates  $Y$  in the first order stochastic dominance sense if and only if  $X \geq PY$ . Then, for the formulation of  $X^* = rw$ , we have  $X^* \geq PYg$  and setting  $P^* = Pg$  we have the optimization problem proposed before. In this problem the number of variables strongly increase of  $T^2 + T + 2$  where  $T^2$  are integer.



Then, we propose a formulation to maximize the STARR with second order stochastic dominance constraints as follow:

$$\begin{aligned}
& \min_{w,g,d,\gamma,z^*} \quad \gamma + \frac{1}{\alpha T} \sum_{t=1}^T d_t \\
& \text{s.t.} \quad \mathbb{E}[X^*] - g\mathbb{E}[Y] = 1 \\
& \quad \quad -r_t w + gy_t - \gamma \leq d_t \quad \forall t = 1, \dots, T \\
& \quad \quad \sum_{n=1}^N w_n = g \\
& \quad \quad X^* \geq Z^* Y \\
& \quad \quad \sum_{r=1}^T z_{r,c}^* = g \quad \forall c = 1, \dots, T \\
& \quad \quad \sum_{c=1}^T z_{r,c}^* = g \quad \forall r = 1, \dots, T \\
& \quad \quad z_{r,c}^* \in \mathbb{R} \quad \forall r, c = 1, \dots, T \\
& \quad \quad 0 \leq z_{r,c}^* \leq g \quad \forall r, c = 1, \dots, T \\
& \quad \quad g \text{ lb} \leq w_n \leq g \text{ ub} \quad \forall n = 1, \dots, N \\
& \quad \quad g \geq 0, d_t \geq 0 \quad \forall t = 1, \dots, T
\end{aligned} \tag{4.17}$$

where  $z_{r,c}^*$  is the modified double stochastic matrix following the same process as in the first order case. This problem is linear programming with an increment of  $T^2 + T + 2$  variables with respect to the problem (4.10).

#### 4.4.2 Mixed-integer linear programming with SD constraints in the maximization of the Rachev Ratio

In the next two portfolio selection problems, we propose a linear optimization model to maximize the Rachev Ratio with first and second order stochastic dominance constraints. In particular, we use the formulation  $P^*$  for the modified permutation matrix and  $Z^*$  for the double stochastic one as in the maximization of the STARR.

Then, the portfolio optimization problem that maximizes the Rachev Ratio,  $RR_{\alpha_1, \alpha_2}$ , with FSD constraints is the following:

$$\begin{aligned}
& \min_{w, g, f, d, \lambda, \gamma, p^*} && - \frac{1}{\lceil \alpha_2 T \rceil} \sum_{t=1}^T f_t \\
\text{s.t.} &&& f_t \leq B \lambda_t && \forall t = 1, \dots, T \\
&&& f_t \geq r_t w - B(1 - \lambda_t) && \forall t = 1, \dots, T \\
&&& f_t \leq r_t w + B(1 - \lambda_t) && \forall t = 1, \dots, T \\
&&& \sum_{t=1}^T \lambda_t = \lceil \alpha_2 T \rceil \\
&&& \gamma + \frac{1}{\lceil \alpha_1 T \rceil} \sum_{t=1}^T d_t \leq 1 \\
&&& -r_t w - \gamma \leq d_t && \forall t = 1, \dots, T \\
&&& \sum_{n=1}^N w_n = g && (4.18) \\
&&& X^* \geq P^* Y \\
&&& \sum_{r=1}^T p_{r,c}^* = g && \forall c = 1, \dots, T \\
&&& \sum_{c=1}^T p_{r,c}^* = g && \forall r = 1, \dots, T \\
&&& g \in \mathbb{N}, p_{r,c}^* \in \mathbb{N} && \forall r, c = 1, \dots, T \\
&&& 0 \leq p_{r,c}^* \leq g && \forall r, c = 1, \dots, T \\
&&& g \text{ lb} \leq w_n \leq g \text{ ub} && \forall n = 1, \dots, N \\
&&& f_t \geq 0, d_t \geq 0 && \forall t = 1, \dots, T \\
&&& \lambda_t \in \{0, 1\} && \forall t = 1, \dots, T
\end{aligned}$$

where  $B$  is a positive high value and the optimal solution is  $\beta = w/g$  and  $X^* = rw$ . Maximizing the Rachev Ratio with first order stochastic dominance constraints we obtain an increment of the number of variable  $T^2 + 3T + 2$  of which  $T^2 + T$  are integer. In this

case the computational complexity of the mixed-integer linear programming problem becomes relevant and we should reduce the number of observations to have feasible solutions.

Then, we propose the following problem for the maximization of the Rachev Ratio with second order stochastic dominance constraints:

$$\begin{aligned}
& \min_{w,g,f,d,\lambda,\gamma,z^*} && - \frac{1}{\lceil \alpha_2 T \rceil} \sum_{t=1}^T f_t \\
\text{s.t.} &&& f_t \leq B \lambda_t && \forall t = 1, \dots, T \\
&&& f_t \geq r_t w - B(1 - \lambda_t) && \forall t = 1, \dots, T \\
&&& f_t \leq r_t w + B(1 - \lambda_t) && \forall t = 1, \dots, T \\
&&& \sum_{t=1}^T \lambda_t = \lceil \alpha_2 T \rceil \\
&&& \gamma + \frac{1}{\lceil \alpha_1 T \rceil} \sum_{t=1}^T d_t \leq 1 \\
&&& -r_t w - \gamma \leq d_k && \forall t = 1, \dots, T \\
&&& \sum_{n=1}^N w_n = g && (4.19) \\
&&& X^* \geq Z^* Y \\
&&& \sum_{r=1}^T z_{r,c}^* = g && \forall c = 1, \dots, T \\
&&& \sum_{c=1}^T z_{r,c}^* = g && \forall r = 1, \dots, T \\
&&& z_{r,c}^* \in \mathbb{R} && \forall r, c = 1, \dots, T \\
&&& 0 \leq z_{r,c}^* \leq g && \forall r, c = 1, \dots, T \\
&&& g \text{ lb} \leq w_n \leq g \text{ ub} && \forall n = 1, \dots, N \\
&&& g \geq 0, d_t \geq 0 && \forall t = 1, \dots, T \\
&&& f_t \geq 0, \lambda_t \in \{0, 1\} && \forall t = 1, \dots, T
\end{aligned}$$

In this case the increment in the number of variable is still relevant since they get  $T^2 + 3T + 2$  of which only  $T$  are integer. Thus, a key aspect in the solution of these MILP problems is the length of the time series data. In the previous section, we discussed the problem to have a relevant window of historical observations while here we analyze the issue related with first order stochastic dominance constraints. In this case, the number of integer variables dramatically increase with the length of the rolling window. To simplify the problem and reduce the computational complexity it is possible to consider the usual length of historical observations and then introduce a shorter window to compute the stochastic dominance constraints. This algorithm results to be very efficient to solve the active strategies maximizing the Rachev Ratio with first and second order stochastic dominance constraints.

#### 4.4.3 Stochastic dominance constraints and maximization of the Sharpe Ratio

Maximizing the Sharpe Ratio ([Sharpe, 1994](#)), we propose to linear problem formulation introducing first and second order stochastic dominance constraints in the selection process. Since the introduction of the modified double stochastic matrix holds the problem linear, the set of constraints which allows to build a portfolio dominating the benchmark in a first stochastic order sense, require integer variables. In this case the linearity of the Sharpe Ratio becomes a relevant aspect in the optimization problem. In this section, we propose the two portfolio selection models to deal with this goal looking for extra-performances in the active strategy framework. However, the dimensionality of the problem and its computational complexity strongly increase. In fact, linearizing the Sharpe Ratio we introduce a new huge set of variables and for stochastic dominance constraints the number of set's components present a further increment.

