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BEREZIN–TOEPLITZ QUANTIZATION, HYPERKÄHLER MANIFOLDS, AND MULTISYMPLECTIC MANIFOLDS

TATYANA BARRON

Department of Mathematics, University of Western Ontario,
London, Ontario N6A 5B7, Canada
e-mail: tatyana.barron@uwo.ca

and BARAN SERAJELAHI

Department of Mathematics, University of Western Ontario,
London, Ontario N6A 5B7, Canada
e-mail: bserajel@uwo.ca

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Abstract. We suggest a way to quantize, using Berezin–Toeplitz quantization, a compact hyperkähler manifold (equipped with a natural 3-plectic form), or a compact integral Kähler manifold of complex dimension n regarded as a $(2n-1)$-plectic manifold. We show that quantization has reasonable semiclassical properties.

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1. Introduction. (Berezin–)Toeplitz quantization, while interesting to study by itself, also has turned out to be a useful tool in several areas of mathematics. Over the years it was found to have applications to deformation quantization (see e.g. [**30, 43**]), to study of the Hitchin connection and TQFT (work of J. Andersen, see in particular [**1, 2**]), L. Polterovich's work on rigidity of Poisson brackets [**38**], and work of Y. Rubinstein and S. Zelditch [**41**] on homogeneous complex Monge–Ampère equation, in connection to geodesics on the space of Kähler metrics. T. Foth (T. Barron) and A. Uribe applied Berezin–Toeplitz quantization to give another proof of Donaldson's "scalar curvature is a moment map" statement [**24**].

In this paper, we discuss how to use Berezin–Toeplitz quantization to quantize hyperkähler manifolds or two types of multisymplectic manifolds.

Geometric quantization and Kähler/Berezin–Toeplitz quantization associate a Hilbert space (say, $\mathcal{H}$) and operators on it to a symplectic manifold (M, ω). In physics' terminology this is a way to pass from classical Hamiltonian mechanics to a quantum system. Let $C^\infty(M)$ denote the space of complex-valued smooth functions on M. Quantization is a linear map $C^\infty(M) \to \{\text{operators on } \mathcal{H}\}, f \mapsto \hat{f}$, satisfying a version of Dirac's quantization conditions:

$1 \mapsto const(\hbar)I$,

$\{f, g\} \mapsto const(\hbar)[\hat{f}, \hat{g}]$.

It is probably fair to say that geometric quantization was developed and mainstreamed in the 1950s–1960s, by representation theorists, including Kostant, Kirillov and others, whose primary agenda was to look for representations of infinite-dimensional Lie algebras with certain properties, and who found this language to be quite convenient.

Berezin–Toeplitz quantization can be regarded as a version of geometric quantization. In the case when the symplectic manifold is, moreover, Kähler, it is also referred to as Kähler quantization. The groundwork for Berezin–Toeplitz quantization was laid in [**10, 13**]. Well-known Theorem 2.9(i) below shows that in the framework of Berezin–Toeplitz quantization the $\{.,.\} \rightsquigarrow [.,.]$ quantization condition is satisfied in the semiclassical limit $\hbar = \frac{1}{k} \to 0$, which is essentially the best one can get, due to various no-go theorems.

There are physical systems whose behaviour is encoded by an *m-plectic* form on M (i.e. a closed non-degenerate $m+1$-form), Ω, for $m \geq 1$. The case $m = 1$ is when Ω is symplectic. Specific examples from physics, with $m \geq 2$, are discussed in [**6, 17, 37**]. See also discussion and references in [**16**]. Multisymplectic geometry has been thoroughly studied by mathematicians. See, in particular, [**6, 7, 15, 16, 34, 35, 39, 45**]. There has been extensive discussion of *quantization* of n-plectic manifolds in physics literature, and substantial amount of work has been done by mathematicians too. See, for example, [**17, 19–22, 37, 40, 42, 45, 46**]. Work of C. Rogers [**40**] addresses quantization of 2-plectic manifolds. It seems that the appropriate quantum-mechanical setting there involves a category, instead of a vector space, and intuitively this makes sense because an (integral) 2-plectic form corresponds to a gerbe and sections of a gerbe form a category, not a vector space.

There have been attempts, informally speaking, "to embed a multisymplectic physical system into Hamiltonian system" [**9, 21, 36**]. As far as we know, there is no known canonical way of doing this.

DeBellis, Sämann and Szabo [**21**] used Berezin–Toeplitz quantization for multisymplectic spheres via embedding them in a certain explicit way into complex projective spaces $\mathbb{CP}^q$ and using Berezin–Toeplitz quantization on $\mathbb{CP}^q$. This is somewhat related to our results in Section 3, only for $M = S^2$ (because among spheres only S^2 admits a Kähler form).

Let (M, ω) be a compact connected integral Kähler manifold of complex dimension n. In this paper, we are looking into two situations when the m-plectic form Ω on (M, ω) is constructed from the Kähler form (or forms):

(I) $m = 2n-1$, $\Omega = \frac{\omega^n}{n!}$

(II) M is, moreover, hyperkähler, $m = 3$,

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$$

where $\omega_1 = \omega, \omega_2, \omega_3$ are the three Kähler forms on M given by the hyperkähler structure.

It is well known (and easy to prove) that a volume form on an oriented N-dimensional manifold is an $(N-1)$-plectic form, and that the 4-form above is a 3-plectic form on a hyperkähler manifold. See, for example, [**16, 39**].

It is intuitively clear that in these two cases the classical multisymplectic system is essentially built from Hamiltonian system(s) and it should be possible to quantize (M, Ω) using the (Berezin–Toeplitz) quantization of (M, ω). We discuss case (I) in Section 3, case (II) in Section 4. Semiclassical asymptotics are the content of Theorems 3.4, 4.5, 4.7, 4.16, Propositions 3.5, 4.6, 4.12, 4.15, Corollary 4.14. In both cases there are natural multisymplectic analogues of the Poisson bracket and the commutator: an almost Poisson bracket $\{., \ldots, .\}$ and the generalized commutator $[., \ldots, .]$. Our discussion mainly revolves around the $\{., \ldots, .\} \rightsquigarrow [., \ldots, .]$ quantization condition.

The main result of Section 3 is Theorem 3.4. It is an analogue, for brackets of order $2n$, of well-known Theorem 2.9(i) (and of its C^l analogue ($l \in \mathbb{N}$) from [**8**]).

In Section 4, we work on a hyperkähler manifold M. For a smooth function f on M we have three Berezin–Toeplitz operators $T^{(k)}_{f;1}$, $T^{(k)}_{f;2}$, $T^{(k)}_{f;3}$, and to four smooth functions f, g, h, t on M we associate three brackets of order 4: $\{f, g, h, t\}_r$, $r = 1, 2, 3$. In Section 4.1, we show that the direct sum of generalized commutators is asymptotic to

$$T^{(k)}_{\{f,g,h,t\}_1;1} \oplus T^{(k)}_{\{f,g,h,t\}_2;2} \oplus T^{(k)}_{\{f,g,h,t\}_3;3}$$

(Theorem 4.5). In Section 4.2, we show that the attempt to formulate everything on *one* vector space (not three), by taking direct sums, goes through all the way in the case when M is the 4-torus with three linear complex structures, where we get a straightforward analogue of Theorem 2.9(i)—see Example 4.8 (7). In Section 4.3, we take the tensor product of the three operators, instead. Tensor product of generalized commutators is asymptotic to

$$T^{(k)}_{\{f,g,h,t\}_1;1} \otimes T^{(k)}_{\{f,g,h,t\}_2;2} \otimes T^{(k)}_{\{f,g,h,t\}_3;3}$$

(Proposition 4.15). Asymptotic properties of commutators and generalized commutators of operators $\mathbb{T}^{(k)}_f = T^{(k)}_{f;1} \otimes T^{(k)}_{f;2} \otimes T^{(k)}_{f;3}$ are captured in Proposition 4.12 and Theorem 4.16.

We note that while, for simplicity, the exposition throughout the paper is for C^∞ symbols, all our results hold, in fact, for C^4 symbols. To modify the proofs in order to get the same statements for C^4 symbols, the estimates from [**12**] should be replaced by estimates from [**8**]—see Section 2.3.2. Results from [**8**] allow to tackle the case of C^2 and C^3 symbols as well, but we do not include the corresponding version of our results (the asymptotics will differ from the C^∞ case).

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2. Preliminaries.

2.1. Some notations and definitions. Throughout the paper, we shall use the following notations:

S_n, for a positive integer n, will denote the symmetric group (i.e. the group of permutations of $1, \ldots, n$), for a finite-dimensional complex vector space V and $A, B \in \mathrm{End}(V)$ $[A, B] = AB - BA$,

I will denote the identity operator on V,

if V is equipped with a norm, then $||A||$ will denote the operator norm of A,

$C^\infty(M)$ will denote the algebra of smooth complex-valued functions on a smooth manifold M,

for $f \in C^\infty(M)$, we write $|f|_\infty = \sup_{x\in M} |f(x)|$.

DEFINITION 2.1. An $(m + 1)$-form Ω on a smooth manifold M is called an **m-plectic form** if it is closed (i.e. $d\Omega = 0$) and non-degenerate (i.e. $v \in T_xM$, $v \lrcorner \Omega_x = 0 \Rightarrow v = 0$). If Ω is an m-plectic form on M, (M, Ω) is called a **multisymplectic**, or **m-plectic**, manifold.

