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# Benoît Mandelbrot and Fractional Brownian Motion[1]

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*Abstract.* Although fractional Brownian motion was not invented by Benoît Mandelbrot, it was he who recognized the importance of this random process and gave it the name by which it is known today. This is a personal account of the history behind fractional Brownian motion and some subsequent developments.

*Key words and phrases:* Long-range dependence, long memory, self-similarity, Hurst statistic, Benoît Mandelbrot.

Since Benoît Mandelbrot's passing in October 2010, many well-deserved tributes have been paid to him.[2] Benoît influenced a great many fields ranging from the physical sciences to economics, and mathematics was certainly among them. Benoît's great gift was his ability to recognize the hidden potential in certain mathematical objects.[3]

I had the good fortune to observe Benoît's mathematical analysis in action, and I would like to tell you about my experience with one of the objects that Benoît worked with, the random process known as a fractional Brownian motion. Although fractional Brownian motion was introduced by Kolmogorov, it was Benoît Mandelbrot who recognized the relevance of this random process and, in his seminal paper with Van Ness [18], derived many important properties. There, he gave this process the name by which it is known today. See [15] for a general review.

Let me recount first how I met Benoît. At the beginning of the seventies, I was a graduate student at Columbia University in the Department of Mathematical Statistics—a small department but home to prominent faculty such as Herbert Robbins, David Siegmund and Yuan Shih Chow. Although I had a fellowship during the academic year, I needed to find summer work—something I failed to do in my first year. I had sent my Curriculum Vitae to many companies in New York City, but I did not receive a single reply.

For my second year, I decided to proceed differently. I asked members of the Department for contacts. This is how I was put in touch with Benoît Mandelbrot, who was then at IBM Research—an hour's drive from New York City—but was also nominally an Adjunct Professor in the Department. In January of my second year, I called him and inquired about potential summer jobs. The conversation began in English but quickly turned to French. I had expected it to last a few minutes, but the conversation lasted an hour with Benoît doing most of the talking (as was often the case). He ended the conversation, saying that he knew of no jobs. But a few months later, he called me back. As things developed, it turned out that he needed a programmer for the summer and asked if I was interested. I accepted. This is how I became acquainted with his research at the time, which involved fractional Brownian motion and its application to hydrology, and how I ended up as Mandelbrot's student.

It started with the so-called "$R/S$ statistic," where $R$ is the range of partial sums of the data, and $S$ is the sample standard deviation. It is a statistic that the British hydrologist Harold Edwin Hurst, in the first half of the twentieth century, had used to study the yearly variation of the levels of Nile river in Egypt [8]. The original work on the subject by Benoît Mandelbrot appeared in 1965 in the Comptes Rendus [12].

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[1]A French version of this article will appear in France in the *Gazette des Mathématiciens*.

[2]There was a special symposium at the École Polytechnique in Paris in March 2011, one at Yale in April 2011 and a number of sessions related to Mandelbrot's work took place at the annual meeting of the American Mathematical Society in Boston in January 2012.

[3]Benoît Mandelbrot studied with Paul Lévy, who is widely acknowledged for his mastery of the Brownian world.

Under the usual assumptions of finite variance and independent and identically distributed observations, the $R/S$ statistic should grow like $n^{1/2}$, where $n$ is the sample size. The Nile data, however, indicated a growth of $n^H$, where $1/2 < H < 1$. The growth $n^{1/2}$ is typically associated with random walk, so $n^H$, with $1/2 < H < 1$ must correspond to something else. This is why Mandelbrot suspected that a process like fractional Brownian motion $B_H(t)$ may perhaps be relevant in this framework[4] since, while the standard deviation of Brownian motion at time $t$ is $t^{1/2}$, that of fractional Brownian motion at time $t$ is $t^H$, where $0 < H < 1$ [19]. The letter $H$, which refers to the hydrologist Hurst and which was used by Mandelbrot, has become standard in this context, and it now labels the fractional Brownian motion.

