The Bergman Kernel on Toric Kähler Manifolds

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Abstract

Let $(L, h) \to (X, \omega)$ be a compact toric polarized Kähler manifold of complex dimension n. For each $k \in \mathbb{N}$, the fibre-wise Hermitian metric h^k on L^k induces a natural inner product on the vector space $\mathcal{C}^\infty(X, L^k)$ of smooth global sections of L^k by integration with respect to the volume form $\frac{\omega^n}{n!}$. The orthogonal projection $P_k : \mathcal{C}^\infty(X, L^k) \to H^0(X, L^k)$ onto the space $H^0(X, L^k)$ of global holomorphic sections of L^k is represented by an integral kernel B_k which is called the Bergman kernel (with parameter $k \in \mathbb{N}$). The restriction $\rho_k : X \to \mathbb{R}$ of the norm of B_k to the diagonal in $X \times X$ is called the density function of B_k.

On a dense subset of X, we describe a method for computing the coefficients of the asymptotic expansion of ρ_k as $k \to \infty$ in this toric setting. We also provide a direct proof of a result which illuminates the off-diagonal decay behaviour of toric Bergman kernels.

We fix a parameter $l \in \mathbb{N}$ and consider the projection $P_{l,k}$ from $\mathcal{C}^\infty(X, L^k)$ onto those global holomorphic sections of L^k that vanish to order at least lk along some toric submanifold of X. There exists an associated toric partial Bergman kernel $B_{l,k}$ giving rise to a toric partial density function $\rho_{l,k} : X \to \mathbb{R}$. For such toric partial density functions, we determine new asymptotic expansions over certain subsets of X as $k \to \infty$. Euler-Maclaurin sums and Laplace's method are utilized as important tools for this. We discuss the case of a polarization of $\mathbb{CP}^n$ in detail and also investigate the non-compact Bargmann-Fock model with imposed vanishing at the origin.

We then discuss the relationship between the slope inequality and the asymptotics of Bergman kernels with vanishing and study how a version of Song and Zelditch's toric localization of sums result generalizes to arbitrary polarized Kähler manifolds.

Finally, we construct families of induced metrics on blow-ups of polarized Kähler manifolds. We relate those metrics to partial density functions and study their properties for a specific blow-up of $\mathbb{C}^n$ and $\mathbb{CP}^n$ in more detail.

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Florian T. Pokorny*)

Contents

Notation

$\mathbb{N}_0$	The natural numbers with 0
$H^0(X,E)$	Holomorphic global sections of a holomorphic vector bundle $E \to X$, where X is a complex manifold
$\mathcal{C}^\infty(X,E)$	Smooth global sections of a smooth vector bundle $E \to X$
$\boldsymbol{z}^{\boldsymbol{\alpha}}$	For $\boldsymbol{z} = (z_1, \cdots, z_n) \in \mathbb{C}^n$ and $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$, $\boldsymbol{z}^{\boldsymbol{\alpha}} \overset{\text{def}}{=} \prod_{i=1}^n z_i^{\alpha_i}$ whenever this is well-defined.
$\boldsymbol{\alpha}!$	For $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, $\boldsymbol{\alpha}! \overset{\text{def}}{=} \prod_{i=1}^n \alpha_i!$.
$\lvert\boldsymbol{\alpha}\rvert$	For $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, $\lvert\boldsymbol{\alpha}\rvert \overset{\text{def}}{=} \sum_{i=1}^n \alpha_i$.
$\binom{j}{\boldsymbol{\alpha}}$	For $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, $j = \lvert\boldsymbol{\alpha}\rvert$ and $n > 1$, we define $\binom{j}{\boldsymbol{\alpha}} \overset{\text{def}}{=} \frac{j!}{\alpha_1! \ldots \alpha_n!}$.
$\frac{\partial}{\partial \boldsymbol{x}}^{\boldsymbol{\alpha}} f$	$\frac{\partial}{\partial \boldsymbol{x}}^{\boldsymbol{\alpha}} f \overset{\text{def}}{=} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \circ \cdots \circ \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f$ for a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^n$.
F_h	Curvature of the Chern connection corresponding to a Hermitian metric h on some holomorphic vector bundle $E \to X$ over a complex manifold X
$\mathrm{Int}(P)$	Interior of P
$\mathrm{RelInt}(P)$	The relative interior of P
$\mathcal{O}\left(k^{-\infty}\right)$	$f(k) = \mathcal{O}\left(k^{-\infty}\right)$ if and only if, for any $n > 0$, there exists $C_n \geq 0$ such that $\lvert f(k)\rvert \leq C_n k^{-n}$ for all $k \gg 0$.
$\mathcal{Z}(f)$	The zero set of some function f
$\mathfrak{g}$	The Lie algebra corresponding to a Lie group G
$M_{n\times n}(\mathbb{Z})$	The vector space of n by n matrices with $\mathbb{Z}$ entries
$A > 0$	For a square matrix A, $A > 0$ denotes that A is positive definite.

Chapter 1

Introduction

1.1 Background and motivation

Let X be a compact Kähler manifold of complex dimension n with Kähler form ω. Let $L \to X$ be a holomorphic line bundle such that $\omega \in 2\pi c_1(L)$. Up to a constant scale factor, there exists a unique Hermitian metric h on L such that the curvature F_h of the corresponding Chern connection ∇_h satisfies

$$\omega = iF_h.$$

We call such an arrangement $(L,h) \to (X,\omega)$ a *polarized Kähler manifold*. For each $k \in \mathbb{N}$, h^k gives a Hermitian fibre-wise inner product

$$(s(x), \hat{s}(x))_{h^k} \in \mathbb{C} \qquad \text{for } x \in X \text{ and } s, \hat{s} \in \mathcal{C}^\infty(X, L^k)$$

and a global inner product

$$\langle s, \hat{s}\rangle_{h^k} \stackrel{\text{def}}{=} \int_X (s,\hat{s})_{h^k} \frac{\omega^n}{n!} \qquad \text{for } s, \hat{s} \in \mathcal{C}^\infty(X, L^k) \text{ and } k \in \mathbb{N}.$$

We will omit the h^k indices if it is clear from the context which power of the line bundle we are considering. For $k \in \mathbb{N}$, let $\{s_{1,k}, \ldots, s_{N_k,k}\}$ denote an orthonormal basis of $(H^0(X, L^k), \langle , \rangle_{h^k})$. We define the Bergman kernel as

$$B_k(x,y) \stackrel{\text{def}}{=} \sum_{i=1}^{N_k} s_{i,k}(x) \otimes \overline{s}_{i,k}(y) \qquad \text{for } x, y \in X.$$

B_k is a smooth section of the line bundle $\pi_1^*(L^k) \otimes \pi_2^*(\overline{L}^k) \to X \times X$, where $\pi_i : X \times X \to X$, for $i \in \{1,2\}$, denotes the projection onto the i^{th} factor. We regard $B_k(.,y)$ as a section of L^k with values in $\overline{L}_y^k$ and have $\langle s, B_k(.,y)\rangle = s(y)$ for $s \in H^0(X, L^k)$ and $y \in X$ (see [Ber03]). B_k can be considered as the integral kernel representing the orthogonal projection P_k from smooth to holomorphic global sections of L^k.

$$P_k : \mathcal{C}^\infty(X, L^k) \to H^0(X, L^k)$$
$$P_k(s)(x) \stackrel{\text{def}}{=} \sum_{j=1}^{N_k} \langle s, s_{j,k}\rangle_{h^k} s_{j,k}(x) = \langle s, B_k(.,x)\rangle$$

for $x \in X, k \in \mathbb{N}$ and $s \in \mathcal{C}^\infty(X, L^k)$.

Note that there exists a natural fibre-wise Hermitian norm on $\pi_1^*(L^k) \otimes \pi_2^*(\overline{L}^k)$ induced by h^k, so that we can talk about $|B_k|_{h^k} : X \times X \to \mathbb{R}$. The norm of the diagonal of the Bergman kernel B_k is called the *density (of states) function*, and we denote it by $\rho_k : X \to \mathbb{R}$. We have

$$\rho_k(x) \stackrel{\text{def}}{=} |B_k(x,x)|_{h^k} = \sum_{i=1}^{N_k} |s_{i,k}(x)|^2_{h^k} \qquad \text{for } x \in X \text{ and } k \in \mathbb{N}.$$

From a geometrical point of view, ρ_k is a very interesting function to study. Of particular interest is the following asymptotic expansion:

Theorem 1.1.1 (Catlin [Cat99], Lu [Lu00], Tian [Tia90], Yau, Zelditch [Zel98])**.** *There is a complete asymptotic expansion*

$$\rho_k(x) \sim \sum_{j=0}^{\infty} a_j(x) k^{n-j} \qquad \textit{for } x \in X \textit{ and as } k \to \infty,$$

for certain smooth functions $\{a_j\}_{j=0}^\infty$ *on* X*, with* $a_0(x) = \frac{1}{(2\pi)^n}$ *and* $a_1(x) = \frac{1}{2}\frac{1}{(2\pi)^n}\operatorname{Scal}(x)$ *for* $x \in X$*. More precisely, for any* $R, r \in \mathbb{N}$*, there exists a constant* $C_{R,r} \geq 0$*, depending on* R, r *and the manifold* (X, ω)*, such that*

$$\left| \rho_k(x) - \sum_{j<R} a_j(x) k^{n-j} \right|_{\mathcal{C}^r(X)} \leq C_{R,r} k^{n-R} \qquad \textit{for all } k \in \mathbb{N}_0 \textit{ and } x \in X.$$

Remark 1.1.2. *Note that we assume* $\omega \in 2\pi c_1(L)$ *and not* $\omega \in c_1(L)$*. Also, we associate the metric* $g = g_{\alpha\overline{\beta}}(dz_\alpha \otimes d\overline{z}_\beta + d\overline{z}_\beta \otimes dz_\alpha)$ *to* $\omega = i g_{\alpha\overline{\beta}} dz_\alpha \wedge d\overline{z}_\beta$ *in local holomorphic coordinates which differs from the normalizations chosen by the authors above in the original version of this theorem. The version of the theorem above is due to Zelditch and Lu, but this result is also related to ideas by Tian [Tia90] and Yau. Note that Catlin [Cat99] proved a version of this theorem independently, while Lu's [Lu00] contribution to the above theorem was to determine the first few terms explicitly. Yet another approach to the asymptotic expansion of* ρ_k *can be found in [BBS08].*

Let us now relate the above theorem to a well-known classical result. The celebrated Hirzebruch-Riemann-Roch theorem yields the following asymptotic expansion:

$$\begin{aligned}
\dim H^0(X, L^k) &= \int_X \left\{ \left(1 + c_1(L^k) + \frac{c_1(L^k)^2}{2!} + \cdots + \frac{c_1(L^k)^n}{n!}\right) \right. \\
&\qquad \left. \left(1 + \frac{1}{2}c_1(X) + \frac{1}{12}\left(c_1(X)^2 + c_2(X)\right) + \ldots \right) \right\} \\
&= k^n \int_X \frac{c_1(L)^n}{n!} + k^{n-1} \int_X \frac{c_1(L)^{n-1} c_1(X)}{2(n-1)!} + \ldots \\
&= \frac{\operatorname{Vol}(X,\omega)}{(2\pi)^n} k^n + \frac{\frac{1}{2}\widehat{\operatorname{Scal}}}{(2\pi)^n} \operatorname{Vol}(X,\omega) k^{n-1} + \mathcal{O}(k^{n-2})
\end{aligned}$$

as $k \to \infty$, where $\widehat{\operatorname{Scal}}$ denotes the average scalar curvature over X. Noting that

$$\int_X \rho_k \frac{\omega^n}{n!} = \dim H^0(X, L^k),$$

we realize that the coefficients a_j in theorem 1.1.1 give back the topological coefficients of the expansion obtained by the Hirzebruch-Riemann-Roch theorem when integrated over X. A local understanding of the functions a_j, such as knowing upper or lower bounds, can hence yield not just geometric, but also *topological* information about $L \to X$. This relationship between the asymptotic expansion of the Bergman kernel and the geometry of X is one of the most interesting aspects of Bergman kernels.

Partial density functions

Let $Y \subset X$ be a complex submanifold and consider, for l and $k \in \mathbb{N}$, an orthonormal basis $\{s_{1,k}, \cdots, s_{M_k,k}\}$ of the space $\mathcal{J}_k^{lk}(Y)$ of global holomorphic sections of L^k vanishing to order at least lk along Y. We define the partial density function $\rho_{l,k} : X \to \mathbb{R}$ with vanishing along Y as

$$\rho_{l,k}(x) \stackrel{\text{def}}{=} \sum_{i=1}^{M_k} |s_{i,k}(x)|^2_{h^k} \quad \text{for } x \in X.$$

Unlike ρ_k, which we now understand relatively well due a sequence of works by Catlin, Lu, Tian, Yau, Zelditch and others (see *e.g.* [Zel98, Lu00]), the asymptotics of $\rho_{l,k}$ - and in fact also the mere existence of an asymptotic expansion similar to theorem 1.1.1 - are open problems. Partial density functions where Y is a divisor have been studied by Keller, Panov, Thomas and Székelyhidi [KPTS] (for Hermitian line bundles with positive curvature) and Berman [Ber07] (in more generality) who discovered that $\rho_{l,k}$ exhibits a “0-1” law in sense that the first order asymptotics of $\rho_{l,k}$ suddenly “switch on” over a certain subset of X.

Theorem 1.1.3 ([Ber07, Theorem 4.3])**.** *Let L be an ample holomorphic line bundle over a compact complex manifold X of dimension n. Fix a smooth volume form* dVol *on X and let $h = e^{-\phi}$ be a smooth Hermitian metric on L, locally represented by $|s_U(z)|_h^2 = e^{-\phi_U(z)}$ for $z \in U$ and a local trivializing section s_U of L over $U \subset X$. Let $Y \subset X$ be a divisor. Assume that the line bundle $L \otimes \mathcal{O}(-Y)$ over X is ample and let $\rho_{k,Y}$ denote the partial density function corresponding to the Hilbert space of sections vanishing to order at least k along Y. Then*

$$k^{-n}\rho_{k,Y} \to \mathbf{1}_{D_Y \cap X(0)} \det(dd^c\phi)$$

in $L^1(X, \mathrm{dVol})$, where $X(0) = \{x \in X : dd^c\phi > 0\}$, $D_Y = \{x \in X : \phi_{e,Y}(x) = \phi(x)\}$ and $\phi_{e,Y}$ is the equilibrium metric with poles along Y (see [Ber07]).

Remark 1.1.4. *Note that in the case where $(L, e^{-\phi}) \to (X, \omega)$ is a polarization and* $\mathrm{dVol} = \frac{\omega^n}{n!}$*, the above result states that*

$$k^{-n}\rho_{k,Y} \to \frac{1}{(2\pi)^n}\mathbf{1}_{D_Y}$$

in $L^1(X, \frac{\omega^n}{n!})$ and we have $X(0) = X$. The interested reader may consult [Ber09b, Ber09a, BBN07] and references therein for more details.

In this thesis, we will explore the asymptotics of partial density functions $\rho_{l,k}$ with vanishing to order at least lk along a submanifold Y. We will mainly focus on the case where X is a toric Kähler manifold with a toric polarization and torus invariant Kähler form and where Y is a toric submanifold of X. The torus symmetry of these manifolds simplifies the study of $\rho_{l,k}$ considerably since we are able to understand the space of holomorphic sections of toric line bundles very well.

Let us now give a quick overview of the chapters of this thesis. Here, we also review some of our main results.

1.2 Overview

Chapter 2 - The Bargmann-Fock Model

We start the discussion of the asymptotics of (partial) density functions in chapter 2 by first considering the *Bargmann-Fock* model, which will serve as an initial example for some of the features that come into play in the toric case. In the Bargmann-Fock model, we consider the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}^n} f(\boldsymbol{z})\overline{g(\boldsymbol{z})}e^{-\frac{k}{2}\|\boldsymbol{z}\|^2}\,\mathrm{dVol},$$

for $k \in \mathbb{N}$ and $f, g \in \mathcal{L}^2_k$, where $\mathcal{L}^2_k$ denotes the space of smooth functions $f : \mathbb{C}^n \to \mathbb{C}$ for which $\|f\|^2 = \langle f, f\rangle < \infty$ and where $\mathrm{dVol} = \frac{1}{n!}\left(i\partial\overline{\partial}\frac{\|\boldsymbol{z}\|^2}{2}\right)^n$ is the Euclidean volume form. We call the space $\mathcal{F}_k$ of holomorphic functions in $\mathcal{L}^2_k$ the Bargmann-Fock space with parameter $k \in \mathbb{N}$.