DEFINITION 2.2. ([**25, 45**]) Let M be a smooth manifold. A multilinear map

$$\{., \ldots, .\} : (C^\infty(M))^{\otimes j} \to C^\infty(M)$$

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is called a **Nambu–Poisson bracket** or **(generalized) Nambu bracket of order j** if it satisfies the following properties:

- (skew-symmetry) $\{f_1, \dots, f_j\} = \mathrm{sign}(\sigma)\{f_{\sigma(1)}, \dots, f_{\sigma(j)}\}$ for any $f_1, \dots, f_j \in C^\infty(M)$ and for any $\sigma \in S_j$,
- (Leibniz rule) $\{f_1, \dots, f_{j-1}, g_1 g_2\} = \{f_1, \dots, f_{j-1}, g_1\}g_2 + g_1\{f_1, \dots, f_{j-1}, g_2\}$ for any $f_1, \dots, f_{j-1}, g_1, g_2 \in C^\infty(M)$,
- (Fundamental Identity)

$$\{f_1, \dots, f_{j-1,}, \{g_1, \dots, g_j\}\} = \sum_{i=1}^{j}\{g_1, \dots, \{f_1, \dots, f_{j-1}, g_i\}, \dots, g_j\},$$

for all $f_1, \dots, f_{j-1}, g_1, \dots, g_j \in C^\infty(M)$.

It is natural to ask how to generalize the Hamiltonian formalism of symplectic geometry to the multisymplectic setting. We don't need the full multisymplectic formalism for the purposes of this paper, and we refer the reader to [**26, 39, 45**].

Definition 2.3. ([**4, 5**]) Let M be a smooth manifold and suppose j is an even positive integer. A multilinear map

$$\{., \dots, .\} : (C^\infty(M))^{\otimes j} \to C^\infty(M)$$

is called a **generalized Poisson bracket** if it satisfies the following properties:

- (skew-symmetry) $\{f_1, \dots, f_j\} = \mathrm{sign}(\sigma)\{f_{\sigma(1)}, \dots, f_{\sigma(j)}\}$ for any $f_1, \dots, f_j \in C^\infty(M)$ and for any $\sigma \in S_j$,
- (Leibniz rule) $\{f_1, \dots, f_{j-1}, g_1 g_2\} = \{f_1, \dots, f_{j-1}, g_1\}g_2 + g_1\{f_1, \dots, f_{j-1}, g_2\}$ for any $f_1, \dots, f_{j-1}, g_1, g_2 \in C^\infty(M)$,
- (Generalized Jacobi Identity)

$$\mathrm{Alt}\{f_1, \dots, f_{j-1}, \{f_j, \dots, f_{2j-1}\}\} =$$

$$\sum_{\sigma \in S_{2j-1}} \mathrm{sign}(\sigma)\{f_{\sigma(1)}, \dots, f_{\sigma(j-1)}, \{f_{\sigma(j)}, \dots, f_{\sigma(2j-1)}\}\} = 0$$

for any $f_1, \dots, f_{2j-1} \in C^\infty(M)$.

Definition 2.4. ([**28**]) A bracket as in Definition 2.3 satisfying only the first two conditions (skew-symmetry and Leibniz rule) is called an **almost Poisson bracket of order j**.

Remark 2.5. A Nambu–Poisson bracket of even order is a generalized Poisson bracket [**28**].

2.2. Generalized commutator. Let $[., ., ., .]$ denote the Nambu generalized commutator ([**17, 37, 45**]): for a finite-dimensional complex vector space V and $A_1, \dots, A_{2n} \in \mathrm{End}(V)$

$$[A_1, \dots, A_{2n}] = \sum_{\sigma \in S_{2n}} \mathrm{sign}(\sigma) A_{\sigma(1)} \dots, A_{\sigma(2n)}.$$

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For example, for $n = 2$

$$[A_1, A_2, A_3, A_4] = \sum_{\sigma \in S_4} \operatorname{sign}(\sigma) A_{\sigma(1)} A_{\sigma(2)} A_{\sigma(3)} A_{\sigma(4)} =$$

$$\begin{aligned} &[A_1, A_2][A_3, A_4] - [A_1, A_3][A_2, A_4] + [A_1, A_4][A_2, A_3] + \\ &[A_3, A_4][A_1, A_2] - [A_2, A_4][A_1, A_3] + [A_2, A_3][A_1, A_4]. \end{aligned} \tag{1}$$

The bracket $[.,.,.,.]$ defines a map $\bigwedge^4 \operatorname{End}(V) \to \operatorname{End}(V)$ which does not satisfy the Leibniz rule and does not satisfy the Fundamental Identity. There has been some discussion of this in physics literature (e.g. [**19**]) and they seem to think that requiring these two conditions is not necessary. There has been investigation into algebraic properties of this bracket—see e.g. [**18**] and [**4**], where some ideas go back to [**14, 23**], and earlier work by Kurosh and his school.

Let us denote, for convenience,

$$\sum_{\sigma \in S_{2n}}{}' = \sum_{\substack{\sigma \in S_{2n} \\ \sigma(1)<\sigma(2),\dots, \\ \sigma(2n-1)<\sigma(2n)}} .$$

LEMMA 2.6.

$$[A_1, \dots, A_{2n}] = \sum_{\sigma \in S_{2n}}{}' \operatorname{sign}(\sigma)[A_{\sigma(1)}, A_{\sigma(2)}][A_{\sigma(3)}, A_{\sigma(4)}] \dots [A_{\sigma(2n-1)}, A_{\sigma(2n)}].$$

Proof. Each monomial from the left-hand side appears in the right-hand side, exactly once, with the same sign. Each term from the right-hand side appears in the left-hand side. Therefore the expressions are identical. □

LEMMA 2.7.

$$[A_1, \dots, A_{2n}] = \frac{1}{2^n} \sum_{\sigma \in S_{2n}} \operatorname{sign}(\sigma)[A_{\sigma(1)}, A_{\sigma(2)}][A_{\sigma(3)}, A_{\sigma(4)}] \dots [A_{\sigma(2n-1)}, A_{\sigma(2n)}].$$

Proof. By straightforward comparison of the polynomials. Observe that each monomial from the left-hand side appears in the sum in the right-hand side exactly 2^n times, with appropriate sign, and this accounts for all the terms in the right-hand side. □

REMARK 2.8. Equality (1) is (93) [**19**]. It is not hard to see that Lemma 2.7 is equivalent to (94) [**19**].

2.3. Berezin–Toeplitz operators. Suppose (M, ω) is a compact connected Kähler manifold and the Kähler form $\frac{\omega}{2\pi}$ is integral. Let L be a holomorphic hermitian line bundle such that the curvature of the hermitian connection is $-i\omega$. Let k be a positive integer. The space $H^0(M, L^{\otimes k})$ of holomorphic sections of $L^{\otimes k}$ is a finite-dimensional complex vector space. Let Π_k denote the orthogonal projection from $L^2(M, L^{\otimes k})$ onto

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$H^0(M, L^{\otimes k})$ (the Hermitian inner product is obtained from the hermitian metric on L).

2.3.1. Smooth symbol. Reference used throughout this subsection is [**12**], where the method is based on the analysis of Toeplitz structures from [**13**]. Results mentioned here and more extensive discussion can be found in surveys on Berezin–Toeplitz quantization, for example in [**44**].

For $f \in C^\infty(M)$ the operator

$$T_f^{(k)} = \Pi_k \circ (mult.\ by\ f) \in \mathrm{End}(H^0(M, L^{\otimes k})),$$

or the operator $\oplus T_f^{(k)}$, is called the *Berezin–Toeplitz operator for* f. Here are some properties of these operators that will be most frequently used in this paper.

For $\alpha, \beta \in \mathbb{C}$ and $f, g \in C^\infty(M)$

$$T^{(k)}_{\alpha f+\beta g} = \alpha T_f^{(k)} + \beta T_g^{(k)}.$$

THEOREM 2.9 ([**12**] Theorem 4.1, 4.2; [**31, 32**]). *For $f, g \in C^\infty(M)$, as $k \to \infty$,*
(i)

$$||ik[T_f^{(k)}, T_g^{(k)}] - T^{(k)}_{\{f,g\}}|| = O\left(\frac{1}{k}\right),$$

(ii) *there is a constant $C = C(f) > 0$ such that*

$$|f|_\infty - \frac{C}{k} \le ||T_f^{(k)}|| \le |f|_\infty.$$

PROPOSITION 2.10 ([**12**] p. 291). *For $f_1, \ldots, f_p \in C^\infty(M)$*

$$||T^{(k)}_{f_1} \ldots, T^{(k)}_{f_p} - T^{(k)}_{f_1 \ldots, f_p}|| = O\left(\frac{1}{k}\right)$$

as $k \to \infty$.

PROPOSITION 2.11 ([**12**] p. 289). *For $f, g \in C^\infty(M)$*

$$\lim_{k\to\infty} ||[T_f^{(k)}, T_g^{(k)}]|| = 0.$$

REMARK 2.12. Proof of this Proposition (it's one line, use Theorem 2.9 and triangle inequality) actually implies that

$$||[T_f^{(k)}, T_g^{(k)}]|| = O\left(\frac{1}{k}\right)$$

as $k \to \infty$.

2.3.2. C^l symbol. The reference for theorems analogous to those above in Section 2.3.1, with $f \in C^l(M)$, is [**8**]. In [**8**], the method is different from [**12**]. It relies on techniques developed in [**31,32**], see also [**33**]. For $l = 4$, statements similar to Theorem 2.9, Proposition 2.10 follow from Corollary 4.5, Remark 5.7(b), Corollary 4.4 of [**8**]. The fact that for $f, g \in C^4(M)$ $||[T_f^{(k)}, T_g^{(k)}]|| = O\left(\frac{1}{k}\right)$ as $k \to \infty$ easily follows too, from Corollary 4.5 and Remark 5.7(b) [**8**].

3. Quantization of the $(2n-1)$**-plectic structure on an** n**-dimensional Kähler manifold.** Let (M, ω) be a compact connected n-dimensional Kähler manifold ($n \geq 1$). We shall denote by $\{.,.\}$ the Poisson bracket for ω. Assume that the Kähler form $\frac{\omega}{2\pi}$ is integral. Let L be a hermitian holomorphic line bundle on M such that the curvature of the hermitian connection is equal to $-i\omega$.

It is clear that the volume form $\Omega = \frac{\omega^n}{n!}$ is a $(2n-1)$-plectic form. The bracket $\{.,\dots,.\} : \bigwedge^{2n} C^\infty(M) \to C^\infty(M)$ defined by

$$df_1 \wedge \dots \wedge df_{2n} = \{f_1, \dots, f_{2n}\}\Omega$$

is a Nambu–Poisson bracket [**25**, Corollary 1 p. 106] .