The term "fractional Brownian motion" was coined by Mandelbrot and Van Ness in the now classical paper [18]. Fractional Brownian motion has a number of nice properties, one of which is "self-similarity." A process $\{X(t), t \in \mathbb{R}\}$ is self-similar with index $H > 0$ if for any $a > 0$, the process $\{X(at), t \in \mathbb{R}\}$ has the same finite-dimensional distributions as $\{a^H X(t), t \in \mathbb{R}\}$. Thus, like a fractal, there is scaling, but it is not the trajectories of the process that scale, but the probability distribution, the "odds." This is why this type of scaling is sometimes called "statistical self-similarity" or, more precisely, "statistical self-affinity."

The fractional Brownian motion process is then characterized by the following three properties:

(1) the process is Gaussian with zero mean;
(2) it has stationary increments;
(3) it is self-similar with index $H$, $0 < H < 1$.

Fractional Brownian motion reduces to Brownian motion when $H = 1/2$, but in contrast to Brownian motion, it has dependent increments when $H \neq 1/2$. Fractional Brownian motion was first introduced in 1940 by Andrei Nikolaevich Kolmogorov [10], who was studying spiral curves in Hilbert space. It was considered by Richard Allen Hunt [7] in the context of random Fourier transforms and by Akiva Moiseevich Yaglom [34], who studied the correlation structure of processes that have stationary $n$th order increments. However, it is undoubtedly the seminal paper of Mandelbrot and Van Ness which put the focus on fractional Brownian motion and gave it its name. Why the term "fractional?" This is because the process can be represented as an integral with respect to Brownian motion $B(t)$, as follows:

$$B_H(t) = \int_{-\infty}^{0} \{(t-s)^{H-1/2} - (-s)^{H-1/2}\}\, dB(s) + \int_0^t (t-s)^{H-1/2}\, dB(s) \tag{1}$$

$$= \int_{-\infty}^{\infty} \{(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\}\, dB(s). \tag{2}$$

The integrals are well defined because the integrands are square integrable with respect to Lebesgue measure. The form of the integrands is also reminiscent of the one that appears in the $n$-fold iterated integral formula,

$$\int_0^t dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_2} dt_1 \int_0^{t_1} g(s)\, ds = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} g(s)\, ds$$

and therefore (1) can be regarded as involving "fractional integrals." This, in fact, turns out to be more than a superficial analogy!

The focus on fractional Brownian motion has proved to be extremely fruitful because it has allowed all kind of extensions, some of which were hinted at by Benoît Mandelbrot.

For example, the Gaussian noise "$dB$" in (1) can be replaced by an infinite variance Lévy-stable noise, giving rise to the linear Lévy fractional stable motion, which is an infinite variance self-similar process with stationary, but dependent, increments [28]. The kernel can also be replaced by a random sum of pulses [3]. From a different perspective, the single integral in (1) can be replaced by a multiple integral, so that it becomes an element of the so-called Wiener chaos [24, 30], of the form

$$\int_{\mathbb{R}^k}' g_t(x_1, \ldots, x_k)\, dB(x_1) \cdots dB(x_k), \tag{3}$$

for a suitable kernel $g_t$ and where prime indicates that one does not integrate on the diagonals. More specifically, if one chooses

$$g_t(x_1, \ldots, x_k) = \left\{ \int_0^t \prod_{j=1}^{k} (s - x_j)_+^{H_0 - 3/2}\, ds \right\}, \tag{4}$$

where

$$H_0 = 1 - \frac{1-H}{k} \in \left(1 - \frac{1}{2k}, 1\right), \tag{5}$$

then the resulting process (3) is also self-similar with index $1/2 < H < 1$ and has stationary increments. It

[4] Some hydrologists argue instead that the $n^H$ behavior may be due to nonstationarity. See [9] and [21].

reduces to fractional Brownian motion if $k = 1$ but is non-Gaussian if $k \geq 2$. The marginal distribution for $k = 2$ is studied in [33].