Let  $M$  be the number linked with the approximation of the fractional integral, we define the following mixed-integer linear problem to maximize the Sharpe Ratio with

first stochastic dominance constraints:

$$\begin{aligned}
& \min_{w,v,u,g,p^*} \sum_{i=1}^{M-1} \frac{1}{T} \sum_{k=1}^T v_{k,i} + u_{k,i} \\
\text{s.t.} \quad & \mathbb{E}[X^*] - g\mathbb{E}[Y] = 1 \\
& \sum_{n=1}^N w_n = g \\
& v_{k,i} \geq c + \frac{i}{M} (\mathbb{E}[X^*] - g\mathbb{E}[Y] - c) - r_k w + g y_k \quad \forall k = 1, \dots, T, i = 1, \dots, M \\
& u_{k,i} \geq c + \frac{i}{M} (-\mathbb{E}[X^*] - g\mathbb{E}[Y] - c) + r_k w - g y_k \quad \forall k = 1, \dots, T, i = 1, \dots, M \\
& X^* \geq P^* Y \quad \forall t = 1, \dots, N \\
& \sum_{r=1}^T p_{r,c}^* = g \quad \forall c = 1, \dots, T \\
& \sum_{c=1}^T p_{r,c}^* = g \quad \forall r = 1, \dots, T \\
& g \in \mathbb{N}, p_{r,c}^* \in \mathbb{N} \quad \forall r, c = 1, \dots, T \\
& 0 \leq p_{r,c}^* \leq g \quad \forall r, c = 1, \dots, T \\
& g lb \leq w_n \leq g ub \quad \forall n = 1, \dots, N \\
& u_{k,i} \geq 0, v_{k,i} \geq 0 \quad \forall k = 1, \dots, T, i = 1, \dots, M
\end{aligned} \tag{4.20}$$

where the optimal portfolio composition is defined as  $\beta_n = \frac{w_n}{g}$  for  $n = 1, \dots, N$ . In this case, we have to introduce  $T^2 + 2T \times M + 1$  variables of whom  $T^2$  are integer. In this case is very important to tuning and define the parameters to have an optimal trade-off between problem dimensionality, approximation accuracy and computational complexity.

Then, we propose a linear programming for active benchmark strategies to build portfolio which maximizes the Sharpe Ratio considering the second order stochastic

dominance constraints. Thus, we solve the following portfolio optimization model:

$$\begin{aligned}
& \min_{w,v,u,g,z^*} \sum_{i=1}^{M-1} \frac{1}{T} \sum_{k=1}^T v_{k,i} + u_{k,i} \\
\text{s.t.} \quad & \mathbb{E}[X^*] - g\mathbb{E}[Y] = 1 \\
& \sum_{n=1}^N w_n = g \\
& v_{k,i} \geq c + \frac{i}{M} (\mathbb{E}[X^*] - g\mathbb{E}[Y] - c) - r_k w + g y_k \quad \forall k = 1, \dots, T, i = 1, \dots, M \\
& u_{k,i} \geq c + \frac{i}{M} (-\mathbb{E}[X^*] - g\mathbb{E}[Y] - c) + r_k w - g y_k \quad \forall k = 1, \dots, T, i = 1, \dots, M \\
& X^* \geq Z^* Y \\
& \sum_{r=1}^T z_{r,c}^* = g \quad \forall c = 1, \dots, T \\
& \sum_{c=1}^T z_{r,c}^* = g \quad \forall r = 1, \dots, T \\
& z_{r,c}^* \in \mathbb{R} \quad \forall r, c = 1, \dots, T \\
& 0 \leq z_{r,c}^* \leq g \quad \forall r, c = 1, \dots, T \\
& g \text{ lb} \leq w_n \leq g \text{ ub} \quad \forall n = 1, \dots, N \\
& g \geq 0, u_{k,i} \geq 0, v_{k,i} \geq 0 \quad \forall k = 1, \dots, T, i = 1, \dots, M
\end{aligned} \tag{4.21}$$

where the optimal portfolio composition is represented by the vector  $\beta = \frac{w}{g}$ . As the previous ones, also this optimization problem presents a huge number of increasing variables  $T^2 + 2T \times M + 1$ .

#### 4.4.4 Portfolio with maximum Mean Absolute Semideviation Ratio with FSD and SSD constraints

To enhance the strategy which maximizes the Mean Absolute Semideviation Ratio, we propose a mixed-integer linear programming and a linear programming portfolio selection models with first and second order stochastic dominance constraints.

Thus, to maximize the MASR with first order stochastic dominance constraints, we introduce the following portfolio optimization problem:

$$\begin{aligned}
& \min_{w,d,g,p^*} \sum_{t=1}^T d_t \\
& \text{s.t.} \quad \mathbb{E}[X^*] - g\mathbb{E}[Y] = 1 \\
& \quad \sum_{n=1}^N w_n = g \\
& \quad d_t \geq gy_t - r_t w \quad \forall t = 1, \dots, T \\
& \quad X^* \geq P^*Y \\
& \quad \sum_{r=1}^T p_{r,c}^* = g \quad \forall c = 1, \dots, T \\
& \quad \sum_{c=1}^T p_{r,c}^* = g \quad \forall r = 1, \dots, T \\
& \quad g \in \mathbb{N}, p_{r,c}^* \in \mathbb{N} \quad \forall r, c = 1, \dots, T \\
& \quad 0 \leq p_{r,c}^* \leq g \quad \forall r, c = 1, \dots, T \\
& \quad g \text{ lb} \leq w_n \leq g \text{ ub} \quad \forall n = 1, \dots, N \\
& \quad d_t \geq 0 \quad \forall t = 1, \dots, T
\end{aligned} \tag{4.22}$$

where the optimal composition is given by the vector  $\beta = w/g$ . This problem is a mixed-integer linear programming with  $T^2$  more integer variables with respect to the linear formulation of the maximization of the Mean Absolute Semideviation Ratio.

Differently, the maximization of the MASR with second order stochastic dominance constraints is still a linear programming problem with an increment of  $T^2$  variables

solved with the following minimization problem:

$$\begin{aligned}
& \min_{w,d,g,z^*} \sum_{t=1}^T d_t \\
& \text{s.t.} \quad \mathbb{E}[X^*] - g\mathbb{E}[Y] = 1 \\
& \quad \sum_{n=1}^N w_n = g \\
& \quad d_t \geq gy_t - r_t w \quad \forall t = 1, \dots, T \\
& \quad X^* \geq Z^* Y \quad \forall t = 1, \dots, N \\
& \quad \sum_{r=1}^T z_{r,c}^* = g \quad \forall c = 1, \dots, T \\
& \quad \sum_{c=1}^T z_{r,c}^* = g \quad \forall r = 1, \dots, T \\
& \quad z_{r,c}^* \in \mathbb{R} \quad \forall r, c = 1, \dots, T \\
& \quad 0 \leq z_{r,c}^* \leq g \quad \forall r, c = 1, \dots, T \\
& \quad g \text{ lb} \leq w_n \leq g \text{ ub} \quad \forall n = 1, \dots, N \\
& \quad g \geq 0, d_t \geq 0 \quad \forall t = 1, \dots, T
\end{aligned} \tag{4.23}$$

where the optimal portfolio composition is  $\beta = w/g$ .

## 4.5 Empirical Application

The proposed methodologies are applied to the active management strategy solving the problem to find a portfolio composition which beat the benchmark in the last ten years. For this reason, we compute empirical applications where the benchmarks stock index are the Russell 1000 and the Nasdaq 100 from 31st December 2002 to 31st December 2013 and we investigate how different portfolio selection problems could outperforms its performances. We consider, as in the previous analyses, a historical moving window of 260 observations while for the mixed-integer linear programming we take into account



120 time series data. Every strategy starts on 12th January 2004. Then, at each optimization step, we compute the  $\text{STARR}_{5\%}$  and the  $\text{RACHEV}_{5\%,2\%}$  for a total number of 125 optimization since we change portfolio composition every month (20 days). Finally, we set an upper bound level of 10% and transaction costs (30bps) are included.