The Bergman kernel $B_k : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ associated to the Bargmann-Fock model is defined to be the integral kernel representing the orthogonal projection $P_k : \mathcal{L}^2_k \twoheadrightarrow \mathcal{F}_k$. The chapter focusses on the density function ρ_k associated to this Bergman kernel as well as on the corresponding density function $\rho_{l,k}$ coming from the projection $P_{l,k}$ onto the integrable holomorphic functions that vanish to order at least lk at the origin. In lemma 2.1.6, we provide a compact formula for this partial density function:

Lemma 2.1.6. The partial density function $\rho_{l,k} : \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ for $l, k \in \mathbb{N}$ is given by

$$\rho_{l,k}(\boldsymbol{z}) = \left(\frac{k}{2\pi}\right)^n \left(1 - \frac{\Gamma(kl, k\frac{\|\boldsymbol{z}\|^2}{2})}{\Gamma(kl, 0)}\right).$$

Proposition 2.2.4 gives a detailed description of the asymptotics of $\rho_{l,k}$ in the Bargman-Fock case. From this, corollary 2.2.6 follows:

Corollary 2.2.6. Suppose $\boldsymbol{z} \in \mathbb{C}^n$ is fixed. Then

$$\rho_{l,k}(\boldsymbol{z}) = \begin{cases} \left(\frac{k}{2\pi}\right)^n + \mathcal{O}\left(k^{-\infty}\right) & \text{if } \|\boldsymbol{z}\| > \sqrt{2l} \\ \frac{1}{2}\left(\frac{k}{2\pi}\right)^n + \sum_{j=0}^{\infty} c_{2j+1}k^{n-\left(j+\frac{1}{2}\right)} + \mathcal{O}\left(k^{-\infty}\right) & \text{if } \|\boldsymbol{z}\| = \sqrt{2l} \\ \mathcal{O}\left(k^{-\infty}\right) & \text{if } \|\boldsymbol{z}\| < \sqrt{2l}, \end{cases}$$

where c_{2j+1}, for $j \in \mathbb{N}_0$, are explicitly computable. In particular, we have

$$c_1 = \frac{1}{6(2\pi)^n}\sqrt{\frac{2}{\pi l}}$$
$$c_3 = \frac{1}{1080(2\pi)^n}\sqrt{\frac{2}{\pi l^3}}.$$

In the above corollary, the expansion for $\|\boldsymbol{z}\| = \sqrt{2l}$ is of particular interest. In chapter 7 we show that the asympotic expansion of the partial density function is of a similar form (see theorem 7.4.2).

Chapter 3 - Toric Geometry

Chapter 3 introduces some terminology and notation that we will need for our discussion of toric Kähler manifolds. We focus on a particularly simple set of coordinate charts determined by the vertices of an integral Delzant polytope P and follow Abreu's [Abr98] discussion of Legendre duality between a symplectic potential on P and a corresponding Kähler potential.

Chapter 4 - Example: $\mathbb{CP}^n$

To familiarize ourselves with the notation introduced in the previous chapter and to explore one of the simplest type of toric partial density function, we consider a polarization of $\mathbb{CP}^n$ with a multiple of the standard Fubini-Study metric.

Chapter 5 - Toric Localization

Chapter 5 introduces some new toric Bergman kernel estimates and a few tools that are helpful for our discussion of the Bergman kernel on polarized toric Kähler manifolds. Theorem 5.1.9, which is strongly inspired by Song and Zelditch's results on the localization of sums [SZ10, lemma 1.2, prop. 5.1], is then obtained by a rather straightforward application of the methods developed in this chapter.

Chapter 6 - Euler-Maclaurin Sums

Another important tool in the discussion of toric density functions are the *Euler-Maclaurin* sums that we explore in chapter 6. Crucially, the Euler-Maclaurin summation result of proposition 6.2.3 allows us to compute the asymptotic expansions of the sums appearing in the definition of toric partial density functions (at least for points lying in a dense subset of the toric variety under consideration).

Chapter 7 - Asymptotics

Here we focus our attention on one of the main results of this thesis. Let P be an integral Delzant polytope and consider a toric polarization $(L_P, h) \to (X_P, \omega)$. We now fix a nontrivial face $F < P$ and, for $s, k \in \mathbb{N}$, consider the corresponding partial density functions $\rho_{F,s,k}$ associated to the projection onto those holomorphic of sections of L_P^k vanishing to order at least sk along the toric submanifold $Y_F \subset X_P$ corresponding to F. Let ρ_k denote the standard density function. By invariance under the real torus action, we can think of ρ_k and $\rho_{F,s,k}$ as functions on $P \subset \mathfrak{t}^*$, where $\mathfrak{t}$ denotes the Lie algebra of the real torus acting on X_P. Suppose that

$$\begin{aligned} P \overset{\text{def}}{=} \Big\{ & \boldsymbol{\alpha} \in \mathfrak{t}^* : l_i(\boldsymbol{\alpha}) \overset{\text{def}}{=} \mu_i - \langle \boldsymbol{\alpha}, \boldsymbol{n}_i \rangle \geq 0 \text{ for some } \mu_i \in \mathbb{R}, \text{ primitive} \\ & \boldsymbol{n}_i \in \operatorname{Ker}(\exp) \subset \mathfrak{t} \text{ and } i \in \{1, \cdots, d\} \Big\}, \end{aligned}$$

and assume without loss of generality that $F = \cap_{i=1}^r \mathcal{Z}(l_i)$ for some $r \in \{1, \cdots, n\}$. For $s \in \mathbb{N}$, we define

$$P_{F,s} \overset{\text{def}}{=} \left\{ \boldsymbol{\alpha} \in P : \sum_{j=1}^r l_j(\boldsymbol{\alpha}) \geq s \right\}, \quad F_s \overset{\text{def}}{=} \left\{ \boldsymbol{\alpha} \in P : \sum_{j=1}^r l_j(\boldsymbol{\alpha}) = s \right\}.$$

Theorem 7.4.2. Let $(L_P, h) \to (X_P, \omega)$ be a toric polarization. Fix a nontrivial face $F < P$ and $s \in \mathbb{N}$. Let $\boldsymbol{\beta} \in \mathrm{Int}(P)$. Then

$$\rho_{F,s,k}(\boldsymbol{\beta}) = \begin{cases} \rho_k(\boldsymbol{\beta}) + \mathcal{O}\left(k^{-\infty}\right) & \text{if } \boldsymbol{\beta} \in \mathrm{Int}(P_{F,s}) \\ \frac{1}{2}\rho_k(\boldsymbol{\beta}) + \sum_{j=0}^{\infty} c_j(\boldsymbol{\beta}) k^{n-\left(j+\frac{1}{2}\right)} + \mathcal{O}\left(k^{-\infty}\right) & \text{if } \boldsymbol{\beta} \in \mathrm{RelInt}(F_s) \\ \mathcal{O}\left(k^{-\infty}\right) & \text{otherwise,} \end{cases}$$

where $c_j \in \mathcal{C}^{\infty}\left(\mathrm{Int}(P)\right)$ are explicitly computable functions. Now let $\mathcal{K} \subset \mathrm{Int}(P)$ be a compact set. For $p \in \mathbb{N}_0$ and $\boldsymbol{\beta} \in \mathcal{K} \cap \mathrm{RelInt}(F_s)$,

$$\rho_{F,s,k}(\boldsymbol{\beta}) = \frac{1}{2}\rho_k(\boldsymbol{\beta}) + \sum_{j=0}^{p} c_j(\boldsymbol{\beta}) k^{n-\left(j+\frac{1}{2}\right)} + S_{p,k}(\boldsymbol{\beta})$$

for all $k \in \mathbb{N}$, and there exists $D \geq 0$ such that $|S_{p,k}(\boldsymbol{\beta})| \leq D k^{n-\left(p+\frac{3}{2}\right)}$ for all $\boldsymbol{\beta} \in \mathcal{K} \cap \mathrm{RelInt}(F_s)$ and $k \gg 0$.

Chapter 8 - The Slope Inequality

In chapter 8, we review the notion of slope stability of a polarization with respect to a submanifold as discussed by Ross and Thomas [RT06]. We discuss this notion of slope stability in the toric setting and reformulate the toric slope inequality slightly in lemma 8.2.4. We then present an argument due to [KPTS] which shows that sufficient asymptotic information about partial density functions could yield a proof of the fact that the existence of a constant scalar curvature Kähler metric in the polarization class implies slope stability with respect to complex submanifolds (Corollary 8.1.3, [Tho06, Corollary 7.4]).

Chapter 9 - General Polarized Kähler Manifolds

In chapter 9, we investigate some ideas related to partial density functions on general polarized Kähler manifolds. This chapter relies heavily on the existence of Tian's "peaked" holomorphic sections, which we also review here. We dwell on Tian's philosophy of using such peaked sections and emphasise the link with the orthogonal complements of vector spaces of sections vanishing to a certain order along a submanifold. The following proposition is a result following this approach:

Proposition 9.1.8. Suppose that $(L, h) \to (X, \omega)$ is a polarized Kähler manifold. Let $l \in \mathbb{N}$, $x_0 \in X$ and let $\{s_k\}_{k=1}^{\infty}$ be a sequence such that $s_k \in \mathcal{J}_k^l(p)^{\perp} \subset H^0(X, L^k)$ and $\|s_k\|_{h^k} = 1$ for all $k \in \mathbb{N}$. Then there exists a constant $C \geq 0$ such that, in local K-coordinates of order 4 centred x_0, we have

$$\left| \int_{X - \left\{ \|\boldsymbol{z}\| \leq \frac{\log(k)}{\sqrt{k}} \right\}} |s_k|_{h^k}^2 \frac{\omega^n}{n!} \right| \leq C k^{-1} \quad \text{for all } k \in \mathbb{N}.$$

In particular, $\{s_k\}_{k=1}^{\infty}$ is peaked at x_0.

The ability to compute the asymptotics of toric (partial) density functions using only an orthonormal basis of a small subspace of holomorphic sections is a useful tool in the toric setting. We explore a localization of sums result in this direction for general polarized Kähler manifolds:

Corollary 9.2.3 (Localization of the density function on a tubular neighbourhod)**.**
Let $(L, h) \to (X, \omega)$ be a polarization of a Kähler manifold (X, ω). Denote by ρ_k the density function for this polarization and let $Y \subset X$ be an embedded complex submanifold of X. There exists $r > 0$ and, for any $l \in \mathbb{N}$, a constant $C_l \geq 0$ such that

$$\left| \rho_k(p) - \sum_{j=1}^{N_k} |s_{k,j}(p)|^2 \right| \leq C_l k^{-l}$$

for all $p \in T_r(Y)$ and $k \in \mathbb{N}$. Here, $\{s_{k,j}\}_{j=1}^{N_k}$ denotes any orthonormal basis of the space $\mathcal{J}_k^k(Y)^{\perp}$ and $|.|$ denotes the fibre-wise norm on L^k. In particular, the asymptotic expansion of $\rho_k(p)$ is equal to the asymptotic expansion of $\sum_{j=1}^{N_k} |s_{k,j}|^2(p)$ for $p \in T_r(Y)$.

Chapter 10 - Induced Metrics on Blow-ups

We consider a polarization $(L^m, h) \to (X, \omega)$ and discuss a sequence of pull-back Fubini-Study metrics $\{\omega_k\}_{k=1}^{\infty} \subset 2\pi c_1(L^m)$ studied by Tian. For some distinct points $p_1, \cdots, p_s \in X$ and $\boldsymbol{l} \in \mathbb{N}^s$, we then consider the blow-up $\pi : \mathrm{Bl}_{p_1,\cdots,p_s}(X) \to X$ with the line bundle

$$\widehat{L} = \widehat{L}_{p_1,\cdots,p_s,\boldsymbol{l},m} \stackrel{\text{def}}{=} \pi^* L^m \otimes \mathcal{O}(-l_1 E_1) \otimes \cdots \otimes \mathcal{O}(-l_s E_s).$$

We assume that m is large enough so that $\widehat{L}$ is very ample. In this chapter, we describe how the polarization induces a sequence of metrics $\{\widehat{\omega}_k\}_{k=1}^{\infty} \subset 2\pi c_1(\widehat{L})$ (see lemma 10.3.1). We then observe that (see lemma 10.3.2) on $X - \bigcup_{i=1}^{s}\{p_i\}$,

$$\pi_* \widehat{\omega}_k - \omega = \frac{i}{k} \partial\overline{\partial} \log \rho_{\boldsymbol{l},k},$$

where $\rho_{\boldsymbol{l},k}$ is a corresponding partial density function on X. In the toric setting, it is relatively easy to study these metrics in detail and we focus on two examples, namely a certain blow-up of $\mathbb{CP}^n$ and $\mathbb{C}^n$ respectively. In these examples, the induced metrics $\{\widehat{\omega}_k\}_{k=1}^{\infty}$ exhibit an interesting behaviour on the exceptional divisor (see lemma 10.5.3 and lemma 10.6.8) and far away from the exceptional divisor (see lemma 10.5.2 and lemma 10.6.7). In addition, the induced metrics on $\mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n)$ turn out to be Asymptotically Euclidean for each $k \in \mathbb{N}$ (see lemma 10.6.6).

Appendices A and B

The two appendices A and B deal with some technical results that we require in this thesis. In particular, we extend Laplace's method to integrals over half-spaces in section B.3 which is essential in chapter 7.

Chapter 2

The Bargmann-Fock Model

In this chapter, we discuss the asymptotics of a partial density function $\rho_{l,k}$ on $\mathbb{C}^n$ equipped with the standard flat metric. We will refer to the setup discussed in this chapter as the "Bargmann-Fock model". Since toric Kähler manifolds have a natural open cover by $\mathbb{C}^n$ charts on which there exist Kähler potentials which share many of the properties of the standard potential function $\phi(\boldsymbol{z}) = \frac{\|\boldsymbol{z}\|^2}{2}$ that appears in the Bargmann-Fock case, we will use the Bargmann-Fock model as a guiding example for our discussion of partial density functions on polarized toric Kähler manifolds in chapter 7. We now also introduce Laplace's method as an important tool for determining the asymptotics of $\rho_{l,k}$.

2.1 Introduction

2.1.1 Definitions

Consider the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}^n} f(\boldsymbol{z})\overline{g(\boldsymbol{z})}e^{-\frac{k}{2}\|\boldsymbol{z}\|^2}\,\mathrm{dVol},$$

for $k \in \mathbb{N}$ and $f, g \in \mathcal{L}^2_k$, where $\mathcal{L}^2_k$ denotes the space of smooth functions $f : \mathbb{C}^n \to \mathbb{C}$ for which $\|f\|^2 = \langle f, f\rangle < \infty$ and where $\mathrm{dVol} = \frac{1}{n!}\left(i\partial\overline{\partial}\frac{\|\boldsymbol{z}\|^2}{2}\right)^n$ is the Euclidean volume form. We call the space $\mathcal{F}_k$ of holomorphic functions in $\mathcal{L}^2_k$ the Bargmann-Fock space with parameter $k \in \mathbb{N}$. $(\mathcal{F}_k, \langle,\rangle)$ is a Hilbert space [Bar61].

Lemma 2.1.1. *The monomials $\{\boldsymbol{z}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathbb{N}_0^n\}$ form an orthogonal basis of $\mathcal{F}_k$ with norm*

$$\|\boldsymbol{z}^{\boldsymbol{\alpha}}\|^2 = \left(\frac{2\pi}{k}\right)^n \left(\frac{2}{k}\right)^{|\boldsymbol{\alpha}|} \boldsymbol{\alpha}!.$$

We denote the orthonormal basis elements by $s_{\boldsymbol{\alpha},k}$, where $s_{\boldsymbol{\alpha},k}(\boldsymbol{z}) \stackrel{\text{def}}{=} \frac{\boldsymbol{z}^{\boldsymbol{\alpha}}}{\|\boldsymbol{z}^{\boldsymbol{\alpha}}\|}$ for $\boldsymbol{z} \in \mathbb{C}^n$, $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ and $k \in \mathbb{N}$.

Proof. It is easy to see in polar coordinates that $\langle \boldsymbol{z}^{\boldsymbol{\alpha}}, \boldsymbol{z}^{\boldsymbol{\beta}}\rangle = 0$ for $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^n$. We have

$$\|\boldsymbol{z}^{\boldsymbol{\alpha}}\|^2 = (2\pi)^n \prod_{j=1}^n \int_0^\infty r^{2\alpha_j+1} e^{-\frac{kr^2}{2}}\,dr$$

$$= (2\pi)^n \prod_{j=1}^{n} \left(\frac{2}{k}\right)^{\alpha_j+1} \frac{\alpha_j!}{2},$$

and it is obvious that $\{\boldsymbol{z}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathbb{N}_0^n\}$ is a complete orthogonal system for $\mathcal{F}_k$. $\square$

The Bergman kernel $B_k : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is defined to be the integral kernel representing the orthogonal projection $P_k : \mathcal{L}_k^2 \twoheadrightarrow \mathcal{F}_k$ in the sense that

$$P_k(f)(\boldsymbol{z}) = \langle f, B_k(.,\boldsymbol{z})\rangle = \int_{\mathbb{C}^n} f(\boldsymbol{w})\overline{B_k(\boldsymbol{w},\boldsymbol{z})} e^{-\frac{k}{2}\|\boldsymbol{w}\|^2}\,\mathrm{dVol} \quad \text{for all } \boldsymbol{z} \in \mathbb{C}^n, f \in \mathcal{L}_k^2.$$

We recall the following classical result.