LEMMA 3.1. *For* $f_1, \dots, f_{2n} \in C^\infty(M)$

$$\{f_1, \dots, f_{2n}\} = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sign}(\sigma) \prod_{j=1}^{n} \{f_{\sigma(2j-1)}, f_{\sigma(2j)}\}. \tag{2}$$

REMARK 3.2. In particular, for $n = 2$

$$\{f_1, f_2, f_3, f_4\} = \{f_1, f_2\}\{f_3, f_4\} - \{f_1, f_3\}\{f_2, f_4\} + \{f_1, f_4\}\{f_2, f_3\}.$$

REMARK 3.3. For $M = \mathbb{R}^{2n}$ with the standard symplectic form equality (2) is (7) in [**19**].

Proof of Lemma 3.1. Let's use Darboux theorem and compare the left-hand side and the right-hand side of (2) in a local chart with coordinates $x_1, \dots, x_{2n}$ such that in this chart $\omega = \sum_{j=1}^n dx_{2j-1} \wedge dx_{2j}$. Locally, in this chart, the Poisson bracket of f_i, f_l ($1 \leq i, l \leq 2n$) is

$$\{f_i, f_l\} = \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_{2j-1}} \frac{\partial f_l}{\partial x_{2j}} - \frac{\partial f_i}{\partial x_{2j}} \frac{\partial f_l}{\partial x_{2j-1}}\right)$$

and $\{f_1, \dots, f_{2n}\} = \det J$, where $J = \left(\frac{\partial f_i}{\partial x_l}\right)$. det is the only function on $(2n) \times (2n)$ complex matrices which takes value 1 on the identity matrix, linear in the rows, and takes value zero on a matrix whose two adjacent rows are equal (axiomatic characterization of the determinant, see e.g. Theorem 1.3.(3.14) [**3**]). The right-hand side of (2) is a polynomial in the entries of J that satisfies these three conditions, therefore it must be equal to $\det J$. □

The following theorem shows that, informally speaking, $\{.,\dots,.\} \to [.,\dots,.]$ as $k \to \infty$.

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THEOREM 3.4. *For* $f_1, \ldots, f_{2n} \in C^\infty(M)$

$$||\frac{(ik)^n}{n!}[T^{(k)}_{f_1}, \ldots, T^{(k)}_{f_{2n}}] - T^{(k)}_{\{f_1,\ldots,f_{2n}\}}|| = O\left(\frac{1}{k}\right)$$

as $k \to \infty$.

Proof. By Theorem 2.9 (i)

$$||ik[T^{(k)}_{f_{2j-1}}, T^{(k)}_{f_{2j}}] - T^{(k)}_{\{f_{2j-1},f_{2j}\}}|| = O\left(\frac{1}{k}\right) \tag{3}$$

for $j = 1, \ldots, n$. Using Proposition 2.10 and the triangle inequality, we get:

$$\begin{aligned}
&||(ik)^n[T^{(k)}_{f_1}, T^{(k)}_{f_2}]\ldots[T^{(k)}_{f_{2n-1}}, T^{(k)}_{f_{2n}}] - T^{(k)}_{\{f_1,f_2\}\ldots\{f_{2n-1},f_{2n}\}}|| \\
&\quad \leq ||(ik)^n[T^{(k)}_{f_1}, T^{(k)}_{f_2}]\ldots[T^{(k)}_{f_{2n-1}}, T^{(k)}_{f_{2n}}] - T^{(k)}_{\{f_1,f_2\}}\ldots T^{(k)}_{\{f_{2n-1},f_{2n}\}}|| \\
&\qquad + ||T^{(k)}_{\{f_1,f_2\}\ldots\{f_{2n-1},f_{2n}\}} - T^{(k)}_{\{f_1,f_2\}}\ldots T^{(k)}_{\{f_{2n-1},f_{2n}\}}|| \\
&\quad = ||\big((ik[T^{(k)}_{f_1}, T^{(k)}_{f_2}] - T^{(k)}_{\{f_1,f_2\}}) + T^{(k)}_{\{f_1,f_2\}}\big)\ldots\big((ik[T^{(k)}_{f_{2n-1}}, T^{(k)}_{f_{2n}}] - T^{(k)}_{\{f_{2n-1},f_{2n}\}} \\
&\qquad + T^{(k)}_{\{f_{2n-1},f_{2n}\}}\big) - T^{(k)}_{\{f_1,f_2\}}\ldots T^{(k)}_{\{f_{2n-1},f_{2n}\}}|| + O\left(\frac{1}{k}\right).
\end{aligned}$$

This is $O\left(\frac{1}{k}\right)$. Indeed, within $||.||$ the term $T^{(k)}_{\{f_1,f_2\}}\ldots T^{(k)}_{\{f_{2n-1},f_{2n}\}}$ cancels and all the other terms are products of factors of the form $(ik[T^{(k)}_{f_{2j-1}}, T^{(k)}_{f_{2j}}] - T^{(k)}_{\{f_{2j-1},f_{2j}\}})$ (at least one of these appears) and of the form $T^{(k)}_{\{f_{2j-1},f_{2j}\}}$. Using the triangle inequality, (3) and Theorem 2.9 (ii), we get $O\left(\frac{1}{k}\right)$. Thus, as $k \to \infty$,

$$||(ik)^n[T^{(k)}_{f_1}, T^{(k)}_{f_2}]\ldots[T^{(k)}_{f_{2n-1}}, T^{(k)}_{f_{2n}}] - T^{(k)}_{\{f_1,f_2\}\ldots\{f_{2n-1},f_{2n}\}}|| = O\left(\frac{1}{k}\right).$$

Exact same proof shows that

$$||(ik)^n[T^{(k)}_{f_{\sigma(1)}}, T^{(k)}_{f_{\sigma(2)}}]\ldots[T^{(k)}_{f_{\sigma(2n-1)}}, T^{(k)}_{f_{\sigma(2n)}}] - T^{(k)}_{\{f_{\sigma(1)},f_{\sigma(2)}\}\ldots\{f_{\sigma(2n-1)},f_{\sigma(2n)}\}}|| = O\left(\frac{1}{k}\right).$$

We note that

$$T^{(k)}_{\{f_1,\ldots,f_{2n}\}} = \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}} \mathrm{sign}(\sigma) T^{(k)}_{\prod_{j=1}^n \{f_{\sigma(2j-1},f_{\sigma(2j)}\}}$$

(by Lemma 3.1). The desired statement now follows from Lemma 2.7 and the triangle inequality. □

The following proposition is similar to Proposition 2.11. It implies that $\lim_{k\to\infty} ||[T^{(k)}_{f_1}, \ldots, T^{(k)}_{f_{2n}}]|| = 0$ (i.e. $T^{(k)}_{f_1}, \ldots, T^{(k)}_{f_{2n}}$ "Nambu-commute as $k \to \infty$").

PROPOSITION 3.5. *For* $f_1, \ldots, f_{2n} \in C^\infty(M)$

$$||[T^{(k)}_{f_1}, \ldots, T^{(k)}_{f_{2n}}]|| = O\left(\frac{1}{k^n}\right)$$

as $k \to \infty$.

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Proof.

$$
\begin{aligned}
&||[T^{(k)}_{f_1},\dots,T^{(k)}_{f_{2n}}]||\\
&= ||\sum_{\sigma\in S_{2n}}{}' \operatorname{sign}(\sigma)[T^{(k)}_{f_{\sigma(1)}},T^{(k)}_{f_{\sigma(2)}}][T^{(k)}_{f_{\sigma(3)}},T^{(k)}_{f_{\sigma(4)}}]\dots[T^{(k)}_{f_{\sigma(2n-1)}},T^{(k)}_{f_{\sigma(2n)}}]||\\
&\le \sum_{\sigma\in S_{2n}}{}' ||[T^{(k)}_{f_{\sigma(1)}},T^{(k)}_{f_{\sigma(2)}}]||\dots||[T^{(k)}_{f_{\sigma(2n-1)}},T^{(k)}_{f_{\sigma(2n)}}]||
\end{aligned}
$$

which is $O(\frac{1}{k^n})$ by Remark 2.12. □

4. Quantization on a hyperkähler manifold. Let (M, g, J_1, J_2, J_3) be a compact connected hyperkähler manifold. Let $4q$ denote the real dimension of M. Denote $\omega_r = g(., J_r.)$ for $r = 1, 2, 3$. The 4-form

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$$

is 3-plectic [**16**]. Define the brackets $\{., ., ., .\}_r$, $\{., ., ., .\}_{\text{hyp}}$ (multilinear maps $\bigwedge^4 C^\infty(M) \to C^\infty(M)$) as follows:

$$\{f_1,f_2,f_3,f_4\}_r = \{f_1,f_2\}_r\{f_3,f_4\}_r - \{f_1,f_3\}_r\{f_2,f_4\}_r + \{f_1,f_4\}_r\{f_2,f_3\}_r,$$

where $\{., .\}_r$ is the Poisson bracket on (M, ω_r), $r = 1, 2, 3$,

$$\{f_1,f_2,f_3,f_4\}_{\text{hyp}} = \sum_{r=1}^{3}\{f_1,f_2,f_3,f_4\}_r.$$

From the properties of the Poisson bracket it immediately follows that the Leibniz rule is satisfied:

$$\{f_1,f_2,f_3,f_4f_5\}_r = f_4\{f_1,f_2,f_3,f_5\}_r + \{f_1,f_2,f_3,f_4\}_r f_5$$

$$\{f_1,f_2,f_3,f_4f_5\}_{\text{hyp}} = f_4\{f_1,f_2,f_3,f_5\}_{\text{hyp}} + \{f_1,f_2,f_3,f_4\}_{\text{hyp}} f_5.$$

Therefore $\{., ., ., .\}_r$, $\{., ., ., .\}_{\text{hyp}}$ are almost Poisson brackets of order 4.