#### 4.5.1 Active Strategies in the Benchmark Tracking Problem with Stochastic Dominance Constraints

We solve the six portfolio selection problem (4.11), (4.12), (4.16), (4.17), (4.18) and (4.19) presented in the previous section proposing portfolio approaches to address with the active management. Then, we compare and empirically test the different optimization methods and the impact to introduce stochastic dominance constraints in the problem formulation.

Figures 4.1 and 4.2 illustrate the out of sample normalized wealth path during the investment period from 12th January 2004 to 31st December 2013. In Figure 4.1 we notice how every active strategy outperforms the Russell 1000. In particular, the classical maximization of the STARR produce an extra-performance of the 30% at the end of the investment period. Whether the introduction of the second order stochastic dominance constraints has not a strong impact in the wealth path, the first order constraints produce an increment in the portfolio gains before the sub-prime crisis. In fact, during this period the portfolio wealth gains more than 70% while after the crisis it is difficult to enhance the performance of the maximization of the STARR. Considering the benchmark represented by the Russell 1000 it is evident the goodness of the active strategies to outperform it but also the increment of the risk component.

These active portfolio strategies with stochastic dominance constraints suffer the sub-prime crisis but in the case of first order stochastic dominance the value of its portfolio does not fall down the initial wealth and it could have solid base to make use of the following financial upturn. Therefore, starting from the 2009 the market increase forcefully and the strategies based on the maximization of the STARR significantly

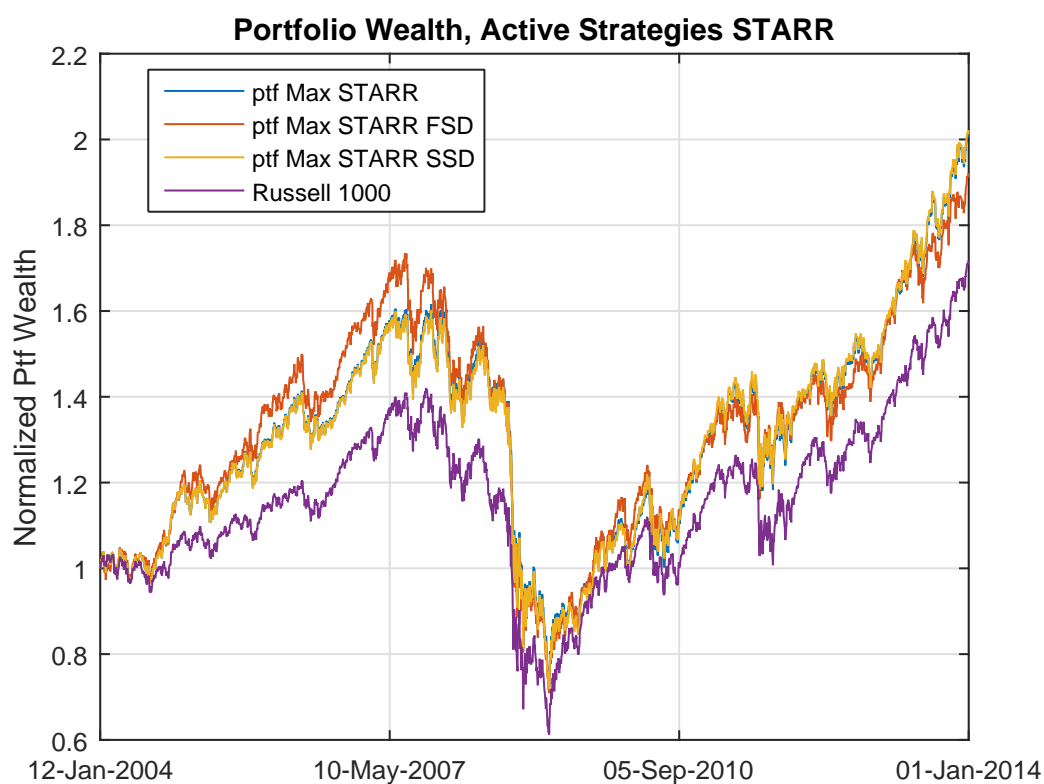


FIGURE 4.1: Portfolio Wealth of Active Strategies STARR, Russell 1000

dominate the benchmark stock index. In particular, the portfolio strategies with first and second order stochastic dominance constraints have a final wealth more than 3.5 and 3.4, respectively.

Neither risk aversion nor risk seeking investors that maximize their utility solving a portfolio problem which involves the maximization of the Rachev Ratio obtain higher final value than the previous strategies. Figure 4.2 shows the normalized wealth path of the Russell 1000 (purple line), the maximization of the Rachev ratio (azure line), the strategy with the introduction of first order stochastic dominance (red line) and the maximization of the Rachev Ratio with second order stochastic dominance constraints (gold line). Also in this case every active managed portfolio outperforms the benchmark stock index for the overall period.

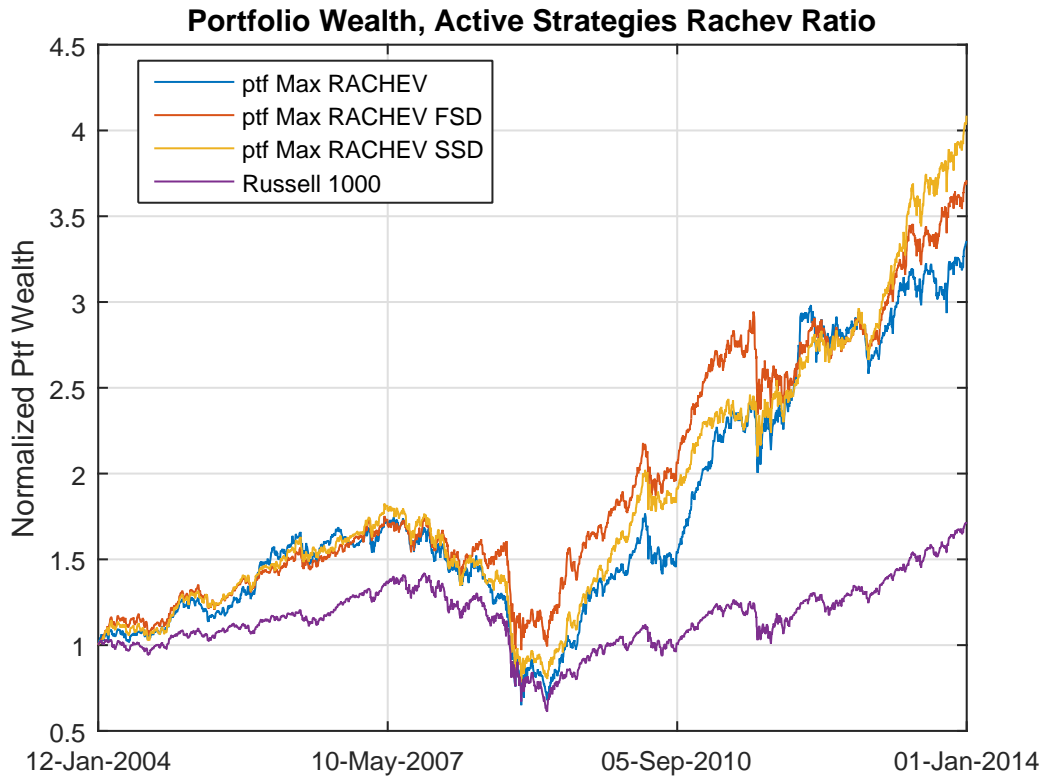


FIGURE 4.2: Portfolio Wealth of Active Strategies Rachev Ratio, Russell 1000

Whether the common maximization of the Rachev Ratio produce a portfolio with more than 3 times the initial wealth allocation, the other two strategies have better results. In particular, the maximization of the performance measure with first order stochastic dominance constraints amplifies the market jumps during the entire period. In fact, this strategy reach 1.7 times the initial wealth in 2008 before the sub-prime crisis while investors holding this portfolio composition have also a peak of about 300% of earnings in 2010 and after a period of stability during the European sovereign debt crisis the portfolio wealth path starts a increasing rally with a final value more than 3.5 times the original invested capital.