Lemma 2.1.2. *The Bergman kernel for the Bargmann-Fock model is given by*

$$\begin{aligned} B_k(\boldsymbol{z},\boldsymbol{w}) &= \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} s_{\boldsymbol{\alpha},k}(\boldsymbol{z})\overline{s_{\boldsymbol{\alpha},k}(\boldsymbol{w})} \\ &= \left(\frac{k}{2\pi}\right)^n \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} \left(\frac{k}{2}\right)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{z}^{\boldsymbol{\alpha}}\overline{\boldsymbol{w}}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \\ &= \left(\frac{k}{2\pi}\right)^n e^{\frac{k}{2}\langle \boldsymbol{z},\boldsymbol{w}\rangle} \quad \textit{for } \boldsymbol{z},\boldsymbol{w} \in \mathbb{C}^n \textit{ and } k \in \mathbb{N}. \end{aligned}$$

Proof. The last simplification above follows from the multinomial theorem, while

$$\langle f, B_k(.,\boldsymbol{z})\rangle = \int_{\mathbb{C}^n} f(\boldsymbol{w})\overline{B_k(\boldsymbol{w},\boldsymbol{z})} e^{-\frac{k}{2}\|\boldsymbol{w}\|^2}\,\mathrm{dVol} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} \langle f, s_{\boldsymbol{\alpha},k}\rangle s_{\boldsymbol{\alpha},k}(\boldsymbol{z})$$

proves that $P(f)(\boldsymbol{z}) = \langle f, B_k(.,\boldsymbol{z})\rangle$ for $f \in \mathcal{L}_k^2$ and $\boldsymbol{z} \in \mathbb{C}^n$. $\square$

Let us note the following well-known result on the translations $T_{k,\boldsymbol{a}}$.

Lemma 2.1.3. *The translations $T_{k,\boldsymbol{a}} : \mathcal{L}_k^2 \to \mathcal{L}_k^2$, for $k \in \mathbb{N}$ and $\boldsymbol{a} \in \mathbb{C}^n$, defined by*

$$(T_{k,\boldsymbol{a}} f)(\boldsymbol{z}) \overset{\text{def}}{=} e^{\frac{k}{2}\left(\langle \boldsymbol{z},\boldsymbol{a}\rangle - \frac{\|\boldsymbol{a}\|^2}{2}\right)} f(\boldsymbol{z}-\boldsymbol{a}),$$

for $f \in \mathcal{L}_k^2$ and $\boldsymbol{z} \in \mathbb{C}^n$, are isometries of $(\mathcal{F}_k, \|.\|)$.

Proof. Let $f \in \mathcal{F}_k$, $\boldsymbol{a} \in \mathbb{C}^n$ and $k \in \mathbb{N}$. We have

$$\begin{aligned} \|T_{k,\boldsymbol{a}} f\|^2 &= \int_{\mathbb{C}^n} |f(\boldsymbol{z}-\boldsymbol{a})|^2 e^{\frac{k}{2}\left(\langle \boldsymbol{z},\boldsymbol{a}\rangle + \langle \boldsymbol{a},\boldsymbol{z}\rangle - \|\boldsymbol{a}\|^2\right)} e^{-\frac{k}{2}\|\boldsymbol{z}\|^2}\,\mathrm{dVol} \\ &= \int_{\mathbb{C}^n} |f(\boldsymbol{z}-\boldsymbol{a})|^2 e^{-\frac{k}{2}\|\boldsymbol{z}-\boldsymbol{a}\|^2}\,\mathrm{dVol} \\ &= \int_{\mathbb{C}^n} |f(\boldsymbol{z})|^2 e^{-\frac{k}{2}\|\boldsymbol{z}\|^2}\,\mathrm{dVol} = \|f\|^2. \end{aligned}$$

$\square$

Remark 2.1.4. *Note that, in the above proof, $\boldsymbol{z} \mapsto e^{-\frac{k}{2}\|\boldsymbol{z}-\boldsymbol{a}\|^2}$ is a Gaussian function which has a peak at $\boldsymbol{a} \in \mathbb{C}^n$. Functions of this type will play an important role in our discussion of the density function on polarized toric Kähler manifolds in chapter 7.*

Although Bergman kernels of the type discussed here are much simpler than the Bergman kernels that we will study in chapter 7, they are nonetheless of independent interest. The interested reader may refer to [Sei92, SW92, BS93] which discuss properties of the Bargmann-Fock space in more detail and which include further background references.

2.1.2 Line bundle interpretation

Let $X = \mathbb{C}^n$ and consider the trivial line bundle $L = \mathbb{C}^n \times \mathbb{C}$ on X. We equip X with the flat Kähler metric $\omega = i\partial\overline{\partial}\phi(\boldsymbol{z})$ with potential $\phi(\boldsymbol{z}) = \frac{1}{2}\|\boldsymbol{z}\|^2$ and L^k with the Hermitian metric h^k which is represented by the positive function $e^{-k\phi(\boldsymbol{z})}$ in the standard holomorphic trivialization. We observe that $iF_h = -i\partial\overline{\partial}\log(h) = \omega$, *i.e.* we have a polarization $(L, h) \to (X, \omega)$. We can think of the Bergman kernel B_k as a section of the line bundle $\pi_1^*(L^k) \otimes \pi_2^*(\overline{L}^k) \to X \times X$, where $\pi_i : X \times X \to X$ denotes the projection onto the i^{th} factor. Then

$$B_k(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} s_{\boldsymbol{\alpha},k}(\boldsymbol{z}) \otimes \overline{s_{\boldsymbol{\alpha},k}(\boldsymbol{w})} \quad \text{for all } \boldsymbol{z}, \boldsymbol{w} \in \mathbb{C}^n.$$

h^k now induces a natural Hermitian inner product on this bundle which we also denote by h^k. If, for example, $f, s \in \mathcal{C}^\infty\left(X, L^k\right)$ and $g, t \in \mathcal{C}^\infty\left(X, \overline{L}^k\right)$, this gives

$$\begin{aligned}(f(\boldsymbol{z}) \otimes g(\boldsymbol{w}), s(\boldsymbol{z}) \otimes t(\boldsymbol{w}))_{h^k} &= (f(\boldsymbol{z}), s(\boldsymbol{z}))_{h^k} \, (g(\boldsymbol{w}), t(\boldsymbol{w}))_{h^k} \\ &= f(\boldsymbol{z})\overline{s(\boldsymbol{z})}e^{-\frac{k}{2}\|\boldsymbol{z}\|^2} g(\boldsymbol{w})\overline{t(\boldsymbol{w})}e^{-\frac{k}{2}\|\boldsymbol{w}\|^2}\end{aligned}$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{C}^n$. We then compute that

$$|B_k(\boldsymbol{z}, \boldsymbol{w})|_{h^k}^2 = \left(\frac{k}{2\pi}\right)^{2n} e^{-\frac{k}{2}\|\boldsymbol{z}-\boldsymbol{w}\|^2} \quad \text{for all } \boldsymbol{z}, \boldsymbol{w} \in \mathbb{C}^n.$$

The *density (of states) function* $\rho_k : X \to \mathbb{R}_{\geq 0}$ is defined to be the restriction of the norm of B_k to the diagonal in $X \times X$. We have

$$\rho_k(\boldsymbol{z}) = \left(\frac{k}{2\pi}\right)^n \quad \text{for all } \boldsymbol{z} \in \mathbb{C}^n.$$

2.1.3 Partial Bergman kernels

For $l, k \in \mathbb{N}$, we are now interested in studying the orthogonal projections $P_{l,k} : \mathcal{L}_k^2 \twoheadrightarrow \mathcal{F}_{l,k}$, where $\mathcal{F}_{l,k} \overset{\text{def}}{=} \mathcal{L}_k^2 \cap \mathcal{J}_{lk}$ and

$$\mathcal{J}_{lk} \overset{\text{def}}{=} \left\{ f : \mathbb{C}^n \to \mathbb{C} : f \text{ is holomorphic and } \left.\frac{\partial}{\partial \boldsymbol{z}}^{\boldsymbol{\alpha}} f\right|_{\boldsymbol{0}} = 0 \text{ for } \boldsymbol{\alpha} \in \mathbb{N}_0^n \text{ such that } |\boldsymbol{\alpha}| < lk \right\}.$$

We call the integral kernel associated to $P_{l,k}$ the *partial Bergman kernel with parameters* (l, k) and denote it by $B_{l,k}$. We have

$$\begin{aligned}B_{l,k}(\boldsymbol{z}, \boldsymbol{w}) &= \sum_{|\boldsymbol{\alpha}| \geq kl} s_{\boldsymbol{\alpha},k}(\boldsymbol{z})\overline{s_{\boldsymbol{\alpha},k}(\boldsymbol{w})} \\ &= \left(\frac{k}{2\pi}\right)^n \sum_{|\boldsymbol{\alpha}| \geq kl} \left(\frac{k}{2}\right)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{z}^{\boldsymbol{\alpha}}\overline{\boldsymbol{w}}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!},\end{aligned}$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{C}^n$, and $|\boldsymbol{\alpha}| \geq kl$ in the sum denotes the index set $\left\{\boldsymbol{\alpha} \in \mathbb{N}_0^n : |\boldsymbol{\alpha}| = \sum_{j=0}^n \alpha_j \geq kl\right\}$. We compute that

$$|B_{l,k}(\boldsymbol{z}, \boldsymbol{w})|_{h^k}^2 = \left(\frac{k}{2\pi}\right)^{2n} \left| \sum_{|\boldsymbol{\alpha}| \geq kl} \left(\frac{k}{2}\right)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{z}^{\boldsymbol{\alpha}} \overline{\boldsymbol{w}}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \right|^2 e^{-\frac{k}{2}\left(\|\boldsymbol{z}\|^2 + \|\boldsymbol{w}\|^2\right)}.$$

Our partial density function $\rho_{l,k}$ is the restriction of the h^k-norm of $B_{l,k}$ to the diagonal in $\mathbb{C}^n \times \mathbb{C}^n$.

$$\begin{aligned} \rho_{l,k}(\boldsymbol{z}) &= \left(\frac{k}{2\pi}\right)^n \sum_{|\boldsymbol{\alpha}| \geq kl} \left(\frac{k}{2}\right)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{z}^{\boldsymbol{\alpha}} \overline{\boldsymbol{z}}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} e^{-\frac{k}{2}\|\boldsymbol{z}\|^2} \\ &= \left(\frac{k}{2\pi}\right)^n \sum_{j \geq kl} \left(\frac{k}{2}\right)^j \frac{\|\boldsymbol{z}\|^{2j}}{j!} e^{-\frac{k}{2}\|\boldsymbol{z}\|^2}, \end{aligned}$$

where we have applied the multinomial identity

$$\begin{aligned} \|\boldsymbol{z}\|^{2j} &= \left(|z_1|^2 + \cdots + |z_n|^2\right)^j \\ &= \sum_{|\boldsymbol{\alpha}| = j} \frac{j!}{\boldsymbol{\alpha}!} |z_1|^{2\alpha_1} \ldots |z_n|^{2\alpha_n} \\ &= \sum_{|\boldsymbol{\alpha}| = j} \frac{j!}{\boldsymbol{\alpha}!} \boldsymbol{z}^{\boldsymbol{\alpha}} \overline{\boldsymbol{z}}^{\boldsymbol{\alpha}} \quad \text{for } \boldsymbol{z} \in \mathbb{C}^n. \end{aligned}$$

We now introduce the variable $s = \frac{\|\boldsymbol{z}\|^2}{2}$ and let $\rho_{l,k}(\boldsymbol{z}) = \eta_{l,k}(s)$, so that

$$\begin{aligned} \eta_{l,k}(s) &= \left(\frac{k}{2\pi}\right)^n \sum_{j \geq kl} \frac{(ks)^j}{j!} e^{-ks} \\ &= \left(\frac{k}{2\pi}\right)^n \left(1 - \sum_{0 \leq j < kl} \frac{(ks)^j}{j!} e^{-ks}\right). \end{aligned}$$

We observe that

$$\begin{aligned} \left.\frac{\partial}{\partial s}\right|_s \eta_{l,k} &= \left(\frac{k}{2\pi}\right)^n k e^{-ks} \left(\sum_{0 \leq j < lk} \frac{(ks)^j}{j!} - \sum_{j=1}^{kl-1} \frac{(ks)^{j-1}}{(j-1)!}\right) \\ &= k e^{-ks} \left(\frac{k}{2\pi}\right)^n \frac{(ks)^{kl-1}}{(kl-1)!}. \end{aligned}$$

Integrating the derivative, we obtain an integral representation for $\rho_{l,k}$.

$$\begin{aligned} \rho_{l,k}(\boldsymbol{z}) = \eta_{l,k}(s) &= \left(\frac{k}{2\pi}\right)^n \frac{k^{kl}}{(kl-1)!} \int_0^s e^{-kt} t^{kl-1} dt \\ &= \left(\frac{k}{2\pi}\right)^n \frac{1}{(kl-1)!} \int_0^{ks} e^{-y} y^{kl-1} dy \\ &= \left(\frac{k}{2\pi}\right)^n \left(1 - \frac{\Gamma(kl, ks)}{\Gamma(kl, 0)}\right). \end{aligned}$$

Here,

$$\Gamma(n,x) \stackrel{\text{def}}{=} \int_x^\infty e^{-t}t^{n-1}dt \quad \text{for } n \in \mathbb{C} \text{ and } x \in \mathbb{R}$$

denotes the *incomplete gamma function* which has the special values $\Gamma(n,0) = (n-1)!$ for $n \in \mathbb{N}$.

Remark 2.1.5. *There exists a rich literature concerned with the asymptotic properties of incomplete gamma functions. The interested reader may consult [Gau98] for an extensive review. The methods that we will employ to understand the asymptotics of $\Gamma(n,x)$ will only apply for real valued n.*

We have now proved the following lemma.

Lemma 2.1.6. *The partial density function $\rho_{l,k} : \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ for $l, k \in \mathbb{N}$ is given by*

$$\rho_{l,k}(\boldsymbol{z}) = \left(\frac{k}{2\pi}\right)^n \left(1 - \frac{\Gamma(kl, k\frac{\|\boldsymbol{z}\|^2}{2})}{\Gamma(kl,0)}\right).$$

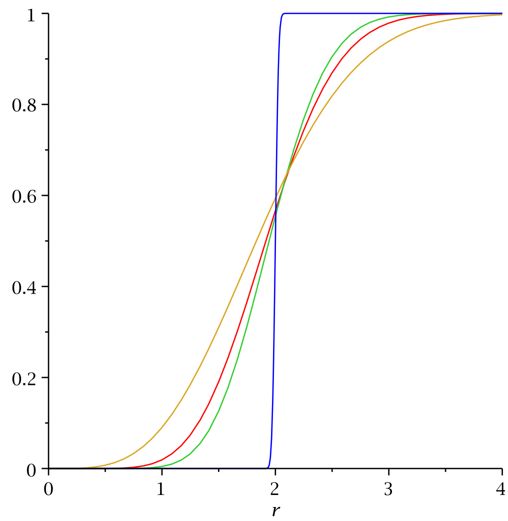

Figure 2.1: Graphs of $1 - \frac{\Gamma(kl, k\frac{r^2}{2})}{(kl-1)!}$ for $r \in [0,4]$, $l = 2$ and $k = 1, 2, 3, 1000$.

2.2 Asymptotics of the partial density function

2.2.1 Laplace's method in one dimension

The traditional form of Laplace's method Traditionally, Laplace's method in one dimension (see [dB81, BH75]) provides a means of determining the asymptotics of an integral of the type

$$I_k = \int_{\mathbb{R}} f(x)e^{-kh(x)}dx,$$

where $k \in \mathbb{R}_{>0}$ tends to infinity, $f, h \in \mathcal{C}^\infty(\mathbb{R})$, h has an absolute minimum which it attains only at $x_0 \in \mathbb{R}$ and where $h''(x_0) > 0$. We assume that $I_k < \infty$ for $k \in \mathbb{R}_{>0}$ and that there exists a $c > 0$ such that $h(x) - h(x_0) \geq c$ outside a compact subset of $\mathbb{R}$. Laplace's method then gives an asymptotic expansion in k,

$$e^{kh(x_0)} I_k = \sum_{j=0}^{\infty} a_j k^{-\left(j+\frac{1}{2}\right)} + \mathcal{O}\left(k^{-\infty}\right),$$

where the coefficients a_j are determined by the derivatives of f and h at x_0. In order to understand the asymptotics of the partial density function in the Bargmann-Fock model, we will now investigate a slightly more subtle version of Laplace's method which depends also on the domain of integration.