For $q = 1$, $\omega_r \wedge \omega_r$ $(r = 1, 2, 3)$ and Ω are volume forms. The standard bracket $\{., ., ., .\}^{(r)}$ is defined by

$$df_1 \wedge df_2 \wedge df_3 \wedge df_4 = \{f_1,f_2,f_3,f_4\}^{(r)}\frac{1}{2}\omega_r \wedge \omega_r.$$

From Lemma 3.1, or by a direct calculation (using Darboux theorem, in local coordinates), we get:

LEMMA 4.1. *For $q = 1$ $\{., ., ., .\}_r$ coincides with $\{., ., ., .\}^{(r)}$.*

From [**25**, Corollary 1 p. 106] it immediately follows that for $q = 1$ (M is 4-dimensional) the Fundamental Identity

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$$\{f_1, f_2, f_3, \{g_1, g_2, g_3, g_4\}_{\rm hyp}\}_{\rm hyp} = \{\{f_1, f_2, f_3, g_1\}_{\rm hyp}, g_2, g_3, g_4\}_{\rm hyp} +$$

$$\{g_1, \{f_1, f_2, f_3, g_2\}_{\rm hyp}, g_3, g_4\}_{\rm hyp} + \{g_1, g_2, \{f_1, f_2, f_3, g_3\}_{\rm hyp}, g_4\}_{\rm hyp} +$$

$$\{g_1, g_2, g_3, \{f_1, f_2, f_3, g_4\}_{\rm hyp}\}_{\rm hyp}$$

is satisfied (similarly for $\{.,.,.,.\}_r$). For $q > 1$ $\{.,.,.,.\}_r$, $\{.,.,.,.\}_{\rm hyp}$ are not necessarily Nambu–Poisson brackets (the Fundamental Identity may not be satisfied if $q > 1$).

Assume that the Kähler forms $\frac{\omega_1}{2\pi}$, $\frac{\omega_2}{2\pi}$, $\frac{\omega_3}{2\pi}$ are integral.

Let L_r be a holomorphic Hermitian line bundle with curvature of the Hermitian connection equal to $-i\omega_r$, for $r = 1, 2, 3$. For a positive integer k and $f \in C^\infty(M)$ denote by $T^{(k)}_{f;r} \in \mathrm{End}(H^0(M, L_r^{\otimes k}))$ the Berezin–Toeplitz operator for f.

There are two obvious ways to form a Hilbert space out of three Hilbert spaces $H^0(M, L_r^{\otimes k})$ $(r = 1, 2, 3)$: by taking direct sum or tensor product. Another way to approach this is to say that the vector space of quantization is $H^0(M, (L_1 \otimes L_2 \otimes L_3)^{\otimes k})$, this would be just the usual Berezin–Toeplitz quantization, with the line bundle $L_1 \otimes L_2 \otimes L_3$. Note: in general $H^0(M, (L_1 \otimes L_2 \otimes L_3)^{\otimes k})$ is not isomorphic to $H^0(M, L_1^{\otimes k}) \otimes H^0(M, L_2^{\otimes k}) \otimes H^0(M, L_3^{\otimes k})$.

Of course, the hyperkähler structure defines a whole S^2 of complex structures (and of Kähler forms) on M, not just three. A. Uribe pointed out to us that maybe an appropriate notion of quantization on a hyperkähler manifold should take into account all $J \in S^2$, and should involve an appropriate vector bundle over the twistor space, with fibres $H^0(M, L_J^{\otimes k})$. We look forward to seeing his work on this.

Note that the twistor space of a hyperkähler manifold is not Kähler (it is generally well known, see for example [**29**] p. 37, or [**27**]), so it's not possible to construct a Berezin–Toeplitz quantization on the twistor space.

REMARK 4.2. Denote by $\pi_r : M \times M \times M \to M$ the projection to the rth factor $(r = 1, 2, 3)$. For sufficiently large k

$$H^0(M \times M \times M, (\pi_1^* L_1 \otimes \pi_2^* L_2 \otimes \pi_3^* L_3)^{\otimes k)}) \cong \mathcal{H}_k.$$

The proof was explained to us by K. Yoshikawa and it goes as follows:

$$\begin{aligned}
&\dim H^0(M \times M \times M, (\pi_1^* L_1 \otimes \pi_2^* L_2 \otimes \pi_3^* L_3)^{\otimes k}) \\
&= \int_{M\times M\times M} \mathrm{Td}(M \times M \times M)\mathrm{ch}((\pi_1^* L_1 \otimes \pi_2^* L_2 \otimes \pi_3^* L_3)^{\otimes k}) \\
&= \int_{M\times M\times M} \pi_1^*\mathrm{Td}(M)\pi_2^*\mathrm{Td}(M)\pi_3^*\mathrm{Td}(M)\pi_1^*\mathrm{ch}(L_1^{\otimes k})\pi_2^*\mathrm{ch}(L_2^{\otimes k})\pi_3^* ch(L_3^{\otimes k}) \\
&= \int_M \mathrm{Td}(M)\mathrm{ch}(L_1^{\otimes k}) \int_M \mathrm{Td}(M)\mathrm{ch}(L_2^{\otimes k}) \int_M \mathrm{Td}(M)\mathrm{ch}(L_3^{\otimes k}) \\
&= \dim H^0(M, L_1^{\otimes k}) \dim H^0(M, L_2^{\otimes k}) \dim H^0(M, L_3^{\otimes k}).
\end{aligned}$$

□

In this paper, we shall work with functions and structures on M, rather than on $M \times M \times M$.

We shall find useful the following statement.

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PROPOSITION 4.3. *For* $f, g, h, t \in C^\infty(M)$, $r = 1, 2, 3$,

$$|| - \frac{k^2}{2}[T^{(k)}_{f;r}, T^{(k)}_{g;r}, T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,g,h,t\}_r;r}|| = O(\frac{1}{k})$$

as $k \to \infty$.

Proof. As $k \to \infty$, for $r = 1, 2, 3$, by Theorem 2.9 (i) for $f, g \in C^\infty(M)$

$$||ik[T^{(k)}_{f;r}, T^{(k)}_{g;r}] - T^{(k)}_{\{f,g\}_r;r}|| = O\left(\frac{1}{k}\right), \tag{4}$$

$$||ik[T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{h,t\}_r;r}|| = O\left(\frac{1}{k}\right). \tag{5}$$

Using Proposition 2.10, we get:

$$\begin{aligned}
&||(ik)^2[T^{(k)}_{f;r}, T^{(k)}_{g;r}][T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,g\}_r\{h,t\}_r;r}|| \\
&\quad \le ||(ik)^2[T^{(k)}_{f;r}, T^{(k)}_{g;r}][T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,g\}_r;r}T^{(k)}_{\{h,t\}_r;r}|| + ||T^{(k)}_{\{f,g\}_r;r}T^{(k)}_{\{h,t\}_r;r} - T^{(k)}_{\{f,g\}_r\{h,t\}_r;r}|| \\
&\quad = ||(ik[T^{(k)}_{f;r}, T^{(k)}_{g;r}] - T^{(k)}_{\{f,g\}_r;r} + T^{(k)}_{\{f,g\}_r;r})(ik[T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{h,t\}_r;r} + T^{(k)}_{\{h,t\}_r;r}) \\
&\qquad - T^{(k)}_{\{f,g\}_r;r}T^{(k)}_{\{h,t\}_r;r}|| + O\left(\frac{1}{k}\right) \\
&\quad = ||(ik[T^{(k)}_{f;r}, T^{(k)}_{g;r}] - T^{(k)}_{\{f,g\}_r;r})(ik[T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{h,t\}_r;r}) \\
&\qquad + (ik[T^{(k)}_{f;r}, T^{(k)}_{g;r}] - T^{(k)}_{\{f,g\}_r;r})T^{(k)}_{\{h,t\}_r;r} + T^{(k)}_{\{f,g\}_r;r}(ik[T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{h,t\}_r;r})|| + O\left(\frac{1}{k}\right) \\
&\quad \le ||ik([T^{(k)}_{f;r}, T^{(k)}_{g;r}] - T^{(k)}_{\{f,g\}_r;r})||\ ||ik[T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{h,t\}_r;r})|| \\
&\qquad + ||ik[T^{(k)}_{f;r}, T^{(k)}_{g;r}] - T^{(k)}_{\{f,g\}_r;r}||\ ||T^{(k)}_{\{h,t\}_r;r}|| + ||T^{(k)}_{\{f,g\}_r;r}||\ ||ik[T^{(k)}_{h;r}, T^{(k)}_{t;r}] \\
&\qquad - T^{(k)}_{\{h,t\}_r;r}|| + O\left(\frac{1}{k}\right) \\
&\quad = O\left(\frac{1}{k}\right)O\left(\frac{1}{k}\right) + |\{h,t\}_r|_\infty O\left(\frac{1}{k}\right) + |\{f,g\}_r|_\infty O\left(\frac{1}{k}\right) + O\left(\frac{1}{k}\right) = O\left(\frac{1}{k}\right).
\end{aligned}$$

In the last line, we used (4), (5), and applied Theorem 2.9 (ii) twice. Similarly we conclude, for f, h and g, t:

$$||(ik)^2[T^{(k)}_{f;r}, T^{(k)}_{h;r}][T^{(k)}_{g;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,h\}_r\{g,t\}_r;r}|| = O\left(\frac{1}{k}\right),$$

etc. (i.e. we get similar asymptotics for f, t and g, h, for h, t and f, g, for g, t and f, h, for g, h and f, t). Note:

$$T^{(k)}_{\{f,g,h,t\}_r;r} = T^{(k)}_{\{f,g\}_r\{h,t\}_r;r} - T^{(k)}_{\{f,h\}_r\{g,t\}_r;r} + T^{(k)}_{\{f,t\}_r\{g,h\}_r;r}.$$

Therefore, by (1) and the triangle inequality,

$$|| - \frac{k^2}{2}[T^{(k)}_{f;r}, T^{(k)}_{g;r}, T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,g,h,t\}_r;r}|| = O\left(\frac{1}{k}\right).$$

□

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4.1. Direct sum. Denote

$$\mathcal{H}_k = H^0(M, L_1^{\otimes k}) \oplus H^0(M, L_2^{\otimes k}) \oplus H^0(M, L_3^{\otimes k})$$

(direct sum of Hilbert spaces) and

$$\mathbf{T}_f^{(k)} = T_{f;1}^{(k)} \oplus T_{f;2}^{(k)} \oplus T_{f;3}^{(k)},$$

($\mathbf{T}_f^{(k)}$ acts on $\mathcal{H}_k$ by $\mathbf{T}_f^{(k)}(s_1, s_2, s_3) = (T_{f;1}^{(k)} s_1, T_{f;2}^{(k)} s_2, T_{f;3}^{(k)} s_3)$).