Analyzing the two strategies with stochastic dominance constraints we notice that the first order stochastic dominance portfolio dominates the other strategies for the

entire period with the exception of the last investment period. Differently from the maximization of the STARR we stress the different wealth path of the three strategies since the maximization of the Rachev Ratio is not consistent with second order stochastic dominance.

Figure 4.3 report the maximization of the Mean Absolute Semideviation Ratio. We observe how the three strategies outperform the benchmark represented by the Russell 1000. Moreover, the red line which illustrates the wealth path of the maximization of the MAS Ratio dominates the other for the overall period with a final gain of more than 2.1 times the initial value.

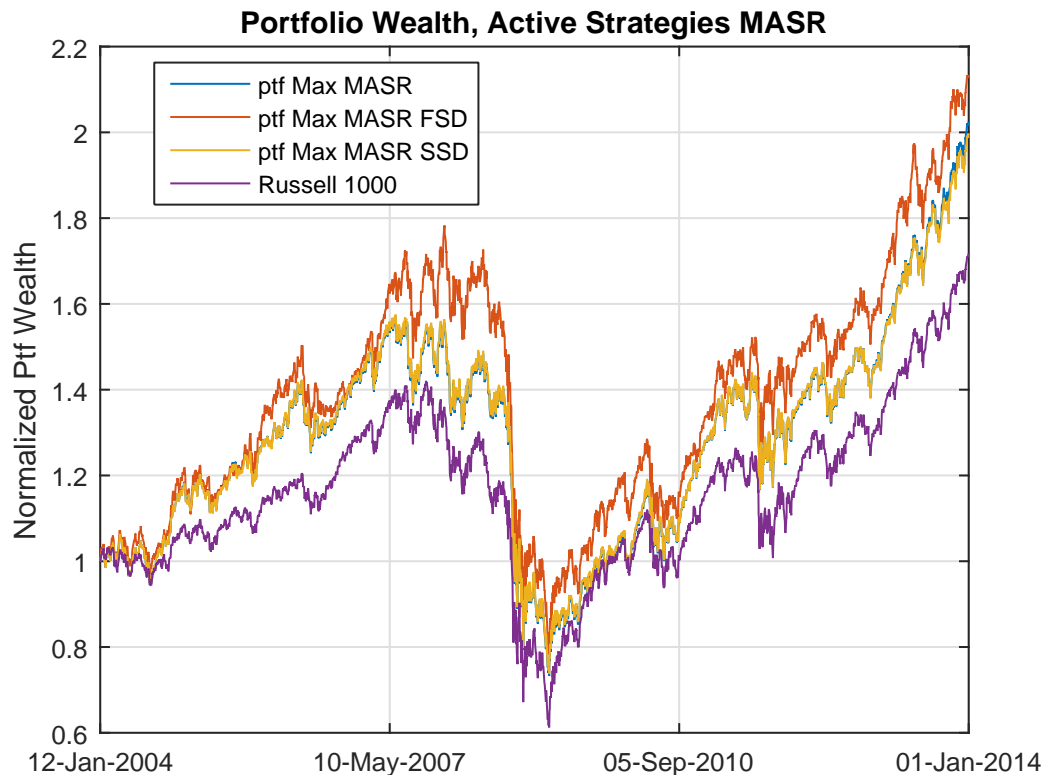


FIGURE 4.3: Portfolio Wealth of Active Strategies Mean Absolute Semideviation Ratio, Russell 1000

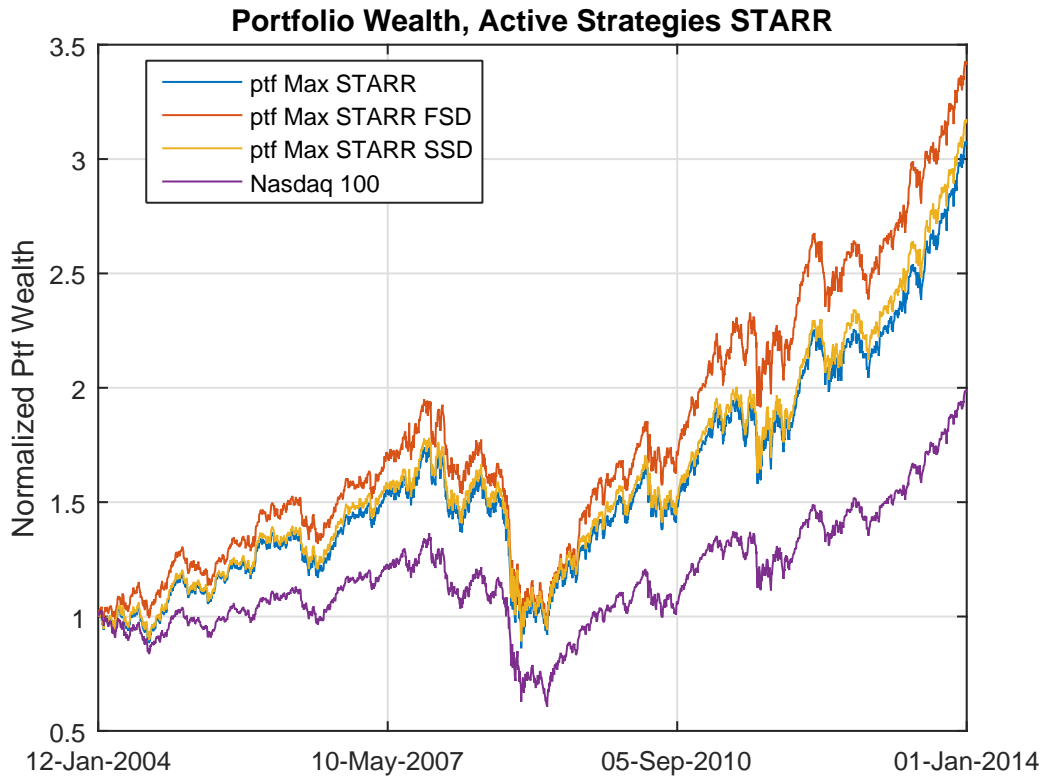


FIGURE 4.4: Portfolio Wealth of Active Strategies STARR, Nasdaq 100

Finally, Figures 4.4 and 4.5 show the maximization of the STARR and the MAS Ratio with the Nasdaq 100 as benchmark. In particular, we notice a strong increment in the portfolio gains during the overall period applying the first order stochastic dominance constraints. In fact, maximizing the STARR we notice that the simple maximization of the performance ratio and the introduction of second order stochastic dominance constraints show the same wealth path with the second one which slightly dominates the first one. They reach final values more than 3 times the initial wealth while the strategy with include first order stochastic dominance constraints dominated the other for the overall period and it show a final gains of about 53% more than the Nasdaq stock index.

Also Figure 4.5 reports the same feature with a higher final values for every active

strategy. In this case, the portfolio which maximize the Mean Absolute Semideviation Ratio with first order stochastic dominance doubles its wealth during the first three years of the investment period and it takes relevant advantages during several upward changes of the financial cycle.