Generalized error functions For $j \in \mathbb{N}_0$, we define the functions $e_j : \mathbb{R} \to \mathbb{R}$,

$$e_j(\lambda) = \frac{\int_0^\lambda x^j e^{-x^2} dx}{\int_0^\infty x^j e^{-x^2} dx} = \frac{2}{\Gamma\left(\frac{j+1}{2}\right)} \int_0^\lambda x^j e^{-x^2} dx \quad \text{for } \lambda \in \mathbb{R}.$$

We observe that e_0 is the standard error function and call e_j the *generalized error function of order* j (see figure 2.2).

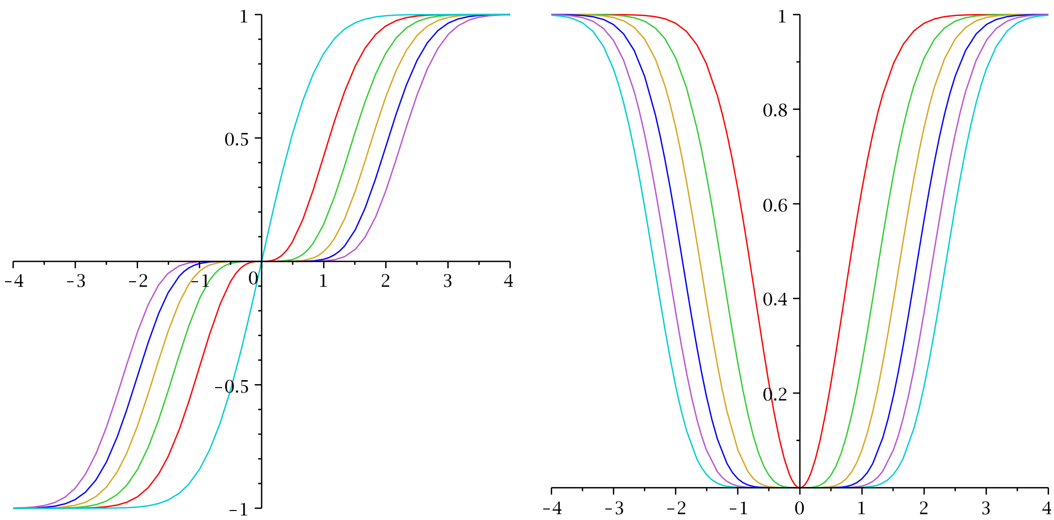

Figure 2.2: Graphs of e_{2j} and e_{2j+1} for $j = 0, \dots, 5$.

Remark 2.2.1. *By the duplication formula for* Γ,

$$\Gamma(i)\Gamma\left(i+\frac{1}{2}\right) = \sqrt{\pi}2^{1-2i}\Gamma(2i) \qquad \text{for } i \in \mathbb{N}_0,$$

we find that,

$$\Gamma\left(i+\frac{1}{2}\right) = \frac{\sqrt{\pi}2^{1-2i}(2i-1)!}{(i-1)!} = \frac{\sqrt{\pi}(2i)!}{2^{2i}i!} \qquad \text{for } i \in \mathbb{N}_0.$$

We have

$$\Gamma\left(\frac{j+1}{2}\right) = \begin{cases} \frac{\sqrt{\pi}(2i)!}{2^{2i}i!} & \text{if } j = 2i,\ i \in \mathbb{N}_0, \\ i! & \text{if } j = 2i+1,\ i \in \mathbb{N}_0. \end{cases}$$

A refinement of Laplace's method The following proposition gives a refined version of Laplace's method in one dimension in the case where $k \in \mathbb{N}$.

Proposition 2.2.2. *Let $f, h \in \mathcal{C}^{\infty}(\mathbb{R})$ and assume that h has an absolute minimum which it attains only at $x_0 \in \mathbb{R}$ and where $h''(x_0) > 0$. Furthermore, assume that there exists $c > 0$ such that $h(x) - h(x_0) > c$ outside a compact subset of $\mathbb{R}$. Suppose that $\int_{\mathbb{R}} |f(x)| e^{-kh(x)} dx < \infty$ for $k \in \mathbb{N}$ and consider the integral*

$$I_k(r) \stackrel{\text{def}}{=} \int_{x_0}^{x_0+r} f(x) e^{-kh(x)} dx \quad \text{for } r \in \mathbb{R} \text{ and } k \in \mathbb{N}.$$

a) Let $\{r_k\}_{k=1}^{\infty} \subset \mathbb{R}$, $m \in \mathbb{N}$ and $M > 0$ such that $|r_k| \leq M k^{-\frac{1}{3}}$ for all $k \in \mathbb{N}$. There exists $C = C(M, m) \geq 0$ such that

$$\left| e^{kh(x_0)} I_k(r_k) - \sum_{j=0}^{m} a_j \left(r_k \sqrt{\frac{kh''(x_0)}{2}} \right) k^{-\frac{j+1}{2}} \right| \leq C k^{-\left(\frac{m}{2}+1\right)},$$

for all $k \in \mathbb{N}$, and where, for $s \in \mathbb{R}$ and $j \in \mathbb{N}_0$, we define

$$a_j(s) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i=0}^{j} \frac{(-1)^i}{i!(j+2i)!} \Gamma\left(\frac{j+1}{2} + i\right) \left(\frac{2}{h''(x_0)}\right)^{\frac{j+1}{2}+i} \frac{\mathrm{d}}{\mathrm{d}x}^{j+2i} \left(h(x) - h(x_0) - \frac{h''(x_0)}{2}(x - x_0)^2 \right)^i f(x) \Bigg|_{x_0} e_{j+2i}(s).$$

b) Let $\{r_k\}_{k=1}^{\infty} \subset \mathbb{R}$, $m \in \mathbb{N}$ and $M, \delta > 0$ such that $|r_k| \geq M k^{-\frac{1}{2}+\delta}$ for all $k \in \mathbb{N}$. There exists $C = C(M, m, \delta) \geq 0$ such that

$$\left| e^{kh(x_0)} I_k(r_k) - \sum_{j=0}^{m} \operatorname{Sgn}(r_k)^{j+1} b_j k^{-\frac{j+1}{2}} \right| \leq C k^{-\left(\frac{m}{2}+1\right)},$$

for all $k \in \mathbb{N}$, and where the constants b_j, for $j \in \mathbb{N}_0$, are given by

$$b_j \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i=0}^{j} \frac{(-1)^i}{i!(j+2i)!} \Gamma\left(\frac{j+1}{2} + i\right) \left(\frac{2}{h''(x_0)}\right)^{\frac{j+1}{2}+i} \frac{\mathrm{d}}{\mathrm{d}x}^{j+2i} \left(h(x) - h(x_0) - \frac{h''(x_0)}{2}(x - x_0)^2 \right)^i f(x) \Bigg|_{x_0}.$$

Proof. **Part a)** By Taylor's theorem, $h(x) - h(x_0) = \frac{h''(x_0)}{2}(x - x_0)^2 + R(x)$. Let $m, k \in \mathbb{N}$ and $M > 0$. There exists a constant $C = C(M) \geq 0$ such that $|R(x)| \leq C |x - x_0|^3$ for $|x - x_0| \leq M$. It then follows that,

$$|kR(x)| \leq CM^3 \quad \text{for } |x - x_0| \leq M k^{-\frac{1}{3}} \text{ and } k \in \mathbb{N}.$$

We have

$$e^{-kR(x)} = \sum_{i=0}^{m} \frac{(-kR(x))^i}{i!} + S_{m+1}(-kR(x)),$$

and there exists a constant $D = D(M, m) \geq 0$ such that

$$|S_{m+1}(-kR(x))| \leq D k^{m+1} |x - x_0|^{3(m+1)}$$

for $k \in \mathbb{N}$ and all $x \in \mathbb{R}$ such that $|x - x_0| \leq Mk^{-\frac{1}{3}}$.

Remark 2.2.3. *Observe that, for $a > 0$ and $k \in \mathbb{N}$,*

$$\int_{x_0}^{\pm\infty} |x - x_0|^j \, e^{-ka(x-x_0)^2} dx = \int_0^{\pm\infty} |x|^j \, e^{-kax^2} dx$$
$$= k^{-\left(\frac{j+1}{2}\right)} \int_0^{\pm\infty} |x|^j \, e^{-ax^2} dx = \mathcal{O}\left(k^{-\left(\frac{j+1}{2}\right)}\right)$$

as $k \to \infty$.

For $s \in \mathbb{N}_0$ such that $s \geq 3i$, we have:

$$\frac{(-R(x))^i}{i!} f(x) = \sum_{j=3i}^{s} \frac{(-1)^i}{i!j!} \left(\frac{\mathrm{d}}{\mathrm{d}\,x}\right)^j R^i(x) f(x)\Big|_{x_0} (x - x_0)^j + T_{s,i}(x), \tag{2.2.1}$$

and there exist constants $E_i = E_i(M, s) \geq 0$ such that $|T_{s,i}(x)| \leq E_i \, |x - x_0|^{s+1}$ for all $s, i \in \mathbb{N}_0$ with $s \geq 3i$ and all $x \in \mathbb{R}$ such that $|x - x_0| \leq M$. Applying remark 2.2.3, we find that, for $|r_k| \leq Mk^{-\frac{1}{3}}$,

$$e^{kh(x_0)} I_k(r_k) = \int_{x_0}^{x_0+r_k} f(x) e^{-k(h(x)-h(x_0))} dx$$
$$= \sum_{i=0}^{m} \int_{x_0}^{x_0+r_k} \frac{(-kR(x))^i}{i!} f(x) e^{-k\frac{h''(x_0)}{2}(x-x_0)^2} dx + \mathcal{O}\left(k^{-\left(\frac{m}{2}+1\right)}\right),$$

where the constant in $\mathcal{O}$ depends on M and m (and of course on h and f). Using equation 2.2.1, we find that

$$e^{kh(x_0)} I_k(r_k) = \sum_{i=0}^{m} \left(\sum_{j=3i}^{m+2i} k^i \frac{(-1)^i}{i!j!} \left(\frac{\mathrm{d}}{\mathrm{d}\,x}\right)^j \left(R^i(x) f(x)\right)\Big|_{x_0} \int_{x_0}^{x_0+r_k} (x - x_0)^j \, e^{-\frac{h''(x_0)}{2}(x-x_0)^2} dx \right.$$
$$\left. + U_{m+2i,i}(x, k) \right) + \mathcal{O}\left(k^{-\left(\frac{m}{2}+1\right)}\right),$$

where

$$|U_{m+2i,i}(x, k)| \leq k^i \int_{x_0}^{x_0+r_k} |T_{m+2i,i}(x)| \, e^{-k\frac{h''(x_0)}{2}(x-x_0)^2} dx$$
$$\leq k^i E_i \int_{x_0}^{x_0+r_k} |x - x_0|^{m+2i+1} \, e^{-k\frac{h''(x_0)}{2}(x-x_0)^2} dx$$
$$\leq E_i \left(\frac{2}{h''(x_0)}\right)^{\frac{m}{2}+i+1} k^{-\left(\frac{m}{2}+1\right)} \int_{\mathbb{R}} |x|^{m+2i+1} \, e^{-x^2} dx$$
$$= \mathcal{O}\left(k^{-\left(\frac{m}{2}+1\right)}\right).$$

Note that

$$\int_{x_0}^{x_0+r_k} (x - x_0)^j e^{-\frac{kh''(x_0)}{2}(x-x_0)^2} dx = \int_0^{r_k} x^j e^{-\frac{kh''(x_0)}{2}x^2} dx$$
$$= k^{-\frac{j+1}{2}} \left(\frac{2}{h''(x_0)}\right)^{\frac{j+1}{2}} \int_0^{\sqrt{\frac{kh''(x_0)}{2}}r_k} y^j e^{-y^2} dy$$

$$= k^{-\frac{j+1}{2}} \left(\frac{2}{h''(x_0)}\right)^{\frac{j+1}{2}} \frac{\Gamma\left(\frac{j+1}{2}\right)}{2} e_j \left(\sqrt{\frac{kh''(x_0)}{2}} r_k\right).$$

We conclude that

$$\begin{aligned}
e^{kh(x_0)} I_k(r_k) &= \frac{1}{2} \sum_{i=0}^{m} \sum_{j=3i}^{m+2i} k^{i-\frac{j+1}{2}} \frac{(-1)^i}{i!j!} \Gamma\left(\frac{j+1}{2}\right) \left(\frac{2}{h''(x_0)}\right)^{\frac{j+1}{2}} \\
&\quad \frac{\mathrm{d}}{\mathrm{d}\,x}^{j} \left(R^i(x) f(x)\right)\Big|_{x_0} e_j \left(r_k \sqrt{\frac{kh''(x_0)}{2}}\right) + \mathcal{O}\left(k^{-\left(\frac{m}{2}+1\right)}\right) \\
&= \frac{1}{2} \sum_{i=0}^{m} \sum_{p=0}^{m-i} k^{-\frac{p+i+1}{2}} \frac{(-1)^i}{i!(p+3i)!} \Gamma\left(\frac{p+3i+1}{2}\right) \left(\frac{2}{h''(x_0)}\right)^{\frac{p+3i+1}{2}} \\
&\quad \frac{\mathrm{d}}{\mathrm{d}\,x}^{p+3i} \left(R^i(x) f(x)\right)\Big|_{x_0} e_{p+3i} \left(r_k \sqrt{\frac{kh''(x_0)}{2}}\right) + \mathcal{O}\left(k^{-\left(\frac{m}{2}+1\right)}\right) \\
&= \frac{1}{2} \sum_{l=0}^{m} k^{-\frac{l+1}{2}} \sum_{i=0}^{l} \frac{(-1)^i}{i!(l+2i)!} \Gamma\left(\frac{l+1}{2}+i\right) \left(\frac{2}{h''(x_0)}\right)^{\frac{l+1}{2}+i} \\
&\quad \frac{\mathrm{d}}{\mathrm{d}\,x}^{l+2i} \left(R^i(x) f(x)\right)\Big|_{x_0} e_{l+2i} \left(r_k \sqrt{\frac{kh''(x_0)}{2}}\right) + \mathcal{O}\left(k^{-\left(\frac{m}{2}+1\right)}\right).
\end{aligned}$$

Part b) Suppose now that $|r_k| \geq Mk^{-\frac{1}{2}+\delta}$ for all $k \in \mathbb{N}$ and let $\mu \overset{\text{def}}{=} -\min\left(\delta - \frac{1}{2}, -\frac{1}{3}\right)$ and $\delta' \overset{\text{def}}{=} \frac{1}{2} - \mu$. Then $\mu \in [\frac{1}{3}, \frac{1}{2})$ and $|r_k| \geq Mk^{-\mu}$, while $\delta' > 0$. By the proof of lemma B.2.3, there exists $C \geq 0$ such that $h(x) - h(x_0) \geq Ck^{2\delta'-1}$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$ such that $|x - x_0| \geq Mk^{-\frac{1}{2}+\delta'}$. It follows that

$$\int_{x_0}^{x_0+r_k} f(x) e^{-k(h(x)-h(x_0))} dx = \int_{x_0}^{x_0+\mathrm{Sgn}(r_k)Mk^{-\frac{1}{2}+\delta'}} f(x) e^{-k(h(x)-h(x_0))} dx + \mathcal{O}\left(k^{-\infty}\right).$$

The constant in $\mathcal{O}$ depends on M and δ' since

$$\begin{aligned}
\int_{x_0+\mathrm{Sgn}(r_k)Mk^{-\frac{1}{2}+\delta'}}^{x_0+r_k} |f(x)|\, e^{-k(h(x)-h(x_0))} dx &\leq e^{-\frac{k-1}{k}Ck^{2\delta'}} \int_{\mathbb{R}} |f(x)|\, e^{-(h(x)-h(x_0))} dx \\
&= \mathcal{O}\left(k^{-\infty}\right).
\end{aligned}$$

We can now expand $\int_{x_0}^{x_0+\mathrm{Sgn}(r_k)Mk^{-\frac{1}{2}+\delta'}} f(x) e^{-k(h(x)-h(x_0))} dx$ using part a) of the result, since $Mk^{-\frac{1}{2}+\delta'} \leq Mk^{-\frac{1}{3}}$ for $k \in \mathbb{N}$. We note that, for $j \in \mathbb{N}$,

$$\begin{aligned}
e_j\left(\mathrm{Sgn}(r_k) M k^{\delta'} \sqrt{\frac{h''(x_0)}{2}}\right) &= \lim_{x \to \mathrm{Sgn}(r_k)\infty} e_j(x) + \mathcal{O}\left(k^{-\infty}\right) \\
&= \mathrm{Sgn}(r_k)^{j+1} + \mathcal{O}\left(k^{-\infty}\right)
\end{aligned}$$

as $k \to \infty$, since for any $a > 0$,

$$\begin{aligned}
\int_{\mathrm{Sgn}(r_k)ak^{\delta'}}^{\mathrm{Sgn}(r_k)\infty} |x|^j e^{-x^2} dx &= k^{\frac{j+1}{2}} \int_{\mathrm{Sgn}(r_k)ak^{\delta'-\frac{1}{2}}}^{\mathrm{Sgn}(r_k)\infty} |y|^j e^{-ky^2} dy \\
&\leq e^{-a^2\frac{k-1}{k}k^{2\delta'}} k^{\frac{j+1}{2}} \int_{\mathbb{R}} |y|^j e^{-x^2} dx
\end{aligned}$$

$$= \mathcal{O}\left(k^{-\infty}\right).$$

□

2.2.2 An application of Laplace's method

Let us now discuss the asymptotics of the partial density function $\rho_{k,l}$ in the Bargmann-Fock model. As we can see from the next lemma, the asymptotics of $\rho_{k,l}(\boldsymbol{z}_k)$ exhibit an interesting transitioning behaviour for $\|\boldsymbol{z}_k\|$ near $\sqrt{2l}$.