REMARK 4.4. Since $||\mathbf{T}_f^{(k)}|| = \max\{||T_{f;1}^{(k)}||, ||T_{f;2}^{(k)}||, ||T_{f;3}^{(k)}||\}$, we immediately have:

- For $f, g \in C^\infty(M)$, as $k \to \infty$,

$$||ik[\mathbf{T}_f^{(k)}, \mathbf{T}_g^{(k)}] - \mathbf{T}_{\{f,g\}}^{(k)}|| = O\left(\frac{1}{k}\right), \quad ||[\mathbf{T}_f^{(k)}, \mathbf{T}_g^{(k)}]|| = O\left(\frac{1}{k}\right).$$

- For $f \in C^\infty(M)$, there is a constant $C = C(f) > 0$ such that, as $k \to \infty$,

$$|f|_\infty - \frac{C}{k} \leq ||\mathbf{T}_f^{(k)}|| \leq |f|_\infty.$$

- For $f_1, \ldots, f_p \in C^\infty(M)$

$$||\mathbf{T}_{f_1}^{(k)} \ldots \mathbf{T}_{f_p}^{(k)} - \mathbf{T}_{f_1 \ldots f_p}^{(k)}|| = O\left(\frac{1}{k}\right)$$

as $k \to \infty$.

For $f, g, h, t \in C^\infty(M)$, we have:

$$[\mathbf{T}_f^{(k)}, \mathbf{T}_g^{(k)}, \mathbf{T}_h^{(k)}, \mathbf{T}_t^{(k)}] = \oplus_{r=1}^3 [T_{f;r}^{(k)}, T_{g;r}^{(k)}, T_{h;r}^{(k)}, T_{t;r}^{(k)}].$$

THEOREM 4.5. *For $f, g, h, t \in C^\infty(M)$*

$$|| - \frac{k^2}{2}[\mathbf{T}_f^{(k)}, \mathbf{T}_g^{(k)}, \mathbf{T}_h^{(k)}, \mathbf{T}_t^{(k)}] - \oplus_{r=1}^3 T_{\{f,g,h,t\}_r;r}^{(k)}|| = O\left(\frac{1}{k}\right)$$

as $k \to \infty$.

Proof. Using Proposition 4.3, we get:

$$|| - \frac{k^2}{2}[\mathbf{T}_f^{(k)}, \mathbf{T}_g^{(k)}, \mathbf{T}_h^{(k)}, \mathbf{T}_t^{(k)}] - \oplus_{r=1}^3 T_{\{f,g,h,t\}_r;r}^{(k)}|| =$$

$$\max_{1 \leq r \leq 3} || - \frac{k^2}{2}[T_{f;r}^{(k)}, T_{g;r}^{(k)}, T_{h;r}^{(k)}, T_{t;r}^{(k)}] - T_{\{f,g,h,t\}_r;r}^{(k)}|| = O\left(\frac{1}{k}\right).$$

□

The following proposition is similar to Proposition 2.11. It implies that $\mathbf{T}_f^{(k)}$, $\mathbf{T}_g^{(k)}$, $\mathbf{T}_h^{(k)}$, $\mathbf{T}_t^{(k)}$ "Nambu-commute as $k \to \infty$".

PROPOSITION 4.6. *For $f_1, f_2, f_3, f_4 \in C^\infty(M)$*

$$||[\mathbf{T}^{(k)}_{f_1}, \mathbf{T}^{(k)}_{f_2}, \mathbf{T}^{(k)}_{f_3}, \mathbf{T}^{(k)}_{f_4}]|| = O(\frac{1}{k^2})$$

as $k \to \infty$.

Proof.

$$||[\mathbf{T}^{(k)}_{f_1}, \mathbf{T}^{(k)}_{f_2}, \mathbf{T}^{(k)}_{f_3}, \mathbf{T}^{(k)}_{f_4}]|| = \max_{1\le r\le 3} ||[T^{(k)}_{f_1;r}, T^{(k)}_{f_2;r}, T^{(k)}_{f_3;r}, T^{(k)}_{f_4;r}]||$$
$$= \max_{1\le r\le 3} ||\sum_{\sigma\in S_4}{}' \operatorname{sign}(\sigma)[T^{(k)}_{f_{\sigma(1)};r}, T^{(k)}_{f_{\sigma(2)};r}][T^{(k)}_{f_{\sigma(3)};r}, T^{(k)}_{f_{\sigma(4)};r}]||$$
$$\le \max_{1\le r\le 3} \sum_{\sigma\in S_4}{}' ||[T^{(k)}_{f_{\sigma(1)};r}, T^{(k)}_{f_{\sigma(2)};r}]|| \; ||[T^{(k)}_{f_{\sigma(3)};r}, T^{(k)}_{f_{\sigma(4)};r}]||.$$

By Remark 2.12 it is $O(\frac{1}{k^2})$. □

4.2. Direct sum: dimension 4. To discuss the correspondence between the bracket on functions and the generalized commutator (as $k \to \infty$) in the hyperkähler case: we showed (Theorem 4.5) that for a hyperkähler manifold M of arbitrary dimension and smooth functions f, g, h, t on M $[\mathbf{T}^{(k)}_f, \mathbf{T}^{(k)}_g, \mathbf{T}^{(k)}_h, \mathbf{T}^{(k)}_t]$ is asymptotic to

$$\begin{pmatrix} T^{(k)}_{\{f,g,h,t\}_1;1} & & \\ & T^{(k)}_{\{f,g,h,t\}_2;2} & \\ & & T^{(k)}_{\{f,g,h,t\}_3;3} \end{pmatrix},$$

not to

$$\mathbf{T}^{(k)}_{\{f,g,h,t\}_{\text{hyp}}} = \begin{pmatrix} T^{(k)}_{\{f,g,h,t\}_{\text{hyp}};1} & & \\ & T^{(k)}_{\{f,g,h,t\}_{\text{hyp}};2} & \\ & & T^{(k)}_{\{f,g,h,t\}_{\text{hyp}};3} \end{pmatrix}.$$

To clarify, we have obtained an asymptotic relation between a map

$$\bigwedge^4 C^\infty(M) \to C^\infty(M) \times C^\infty(M) \times C^\infty(M)$$

$$f, g, h, t \mapsto (\{f, g, h, t\}_1, \{f, g, h, t\}_2, \{f, g, h, t\}_3)$$

and the Nambu generalized commutator $[., ., ., .]$. It is not the same as a correspondence between $\{., ., ., .\}_{\text{hyp}} : \bigwedge^4 C^\infty(M) \to C^\infty(M)$ and $[., ., ., .]$.

From now on M will be of real dimension 4 (hence M is isomorphic to a K3-surface or a torus [**11**] 14.22). In this case we get Theorem 4.7 below, and in the case when M is a 4-torus with three standard linear complex structures (Example 4.8 below)—we get that $[\mathbf{T}^{(k)}_f, \mathbf{T}^{(k)}_g, \mathbf{T}^{(k)}_h, \mathbf{T}^{(k)}_t]$ is asymptotic to $\mathbf{T}^{(k)}_{\{f,g,h,t\}_{\text{hyp}}}$.

We have: for $r = 1, 2, 3$

$$\Omega = \frac{\mu_r}{2}\omega_r \wedge \omega_r,$$

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where μ_r is a smooth non-vanishing function on M. Denote by $\{.,.,.,.\}$ the Nambu–Poisson bracket defined by

$$df_1 \wedge df_2 \wedge df_3 \wedge df_4 = \{f_1, f_2, f_3, f_4\}\Omega.$$

Therefore

$$\{f_1, f_2, f_3, f_4\}_r = \{f_1, f_2, f_3, f_4\}^{(r)} = \mu_r\{f_1, f_2, f_3, f_4\}.$$

Denote

$$\mathbf{T}^{(k)}_{\mu} = \begin{pmatrix} T^{(k)}_{\mu_1;1} & & \\ & T^{(k)}_{\mu_2;2} & \\ & & T^{(k)}_{\mu_3;3} \end{pmatrix}.$$

The following theorem shows that $[\mathbf{T}^{(k)}_{f_1}, \mathbf{T}^{(k)}_{f_2}, \mathbf{T}^{(k)}_{f_3}, \mathbf{T}^{(k)}_{f_4}]$ is asymptotic to $\mathbf{T}^{(k)}_{\{f_1,f_2,f_3,f_4\}}\mathbf{T}^{(k)}_{\mu}$.

THEOREM 4.7. *For* $f, g, h, t \in C^{\infty}(M)$

$$|| -\frac{k^2}{2}[\mathbf{T}^{(k)}_f, \mathbf{T}^{(k)}_g, \mathbf{T}^{(k)}_h, \mathbf{T}^{(k)}_t] - \mathbf{T}^{(k)}_{\{f,g,h,t\}}\mathbf{T}^{(k)}_{\mu}|| = O\left(\frac{1}{k}\right)$$

as $k \to \infty$.