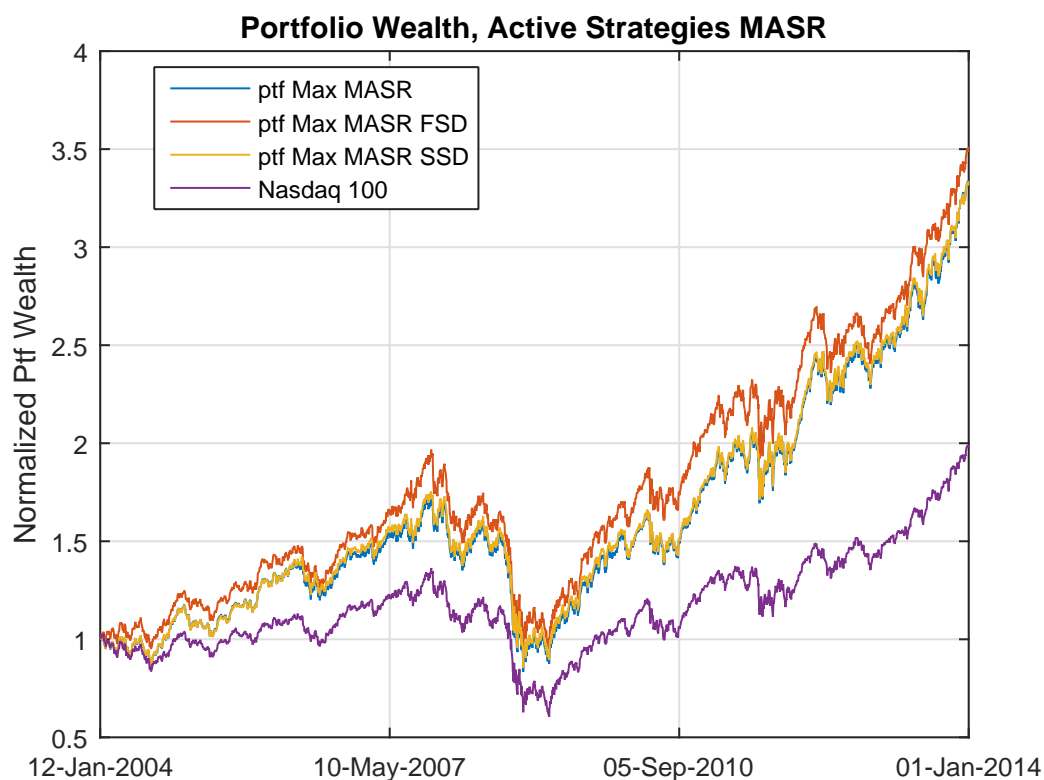


FIGURE 4.5: Portfolio Wealth of Active Strategies Mean Absolute Semideviation Ratio, Nasdaq 100

## 4.6 Final Remarks

In this chapter, we treat active management portfolio strategy proposing a portfolio selection problem consistent with the maximization of a performance measure and stochastic dominance constraints. Following the linear formulation proposed by [Stoyanov et al.](#)

(2007), we implement LP portfolio selection problem adding stochastic dominance constraints. Finally, we empirically test this approach and we could notice how there is an increment in the portfolio wealth not only with respect to the index tracking and enhanced indexation strategies but also adding different orders of stochastic dominance in the active strategy management problem. Future researches will focus on the introduction of different measures such as the Sharpe Ratio and the Mean Absolute Semideviation Ratio in a linear formulation problem. Then, a comparison with the portfolio selection problem proposed in this essay could completely cover the area of maximization of performance measures. Moreover, the investigation of linear third order stochastic dominance constraints will also lead to find another important specification in the optimization problem.

## Chapter 5

# Conclusion and Future Research.

### 5.1 Conclusion

In this essay, we describe the three main areas of the benchmarking problem. This kind of problem is related to the construction of an invested portfolio which compares its performance with a given index. Considering the assets that compose the stock index, we develop index tracking, enhanced indexation and active strategies. The aim of this work is to propose theoretical and methodological approaches to cover different portfolio managers goals. In particular, we grounded our analysis on the definition of linear portfolio selection models with the introduction of different stochastic dominance constraints in the decisional problem and evaluating their benefits in terms of risk reduction and increasing gains.

After a review of the literature, in [Chapter 2](#), we develop a new measure for the index tracking based on the quantile regression which aims to mimic the performance of a benchmark in several phases of the financial cycle considering medium and big tracked portfolios. Then, we propose a realistic model introducing a penalty function with transaction costs and turnover constraints to limit the changes in portfolio active



assets. In this model, we add stochastic dominance constraints to enhance the performance of invested portfolios obtaining strategies that minimize linear and asymmetric dispersion measure, as the tracking error quantile regression, and outperform the benchmark. Empirical applications dictate the benefits of this approach in a reduction of the active assets and portfolio turnover. Moreover, stochastic dominance allows to obtain attractive portfolio returns controlling the risk.

Then, [Chapter 3](#) presents a generalization of the functional measure in the benchmark tracking problem reviewing the Gini tail measure and the  $L^p$  metric for this kind of problem. We develop a theoretical and methodological formulation to take advantages from different orders of stochastic dominance introducing investment chain to increase the portfolio wealth or to improve the risk premium. In this framework, we linearize the portfolio problem to maximize utility functions and we introduce different types of stochastic dominance. Considering three levels to improve the portfolio construction process, we create investment strategies focusing on the behavior of non satiable risk averse investor with positive skewness. Future research will focus on an extension of this concept analyzing other possible development of the stochastic investment chain through the introduction of other order of stochastic dominance and utility functions.

Finally in [Chapter 4](#), we address with problem to actively manage the invested portfolio to outperform the benchmark. Facing high dimensionality problem and evaluating the impact of the introduction of stochastic dominance constraints, we develop linear portfolio selection models that maximize some performance measures which are consistent with different investor's profiles. Empirical applications show the out of sample wealth paths of these strategies and they highlight the importance of stochastic dominance in the decisional problem to obtain portfolios with a strong behavior capable to strongly produce consistent and permanent gains with respect to the benchmark.

## Appendix A

# Econometrics Model to Generate Future Scenarios of the Asset Returns

The portfolio selection problem could be solved considering different decisional sets of variable. In this essay, we consider the historical observation but several methodologies based on the forecast of future returns are proposed in the financial literature [Ortobelli Lozza et al. \(2011\)](#); [Biglova et al. \(2009\)](#); [Andersen et al. \(2001\)](#); [Black and Litterman \(1992\)](#); [Breen et al. \(1989\)](#). In this Appendix, we describe the factor model used to reduce complexity of the problem searching for an acceptable model to describe the dependence structure. Thus, we perform a principal components analysis (PCA) to identify the main portfolio factors whose variance is significantly different from zero ([Biglova et al., 2009](#)). With this approach, we obtain the few components that explain the majority of the return volatility, resulting in a reduction of the dependence structure dimension. This step allow to shrink the complexity of the problem approximating the historical return and reducing the dimensionality in problems with numerous index components.

To simulate realistic future return scenarios, we distinguish between the approximation of PCA-residuals and PCA-factors. The sample residuals obtained from the factor model are well approximated with an ARMA(1,1)-GARCH(1,1) model with different distributional hypotheses on the innovations while we independently simulate the factors with the same econometric process but we model their dependencies structure with an asymmetric Student t-copula (Biglova et al., 2009; Sun et al., 2008) with stable marginal distributions. This approach allows to consider the stylized facts observed in financial markets such as clustering of the volatility effect, heavy tails, and skewness and investigate the nature of the PCA-residuals during different phases on the financial cycle.

Several issues need to be addressed in order to model, control and forecast portfolios in equity markets. Firstly, the reduction of the dimensionality of the problem gets robust estimations in a multivariate framework modeling the dependence structure of the returns with a copula approach. Secondly, the proposed modelization allows to consider the main features of the stock returns in the scenario generation: heavy-tailed distributions, volatility clustering, and non-Gaussian copula dependence.