The representation (3) with the kernel $g_t$ in (4) is called a "time representation," but there are also other representations, for example, a "spectral representation" or a "finite interval representation" [26].

One can also try to define stochastic integrals, where the integrator is $dB_H$, even though fractional Brownian motion $B_H$ does not have independent increments. One then needs to define integrals of the type

$$\int_{\mathbb{R}} g(x)\, dB_H(x), \tag{6}$$

first for nonrandom functions $g$ [20, 25], and then for random functions $g$ [2]. One can also consider stochastic differential equations driven by fractional Brownian motion [22].

In a more applied vein, one can focus on the increments

$$X(n) = B_H(n) - B_H(n-1), \quad n \geq 1, \tag{7}$$

which form a stationary time series with covariance

$$r(k) = \mathbb{E}[X(0)X(k)] \sim Ck^{2H-2} \tag{8}$$

as $k \to \infty$. The Fourier transform of the covariance (spectral density),

$$f(\lambda) = \sum_{k=-\infty}^{\infty} r(k)e^{ik\lambda} \tag{9}$$

blows up at the origin if $1/2 < H < 1$ since $f(0) = \sum_{k=-\infty}^{\infty} r(0) = \infty$. This type of dependence is called *long memory*, *long-range dependence* or *strong dependence* [31]. Time series with long-memory are important in modeling, particularly in econometrics. Financial returns, for example, appear uncorrelated, but their squares often display long-memory [1, 4].

Mandelbrot was very interested in finance and in developing suitable models for financial returns [14]. Together with his students Adlai Fisher and Laurent Calvet [16], he introduced a multifractal model of assets returns, $B_H(\theta(t))$, where $B_H$ is fractional Brownian motion, and $\theta(t)$ is an independent multifractal process corresponding to "activity time." A multifractal process has stationary increments and satisfies

$$\mathbb{E}|\theta(t)|^q = c(q)t^{\tau(q)+1}, \quad t \geq 0, \tag{10}$$

where $\tau(q)$ is not necessarily a linear function of $q$. Observe that fractional Brownian motion itself is multifractal, more precisely monofractal, because by self-similarity, $\mathbb{E}|B_H(t)|^q = t^{qH}$, corresponding to the linear function $\tau(q) = qH - 1$. The idea of "subordination," replacing "physical time" by "activity time" can already be found as early as 1967 in his pioneering paper with Howard Taylor, with Brownian motion instead of fractional Brownian motion [17].

Statistics also enters into the picture. Given a time series, how does one check that it displays long-memory? And if it does, how does one estimate $H$? There is quite a large literature on the subject [27]. Many of the available tests are graphical [32] or asymptotic in nature. The asymptotic ones are related to central limit theorems but also to so-called noncentral limit theorems which arise as follows.

Consider a time series $\{X_n, n \in \mathbb{Z}\}$, and let $s_n^2 = \operatorname{Var}(\sum_{k=1}^n X_k)$. What is the limit of the normalized sum

$$\frac{1}{s_n} \sum_{k=1}^{[nt]} X_k, \quad t \geq 0, \tag{11}$$

as $n \to \infty$? It is typically Brownian motion in the case of weak dependence. But if $\{X_n, n \in \mathbb{Z}\}$ is long-range dependent, it could be Brownian motion, fractional Brownian motion or a non-Gaussian process. The study of limit theorems in this context is thus very important, and while there is a large literature about it, there are still many open problems [5, 6, 11, 13, 23, 29, 30].

This story illustrates one of Benoît Mandelbrot's contributions to mathematics. He tended to focus on a concept or mathematical object whose importance was not recognized. He studied and developed it—at times rigorously, at times heuristically—with the frequent consequence that the object of his attention became the basis of major subsequent developments. Because he challenged accepted views, many of his ideas met with initial resistance. Ultimately though, Benoît Mandelbrot's influence on the course of mathematical thinking has been far-reaching. He will be greatly missed.

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