Proof. For $r = 1, 2, 3$, the same argument as in the proof of Proposition 4.3 gives:

$$|| -\frac{k^2}{2}[T^{(k)}_{f;r}, T^{(k)}_{g;r}, T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,g,h,t\}_r;r}|| = O\left(\frac{1}{k}\right). \tag{6}$$

We have:

$$\begin{aligned} &|| -\frac{k^2}{2}[T^{(k)}_{f;r}, T^{(k)}_{g;r}, T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,g,h,t\};r}T^{(k)}_{\mu_r;r}|| \\ &\quad \leq || -\frac{k^2}{2}[T^{(k)}_{f;r}, T^{(k)}_{g;r}, T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,g,h,t\}\mu_r;r}|| + ||T^{(k)}_{\{f,g,h,t\}\mu_r;r} - T^{(k)}_{\{f,g,h,t\};r}T^{(k)}_{\mu_r;r}||. \end{aligned}$$

This is $O\left(\frac{1}{k}\right)$ by (6) and Proposition 2.10. Hence,

$$\begin{aligned} &|| -\frac{k^2}{2}[\mathbf{T}^{(k)}_f, \mathbf{T}^{(k)}_g, \mathbf{T}^{(k)}_h, \mathbf{T}^{(k)}_t] - \mathbf{T}^{(k)}_{\{f,g,h,t\}}\mathbf{T}^{(k)}_{\mu}|| \\ &\quad = \max_{1\leq r\leq 3} || -\frac{k^2}{2}[T^{(k)}_{f;r}, T^{(k)}_{g;r}, T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,g,h,t\};r}T^{(k)}_{\mu_r;r}|| = O\left(\frac{1}{k}\right). \end{aligned}$$

□

EXAMPLE 4.8. Denote $\tilde{M} = \mathbb{R}^4$, with coordinates x_1, x_2, x_3, x_4, and equipped with three (linear) complex structures

$$J_1 = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ -1 & 0 & & \\ 0 & -1 & & \end{pmatrix}, \quad J_3 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

We have: $J_1J_2 = J_3$ and, of course, $J_1^2 = J_2^2 = J_3^2 = -I$.

Note: if we regard $\tilde{M}$ as the one-dimensional quaternionic vector space, with basis 1, **i, j, k** ($\mathbf{i}^2$=$\mathbf{j}^2$=$\mathbf{k}^2$=-1, **ij=k**), then J_1, J_2, J_3 correspond to left multiplication by **i, j, k,** respectively.

For the standard Riemannian metric on $\tilde{M}$, with the metric tensor $g = I$, the symplectic forms are as follows:

$$\omega_1 = -dx_1 \wedge dx_2 - dx_3 \wedge dx_4,$$

$$\omega_2 = -dx_1 \wedge dx_3 + dx_2 \wedge dx_4,$$

$$\omega_3 = -dx_1 \wedge dx_4 - dx_2 \wedge dx_3.$$

For $r = 1, 2, 3$,

$$\frac{1}{2}\omega_r \wedge \omega_r = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4,$$

$$\Omega = \sum_{r=1}^{3} \omega_r \wedge \omega_r = 6dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

Everything is $\mathbb{Z}^4$-invariant and g, J_1, J_2, J_3, ω_1, ω_2, ω_3, Ω descend to $M = \tilde{M}/\mathbb{Z}^4$. We get: $\mu_1 = \mu_2 = \mu_3 = 6$ and

$$6\{., ., ., .\} = \{., ., ., .\}_r = \{., ., ., .\}^{(r)} = \frac{1}{3}\{., ., ., .\}_{\text{hyp}}.$$

Theorem 4.7 gives: for $f, g, h, t \in C^\infty(M)$

$$|| - ck^2[\mathbf{T}_f^{(k)}, \mathbf{T}_g^{(k)}, \mathbf{T}_h^{(k)}, \mathbf{T}_t^{(k)}] - \mathbf{T}^{(k)}_{\{f,g,h,t\}_{\text{hyp}}}|| = O\left(\frac{1}{k}\right) \tag{7}$$

as $k \to \infty$, where c is a positive constant.

4.3. Tensor product. Denote

$$\mathcal{H}_k = H^0(M, L_1^{\otimes k}) \otimes H^0(M, L_2^{\otimes k}) \otimes H^0(M, L_3^{\otimes k})$$

(tensor product of Hilbert spaces) and

$$\mathbb{T}_f^{(k)} = T_{f;1}^{(k)} \otimes T_{f;2}^{(k)} \otimes T_{f;3}^{(k)},$$

$(\mathbb{T}_f^{(k)}(s_1 \otimes s_2 \otimes s_3) = T_{f;1}^{(k)}s_1 \otimes T_{f;2}^{(k)}s_2 \otimes T_{f;3}^{(k)}s_3$ and the action extends to $\mathcal{H}_k$ by linearity, also note: $||\mathbb{T}_f^{(k)}|| = ||T_{f;1}^{(k)}||\ ||T_{f;2}^{(k)}||\ ||T_{f;3}^{(k)}||)$.

In the proofs below, we shall need the following elementary statement.

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LEMMA 4.9. *If M_j, N_j are linear operators on a finite dimensional Hilbert space V_j $(j = 1, 2, 3)$, then*

$$\begin{aligned}||M_1 \otimes M_2 \otimes M_3 - N_1 \otimes N_2 \otimes N_3|| &\leq ||M_1 - N_1||\, ||M_2 - N_2||\, ||M_3 - N_3|| \\ &+ ||M_1 - N_1||\, ||M_2||\, ||N_3|| + ||M_1||\, ||N_2||\, ||M_3 - N_3|| + ||N_1||\, ||M_2 - N_2||\, ||M_3||.\end{aligned}$$

Proof. This immediately follows from the equality

$$\begin{aligned}(M_1 - N_1) \otimes (M_2 - N_2) \otimes (M_3 - N_3) &= M_1 \otimes M_2 \otimes M_3 - N_1 \otimes N_2 \otimes N_3 \\ &- (M_1 - N_1) \otimes M_2 \otimes N_3 - M_1 \otimes N_2 \otimes (M_3 - N_3) - N_1 \otimes (M_2 - N_2) \otimes M_3.\end{aligned}$$

□

We also note the following identity for tensor products of operators:

$$\begin{aligned}[A_1 \otimes A_2 \otimes A_3, B_1 \otimes B_2 \otimes B_3] &= [A_1, B_1] \otimes [A_2, B_2] \otimes [A_3, B_3] \\ &+ [A_1, B_1] \otimes B_2A_2 \otimes A_3B_3 + A_1B_1 \otimes [A_2, B_2] \otimes B_3A_3 + B_1A_1 \otimes A_2B_2 \otimes [A_3, B_3].\end{aligned} \tag{8}$$

REMARK 4.10.

- For $f \in C^\infty(M)$, there is a constant $C = C(f) > 0$ such that, as $k \to \infty$,

$$(|f|_\infty - \frac{C}{k})^3 \leq ||\mathbb{T}_f^{(k)}|| \leq (|f|_\infty)^3.$$

- For $f_1, \ldots, f_p \in C^\infty(M)$

$$||\mathbb{T}_{f_1}^{(k)} \ldots \mathbb{T}_{f_p}^{(k)} - \mathbb{T}_{f_1 \ldots f_p}^{(k)}|| = O\left(\frac{1}{k}\right)$$

  as $k \to \infty$.

The last statement holds for $p = 2$ by Lemma 4.9, Theorem 2.9 and Proposition 2.10. It follows for arbitrary p by induction.

PROPOSITION 4.11. *For $f, g \in C^\infty(M)$*

$$||(ik)^3[T_{f;1}^{(k)}, T_{g;1}^{(k)}] \otimes [T_{f;2}^{(k)}, T_{g;2}^{(k)}] \otimes [T_{f;3}^{(k)}, T_{g;3}^{(k)}] - T_{\{f,g\}_1;1}^{(k)} \otimes T_{\{f,g\}_2;2}^{(k)} \otimes T_{\{f,g\}_3;3}^{(k)}|| = O\left(\frac{1}{k}\right)$$

as $k \to \infty$.

Proof. This follows from Lemma 4.9, Theorem 2.9 and Remark 2.12. □

PROPOSITION 4.12. *For $f, g \in C^\infty(M)$*

$$\begin{aligned}||ik[\mathbb{T}_f^{(k)}, \mathbb{T}_g^{(k)}] - (T_{\{f,g\}_1;1}^{(k)} \otimes T_{fg;2}^{(k)} \otimes T_{fg;3}^{(k)} + T_{fg;1}^{(k)} \otimes T_{\{f,g\}_2;2}^{(k)} \otimes T_{fg;3}^{(k)} \\ + T_{fg;1}^{(k)} \otimes T_{fg;2}^{(k)} \otimes T_{\{f,g\}_3;3}^{(k)})|| = O\left(\frac{1}{k}\right)\end{aligned}$$

as $k \to \infty$.

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Proof. Using (8), we get:

$$\begin{aligned}
&||ik[\mathbb{T}_f^{(k)}, \mathbb{T}_g^{(k)}] - (T^{(k)}_{\{f,g\}_1;1} \otimes T^{(k)}_{fg;2} \otimes T^{(k)}_{fg;3} + T^{(k)}_{fg;1} \otimes T^{(k)}_{\{f,g\}_2;2} \otimes T^{(k)}_{fg;3} \\
&\quad + T^{(k)}_{fg;1} \otimes T^{(k)}_{fg;2} \otimes T^{(k)}_{\{f,g\}_3;3})|| \\
&\le ||ik[T^{(k)}_{f;1}, T^{(k)}_{g;1}] \otimes T^{(k)}_{g;2} T^{(k)}_{f;2} \otimes T^{(k)}_{f;3} T^{(k)}_{g;3} - T^{(k)}_{\{f,g\}_1;1} \otimes T^{(k)}_{fg;2} \otimes T^{(k)}_{fg;3}|| \\
&\quad + ||T^{(k)}_{f;1} T^{(k)}_{g;1} \otimes ik[T^{(k)}_{f;2}, T^{(k)}_{g;2}] \otimes T^{(k)}_{g;3} T^{(k)}_{f;3} - T^{(k)}_{fg;1} \otimes T^{(k)}_{\{f,g\}_2;2} \otimes T^{(k)}_{fg;3}|| \\
&\quad + ||T^{(k)}_{g;1} T^{(k)}_{f;1} \otimes T^{(k)}_{f;2} T^{(k)}_{g;2} \otimes ik[T^{(k)}_{f;3}, T^{(k)}_{g;3}] - T^{(k)}_{fg;1} \otimes T^{(k)}_{fg;2} \otimes T^{(k)}_{\{f,g\}_3;3}|| \\
&\quad + k||[T^{(k)}_{f;1}, T^{(k)}_{g;1}] \otimes [T^{(k)}_{f;2}, T^{(k)}_{g;2}] \otimes [T^{(k)}_{f;3}, T^{(k)}_{g;3}]||.
\end{aligned}$$

Each of the first three terms is $O\left(\frac{1}{k}\right)$ by Lemma 4.9, Theorem 2.9, Proposition 2.10 and Remark 2.12. The last term is $O(\frac{1}{k^2})$ by Remark 2.12. □

COROLLARY 4.13. *For $f, g \in C^\infty(M)$*

$$||[\mathbb{T}_f^{(k)}, \mathbb{T}_g^{(k)}]|| = O\left(\frac{1}{k}\right)$$

as $k \to \infty$.