### A.0.1 Regression Model and Dimensionality Reduction

One methodology to reduce the dimensionality of the problem is to approximate the return series with a regression-type model (such as a  $k$ -fund separation model) that depends on an adequate number of parameters (Ross, 1978). Thus, we compute a principal component analysis (PCA) of the returns of the  $N$  stock index components in order to identify few factors (portfolios) with the highest variability. Replacing the original  $N$  correlated time series  $r_n$  with  $N$  uncorrelated time series  $P_n$ , we assume that each  $r_n$  is a linear combination of the  $P_n$ . Then we implement a dimensionality reduction by choosing only those portfolios whose variance is significantly different from zero. In particular, we call portfolios factors  $f_n$  the  $p$  portfolios  $P_n$  with a significant variance, while the remaining  $N - p$  portfolios with very small variances are summarized by an

error  $\epsilon$ . In conclusion, we obtain that each series  $r_n$  is a linear combination of the factors plus a small uncorrelated noise:

$$r_n = \sum_{n=1}^p c_n f_n + \sum_{n=p+1}^N d_n P_n = \sum_{n=1}^p c_n f_n + \epsilon \quad (\text{A.1})$$

Generally, we can apply the PCA either to the variance-covariance matrix or to the correlation matrix. Since returns are heavy-tailed dimensionless quantities, we apply PCA to the correlation matrix obtaining  $N$  principal components, which are linear combinations of the original series,  $r = (r_1, \dots, r_N)$ . At each time of our approach, we select the first  $p$  component analyzing their global portfolio variance. This approach wants to capture and take advantage from the increasing correlation during period financial distress (Chiang et al., 2007; Dungey and Martin, 2007; Veldkamp, 2006; Hartmann et al., 2004; Loretan and English, 2000) when the residuals show particular feature and a dynamic approach to describe and capture their distribution could improve the estimation process.

As a consequence of this principal component analysis, each series  $r = (r_1, \dots, r_N)$  can be represented as a linear combination of  $p$  factors plus a small uncorrelated noise. Once we have identified the factors, we can generate the future returns  $r_n$  using the factor model:

$$r_{t,n} = \alpha_n + \sum_{j=1}^p \beta_{j,n} f_{t,j} + \epsilon_{t,n} \quad (\text{A.2})$$

where  $n = 1, \dots, N$  is the  $n$ -th components of the stock index and  $t$  the historical observations. Formula A.2 shows how to approximate the time series through a factor model with  $p$  factors. This model will be the base also to compute the future simulated assets' returns.

In fact, the generation of future scenarios should consider three main feature: the empirical evidence observed in equity returns; the time evolution of factor  $f_{t,j}$  and of

errors  $\epsilon_{t,n}$  and the comovements of the vector of the returns considering the skewness and kurtosis of the joint distribution. In particular, this last feature is solved introducing a skewed copula with heavy tails. A copula function  $C$  associated to random vector  $u = (u_1, \dots, u_N)$  is a probability distribution function on the  $n$ -dimensional hypercube, such that:

$$\begin{aligned} F_u(y_1, \dots, y_N) &= \mathbb{P}(u_1 \leq y_1, \dots, u_N \leq y_N) = \\ &C(\mathbb{P}(u_1 \leq y_1), \dots, \mathbb{P}(u_N \leq y_N)) = \\ &= C(F_{u_1}(y_1), \dots, F_{u_N}(y_N)) \end{aligned} \tag{A.3}$$

where  $F_{u_n}$  is the marginal distribution of the  $n$ -th component (Sklar, 1959). Once we have generated scenarios with the copula  $C(u_1, \dots, u_N) = F_u(F_{u_1}^{-1}(u_1), \dots, F_{u_N}^{-1}(u_N))$  (where  $F_{u_n}^{-1}$  is the inverse cumulative function of the  $n$ -th marginal derived from the multivariate distributional assumption  $F_u$ ) that summarizes the dependence structure of returns, then we can easily generate joint observations using the most opportune inverse distribution functions  $F_{u_n}^{-1}$  of the single components applied to the points generated by the copula. In particular, we consider a multivariate skewed Student's  $t$ -copula for the joint generation of innovations of the  $p$  factors.

In the following part of this section, we summarize the algorithm proposed to generate future return scenarios according to Biglova et al. (2009). Assuming that the log-returns follow the factor model A.2, we firstly approximate each factor  $f_{t,j}$  with an ARMA(1,1)-GARCH(1,1) process with stable Paretian innovations. Then, we provide the marginal distributions for standardized innovations of each factor used to simulate the next-period returns. Secondly, we estimate the dependence structure of the vector of standardized innovations with a skewed Student- $t$  copula with stable marginal distributions.

Thirdly, combining the marginal distributions and the scenarios for the copula into scenarios for the vector of factors, we generate the vector of the standardized innovation assuming that the marginal distributions are  $\alpha_j$ -stable distributions and considering

an asymmetric  $t$ -copula to summarize the dependence structure. Finally, we obtain the vector of factors and combining the simulated factor with the simulated residuals of the model A.2, we generate future returns. The algorithm is as follows. Firstly, we compute the maximum likelihood parameter estimation of ARMA(1,1)-GARCH(1,1) for each factor  $f_{t,j}$  ( $j = 1, \dots, p$ ).

$$f_{t,j} = a_{0,j} + a_{1,j}f_{t-1,j} + b_{1,j}\epsilon_{t-1,j} + \epsilon_{t,j} \quad (\text{A.4})$$

$$\epsilon_{t,j} = \sigma_{t,j}u_{t,j} \quad (\text{A.5})$$

$$\sigma_{t,j}^2 = c_{0,j} + c_{1,j}\sigma_{t-1,j}^2 + d_{1,j}\epsilon_{t-1,j}^2 \quad (\text{A.6})$$

for  $j = 1, \dots, p$  and  $t = 1, \dots, T$  number of historical moving window observations.

Approximate with  $\alpha_j$ -stable distribution  $S_{\alpha_j}(\sigma_j, \beta_j, \mu_j)$  (Samoradnitsky and Taqqu, 1994; Rachev and Mittnik, 2000) (see Appendix B) the empirical standardized innovations  $\hat{u}_{t,j} = \hat{\epsilon}_{t,j}/\sigma_{t,j}$  where the innovations  $\hat{\epsilon}_{t,j} = f : t, j - a_{0,j} - a_{1,j}f_{t-1,j} - b_{1,j}\epsilon_{t-1,j}$  for  $j = 1, \dots, p$ . In order to value the marginal distribution of each innovation, we first simulate  $S$  stable distributed scenarios for each of the future standardized innovations series. Then, we compute the sample distribution functions of these simulated series:

$$F_{\hat{u}_{T+h,j}}(x) = \frac{1}{S} \sum_{s=1}^S I_{[\hat{u}_{T+h,j}^s \leq x]} \quad (\text{A.7})$$

$$x \in \mathbb{R}, j = 1, \dots, p$$

where  $\hat{u}_{T+h,j}$  ( $1 \leq s \leq S$ ) is the  $s$ -th value simulated with the fitted  $\alpha_j$ -stable distribution for future standardized innovation (valued in  $T + h$ ) of the  $j$ -th factor and  $h$  is the investment horizon when the investor recalibrate its portfolio.