Proposition 2.2.4. *Suppose that $\{\boldsymbol{z}_k\}_{k=1}^{\infty} \subset \mathbb{C}^n$ is a sequence of complex vectors and let $l \in \mathbb{N}$.*

a) *Suppose that $M, \delta > 0$ and $\left|\frac{\|\boldsymbol{z}_k\|^2}{2} - l\right| \geq M k^{-\frac{1}{2}+\delta}$ for all $k \in \mathbb{N}$. Then*

$$\rho_{l,k}\left(\boldsymbol{z}_k\right) = \begin{cases} \left(\frac{k}{2\pi}\right)^n + \mathcal{O}\left(k^{-\infty}\right) & \text{if } \|\boldsymbol{z}_k\| > \sqrt{2l} \text{ for all } k \in \mathbb{N} \\ \mathcal{O}\left(k^{-\infty}\right) & \text{if } \|\boldsymbol{z}_k\| < \sqrt{2l} \text{ for all } k \in \mathbb{N}. \end{cases}$$

b) *Define $h : (-l, \infty) \to \mathbb{R}$ by $h(x) \stackrel{\text{def}}{=} x - l\log(x+l) + l\log(l)$, for $x \in (-l, \infty)$, and let $M \geq 0$ and $m \in \mathbb{N}$. Suppose that $\left|\frac{\|\boldsymbol{z}_k\|^2}{2} - l\right| \leq M k^{-\frac{1}{3}}$ for all $k \in \mathbb{N}$. Then there exists $C = C(M, m) \geq 0$ such that*

$$\left|\rho_{l,k}\left(\boldsymbol{z}_k\right) - \left(\frac{k}{2\pi}\right)^n \frac{(kl)^{kl}}{(kl-1)!} e^{-kl} \sum_{j=0}^{m} \left(a_j\left(\left(\frac{\|\boldsymbol{z}_k\|^2}{2} - l\right)\sqrt{\frac{k}{2l}}\right) + (-1)^j b_j\right) k^{-\frac{j+1}{2}}\right| \leq C k^{-\left(\frac{m}{2}+1\right)},$$

for all $k \in \mathbb{N}$, and where

$$b_j \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i=0}^{j} \frac{(-1)^i}{i!(j+2i)!} \Gamma\left(\frac{j+1}{2} + i\right) (2l)^{\frac{j+1}{2}+i} \frac{\mathrm{d}}{\mathrm{d}\,x}^{j+2i} \left(h(x) - \frac{x^2}{2l}\right)^i \frac{1}{x+l}\Bigg|_0$$

and

$$a_j(s) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i=0}^{j} \frac{(-1)^i}{i!(j+2i)!} \Gamma\left(\frac{j+1}{2} + i\right) (2l)^{\frac{j+1}{2}+i} \frac{\mathrm{d}}{\mathrm{d}\,x}^{j+2i} \left(h(x) - \frac{x^2}{2l}\right)^i \frac{1}{x+l}\Bigg|_0 e_{j+2i}(s).$$

c) *Suppose $\frac{\|\boldsymbol{z}_k\|^2}{2} = \frac{s}{\sqrt{k}} + l$ for all $k \in \mathbb{N}$ and some constant $s \in \mathbb{R}$. Then part b) yields the expansion*

$$\begin{aligned} \rho_{l,k}\left(\boldsymbol{z}_k\right) &= \left(\frac{k}{2\pi}\right)^n \frac{(kl)^{kl}}{(kl-1)!} e^{-kl} \sum_{j=0}^{m} \left(a_j\left(\frac{s}{\sqrt{2l}}\right) + (-1)^j b_j\right) k^{-\frac{j+1}{2}} + \mathcal{O}\left(k^{-\infty}\right) \\ &= \sum_{j=0}^{\infty} c_j(s) k^{n-\frac{j}{2}} + \mathcal{O}\left(k^{-\infty}\right), \end{aligned}$$

where a_j and b_j are the same as in part b), and the coefficients $c_j(s)$ are obtained by expanding

$$\frac{(kl)^{kl}}{(kl-1)!} e^{-kl} \sim \sum_{j=0}^{\infty} d_j k^{-j+\frac{1}{2}} = \sqrt{\frac{kl}{2\pi}} \left(1 - \frac{1}{12l} k^{-1} + \cdots\right)$$

via Stirling's series and by collecting powers of k in the resulting sum.

Proof. For $r = \sqrt{\frac{\|\boldsymbol{z}\|^2}{2}}$, we have

$$\begin{aligned}\rho_{l,k}(\boldsymbol{z}) &= \left(\frac{k}{2\pi}\right)^n \frac{1}{(kl-1)!}\int_0^{kr^2} y^{kl-1}e^{-y}dy \\ &= \left(\frac{k}{2\pi}\right)^n \frac{(kl)^{kl}}{(kl-1)!}e^{-kl}\int_{-l}^{r^2-l}\frac{1}{y+l}e^{-kh(y)}dy,\end{aligned} \tag{2.2.2}$$

where $h(y) \overset{\text{def}}{=} y - l\log(y+l) + l\log(l)$ is a smooth strictly convex function on $(-l,\infty)$ with unique minimum $h(0) = 0$. Observe that the singularity of $\frac{1}{y+l}$ at $y = -l$ is no cause for concern, since $l \geq 1$ and

$$\int_{-l}^{-\frac{1}{2}} \frac{1}{y+l}e^{-kh(y)}dy \leq e^{-(k-1)h\left(-\frac{1}{2}\right)}\int_{-l}^{-\frac{1}{2}}\frac{1}{y+l}e^{-h(y)}dy = \mathcal{O}\left(k^{-\infty}\right).$$

Multiplying the integrand with a smooth cut-off function ξ such that $\xi(x) = 1$ for $x \geq -\frac{1}{2}$ and $\xi(x) = 0$ for $x \leq -\frac{2}{3}$ does not change the asymptotic expansion of the integral.

a) Let $r_k \overset{\text{def}}{=} \sqrt{\frac{\|\boldsymbol{z}_k\|^2}{2}}$ for all $k \in \mathbb{N}$. If $r_k^2 \geq l + Mk^{-\frac{1}{2}+\delta}$ for all $k \in \mathbb{N}$, we can use the proof of corollary B.2.4 which shows that

$$\int_{Mk^{-\frac{1}{2}+\delta}}^{\infty}\frac{1}{y+l}e^{-kh(y)}dy = \mathcal{O}\left(k^{-\infty}\right).$$

Hence $\rho_{l,k}(\boldsymbol{z}_k) - \rho_k(\boldsymbol{z}_k) = \mathcal{O}\left(k^{-\infty}\right)$ in that case. The proof in the case where $r_k^2 \leq l - Mk^{-\frac{1}{2}+\delta}$ uses similar reasoning.

b) We have

$$\int_{-l}^{r_k^2-l}\frac{1}{y+l}e^{-kh(y)}dy = \left(\int_0^{r_k^2-l} - \int_0^{-l}\right)\frac{1}{y+l}e^{-kh(y)}dy.$$

We apply proposition 2.2.2 b) to the integral

$$\int_0^{-l}\frac{1}{y+l}e^{-kh(y)}dy$$

and obtain

$$\int_0^{-l}\frac{1}{y+l}e^{-kh(y)}dy \sim \sum_{j=0}^{\infty}(-1)^{j+1}b_j k^{-\frac{j+1}{2}},$$

where

$$b_j = \frac{1}{2}\sum_{i=0}^{j}\frac{(-1)^i}{i!(j+2i)!}\Gamma\left(\frac{j+1}{2}+i\right)(2l)^{\frac{j+1}{2}+i}\,\frac{\mathrm{d}}{\mathrm{d}\,x}^{j+2i}\left(h(x)-\frac{x^2}{2l}\right)^i\frac{1}{x+l}\Bigg|_0.$$

We also have

$$\int_0^{r_k^2-l} \frac{1}{y+l} e^{-kh(y)} dy \sim \sum_{j=0}^{\infty} a_j \left(\left(r_k^2 - l\right) \sqrt{\frac{k}{2l}} \right) k^{-\frac{j+1}{2}}$$

by proposition 2.2.2 a), where, for $s \in \mathbb{R}$, we define

$$a_j(s) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i=0}^{j} \frac{(-1)^i}{i!(j+2i)!} \Gamma\left(\frac{j+1}{2} + i\right) (2l)^{\frac{j+1}{2}+i} \frac{\mathrm{d}}{\mathrm{d}\,x}^{j+2i} \left(h(x) - \frac{x^2}{2l}\right)^i \frac{1}{x+l}\bigg|_0 e_{j+2i}(s).$$

Combining these two expansions implies that, for $m \in \mathbb{N}$ and $M \geq 0$, there exists a constant $C = C(M, m) \geq 0$ such that

$$\left| \rho_{l,k}\left(\boldsymbol{z}_k\right) - \left(\frac{k}{2\pi}\right)^n \frac{(kl)^{kl}}{(kl-1)!} e^{-kl} \left(\sum_{j=0}^{m} \left(a_j \left(\left(\frac{\|\boldsymbol{z}_k\|^2}{2} - l \right) \sqrt{\frac{k}{2l}} \right) + (-1)^j b_j \right) k^{-\frac{j+1}{2}} \right) \right| \\ \leq Ck^{-\left(\frac{m}{2}+1\right)} \tag{2.2.3}$$

for all $k \in \mathbb{N}$.

c) This follows directly from b). □

Remark 2.2.5. *We compute that* $b_1 = -\frac{1}{3l}$, $b_2 = \frac{\sqrt{2\pi}}{24l^{\frac{3}{2}}}$, $b_3 = -\frac{4}{135l^2}$ *and* $b_4 = \frac{\sqrt{2\pi}}{575l^{\frac{5}{2}}}$.

Corollary 2.2.6. *Suppose* $\boldsymbol{z} \in \mathbb{C}^n$ *is fixed. Then*

$$\rho_{l,k}(\boldsymbol{z}) = \begin{cases} \left(\frac{k}{2\pi}\right)^n + \mathcal{O}\left(k^{-\infty}\right) & \text{if } \|\boldsymbol{z}\| > \sqrt{2l} \\ \frac{1}{2}\left(\frac{k}{2\pi}\right)^n + \sum_{j=0}^{\infty} c_{2j+1}(0) k^{n-\left(j+\frac{1}{2}\right)} + \mathcal{O}\left(k^{-\infty}\right) & \text{if } \|\boldsymbol{z}\| = \sqrt{2l} \\ \mathcal{O}\left(k^{-\infty}\right) & \text{if } \|\boldsymbol{z}\| < \sqrt{2l}, \end{cases}$$

where c_j, *for* $j \in \mathbb{N}_0$, *denote the same functions as in the previous lemma. In particular, we have*

$$c_1(0) = \frac{1}{6(2\pi)^n} \sqrt{\frac{2}{\pi l}}$$
$$c_3(0) = \frac{1}{1080(2\pi)^n} \sqrt{\frac{2}{\pi l^3}}.$$

Proof. Note that

$$(kl-1)! = \int_0^{\infty} y^{kl-1} e^{-y} dy = (kl)^{kl} e^{-kl} \int_{-l}^{\infty} \frac{1}{y+l} e^{-kh(y)} dy$$
$$= (kl)^{kl} e^{-kl} \left(\int_0^{\infty} - \int_0^{-l} \right) \left(\frac{1}{y+l} e^{-kh(y)} \right) dy.$$

Let a_j and b_j, for $j \in \mathbb{N}_0$, be defined as in the previous proposition. It follows from proposition

2.2.2 that

$$\begin{aligned}(kl-1)!(kl)^{-kl}e^{kl} &\sim \sum_{i=0}^{\infty}\left(1+(-1)^i\right)b_i k^{-\frac{i+1}{2}} \\ &\sim 2\sum_{i=0}^{\infty} b_{2i}k^{-\left(i+\frac{1}{2}\right)}.\end{aligned}$$

Also noting that $a_j(0)=0$ for $j \in \mathbb{N}_0$, we find that

$$\rho_{l,k}(\boldsymbol{z}) \sim \frac{1}{2}\left(\frac{k}{2\pi}\right)^n - \left(\frac{k}{2\pi}\right)^n \frac{(kl)^{kl}}{(kl-1)!}e^{-kl}\sum_{i=0}^{\infty} b_{2i+1}k^{-(i+1)}$$

if $\|\boldsymbol{z}\| = \sqrt{2l}$. The other cases follow from proposition 2.2.4. By Stirling's series we have

$$\frac{1}{(kl)!} = \frac{1}{\sqrt{2\pi kl}}(kl)^{-kl}e^{kl}\left(1-\frac{1}{12kl}+\mathcal{O}\left(k^{-2}\right)\right),$$

so that

$$\rho_{l,k}(\boldsymbol{z}) \sim \frac{1}{2}\left(\frac{k}{2\pi}\right)^n + \left(\frac{k}{2\pi}\right)^n\sqrt{\frac{l}{2\pi}}\left\{-b_1k^{-\frac{1}{2}} + \left(\frac{1}{12l}b_1 - b_3\right)k^{-\frac{3}{2}}\right\} + \mathcal{O}\left(k^{-\frac{5}{2}}\right)$$

for $\|\boldsymbol{z}\| = \sqrt{2l}$.

$\square$

Chapter 3

Toric Geometry

In this chapter, we review some standard terminology and notation needed for our discussion of toric varieties. We then specialize to the case of polarized toric Kähler manifolds and explicitly describe a natural open cover of such manifolds by $\mathbb{C}^n$ charts. We describe the torus action as well as the Kähler form in these coordinates. Since there are at least two ways of thinking about toric Kähler manifolds, one can approach the subject from several directions.

From the algebraic geometry point of view, one is interested in studying certain algebraic varieties of complex dimension n admitting an action of a complex n-dimensional torus $T_N \cong (\mathbb{C}^*)^n$ with an open dense orbit. Associated to this approach is the notion of a fan, a combinatorial object describing the way in which such a variety is glued together from various charts. We will recall a pedestrian approach to understanding the transition functions for smooth complex toric varieties in detail in 3.1.

Alternatively, one can think of a toric Kähler manifold (X_P, ω) as a symplectic manifold with additional structure. In this approach, such manifolds are classified by their image under their moment map. These images turn out to be special kinds of polytopes which are called *Delzant polytopes*. We describe how these approaches are related in 3.2.

3.1 Complex algebraic approach

3.1.1 Construction

From an algebraic geometry point of view, it is easiest to specify a toric variety by a *fan*. For the algebraic approach, we follow the notation of [Oda88]. We start with a free module $N \cong \mathbb{Z}^r$ of rank r and let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Let us review the basic objects required:

Definition 3.1.1 ([Oda88])**.** A subset σ of $N_{\mathbb{R}} \stackrel{\text{def}}{=} N \otimes_{\mathbb{Z}} \mathbb{R}$ is called a *convex polyhedral cone* if there exists a finite number of elements $n_1, \ldots, n_s$ in $N_{\mathbb{R}}$ such that

$$\sigma = \mathbb{R}_{\geq 0} n_1 + \ldots + \mathbb{R}_{\geq 0} n_s.$$

A convex polyhedral cone $\sigma \subset N_{\mathbb{R}}$ is called *rational* if we can choose the elements $n_1, \cdots, n_s$ above to lie in N, and it is called *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$.

We define the *dual cone* of a convex polyhedral cone σ in $N_{\mathbb{R}}$ by

$$\sigma^\vee = \{x \in M_{\mathbb{R}} : \langle x, y \rangle \geq 0 \text{ for all } y \in \sigma\}.$$

A subset τ of σ is called a *face* of σ, denoted $\tau < \sigma$, if there exists $m_0 \in \sigma^\vee$ such that $\tau = \sigma \cap \{m_0\}^\perp \stackrel{\text{def}}{=} \{y \in \sigma : \langle m_0, y\rangle = 0\}$.

Definition 3.1.2 ([Oda88, p.2])**.** A *fan* in N is a nonempty collection Δ of strongly convex rational polyhedral cones in $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ satisfying the following conditions:

i) Every face of any $\sigma \in \Delta$ is contained in Δ.

ii) For any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

The union $|\Delta| \stackrel{\text{def}}{=} \bigcup_{\sigma\in\Delta} \sigma$ is called the *support* of Δ.