Proof. Follows from Proposition 4.12 and Theorem 2.9(ii) by triangle inequality. □

COROLLARY 4.14. *For $f, g, h, t \in C^\infty(M)$*

$$||[\mathbb{T}_f^{(k)}, \mathbb{T}_g^{(k)}, \mathbb{T}_h^{(k)}, \mathbb{T}_t^{(k)}]|| = O(\frac{1}{k^2})$$

as $k \to \infty$.

Proof. Follows from equality (1) and Corollary 4.13 by triangle inequality. □

PROPOSITION 4.15. *For $f, g, h, t \in C^\infty(M)$*

$$\begin{aligned}
&|| - \frac{k^6}{8}[T^{(k)}_{f;1}, T^{(k)}_{g;1}, T^{(k)}_{h;1}, T^{(k)}_{t;1}] \otimes [T^{(k)}_{f;2}, T^{(k)}_{g;2}, T^{(k)}_{h;2}, T^{(k)}_{t;2}] \otimes [T^{(k)}_{f;3}, T^{(k)}_{g;3}, T^{(k)}_{h;3}, T^{(k)}_{t;3}] \\
&\quad - T^{(k)}_{\{f,g,h,t\}_1;1} \otimes T^{(k)}_{\{f,g,h,t\}_2;2} \otimes T^{(k)}_{\{f,g,h,t\}_3;3}|| = O(\frac{1}{k^2})
\end{aligned}$$

as $k \to \infty$.

Proof. For $r = 1, 2, 3$,

$$||[T^{(k)}_{f;r}, T^{(k)}_{g;r}, T^{(k)}_{h;r}, T^{(k)}_{t;r}]|| = O(\frac{1}{k^2})$$

as $k \to \infty$ (this follows by triangle inequality from (1) and Remark 2.12). The statement now follows from Lemma 4.9, Proposition 4.3 and Theorem 2.9 (ii). □

It is natural to ask about asymptotics of $[\mathbb{T}_f^{(k)}, \mathbb{T}_g^{(k)}, \mathbb{T}_h^{(k)}, \mathbb{T}_t^{(k)}]$ for given $f, g, h, t \in C^\infty(M)$. Proposition 4.12 dictates the following very technical statement.

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THEOREM 4.16. *For* $f_1, f_2, f_3, f_4 \in C^\infty(M)$

$$||-\frac{k^2}{2}[\mathbb{T}^{(k)}_{f_1}, \mathbb{T}^{(k)}_{f_2}, \mathbb{T}^{(k)}_{f_3}, \mathbb{T}^{(k)}_{f_4}] - \mathbb{W}^{(k)}_{f_1,f_2,f_3,f_4}|| = O\left(\frac{1}{k}\right)$$

as $k \to \infty$, *where*

$$\begin{aligned}
&\mathbb{W}^{(k)}_{f_1,f_2,f_3,f_4} = T^{(k)}_{\{f_1,f_2,f_3,f_4\}_1;1} \otimes T^{(k)}_{f_1f_2f_3f_4;2} \otimes T^{(k)}_{f_1f_2f_3f_4;3}\\
&\quad + T^{(k)}_{f_1f_2f_3f_4;1} \otimes T^{(k)}_{\{f_1,f_2,f_3,f_4\}_2;2} \otimes T^{(k)}_{f_1f_2f_3f_4;3} + T^{(k)}_{f_1f_2f_3f_4;1} \otimes T^{(k)}_{f_1f_2f_3f_4;2} \otimes T^{(k)}_{\{f_1,f_2,f_3,f_4\}_3;3}\\
&\quad + \sum_{\substack{(i,j,m,l)=(1,2,3,4),\\(1,3,2,4),(1,4,2,3)}} \mathrm{sign}(i,j,m,l)\Big[T^{(k)}_{f_if_j\{f_m,f_l\}_1;1} \otimes (T^{(k)}_{f_mf_l\{f_i,f_j\}_2;2} \otimes T^{(k)}_{f_if_jf_mf_l;3}\\
&\quad + T^{(k)}_{f_if_jf_mf_l;2} \otimes T^{(k)}_{f_mf_l\{f_i,f_j\}_3;3}) + T^{(k)}_{f_mf_l\{f_i,f_j\}_1;1}\\
&\quad \otimes (T^{(k)}_{f_if_j\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_if_jf_mf_l;3} + T^{(k)}_{f_if_jf_mf_l;2} \otimes T^{(k)}_{f_if_j\{f_m,f_l\}_3;3})\\
&\quad + T^{(k)}_{f_if_jf_mf_l;1} \otimes (T^{(k)}_{f_if_j\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_mf_l\{f_i,f_j\}_3;3} + T^{(k)}_{f_mf_l\{f_i,f_j\}_2;2} \otimes T^{(k)}_{f_if_j\{f_m,f_l\}_3;3})\Big].
\end{aligned}$$

Proof. First, we observe: as $k \to \infty$

$$\begin{aligned}
&||(ik)^2[\mathbb{T}^{(k)}_{f_i}, \mathbb{T}^{(k)}_{f_j}][\mathbb{T}^{(k)}_{f_m}, \mathbb{T}^{(k)}_{f_l}] - (T^{(k)}_{\{f_i,f_j\}_1;1} \otimes T^{(k)}_{f_if_j;2} \otimes T^{(k)}_{f_if_j;3}\\
&\quad + T^{(k)}_{f_if_j;1} \otimes T^{(k)}_{\{f_i,f_j\}_2;2} \otimes T^{(k)}_{f_if_j;3} + T^{(k)}_{f_if_j;1} \otimes T^{(k)}_{f_if_j;2} \otimes T^{(k)}_{\{f_i,f_j\}_3;3})\\
&\quad \times (T^{(k)}_{\{f_m,f_l\}_1;1} \otimes T^{(k)}_{f_mf_l;2} \otimes T^{(k)}_{f_mf_l;3} + T^{(k)}_{f_mf_l;1} \otimes T^{(k)}_{\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_mf_l;3}\\
&\quad + T^{(k)}_{f_mf_l;1} \otimes T^{(k)}_{f_mf_l;2} \otimes T^{(k)}_{\{f_m,f_l\}_3;3})|| = O\left(\frac{1}{k}\right).
\end{aligned}$$

This follows from the elementary inequality

$$\begin{aligned}
&||M_1M_2 - N_1N_2|| = ||M_1M_2 - M_2N_1 + M_2N_1 - N_1N_2||\\
&\quad \leq ||M_2||||M_1 - N_1|| + ||N_1||||M_2 - N_2||
\end{aligned}$$

by setting

$$\begin{aligned}
&M_1 = ik[\mathbb{T}^{(k)}_{f_i}, \mathbb{T}^{(k)}_{f_j}],\ M_2 = ik[\mathbb{T}^{(k)}_{f_m}, \mathbb{T}^{(k)}_{f_l}]\\
&N_1 = T^{(k)}_{\{f_i,f_j\}_1;1} \otimes T^{(k)}_{f_if_j;2} \otimes T^{(k)}_{f_if_j;3} + T^{(k)}_{f_if_j;1} \otimes T^{(k)}_{\{f_i,f_j\}_2;2} \otimes T^{(k)}_{f_if_j;3}\\
&\quad + T^{(k)}_{f_if_j;1} \otimes T^{(k)}_{f_if_j;2} \otimes T^{(k)}_{\{f_i,f_j\}_3;3},\\
&N_2 = T^{(k)}_{\{f_m,f_l\}_1;1} \otimes T^{(k)}_{f_mf_l;2} \otimes T^{(k)}_{f_mf_l;3} + T^{(k)}_{f_mf_l;1} \otimes T^{(k)}_{\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_mf_l;3}\\
&\quad + T^{(k)}_{f_mf_l;1} \otimes T^{(k)}_{f_mf_l;2} \otimes T^{(k)}_{\{f_m,f_l\}_3;3},
\end{aligned}$$

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with the use of Theorem 2.9 (ii), Proposition 4.12 and Corollary 4.13. Next, using Lemma 4.9, Theorem 2.9 (ii) and Proposition 2.10, we get:

$$\begin{aligned}
&|| - k^2[\mathbb{T}^{(k)}_{f_i}, \mathbb{T}^{(k)}_{f_j}][\mathbb{T}^{(k)}_{f_m}, \mathbb{T}^{(k)}_{f_l}] - \Big[T^{(k)}_{\{f_i,f_j\}_1\{f_m,f_l\}_1;1} \otimes T^{(k)}_{f_if_jf_mf_l;2} \otimes T^{(k)}_{f_if_jf_mf_l;3} \\
&\quad + T^{(k)}_{f_if_jf_mf_l;1} \otimes T^{(k)}_{\{f_i,f_j\}_2\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_if_jf_mf_l;3} + T^{(k)}_{f_if_jf_mf_l;1} \otimes T^{(k)}_{f_if_jf_mf_l;2} \otimes T^{(k)}_{\{f_i,f_j\}_3\{f_m,f_l\}_3;3} \\
&\quad + T^{(k)}_{f_if_j\{f_m,f_l\}_1;1} \otimes (T^{(k)}_{f_mf_l\{f_i,f_j\}_2;2} \otimes T^{(k)}_{f_if_jf_mf_l;3} + T^{(k)}_{f_if_jf_mf_l;2} \otimes T^{(k)}_{f_mf_l\{f_i,f_j\}_3;3}) \\
&\quad + T^{(k)}_{f_mf_l\{f_i,f_j\}_1;1} \otimes (T^{(k)}_{f_if_j\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_if_jf_mf_l;3} + T^{(k)}_{f_if_jf_mf_l;2} \otimes T^{(k)}_{f_if_j\{f_m,f_l\}_3;3}) \\
&\quad + T^{(k)}_{f_if_jf_mf_l;1} \otimes (T^{(k)}_{f_if_j\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_mf_l\{f_i,f_j\}_3;3} + T^{(k)}_{f_mf_l\{f_i,f_j\}_2;2} \otimes T^{(k)}_{f_if_j\{f_m,f_l\}_3;3})\Big]|| = O\left(\frac{1}{k}\right).
\end{aligned}$$