Secondly, fitting the  $p$ -dimensional vector of empirical standardized innovations  $\hat{u} = [\hat{u}_1, \dots, \hat{u}_p]'$  with an asymmetric  $t$ -distribution  $V = [V_1, \dots, V_p]$  with  $v$  degree of freedom; i.e.,

$$\mathbf{V} = \boldsymbol{\mu} + \gamma Y + \sqrt{Y} \mathbf{Z} \quad (\text{A.8})$$

where  $\mu$  and  $\gamma$  are constant vectors and  $Y$  is inverse  $\gamma$ -distributed  $IG(v/2; v/2)$  (Rachev and Mittnik, 2000) independent of the vector  $\mathbf{Z}$  that is normally distributed with zero mean and covariance matrix  $\Sigma = [\sigma_{i,j}]$ . We use the maximum likelihood method to estimate the parameters  $(v, \hat{\mu}_i, \hat{\sigma}_{i,i}, \hat{\gamma}_i)$  of each component. Then, an estimator of matrix  $\Sigma$  is given by

$$\hat{\Sigma} = \left( \text{cov}(\mathbf{V}) - \frac{2v^2}{(v-2)^2(v-4)} \hat{\gamma} \hat{\gamma}' \right) \frac{v-2}{2} \quad (\text{A.9})$$

where  $\hat{\gamma} = (\gamma_1, \dots, \gamma_p)$  and  $\text{cov}(\mathbf{V})$  is the variance-covariance matrix of  $\mathbf{V}$ . Since we have estimated all the parameters of  $Y$  and  $\mathbf{Z}$ , we can generate  $S$  scenarios for  $Y$  and, independently,  $S$  scenarios for  $\mathbf{Z}$ , and using Equation A.8 we obtain  $S$  scenarios for the vector of standardized innovations  $\hat{u} = [\hat{u}_1, \dots, \hat{u}_p]$  that is asymmetric  $t$ -distributed.

Denote these scenarios by  $(V_1^{(s)}, \dots, V_p^{(s)})$  for  $s = 1, \dots, S$  and denote the marginal distributions  $F_{V_j}(x)$  for  $1 \leq j \leq p$  of the estimated  $p$ -dimensional asymmetric  $t$ -distribution by  $F_V(x_1, \dots, x_p) = \mathbb{P}(V_1 \leq x_1, \dots, V_p \leq x_p)$ . Then, considering  $U_j^{(s)} = F_{V_j}(V_j(s))$ ,  $1 \leq j \leq p$ ;  $1 \leq s \leq S$ , we can generate  $S$  scenarios  $(U_1(s), \dots, U_p(s))$ ,  $s = 1, \dots, S$  of the uniform random vector  $(U_1, \dots, U_p)$  (with support on the  $p$ -dimensional unit cube) and whose distribution is given by the copula:

$$\begin{aligned} C(t_1, \dots, t_p) &= F_V(F_{V_1}^{-1}(t_1), \dots, F_{V_p}^{-1}(t_p)) \\ 0 &\leq t_i \leq 1; \\ 1 &\leq i \leq p \end{aligned} \quad (\text{A.10})$$

Considering the stable distributed marginal sample distribution function of the  $j$ -th standardized innovation  $F_{\hat{u}_j, T+h}$ ;  $j = 1, \dots, p$  (see Equation A.8) and the scenarios  $U_j^{(s)}$  for  $1 \leq j \leq p$ ;  $1 \leq s \leq S$ , then we can generate  $S$  scenarios of the vector of standardized innovations (taking into account the dependence structure of the vector)

$u_{T+h}^{(s)} = \left( u_{T+h}^{(1,s)}, \dots, u_{T+h}^{(p,s)} \right)$ ,  $s = 1 \dots, S$  valued at time  $T + h$  assuming:

$$\begin{aligned} u_{T+h}^{(j,s)} &= \left( F_{u_{T+h,j}} \right)^{-1} \\ 1 &\leq j \leq p \\ 1 &\leq s \leq S \end{aligned} \tag{A.11}$$

Once we have described the multivariate behavior of the standardized innovation at time  $T + h$  using relation A.6, we can generate  $S$  scenarios of the vector of innovation:

$$\epsilon_{T+h}^{(s)} = \left( \epsilon_{T+h}^{(1,s)}, \dots, \epsilon_{T+h}^{(p,s)} \right) = \left( \sigma_{T+h,1} u_{T+h}^{(1,s)}, \dots, \sigma_{T+h,p} u_{T+h}^{(p,s)} \right) \tag{A.12}$$

where  $\sigma_{T+h,j}$ , are still defined by Equation A.6. Thus, using relation A.6, we can generate  $S$  scenarios of the vector of factors valued at time  $T + h$ .

Finally, we estimate a model ARMA(1,1)-GARCH(1,1) for the residuals of the factor model A.2. That is, we consider the empirical residuals:

$$\hat{\epsilon}_{t,n} = r_{t,n} - \hat{\alpha}_n - \sum_{j=1}^p \hat{\beta}_{j,n} f_{t,j} \tag{A.13}$$

We assume that also the residuals  $\hat{\epsilon}_{t,n}$  follow an ARMA(1,1)-GARCH(1,1) model and then we estimate its parameters  $g_{0,n}, g_{1,n}, h_{1,n}, k_{0,n}, k_{1,n}, p_{1,n}$  for all  $n = 1, \dots, N$ :

$$\begin{aligned} \hat{\epsilon}_{t,n} &= g_{0,n} + g_{1,n} \hat{\epsilon}_{t-1,n} + h_{1,n} q_{t-1,n} + q_{t,n} \\ q_{t,n} &= v_{t,n} z_{t,n} \\ v_{t,n}^2 &= k_{0,n} + k_{1,n} v_{t-1,n}^2 + p_{1,n} q_{t-1,n}^2 \end{aligned} \tag{A.14}$$

for  $n = 1, \dots, N$  and  $t = 1, \dots, T$ .



### A.0.2 Dynamic Innovation Hypotheses. Distributions and Statistical Tests

To assess the goodness of fit, we consider a classical statistic. It might be of interest to test the ability to model extreme events and to test which of three different hypotheses could better capture the distribution of the innovation  $z_{i,t}$ . To this end, we introduce the Anderson-Darling statistic (AD-statistic) (Razali and Wah, 2011; Anderson and Darling, 1954; Scholz and Stephens, 1987). The Anderson-Darling test is commonly used to test whether a data sample comes from a given distribution. The test statistic belongs to the family of quadratic empirical distribution function statistics, which measure the distance between the hypothesized distribution,  $F(x)$  and the empirical c.d.f.,  $F_n(x)$  as

$$n \int_{-\infty}^{\infty} (F_n(x) - F(x))^2 g(x) dF(x) \quad (\text{A.15})$$

over the ordered sample values  $x_1 < x_2 < \dots < x_n$  where  $g(x)$  is a weight function and  $n$  is the number of data points in the sample. The weight function for the Anderson-Darling test is

$$g(x) = [F(x)(1 - F(x))]^{-1} \quad (\text{A.16})$$

which places greater weight on the observations in the tails of the distribution. The Anderson-Darling test statistic is:

$$A_n^2 = -n - \sum_{i=1}^n \frac{2i-1}{n} [\ln(F(X_i)) + \ln(1 - F(X_{n+1-i}))] \quad (\text{A.17})$$

where  $\{X_1 < \dots < X_n\}$  are the ordered sample data points and  $n$  is the number of data points in the sample.

In this work, we consider three hypotheses to capture the distributional behavior of the innovations (see Appendix B for a review of these distributions):

1. Gaussian or Normal distribution.

2. Student-t distribution (Blattberg and Gonedes, 1974).
3. Alpha Stable distribution (Samorodnitsky and Taqqu, 1994 and Rachev and Mitnik, 2000).

Thus, we test using the A-D test the three different distributional hypothesis and we select the minimum level of the statistic to simulate future scenarios.