To each face σ of a fan Δ, we associate an additive semigroup $\mathcal{S}_\sigma \stackrel{\text{def}}{=} M \cap \sigma^\vee$ which turns out to be saturated (If $cm \in \mathcal{S}_\sigma$ for $m \in M$ and a positive integer c, then $m \in \mathcal{S}_\sigma$.), finitely generated, and it satisfies $\mathcal{S}_\sigma + (-\mathcal{S}_\sigma) = M$. For each face $\sigma \in \Delta$, we get a set

$$\mathcal{U}_\sigma \stackrel{\text{def}}{=} \{u : \mathcal{S}_\sigma \to \mathbb{C} : u(0) = 1, u(m + m') = u(m)u(m') \text{ for all } m, m' \in \mathcal{S}_\sigma\},$$

and a choice of generators $m_1, \ldots, m_p$ of the semigroup $\mathcal{S}_\sigma$ yields an injective map

$$(e(m_1), \ldots, e(m_p)) : \mathcal{U}_\sigma \to \mathbb{C}^p,$$

where $e(m)(u) \stackrel{\text{def}}{=} u(m)$ for $m \in \mathcal{S}_\sigma$ and $\mathcal{U}_\sigma$. We identify $\mathcal{U}_\sigma$ with its image under the above map. Equivalently, following Fulton [Ful93], we could have defined $\mathcal{U}_\sigma \stackrel{\text{def}}{=} \text{Spec}_{max}\,(\mathbb{C}[S_\sigma])$, where $\mathbb{C}[S_\sigma]$ denotes the group ring corresponding to the semigroup S_σ. $\mathbb{C}[S_\sigma]$ is a commutative $\mathbb{C}$-algebra, and $\text{Spec}_{max}(\mathbb{C}[S_\sigma])$ denotes the set of maximal ideals in $\mathbb{C}[S_\sigma]$. The faces of a cone σ in the fan now naturally correspond to subsets of $\mathcal{U}_\sigma$:

Proposition 3.1.3 ([Oda88, p.7])**.** *For a strongly convex rational polyhedral cone σ in $N_\mathbb{R}$, its dual cone $\sigma^\vee$ is a rational polyhedral cone in $M_\mathbb{R}$. If τ is a face of σ, then there exists $m_0 \in M \cap \sigma^\vee$ such that $\tau = \sigma \cap \{m_0\}^\perp$. Hence τ is also a strongly convex rational polyhedral cone in $N_\mathbb{R}$. In this case, we have $\mathcal{S}_\tau = \mathcal{S}_\sigma + \mathbb{N}_0(-m_0)$ and*

$$\mathcal{U}_\tau = \{u \in \mathcal{U}_\sigma : u(m_0) \neq 0\},$$

which is an open subset of $\mathcal{U}_\sigma$.

We now obtain toric varieties by gluing these sets $\mathcal{U}_\sigma$ for $\sigma \in \Delta$:

Theorem 3.1.4 ([Oda88, p.7])**.** *For a fan Δ in N, we can naturally glue $\{\mathcal{U}_\sigma : \sigma \in \Delta\}$ together to obtain a Hausdorff complex analytic space*

$$T_N emb(\Delta) \stackrel{\text{def}}{=} \coprod_{\sigma\in\Delta} \mathcal{U}_\sigma \Big/ \sim,$$

which is irreducible and normal with dimension equal to $r = \text{rank}(N)$. We call $T_N emb(\Delta)$ the toric variety *associated to the fan $\Delta \subset N_\mathbb{R}$.*

3.1.2 The torus action

We define the algebraic torus

$$T_N \stackrel{\text{def}}{=} \text{Hom}_\mathbb{Z}(M, \mathbb{C}^*)$$

and, for $m \in M$, a character $e(m) : T_N \to \mathbb{C}^*$ given by $e(m)(t) = t(m)$ for $t \in T_N$. $T_N \subset T_N emb(\Delta)$, and the action of T_N on $\mathcal{U}_\sigma \subset T_N emb(\Delta)$, for $\sigma \in \Delta$, is defined by

$$(t.u)(m) \stackrel{\text{def}}{=} t(m)u(m) \quad \text{for } t \in T_N, u \in \mathcal{U}_\sigma \text{ and } m \in M.$$

3.1.3 Integral Delzant polytopes

Introduction

Let us now discuss Delzant polytopes in some detail. We recall the following definitons.

Definition 3.1.5. A *convex polytope* in $M_\mathbb{R}$ is a convex hull of a finite set of points in $M_\mathbb{R}$. A subset F of a convex polytope $P \subset M_\mathbb{R}$ is called a *face* of P, denoted $F < P$, if there exists $\boldsymbol{u} \in M_\mathbb{R}^*$ and $b \in \mathbb{R}$ such that

$$\begin{aligned} &P \subset H^+_{\boldsymbol{u},b} \stackrel{\text{def}}{=} \{\boldsymbol{v} \in M_\mathbb{R} : \langle \boldsymbol{v}, \boldsymbol{u} \rangle \geq b\} \qquad \text{and} \\ &F = P \cap \partial H^+_{\boldsymbol{u},b} = \{\boldsymbol{v} \in P : \langle \boldsymbol{v}, \boldsymbol{u} \rangle = b\}\,. \end{aligned}$$

Definition 3.1.6. A convex polytope $P \subset M_\mathbb{R}$ is *Delzant* if

1. There are n edges meeting in each vertex v.
2. The edges meeting in the vertex v are rational; i.e., each edge is of the form $v + te_i$, with $t \geq 0$, $t \in \mathbb{R}$ and $e_i \in M$.
3. The $e_1, \dots, e_n$ in (2) can be chosen to form a basis of M.

An *integral Delzant polytope* in $M_\mathbb{R}$ is a Delzant polytope whose vertices lie in M.

Now let $P \subset M_\mathbb{R}$ be an integral Delzant polytope. To P we can associate a set of cones as follows: To any face F of the polytope P we associate its *tangent cone* (see also [Aud04])

$$s_F \stackrel{\text{def}}{=} \bigcup_{r \geq 0} r(P - m),$$

where m is any point in the relative interior of F. Recall that the relative interior of a subset $S \subset \mathbb{R}^n$ is its interior considered as a subset of its affine hull $\mathrm{Aff}(S)$. The dual cones

$$\sigma_F \stackrel{\text{def}}{=} s_F^\vee = \{n \in N_\mathbb{R} : \langle n, m \rangle \geq 0 \text{ for all } m \in s_F\},$$

for $F < P$, form a fan $\Delta(P)$ defining the toric variety $T_N emb(\Delta(P))$ which we denote by X_P (see also [Aud04]). The cones σ_v, for $v \in \text{vertices}(P)$, are of particular importance because

$$\bigcup_{v \in \text{vertices}(P)} \mathcal{U}_{\sigma_v} = X_P.$$

In this way, a Delzant polytope defines a special type of toric variety X_P. It is well known that X_P is smooth, compact, n-dimensional and projective.

Deviating from the standard textbooks on toric varieties, we will from now on only consider integral Delzant polytopes and their corresponding fans. First, we check that the intersections of any two top dimensional cones of the fan $\Delta(P)$ corresponding to such a polytope are particularily easy to describe:

Lemma 3.1.7. *Let P be an integral Delzant polytope and let v, v' be vertices of P. Then*

$$\sigma_v \cap \sigma_{v'} = \sigma_v \cap \{v - v'\}^\perp.$$

Proof. Let $x \in \sigma_v \cap \sigma_{v'}$. Then, for $m \in s_v$, $\langle x, m\rangle \geq 0$, but $v' - v \in P - v \subset s_v$. Hence $\langle x, v' - v\rangle \geq 0$ and similarly, for $m \in s_{v'}$, $\langle x, m\rangle \geq 0$, but $v - v' \in P - v' \subset s_{v'}$. It follows that $\langle x, v - v'\rangle \geq 0$. We conclude that $x \in \{v - v'\}^\perp$. Conversely, let $x \in \sigma_v \cap \{v - v'\}^\perp$. Then, for $m \in s_{v'}$, $m = \sum_{i=1}^p \lambda_i m_i$ for some $\lambda_i \geq 0$ and $m_i \in P - v' = (P - v) + (v - v')$. We have $P - v \subset s_v$ and $x \in \{v - v'\}^\perp$. Therefore $\langle x, m_i\rangle \geq 0$ for all $i \in \{1, \cdots, p\}$ and so $\langle x, m\rangle \geq 0$. □

This gives us an explicit description of $\mathcal{U}_{\sigma_v \cap \sigma_{v'}}$ as follows:

Corollary 3.1.8. *Let P be an integral Delzant polytope and let v, v' be vertices of P. Then*

$$\mathcal{S}_{\sigma_v \cap \sigma_{v'}} = \mathcal{S}_{\sigma_v} - \mathbb{N}_0(v' - v).$$

In particular, $\mathcal{U}_{\sigma_v \cap \sigma_{v'}} \subset \mathcal{U}_{\sigma_v}$ is given by $\{u \in \mathcal{U}_{\sigma_v} : u(v' - v) \neq 0\}$.

Proof. This follows from [Ful93, proposition 2]. □

The line bundle L_P

Note that so far we have not used all the information contained in the polytope P. We will now see that P also determines a very ample holomorphic line bundle L_P whose bases of global sections give embeddings of X_P into projective space.

Definition 3.1.9 ([Oda88, p.66])**.** Let $\Delta \subset N$ be a fan. A function $h : |\Delta| \to \mathbb{R}$ on the support $|\Delta| \overset{\text{def}}{=} \bigcup_{\sigma \in \Delta} \sigma$ of Δ is called a Δ-linear support function if it is $\mathbb{Z}$-valued on $N \cap |\Delta|$ and is linear on each $\sigma \in \Delta$.

We can associate to P a $\Delta(P)$-linear support function $h : N_\mathbb{R} \to \mathbb{R}$ by

$$h(n) \overset{\text{def}}{=} \inf\{\langle m, n\rangle : m \in P\} \quad \text{for } n \in N_\mathbb{R},$$

as described in [Oda88, A.3]. Note that the tangent cone s_F for a face F of P satisfies

$$s_F = \bigcup_{r \geq 0} r(P - m) = \bigcup_{r \geq 0} r(P - F) = \bigcup_{r \geq 0, p \in P, f \in F} r(p - f),$$

where m is any point in the relative interior of F. The dual cone σ_F to s_F is hence given by

$$\sigma_F = \{n \in N_\mathbb{R} : \langle p - f, n\rangle \geq 0, \text{ for all } f \in F, p \in P\}.$$

Lemma 3.1.10. *Let F be a face of P. Then*

$$\sigma_F = \{w \in N_\mathbb{R} : \langle f, w\rangle = h(w) \textit{ for all } f \in F\}.$$

Proof. Let $n \in \{w \in N_\mathbb{R} : \langle f, w\rangle = h(w)$ for all $f \in F\}$. Then $\langle f, n\rangle \leq \langle p, n\rangle$ for all $p \in P$ and $f \in F$. Hence $\langle p - f, n\rangle \geq 0$ for all $p \in P$ and $f \in F$. If $n \in \sigma_F$, then $\langle f, n\rangle \leq \langle p, n\rangle$ for all $p \in P$ and $f \in F$. But $f \in P$, so that $\langle f, n\rangle = h(n)$ for all $f \in F$. □

We observe that $\Delta(P)$-linear support functions take the following simple form: For $n \in \sigma_v$ and $v \in \text{vertices}(P)$, we have $h(n) = \langle n, v \rangle$. Since $\bigcup_{v \in \text{vertices}(P)} \sigma_v = N_{\mathbb{R}}$, this gives an easy description of h. Following [Oda88, §2], we can now define a very ample line bundle

$$L_P \stackrel{\text{def}}{=} \coprod_{v \in \text{vertices}(P)} \mathcal{U}_{\sigma_v} \times \mathbb{C} \Big/ \sim,$$

where we glue $\mathcal{U}_{\sigma_v} \times \mathbb{C}$ to $\mathcal{U}_{\sigma_{v'}} \times \mathbb{C}$ along $\mathcal{U}_{\sigma_v \cap \sigma_{v'}}$ via the isomorphism

$$g_{v'v} : \mathcal{U}_{\sigma_v \cap \sigma_{v'}} \times \mathbb{C} \to \mathcal{U}_{\sigma_v \cap \sigma_{v'}} \times \mathbb{C}$$

given by $g_{v'v}(x, c) = (x, e(v - v')(x)c)$ for $x \in \mathcal{U}_{\sigma_v \cap \sigma_{v'}}$ and $c \in \mathbb{C}$. On the line bundle L_P, we define an action of T_N on $\mathcal{U}_{\sigma_v} \times \mathbb{C}$ by $t.(u, c) \stackrel{\text{def}}{=} (t.u, e(-v)(t)c)$ for $v \in \text{vertices}(P), t \in T_N$, $u \in \mathcal{U}_{\sigma_v}$ and $c \in \mathbb{C}$. It is a standard result in toric geometry that the global holomorphic sections of L_P have a basis corresponding to the integral points of P. Let us elaborate this point a little bit. The actions of T_N on X_P and L_P induce an action of T_N on $H^0(X_P, L_P)$ as follows:

$$(t \circ s)(p) \stackrel{\text{def}}{=} t^{-1}.s(t.p) \quad \text{for } s \in H^0(X_P, L_P), p \in X_P \text{ and } t \in T_N.$$

The vector space $H^0(X_P, L_P) = \oplus_{\alpha \in P \cap M} V_\alpha$ then decomposes as a direct sum of one-dimensional weight-spaces V_α for $\alpha \in P \cap M$ for this representation, where

$$V_\alpha \stackrel{\text{def}}{=} \left\{ s \in H^0(X_P, L_P) : t \circ s = e(\alpha)(t)s \text{ for } t \in T_N \right\} \quad \text{for } \alpha \in P \cap M.$$

We define $s_{\alpha,v} : \mathcal{U}_{\sigma_v} \to \mathcal{U}_{\sigma_v} \times \mathbb{C}$ by $s_{\alpha,v}(u) \stackrel{\text{def}}{=} (u, e(\alpha - v)(u))$ for $\alpha \in P \cap M$, $v \in \text{vertices}(P)$ and $u \in \mathcal{U}_{\sigma_v}$. One can check that such a collection $\{s_{\alpha,v} : v \in \text{vertices}(P)\}$, for $\alpha \in P \cap M$, descends to give a global non-trivial section $s_\alpha \in H^0(X_P, L_P)$ and that $s_\alpha \in V_\alpha$. It is also not hard to see that the scaled polytope kP gives rise to the line bundle L_P^k over X_P for $k \in \mathbb{N}$.

Coordinates

Let P be an integral Delzant polytope in $M_{\mathbb{R}}$. Since we will later work in concrete local coordinates on X_P and L_P, we will now give a very explicit description of toric coordinates, transition functions and the torus action in terms of the polytope P. The description of these coordinates on X_P is also sketched in less detail in [Don08].

A choice of an ordered reference basis $(e_1, \ldots, e_n)$ for M gives rise to an isomorphism of groups $f : T_N \cong (\mathbb{C}^*)^n$, $f : u \mapsto (u(e_1), \ldots, u(e_n))$ for $u \in T_N$. For any two vertices $\boldsymbol{v}, \boldsymbol{v'} \in P$, we define $A_{\boldsymbol{v'v}} : M_{\mathbb{R}} \to M_{\mathbb{R}}$ to be the linear map such that

$$A_{\boldsymbol{v'v}} \left(m(\boldsymbol{v})_i \right) = m \left(\boldsymbol{v'} \right)_i \quad \text{for all } i \in \{1, \cdots, n\},$$

where $(m(\boldsymbol{v})_1, \ldots, m(\boldsymbol{v})_n)$, for a vertex $\boldsymbol{v}$ of P, denotes an ordered basis for M given by primitive integral vectors along the edges of P emanating from $\boldsymbol{v}$. Let us assume that we have fixed an ordering of these edge vectors for all vertices of P at the beginning.