After that, we note:

$$[\mathbb{T}^{(k)}_{f_1}, \mathbb{T}^{(k)}_{f_2}, \mathbb{T}^{(k)}_{f_3}, \mathbb{T}^{(k)}_{f_4}] = \sum_{\substack{(i,j,m,l)= \\ (1,2,3,4),(1,3,2,4),(1,4,2,3), \\ (3,4,1,2),(2,4,1,3),(2,3,1,4)}} \mathrm{sign}(i,j,m,l)[\mathbb{T}^{(k)}_{f_i}, \mathbb{T}^{(k)}_{f_j}][\mathbb{T}^{(k)}_{f_m}, \mathbb{T}^{(k)}_{f_l}]$$

(see (1)). Taking the sum, we get:

$$|| - k^2[\mathbb{T}^{(k)}_{f_1}, \mathbb{T}^{(k)}_{f_2}, \mathbb{T}^{(k)}_{f_3}, \mathbb{T}^{(k)}_{f_4}] - 2\mathbb{W}^{(k)}_{f_1,f_2,f_3,f_4}|| = O\left(\frac{1}{k}\right).$$

□

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REFERENCES

1. J. Andersen, Asymptotic faithfulness of the quantum SU(n) representations of the mapping class groups, *Ann. Math.* (2) **163**(1) (2006), 347–368.

2. J. Andersen, Toeplitz operators and Hitchin's projectively flat connection, in *The many facets of geometry* (Oxford Univ. Press, Oxford, 2010), 177–209.

3. M. Artin, *Algebra* (Prentice Hall, Inc., Englewood Cliffs, NJ, 1991).

4. J. A. de Azcárraga, J. M. Izquierdo, n-ary algebras: A review with applications, *J. Phys. A: Math. Theor.* **43** (2010), 293001.

5. J. A. de Azcárraga, A. M. Perelomov and J. C. Pérez Bueno, New generalized Poisson structures, *J. Phys. A* **29**(7) (1996), L151–L157.

6. J. Baez, A. Hoffnung and C. Rogers, Categorified symplectic geometry and the classical string, *Comm. Math. Phys.* **293**(3) (2010), 701–725.

7. J. Baez and C. Rogers. Categorified symplectic geometry and the string Lie 2-algebra, *Homology, Homotopy Appl.* **12**(1) (2010), 221–236.

8. T. Barron, X. Ma, G. Marinescu and M. Pinsonnault, Semi-classical properties of Berezin-Toeplitz operators with C^k-symbol, *J. Math. Phys.* **55** (2014), 042108.

9. F. Bayen and M. Flato, Remarks concerning Nambu's generalized mechanics, *Phys. Rev. D* (3) **11** (1975), 3049–3053.

10. F. Berezin, Quantization, *Math. USSR-Izv.* **8**(5) (1974), 1109–1165 (1975).

11. A. Besse, *Einstein manifolds* (Springer-Verlag, Berlin, 1987).

12. M. Bordemann, E. Meinrenken and M. Schlichenmaier, Toeplitz quantization of Kähler manifolds and gl(N), $N \to \infty$ limits, *Comm. Math. Phys.* **165**(2) (1994), 281–296.

13. L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Annals of Mathematics Studies, vol. 99 (Princeton University Press, Princeton, NJ, University of Tokyo Press, Tokyo, 1981).

14. M. Bremner, Identities for the ternary commutator, *J. Algebra* **206**(2) (1998), 615–623.

15. H. Bursztyn, A. Cabrera and D. Iglesias, Multisymplectic geometry and Lie groupoids, in *Geometry, mechanics and dynamics.* Fields Inst. Commun., vol. 73 (Springer, New York, 2015), 57–73.

16. F. Cantrijn, A. Ibort and M. de León, On the geometry of multisymplectic manifolds, *J. Austral. Math. Soc. Ser. A* **66**(3) (1999), 303–330.

17. R. Chatterjee and L. Takhtajan, Aspects of classical and quantum Nambu mechanics, *Lett. Math. Phys.* **37**(4) (1996), 475–482.

18. T. Curtright, X. Jin and L. Mezincescu, Multi-operator brackets acting thrice, *J. Phys. A* **42**(46) (2009), 462001.

19. T. Curtright and C. Zachos, Classical and quantum Nambu mechanics, *Phys. Rev. D.* **68** (2003), 085001, 1–29.

20. T. Curtright and C. Zachos, Nambu dynamics, deformation quantization, and superintegrability, in *Superintegrability in classical and quantum systems,* CRM Proc. Lecture Notes, vol. 37 (Amer. Math. Soc., Providence, RI, 2004), 29–46.

21. J. DeBellis, C. Sämann and R. Szabo, Quantized Nambu-Poisson manifolds and n-Lie algebras, *J. Math. Phys.* **51**(12) (2010), 122303.

22. G. Dito, M. Flato, D. Sternheimer and L. Takhtajan, Deformation quantization and Nambu mechanics, *Comm. Math. Phys.* **183**(1) (1997), 1–22.

23. V. Filippov, n-Lie algebras, *Siberian Math. J.* **26**(6) (1985), 879–891.

24. T. Foth and A. Uribe, The manifold of compatible almost complex structures and geometric quantization, *Comm. Math. Phys.* **274**(2) (2007), 357–379.

25. P. Gautheron, Some remarks concerning Nambu mechanics, *Lett. Math. Phys.* **37**(1) (1996), 103–116.

26. F. Hélein, Hamiltonian formalisms for multidimensional calculus of variations and perturbation theory, in *Noncompact problems at the intersection of geometry, analysis, and topology,* Contemp. Math., vol. 350 (Amer. Math. Soc., Providence, RI, 2004).

27. D. Huybrechts, Hyperkähler manifolds and sheaves, in *Proceedings of the International Congress of Mathematicians*, vol. II, (Hindustan Book Agency, New Delhi, 2010), 450–460.

28. R. Ibàñez, M. de León, J. Marrero and D. Martín de Diego, Dynamics of generalized Poisson and Nambu-Poisson brackets, *J. Math. Phys.* **38**(5) (1997), 2332–2344.

29. M. Verbitsky and D. Kaledin, *Hyperkahler manifolds*, Mathematical Physics (Somerville), vol. 12 (International Press, Somerville, MA, 1999).

30. A. Karabegov and M. Schlichenmaier, Identification of Berezin-Toeplitz deformation quantization, *J. Reine Angew. Math.* **540** (2001), 49–76.

31. X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, Progress in Mathematics, vol. 254 (Birkhäuser Verlag, Basel, 2007).

32. X. Ma and G. Marinescu, Toeplitz operators on symplectic manifolds, *J. Geom. Anal.* **18**(2) (2008), 565–611.

33. X. Ma and G. Marinescu, Berezin-Toeplitz quantization and its kernel expansion, *Geometry and quantization*, Trav. Math., vol. 19 (Univ. Luxemb., Luxembourg, 2011), 125–166.

34. T. Madsen and A. Swann, Multi-moment maps, *Adv. Math.* **229**(4) (2012), 2287–2309.

35. G. Martin, A Darboux theorem for multi-symplectic manifolds, *Lett. Math. Phys.* **16**(2) (1988), 133–138.

36. N. Mukunda and E. Sudarshan, Relation between Nambu and Hamiltonian mechanics, *Phys. Rev. D (3)* **13**(10) (1976), 2846–2850.

37. Y. Nambu, Generalized Hamiltonian dynamics, *Phys. Rev. D* (3) **7** (1973), 2405–2412.

38. L. Polterovich, Quantum unsharpness and symplectic rigidity, *Lett. Math. Phys.* **102**(3) (2012), 245–264.

39. C. Rogers, m L_∞-algebras from multisymplectic geometry, *Lett. Math. Phys.* **100**(1) (2012), 29–50.

40. C. Rogers, 2-plectic geometry, Courant algebroids, and categorified prequantization, *J. Symplectic Geom.* **11**(1) (2013), 53–91.

41. Y. Rubinstein and S. Zelditch, The Cauchy problem for the homogeneous Monge-Ampère equation, I. Toeplitz quantization, *J. Differential Geom.* **90**(2) (2012), 303–327.

42. C. Sämann and R. Szabo, Groupoids, loop spaces and quantization of 2-plectic manifolds, *Rev. Math. Phys.* **25**(03) (2013), 1330005.

43. M. Schlichenmaier, *Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization*, Conférence Moshé Flato 1999, Vol. II (Dijon), Math. Phys. Stud., vol. 22 (Kluwer Acad. Publ., Dordrecht, 2000), 289–306.

44. M. Schlichenmaier, Berezin-Toeplitz quantization for compact Kähler manifolds. A review of results, *Adv. Math. Phys.* (2010), Article ID 927280, 38 pages.

45. L. Takhtajan, On foundation of the generalized Nambu mechanics, *Comm. Math. Phys.* **160**(2) (1994), 295–315.

46. I. Vaisman, A survey on Nambu-Poisson brackets, *Acta Math. Univ. Comenian. (N.S.)* **68**(2) (1999), 213–241.