### A.0.3 Scenario Generation Process

Moreover, as for the factor innovation, we approximate with  $\alpha_j$ -stable distribution  $S_{\alpha_j}(\sigma_n, \beta_n, \mu_n)$  for any  $n = 1, \dots, N$  the empirical standardized innovations  $z_{t,n}^{\hat{}} = \hat{q}_{t,n}/v_{t,n}$ , where the innovations  $\hat{q}_{t,n} = \hat{e}_{t,n} - g_{0,n} - g_{1,n}\hat{e}_{t-1,n} - h_{1,n}q_{t-1,n}$ . Then, we can generate  $S$  scenarios  $\alpha_j$ -stable distributed for the standardized innovations  $z_{T+1,n}^{(s)}$ ,  $s = 1, \dots, S$  and from Equation A.15 we get  $S$  possible scenarios for the residuals  $e_{T+h,n}^{(s)} = v_{T+h,n}z_{T+h,n}^{(s)}$ ,  $s = 1, \dots, S$ . Therefore, combining the simulation of the factor with the simulation of the residuals we get  $S$  possible scenarios of returns:

$$r_{T+h,n}^{(s)} = \hat{\alpha}_n + \sum_{j=1}^p \hat{\beta}_{j,n} f_{t,n}^{(s)} + \epsilon_{t,n}^{(s)} \quad (\text{A.18})$$

The procedure illustrated here permits one to generate  $S$  scenarios at time  $T + h$  of the vector of returns.

## Appendix B

# Gaussian, Student-t and Alpha Stable Distributions

In this Appendix, we review the three main principal distributional assumption considered in this work.

### Gaussian Distribution

The class of normal distributions, or Gaussian distributions, is certainly one of the most important probability distributions in statistics and, due to some of its appealing properties, also the class that is used in most applications in finance. Here we introduce some of its basic properties.

The random variable  $X$  is said to be normally distributed with parameters  $\mu$  and  $\sigma$ , abbreviated by  $X \sim N(\mu, \sigma)$ , if the density function of the random variable is given by the formula:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\} \quad (\text{B.1})$$

A normal distribution with  $\mu = 0$  and  $\sigma = 1$  is called a standard normal distribution. Notice the following characteristics of the normal distribution. First, the middle of the distribution equals  $\mu$ . Second, the distribution is symmetric around  $\mu$ . This second characteristic justifies the name location parameter for  $\mu$ . For small values of  $\sigma$ , the density function becomes more narrow and peaked whereas for larger values of  $\sigma$  the shape of the density widens. These observations lead to the name shape parameter or scale parameter for  $\sigma$ .

### **Student-t Distribution**

The Student-t distribution has become the mainstream alternative of the normal distribution, when attempting to address asset returns' heavy-tailedness. It is a symmetric and mound-shaped, like the normal distribution. However, it is more peaked around the center and has fatter tails (Blattberg and Gonedes, 1974). This makes it better suited for return modeling than the Gaussian distribution (Theodossiou, 1998 and Andersen et al., 2001).

Additionally, numerical methods for the t-distribution are widely available and easy to implement (McNeil and Frey, 2000). The t-distribution has a single parameter, called degrees of freedom (DOF), that controls the heaviness of the tails and, therefore, the likelihood for extreme returns. The DOF takes only positive values, with lower values signifying heavier tails. Values less than 2 imply infinite variance, while values less than 1 imply infinite mean since given  $\nu$  the DOF, every moments lower than  $\nu$  do not exist. The t-distribution becomes arbitrarily close to the normal distribution as DOF increases above 30. Generally, a random variable  $X$  (taking any real value) distributed with the Student-t distribution with  $\nu$  degrees of freedom has a density function given by

$$f(x|\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2} \quad (\text{B.2})$$

where  $\Gamma$  is the Gamma function. Usually, this distribution is denoted by  $t_\nu$ . The mean of  $X$  is zero and its variance is given by

$$\text{var}(X) = \frac{\nu}{\nu - 2} \quad (\text{B.3})$$

In financial applications, it is often necessary to define the Student's t-distribution in a more general manner so that we allow for the mean (location) and scale to be different from zero and one, respectively (Rachev et al., 2005 and Aas and Haff, 2006). The density function of such a "scaled" Student's t-distribution is described by

$$f(x|\nu, \mu, \sigma) = \frac{\Gamma(\frac{\nu+1}{2})}{\sigma\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}} \left(1 + \frac{1}{\nu} \left(\frac{x-\mu}{\sigma}\right)^2\right)^{-(\nu+1)/2} \quad (\text{B.4})$$

where the mean  $\mu \in \mathbb{R}$  and the standard deviation  $\sigma > 0$ . Finally, we make a note of an equivalent representation of the Student's t-distribution which is useful for obtaining simulations from it (Bigliova et al., 2009). The  $t_\nu(\mu, \sigma)$  distribution is equivalently expressed as a scale mixture of the normal distribution where the mixing variable distributed with the inverse-gamma distribution,

$$X \sim N(\mu, \sqrt{W}\sigma) \quad (\text{B.5})$$

$$W \sim \text{Inv} - \text{Gamma} \left(\frac{\nu}{2}, \frac{\nu}{2}\right)$$

### Alpha Stable Distribution

Research on stable distributions in the field of finance has a long history (Samorodnitsky and Taqqu, 1994 and Rachev and Mittnik, 2000). In 1963, the mathematician Benoit Mandelbrot first used the stable distribution to model empirical distributions that have skewness and fat tails. The practical implementation of stable distributions to risk modeling, however, has only recently been developed. Reasons for the late penetration are the complexity of the associated algorithms for fitting and simulating stable models, as well as the multivariate extensions.

To distinguish between Gaussian and non-Gaussian stable distributions, the latter are commonly referred to as stable Paretian, Levy stable, or  $\alpha$ -stable distributions. Stable Paretian tails decay more slowly than the tails of the normal distribution and therefore better describe the extreme events present in the data (Rachev et al., 2011). Like the Student's t-distribution, stable Paretian distributions have a parameter responsible for the tail behavior, called tail index or index of stability.

It is possible to define the stable Paretian distribution in two ways. The first one establishes the stable distribution as having a domain of attraction. That is, (properly normalized) sums of *IID* random variables are distributed with the  $\alpha$ -stable distribution as the number of summands  $n$  goes to infinity. Formally, let  $Y_1, Y_2, \dots, Y_n$  be *IID* random variables and  $a_n$  and  $b_n$  be sequences of real and positive numbers, respectively. A variable  $X$  is said to have the stable Paretian distribution if

$$\frac{\sum_{i=1}^n Y_i - a_n}{b_n} \xrightarrow{d} X \quad (\text{B.6})$$

The density function of the stable Paretian distribution is not available in a closed-form expression in the general case. Therefore, the distribution of a stable random variable  $X$  is alternatively defined through its characteristic function. The density function can be obtained through a numerical method. The characteristic function of the  $\alpha$ -stable distribution is given by

$$\varphi_X(t|\alpha, \sigma, \beta, \mu) = \mathbb{E} [e^{itX}] = \begin{cases} \exp(i\mu t - |\sigma t|^\alpha (1 - i\beta(\text{sign } t) \tan \frac{\pi\alpha}{2})), & \alpha \neq 1 \\ \exp(i\mu t - \sigma |t| (1 - i\beta \frac{2}{\pi}(\text{sign } t) \ln |t|)), & \alpha = 1 \end{cases} \quad (\text{B.7})$$

where

$$\text{sign } t = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases} \quad (\text{B.8})$$

The distribution is characterized by four parameters:

- $\alpha \in (0, 2)$ : the index of stability or the shape parameter.
- $\beta \in [-1, +1]$ : the skewness parameter.
- $\sigma \in (0, +\infty)$ : the scale parameter.
- $\mu \in (-\infty, +\infty)$ : the location parameter.

Because of the four parameters, the  $\alpha$ -stable distribution is highly flexible and suitable for modeling non-symmetric, highly kurtotic, and heavy-tailed data. When a random variable  $X$  follows the  $\alpha$ -stable distribution characterized by those parameters, then we denote  $X \sim S_\alpha(\sigma, \beta, \mu)$ . The three special cases where there is a closed-form solution for the densities are the Gaussian case ( $\alpha = 2$ ), the Cauchy case ( $\alpha = 1, \beta = 0$ ) and the Lévy case ( $\alpha = \frac{1}{2}, \beta = \pm 1$ ).

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