Note that any such choice of ordered edge-vectors at a vertex v precisely corresponds to an isomorphism $\mathcal{U}_{\sigma_v} \cong \mathbb{C}^n$, given by $u \mapsto (u(m(\boldsymbol{v})_1), \ldots, u(m(\boldsymbol{v})_n))$ for $u \in \mathcal{U}_{\sigma_v}$. We denote by

$(A_{\boldsymbol{v'v}})_{ij}$ the representation of $A_{\boldsymbol{v'v}}$ in the ordered basis $(m(\boldsymbol{v})_1, \ldots, m(\boldsymbol{v})_n)$. We have

$$A_{\boldsymbol{v'v}}(m(\boldsymbol{v})_l) = \sum_j (A_{\boldsymbol{v'v}})_{jl} m(\boldsymbol{v})_j.$$

As a matrix, we then have

$$A_{\boldsymbol{v'v}} = ([m(\boldsymbol{v'})_1]_{\boldsymbol{v}}, \ldots, [m(\boldsymbol{v'})_n]_{\boldsymbol{v}}),$$

where $[\boldsymbol{x}]_{\boldsymbol{v}}$ denotes the coordinates of $\boldsymbol{x} \in M_{\mathbb{R}}$ in the basis $(m(\boldsymbol{v})_1, \ldots, m(\boldsymbol{v})_n)$. We define, for $A \in M_{n\times n}(\mathbb{Z})$, an open set

$$\begin{aligned} \mathcal{U}_A \overset{\text{def}}{=} \big\{&(z_1, \ldots, z_n) : z_j \in \mathbb{C} \text{ if } A_{jk} \geq 0 \text{ for all } k \in \{1, \cdots, n\} \\ &\text{and } z_j \in \mathbb{C}^* \text{ if there exists } k \in \{1, \cdots, n\} \text{ such that } A_{jk} < 0\big\} \end{aligned}$$

and similarly for $\boldsymbol{b} \in \mathbb{Z}^n$:

$$\mathcal{U}_{\boldsymbol{b}} \overset{\text{def}}{=} \big\{(z_1, \ldots, z_n) : z_j \in \mathbb{C} \text{ if } b_j \geq 0 \text{ and } z_j \in \mathbb{C}^* \text{ if } b_j < 0\big\}.$$

For $A \in M_{n\times n}(\mathbb{Z})$, we define $\phi_A : \mathcal{U}_A \to \mathbb{C}^n$ by

$$\phi_A : \boldsymbol{z} \mapsto \left(\prod_{j=1}^n z_j^{A_{j1}}, \ldots, \prod_{j=1}^n z_j^{A_{jn}}\right).$$

We observe that, for $A, B \in M_{n\times n}(\mathbb{Z})$ and $\boldsymbol{z} \in \mathcal{U}_A \cap \phi_A^{-1}(\mathcal{U}_B)$,

$$\begin{aligned} \phi_B \circ \phi_A(\boldsymbol{z}) &= \phi_B\left(\prod_{j=1}^n z_j^{A_{j1}}, \ldots, \prod_{j=1}^n z_j^{A_{jn}}\right) \\ &= \left(\prod_{i,j=1}^n \left(z_j^{A_{ji}}\right)^{B_{i1}}, \ldots, \prod_{i,j=0}^n \left(z_j^{A_{ji}}\right)^{B_{in}}\right) \\ &= \left(\prod_{j=1}^n z_j^{(AB)_{j1}}, \ldots, \prod_{j=1}^n z_j^{(AB)_{jn}}\right) \\ &= \phi_{AB}(\boldsymbol{z}). \end{aligned}$$

Similarly, for $\boldsymbol{b} \in \mathbb{Z}^n$ and $\boldsymbol{z} \in \mathcal{U}_{\boldsymbol{b}}$, define $\phi_{\boldsymbol{b}}(\boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{b}} = \prod_{j=1}^n z_j^{b_j}$. Then, for $\boldsymbol{z} \in \mathcal{U}_A \cap \phi_A^{-1}(\mathcal{U}_{\boldsymbol{b}})$, we have

$$\phi_{\boldsymbol{b}} \circ \phi_A(\boldsymbol{z}) = \prod_{i,j=1}^n z_j^{A_{ji}b_i} = \phi_{A\boldsymbol{b}}(\boldsymbol{z}).$$

Lemma 3.1.11.

$$\mathcal{U}_{A_{\boldsymbol{v'v}}} = \{\boldsymbol{z} \in \mathbb{C}^n : \boldsymbol{z}^{[\boldsymbol{v'}-\boldsymbol{v}]_{\boldsymbol{v}}} \neq 0\}$$

In particular, $\mathcal{U}_{A_{\boldsymbol{v'v}}}$ *is the image of* $\mathcal{U}_{\sigma_v \cap \sigma_{v'}} \subset \mathcal{U}_{\sigma_v}$ *under the isomorphism* $\mathcal{U}_{\sigma_v} \cong \mathbb{C}^n$ *obtained by choosing a* $\mathbb{Z}^n$ *basis of edge-vectors at* v *as described.* $\phi_{A_{\boldsymbol{v'v}}}$ *is the local coordinate description of the identity map* $id : \mathcal{U}_{\sigma_v} \supset \mathcal{U}_{\sigma_v\cap\sigma_{v'}} \to \mathcal{U}_{\sigma_v\cap\sigma_{v'}} \subset \mathcal{U}_{\sigma_{v'}}$*, and, for* $\boldsymbol{m} \in \mathcal{S}_{\sigma_v}$*,* $\phi_{[\boldsymbol{m}]_{\boldsymbol{v}}}$ *is the local coordinate version of* $e(\boldsymbol{m}) : \mathcal{U}_{\sigma_v} \to \mathbb{C}$.

Proof. Let $i \in \{1, \cdots, n\}$. We need to show that $([\boldsymbol{v'} - \boldsymbol{v}]_{\boldsymbol{v}})_i = 0$ if and only if $([m(\boldsymbol{v'})_j]_{\boldsymbol{v}})_i \geq 0$

for all $j \in \{1, \cdots, n\}$. Note that

$$P \subset \boldsymbol{v} + \sum_{j=1}^{n} \mathbb{R}_{\geq 0} m(\boldsymbol{v})_j \qquad \text{for any vertex } \boldsymbol{v} \text{ of } P.$$

In particular, in the basis $(m(\boldsymbol{v})_1, \ldots, m(\boldsymbol{v})_n)$, we have

$$\begin{aligned} P &\subset [\boldsymbol{v}]_{\boldsymbol{v}} + \mathbb{R}^n_{\geq 0} \\ P &\subset [\boldsymbol{v'}]_{\boldsymbol{v}} + \sum_{j=1}^{n} \mathbb{R}_{\geq 0} [m(\boldsymbol{v'})_j]_{\boldsymbol{v}}. \end{aligned}$$

Suppose $([\boldsymbol{v'} - \boldsymbol{v}]_{\boldsymbol{v}})_i = 0$. For any $j \in \{1, \cdots, n\}$, $[\boldsymbol{v'} + m(\boldsymbol{v'})_j]_{\boldsymbol{v}} \in P$, so $([m(\boldsymbol{v'})_j]_{\boldsymbol{v}})_i \geq 0$ if $([\boldsymbol{v'} - \boldsymbol{v}]_{\boldsymbol{v}})_i = 0$. Now suppose $([m(\boldsymbol{v'})_j]_{\boldsymbol{v}})_i \geq 0$ for all $j \in \{1, \cdots, n\}$. We have $([\boldsymbol{v}]_{\boldsymbol{v}})_i \in ([\boldsymbol{v'}]_{\boldsymbol{v}})_i + \sum_{j=1}^{n} \mathbb{R}_{\geq 0}([m(\boldsymbol{v'})_j]_{\boldsymbol{v}})_i$ which implies $([\boldsymbol{v} - \boldsymbol{v'}]_{\boldsymbol{v}})_i \geq 0$, but $([\boldsymbol{v'}]_{\boldsymbol{v}})_i \in ([\boldsymbol{v}]_{\boldsymbol{v}})_i + \mathbb{R}_{\geq 0}$ implies $([\boldsymbol{v'} - \boldsymbol{v}]_{\boldsymbol{v}})_i \geq 0$. ☐

We can now reformulate our description of X_P and L_P as follows:

$$X_P = \left. \coprod_{\boldsymbol{v} \in \mathrm{vertices}(P)} \mathbb{C}^n \times \{\boldsymbol{v}\} \middle/ \sim \right. ,$$

where

$$\mathbb{C}^n \times \{\boldsymbol{v}\} \supset \mathcal{U}_{A_{\boldsymbol{v'}\boldsymbol{v}}} \times \{\boldsymbol{v}\} \ni (\boldsymbol{z}, \boldsymbol{v}) \sim (\phi_{A_{\boldsymbol{v'}\boldsymbol{v}}}(\boldsymbol{z}), \boldsymbol{v'}) \in \mathcal{U}_{A_{\boldsymbol{v}\boldsymbol{v'}}} \times \{\boldsymbol{v'}\} \subset \mathbb{C}^n \times \{\boldsymbol{v'}\}$$

for $\boldsymbol{v}, \boldsymbol{v'} \in \mathrm{vertices}(P)$. Similarly, we have

$$L_P = \left. \coprod_{\boldsymbol{v} \in \mathrm{vertices}(P)} \mathbb{C}^n \times \mathbb{C} \times \{\boldsymbol{v}\} \middle/ \sim \right. ,$$

where

$$\begin{aligned} \mathbb{C}^n \times \mathbb{C} \times \{\boldsymbol{v}\} \supset \mathcal{U}_{A_{\boldsymbol{v'}\boldsymbol{v}}} \times \mathbb{C} \times \{\boldsymbol{v}\} \ni (\boldsymbol{z}, \lambda, \boldsymbol{v}) \sim \\ (\phi_{A_{\boldsymbol{v'}\boldsymbol{v}}}(\boldsymbol{z}), \phi_{[\boldsymbol{v}-\boldsymbol{v'}]_{\boldsymbol{v}}}(\boldsymbol{z})\lambda, \boldsymbol{v'}) \in \mathcal{U}_{A_{\boldsymbol{v}\boldsymbol{v'}}} \times \mathbb{C} \times \{\boldsymbol{v'}\} \subset \mathbb{C}^n \times \mathbb{C} \times \{\boldsymbol{v'}\} \end{aligned}$$

for $\boldsymbol{v}, \boldsymbol{v'} \in \mathrm{vertices}(P)$. We call these coordinate charts $\mathcal{U}_{\boldsymbol{v}} \cong \mathbb{C}^n \times \{\boldsymbol{v}\}$ of X_P (and $\mathcal{U}_{\boldsymbol{v}} \times \mathbb{C} \cong \mathbb{C}^n \times \mathbb{C} \times \{\boldsymbol{v}\}$ of L_P) for $\boldsymbol{v} \in$ *vertices*(P) the *toric defining charts* (*trivializations*).

Holomorphic sections in coordinates We observe that, for $\boldsymbol{\alpha} \in P \cap M$, the section $s_{\boldsymbol{\alpha}} \in H^0(X_P, L)$ is determined in local coordinates by a collection of functions $s_{\boldsymbol{\alpha},\boldsymbol{v}} : \mathcal{U}_{\sigma_v} \cong \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C} \cong \mathcal{U}_{\sigma_v} \times \mathbb{C}$ for each $\boldsymbol{v} \in \mathrm{vertices}(P)$, where $s_{\boldsymbol{\alpha},\boldsymbol{v}}(\boldsymbol{z}) \stackrel{\mathrm{def}}{=} (\boldsymbol{z}, \phi_{[\boldsymbol{\alpha}-\boldsymbol{v}]_{\boldsymbol{v}}}(\boldsymbol{z}))$.

Torus action in coordinates Recall that, at the beginning, we fixed a reference basis $(e_1, \ldots, e_n)$ for M when we chose coordinates for the torus T_N. For $t \in T_N$ and $u \in \mathcal{U}_{\sigma_v}$, $t.u \in \mathcal{U}_{\sigma_v}$ is identified with

$$(t(m(v)_1)u(m(v)_1), \ldots, t(m(v)_n)u(m(v)_n)),$$

so that, in local coordinates, the action is given by

$$(t_1, \dots, t_n).(z_1, \dots, z_n) = (\boldsymbol{t}^{[m(\boldsymbol{v})_1]_e} z_1, \dots, \boldsymbol{t}^{[m(\boldsymbol{v})_n]_e} z_n),$$

where $[\boldsymbol{x}]_e$ denotes the coordinates of $\boldsymbol{x} \in M_{\mathbb{R}}$ in the basis $(e_1, \dots, e_n)$ and where $t_j \in \mathbb{C}^*$, $z_j \in \mathbb{C}$ for all $j \in \{1, .., n\}$. If we denote $A_{\boldsymbol{ve}} = ([m(\boldsymbol{v})_1]_e, \dots, [m(\boldsymbol{v})_n]_e)$, so that $A_{\boldsymbol{ve}}(e_i) = m(\boldsymbol{v})_i$, then $\boldsymbol{t}.\boldsymbol{z} = \phi_{A_{\boldsymbol{ve}}}(\boldsymbol{t}) ._{diag} \boldsymbol{z}$, where $._{diag}$ denotes the standard diagonal action. Similarly, the torus action on the line bundle is given by $\boldsymbol{t}.(\boldsymbol{z}, \lambda) = (\phi_{A_{\boldsymbol{ve}}}(\boldsymbol{t}) ._{diag} \boldsymbol{z}, \phi_{[-\boldsymbol{v}]_e}(\boldsymbol{t})\lambda)$ for $(\boldsymbol{z}, \lambda) = (\boldsymbol{z}, \lambda, \boldsymbol{v}) \in \mathbb{C}^n \times \mathbb{C} \times \{\boldsymbol{v}\}$, $\boldsymbol{v} \in \text{vertices}(P)$ and $\boldsymbol{t} \in (\mathbb{C}^*)^n$.

Convenient charts When we work in any of the charts $\mathcal{U}_{\sigma_v}$, for $v \in \text{vertices}(P)$, note that we can pick corresponding coordinates of the torus T_N such that the reference basis $(e_1, \dots, e_n)$ and the basis of edge-vectors $(m(v)_1, \cdots, m(v)_n)$ agree. The action of T_N on $\mathcal{U}_{\sigma_v}$ then corresponds to the standard diagonal action on $\mathbb{C}^n \cong \mathcal{U}_{\sigma_v}$ in these coordinates.

3.2 Symplectic approach

Having discussed the complex algebraic approach to toric Kähler manifolds, let us now consider the subject through the eyes of a symplectic geometer (see e.g. [CdS03]).

From the symplectic point of view, we define:

Definition 3.2.1 ([CdS03, definitions 1.6.1 and 1.6.2]). A *symplectic toric manifold* $(X, \omega, \mathbb{T}, \mu)$ is a compact connected symplectic manifold (X, ω) equipped with an effective Hamiltonian action of a real torus $\mathbb{T}$ of dimension $\frac{1}{2}\dim(X)$ and with a choice of moment map $\mu : X \to \mathfrak{t}^*$, where $\mathfrak{t}$ denotes the Lie algebra of $\mathbb{T}$.

We call two symplectic toric manifolds $(X_i, \omega_i, \mathbb{T}_i, \mu_i)$, $i = 1, 2$ *equivalent* if there exists an isomorphism $\lambda : \mathbb{T}_1 \to \mathbb{T}_2$ and a λ-equivariant symplectomorphism $\phi : X_1 \to X_2$ such that $\mu_1 = \mu_2 \circ \phi$.

The central classification result is:

Theorem 3.2.2 ([CdS03, Theorem 2.1.2, Delzant's theorem]). *Symplectic toric manifolds are classified by Delzant polytopes. The bijective correspondence is given by mapping the symplectic toric manifold to its image under the moment map.*

One direction of the theorem is proved by a procedure called "Delzant's construction" which associates to each Delzant polytope P a symplectic toric manifold which is constructed as a symplectic reduction of the standard flat space $(\mathbb{C}^d, \omega_{std})$, where d is equal to the number of codimension 1 faces of P. Interestingly, one can also construct the complex manifold X_P via a GIT quotient of $\mathbb{C}^d$ and the complex and symplectic structures obtained from these two quotients turn out to be compatible giving (X_P, ω) the structure of a Kähler manifold. Symplectic toric manifolds with $2\pi\omega \in H^2(X_P, \mathbb{Z})$ are smooth polarized toric varieties with integral Delzant polytopes and $2\pi[\omega] = c_1(L_P)$. The interested reader may consult [Gui94a, Gui94b, CdS03] for more details on this story.

Note that, via the moment map $\mu : X_P \to \mathfrak{t}^*$, we think of the Delzant polytope as lying inside $\mathfrak{t}^*$. The group lattice $L \overset{\text{def}}{=} \text{Ker}(\exp : \mathfrak{t} \to \mathbb{T}) \subset \mathfrak{t}$ is a natural lattice in $\mathfrak{t}$, and we can consider its dual lattice $L^* = \{\phi \in \mathfrak{t}^* : \phi(l) \in \mathbb{Z} \text{ for all } l \in L\}$. L and L^* are the lattices N and M that we discussed previously as seen from the symplectic point of view.

3.3 Toric Kähler geometry

3.3.1 Abreu's work

Let $(L_P, h) \to (X_P, \omega)$ be a polarized toric Kähler manifold such that $[\omega] = 2\pi c_1(L)$, and let $\mu : X_P \to \mathfrak{t}^* = M_{\mathbb{R}} \cong \mathbb{R}^n$ denote a choice of moment map that has P as its image.

Legendre duality

We will now recall the notion of Legendre duality for strictly convex functions (see also [Gui94b, Appendix 1]). Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex smooth function and suppose that ϕ has a global minimum at $x_0 \in \mathbb{R}^n$. We define the Legendre transform of ϕ to be the map $\mu : \mathbb{R}^n \to \mathbb{R}^n$ given by $\mu(\boldsymbol{t}) \stackrel{\text{def}}{=} \nabla\phi|_{\boldsymbol{t}} = \left(\frac{\partial\phi}{\partial x_1}\Big|_{\boldsymbol{t}}, \cdots, \frac{\partial\phi}{\partial x_n}\Big|_{\boldsymbol{t}} \right)$ for $\boldsymbol{t} \in \mathbb{R}^n$. μ is a diffeomorphism onto its image which is an open convex subset of $\mathbb{R}^n$ and which we denote by $\mathcal{U}$. There exists a function $u : \mathcal{U} \to \mathbb{R}$, dual to ϕ, such that

$$\phi(\boldsymbol{t}) + u(\boldsymbol{\alpha}) = \langle \boldsymbol{t}, \boldsymbol{\alpha} \rangle \qquad \text{if and only if } \boldsymbol{\alpha} = \mu(\boldsymbol{t}).$$

We define, for $\boldsymbol{t} \in \mathbb{R}^n$ and $\boldsymbol{\alpha} \in \mathcal{U}$,

$$h(\boldsymbol{\alpha}, \boldsymbol{t}) \stackrel{\text{def}}{=} \phi(\boldsymbol{t}) + u(\boldsymbol{\alpha}) - \langle \boldsymbol{t}, \boldsymbol{\alpha} \rangle.$$

$h : \mathcal{U} \times \mathbb{R}^n \to \mathbb{R}$, defined in this way, is a smooth function that has a nice geometric interpretation. We consider the graph:

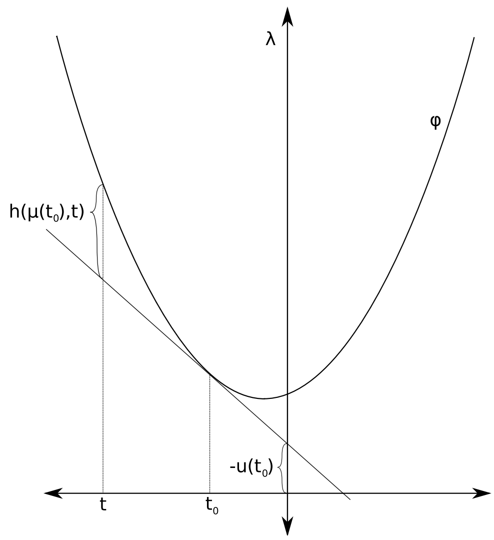

Figure 3.1: Legendre Duality

$$\Gamma_\phi \stackrel{\text{def}}{=} \{(\boldsymbol{t}, \phi(\boldsymbol{t})) \in \mathbb{R}^{n+1} : \boldsymbol{t} \in \mathbb{R}^n\}.$$

Let us pick coordinates $(\boldsymbol{t}, \lambda) \in \mathbb{R}^{n+1}$ and let $\gamma : \mathbb{R}^n \to \Gamma_\phi$, $\gamma(\boldsymbol{t}) \stackrel{\text{def}}{=} (\boldsymbol{t}, \phi(\boldsymbol{t}))$ for $\boldsymbol{t} \in \mathbb{R}^n$. At the point $(\boldsymbol{t}_0, \phi(\boldsymbol{t}_0)) \in \Gamma_\phi$, we have the tangent hyperplane

$$T_{(\boldsymbol{t}_0, \phi(\boldsymbol{t}_0))}\Gamma_\phi = \operatorname{Span}\left(\gamma_* \frac{\partial}{\partial t_1}\Big|_{\boldsymbol{t}_0}, \ldots, \gamma_* \frac{\partial}{\partial t_n}\Big|_{\boldsymbol{t}_0} \right)$$

whose orthogonal complement is given by

$$N_{(\boldsymbol{t}_0,\phi(\boldsymbol{t}_0))}\Gamma_\phi = \mathrm{Span}\left(-\sum_{j=1}^{n} \frac{\partial \phi}{\partial t_j}\bigg|_{\boldsymbol{t}_0} \frac{\partial}{\partial t_j}\bigg|_{(\boldsymbol{t}_0,\phi(\boldsymbol{t}_0))} + \frac{\partial}{\partial \lambda}\bigg|_{(\boldsymbol{t}_0,\phi(\boldsymbol{t}_0))} \right)$$
$$\cong \mathrm{Span}\left(\left(-\frac{\partial \phi}{\partial t_1}\bigg|_{\boldsymbol{t}_0}, \dots, -\frac{\partial \phi}{\partial t_n}\bigg|_{\boldsymbol{t}_0}, 1 \right) \right).$$

Concretely, we have $T_{(\boldsymbol{t}_0,\phi(\boldsymbol{t}_0))}\Gamma_\phi$ as a subset of $\mathbb{R}^{n+1}$ given by:

$$T_{(\boldsymbol{t}_0,\phi(\boldsymbol{t}_0))}\Gamma_\phi = \left\{ (\boldsymbol{t}, s) \in \mathbb{R}^{n+1} : \left\langle (\boldsymbol{t}, s) - (\boldsymbol{t}_0, \phi(\boldsymbol{t}_0)), \left(-\frac{\partial \phi}{\partial t_1}\bigg|_{\boldsymbol{t}_0}, \dots, -\frac{\partial \phi}{\partial t_n}\bigg|_{\boldsymbol{t}_0}, 1 \right) \right\rangle = 0 \right\}.$$

For each $\boldsymbol{t}_0 \in \mathbb{R}^n$, we hence have an affine hyperplane $H_{\boldsymbol{t}_0} \stackrel{\text{def}}{=} T_{(\boldsymbol{t}_0,\phi(\boldsymbol{t}_0))}\Gamma_\phi$, and the point in $H_{\boldsymbol{t}_0}$ with the first n coordinates given by $\boldsymbol{t} \in \mathbb{R}^n$ is given by $\big(\boldsymbol{t}, \phi(\boldsymbol{t}_0) + \langle \boldsymbol{t} - \boldsymbol{t}_0, \mu(\boldsymbol{t}_0)\rangle\big) = \big(\boldsymbol{t}, \langle \boldsymbol{t}, \mu(\boldsymbol{t}_0)\rangle - u(\mu(\boldsymbol{t}_0))\big)$. In particular, for $\boldsymbol{t} = \boldsymbol{0}$, we get $\big(\boldsymbol{0}, -u(\mu(\boldsymbol{t}_0))\big)$ which gives us a geometric interpretation for the function u. Note that, as a byproduct, $h\left(\mu(\boldsymbol{t}_0), \boldsymbol{t}\right) = \phi(\boldsymbol{t}) - \left(\langle \boldsymbol{t}, \mu(\boldsymbol{t}_0)\rangle - u(\mu(\boldsymbol{t}_0))\right)$, so that we can recover an interpretation of Young's inequality

$$\phi(\boldsymbol{t}) + u(\boldsymbol{\alpha}) \geq \langle \boldsymbol{t}, \boldsymbol{\alpha}\rangle \iff h(\boldsymbol{\alpha}, \boldsymbol{t}) \geq 0, \qquad \text{for } \boldsymbol{t} \in \mathbb{R}^n \text{ and } \boldsymbol{\alpha} \in \mathcal{U},$$

as simply stating that Γ_ϕ lies above each tangent hyperplane (which follows by convexity). We will see later that functions like h are related to the asymptotic expansion of the Bergman kernel on polarized toric Kähler manifolds.

Coordinates on the open orbit

Following Abreu [Abr98] and using the notation developed in this chapter, we consider the open dense T_N-orbit in X_P, where X_P is a complex n-dimensional toric manifold corresponding to a Delzant polytope $P \subset M_{\mathbb{R}} \cong \mathbb{R}^n$. Pick $\boldsymbol{v} \in \mathrm{vertices}(P)$ and choose $(m(\boldsymbol{v})_1, \dots, m(\boldsymbol{v})_n)$ as a reference basis for M giving an isomorphism $U_{\sigma_{\boldsymbol{v}}} \cong \mathbb{C}^n$ and, due to the inclusion $T_N \subset U_{\sigma_{\boldsymbol{v}}}$, also an isomorphism $T_N \cong (\mathbb{C}^*)^n \subset \mathbb{C}^n$. We have a holomorphic surjection

$$\begin{array}{rcl} \mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n & \stackrel{\pi}{\twoheadrightarrow} & (\mathbb{C}^*)^n \\ \boldsymbol{w} = \boldsymbol{t} + i\boldsymbol{\theta} & \mapsto & e^{\boldsymbol{t}+i\boldsymbol{\theta}} \stackrel{\text{def}}{=} (e^{t_1+i\theta_1}, \dots, e^{t_n+i\theta_n}). \end{array}$$

Slightly abusing notation, we will denote local holomorphic coordinates on the quotient $(\mathbb{C}^*)^n = \mathbb{C}^n/\operatorname{Ker}\pi = \mathbb{R}^n + i\mathbb{R}^n/(2\pi\mathbb{Z})^n$ by $\boldsymbol{t} + i\boldsymbol{\theta}$ as well. On $T_N \cong (\mathbb{C}^*)^n \subset X_P$, there exists a $\mathbb{T}^n$-invariant real Kähler potential $\phi : (\mathbb{C}^*)^n \to \mathbb{R}$ which we think of as a function in the variables $\boldsymbol{t} = (t_1, \cdots, t_n) \in \mathbb{R}^n$. We have

$$\begin{aligned} \omega &= 2i\partial\overline{\partial}\phi = \mathrm{Hess}(\phi)_{ij} dt_i \wedge d\theta_j \\ g &= \mathrm{Hess}(\phi)_{ij} \left(dt_i dt_j + d\theta_i d\theta_j\right). \end{aligned}$$

Note that $\phi : \mathbb{R}^n \to \mathbb{R}$ is unique only up to the addition of some affine function $\boldsymbol{t} \mapsto \langle \boldsymbol{t}, \boldsymbol{c}\rangle + \lambda$ for $\boldsymbol{t} \in \mathbb{R}^n$, some $\boldsymbol{c} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. If, however, we think of the polytope P as fixed, we can require that $\mu = \nabla\phi$ in these coordinates with $\mu(\mathbb{R}^n) = \mathrm{Int}(P)$. This determines ϕ up to the addition of a constant $\lambda \in \mathbb{R}$. Suppose that we have normalized ϕ in this way and suppose that we have fixed a Hermitian fibre-wise metric h on L_P such that $iF_h = \omega$ (Recall that such a fibre-wise metric is unique up to multiplication by a positive constant $e^{-\lambda}$, where $\lambda \in \mathbb{R}$). We can now fix the remaining ambiguity in ϕ by demanding that there exists a holomorphic trivialization of L_P over T_N such that the canonical section $s_{\boldsymbol{\alpha}} \in H^0(X_P, L_P)$ for $\boldsymbol{\alpha} \in P \cap M$ is represented by the function $\boldsymbol{z} \mapsto \boldsymbol{z}^{\boldsymbol{\alpha}}$, for $\boldsymbol{z} \in (\mathbb{C}^*)^n$, and we have $|s_{\boldsymbol{\alpha}}(\boldsymbol{t})|_h^2 = e^{-2(\phi(\boldsymbol{t}) - \langle \boldsymbol{t}, \boldsymbol{\alpha} \rangle)}$ for $\boldsymbol{z} = (z_1, \cdots, z_n) = \left(e^{t_1 + i\theta_1}, \cdots, e^{t_n + i\theta_n}\right) \in (\mathbb{C}^*)^n$.

Since the Kähler metric $g(.,.) = \omega(., J.)$ is non-degenerate, $\phi : \mathbb{R}^n \to \mathbb{R}$ is strongly convex. μ is a diffeomorphism from $\mathbb{R}^n$ onto the interior of P. Furthermore, we call the strongly convex function $u : \mathrm{Int}(P) \to \mathbb{R}$ which is Legendre dual to ϕ the *symplectic potential* corresponding to ϕ. We have

$$u(\boldsymbol{x}) + \phi(\boldsymbol{t}) \geq \langle \boldsymbol{x}, \boldsymbol{t} \rangle,$$

for $\boldsymbol{t} \in \mathbb{R}^n$ and $\boldsymbol{x} \in \mathrm{Int}(P)$, with equality if and only if $\boldsymbol{x} = \mu(\boldsymbol{t})$.

Chapter 4

Example: $\mathbb{CP}^n$

We now familiarize ourselves with some of the toric geometry that we introduced in the previous chapter by studying a partial density function with vanishing at a point in $\mathbb{CP}^n$. Investigating this example sheds some light on the more general asymptotics that we will study in chapter 7. The results obtained here will also be helpful for our discussion of blow-ups of $\mathbb{CP}^n$ in chapter 10.

4.1 A polarization of $\mathbb{CP}^n$

For a parameter $m \in \mathbb{N}$, we consider the polarization

$$(\mathcal{O}(m), h_{FS}^m) \to (\mathbb{CP}^n, m\omega_{FS}).$$

We normalize the Fubini-Study Kähler form ω_{FS} such that $\omega_{FS} \in 2\pi c_1(\mathcal{O}(1))$. On the coordinate chart $\mathcal{U}_i = \{[t_0 : \cdots : t_n] \in \mathbb{CP}^n : t_i \neq 0\}$, for $i \in \{0, \cdots, n\}$, we pick coordinates $\psi_i : \mathbb{C}^n \to \mathcal{U}_i$,

$$\psi_i : \boldsymbol{z} = (z_1, \cdots, z_n) \mapsto [z_1 : \cdots : z_{i-1} : \underbrace{1}_{i^{th}} : z_i : \cdots : z_n].$$

On $\mathcal{U}_0$, $\omega_{FS} \overset{\text{def}}{=} i\partial\overline{\partial} \log\left(1 + \sum_{i=1}^n |z_i|^2\right)$ and $h_{FS}(\boldsymbol{z}) \overset{\text{def}}{=} \left(1 + \sum_{i=1}^n |z_i|^2\right)^{-1}$ in the standard trivialization of $\mathcal{O}(1)$ over $\mathcal{U}_0$. We define the $\mathcal{L}^2$ inner product induced by h_{FS}^{mk} on $H^0(\mathbb{CP}^n, \mathcal{O}(mk))$ by

$$\langle s, s' \rangle_{h_{FS}^{mk}} \overset{\text{def}}{=} \int_{\mathbb{CP}^n} (s, s')_{h_{FS}^{mk}} \frac{(m\omega_{FS})^n}{n!} \quad \text{for } s, s' \in H^0(\mathbb{CP}^n, \mathcal{O}(mk)).$$

Recall that, as a toric variety, the polarization above is determined by the simplex $Simp_n(m)$ in $\mathbb{R}^n$ with vertices at $(0, \cdots, 0), (m, 0, \cdots, 0), \cdots, (0, \cdots, 0, m)$. These vertices correspond to the charts $\mathcal{U}_0, \cdots, \mathcal{U}_n$ respectively. Similarly, $\mathcal{O}(mk)$ has a defining toric trivialization over each $\mathcal{U}_i$.

Lemma 4.1.1. *For $m, k \in \mathbb{N}$, an orthonormal basis of $\left(H^0(\mathbb{CP}^n, \mathcal{O}(mk)), \langle ., . \rangle_{h_{FS}^{mk}}\right)$ is given by*

$$\{s_{\boldsymbol{\alpha},m,k} \in H^0(\mathbb{CP}^n, \mathcal{O}(mk)) : \boldsymbol{\alpha} \in \mathbb{N}_0^n, |\boldsymbol{\alpha}| \leq mk\},$$

where, on the toric defining trivialization $\mathcal{O}(mk)|_{\mathcal{U}_0} \cong \mathcal{U}_0 \times \mathbb{C}$, $s_{\boldsymbol{\alpha},m,k}$ takes the form

$$s_{\boldsymbol{\alpha},m,k} : \boldsymbol{z} \mapsto (\boldsymbol{z}, a_{\boldsymbol{\alpha},m,k}\boldsymbol{z}^{\boldsymbol{\alpha}}), \ \textit{for } \boldsymbol{z} \in \mathbb{C}^n,$$

and

$$a_{\boldsymbol{\alpha},m,k} \stackrel{\text{def}}{=} \sqrt{\frac{(mk+n)!}{(2\pi)^n (mk)! m^n} \binom{mk}{mk - |\boldsymbol{\alpha}|, \boldsymbol{\alpha}}}.$$

Proof. As usual, the basis above is (up to scaling) the standard toric basis of sections corresponding to integral points of a polytope P. In this case, P is the simplex $Simp_n(mk)$ with vertices

$$(0, \cdots, 0), (mk, 0, \cdots, 0), \cdots, (0, \cdots, 0, mk).$$

We remark that the resulting basis for $m = 1$ has also previously been discussed *e.g.* in [AL04]. For $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ and $j \in \mathbb{N}$ such that $j \geq |\boldsymbol{\alpha}| + n + 1$, we compute that

$$I_n(\boldsymbol{\alpha}, j) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n_{\geq 0}} \frac{r_1^{2\alpha_1+1} \cdots r_n^{2\alpha_n+1}}{(1 + \sum_{i=1}^n r_i^2)^j} d\boldsymbol{r} = \frac{\boldsymbol{\alpha}!(j - (|\boldsymbol{\alpha}| + n + 1))!}{(j-1)!}.$$

For $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ and $|\boldsymbol{\alpha}| \leq mk$, consider the section $s'_{\boldsymbol{\alpha},m,k} \in H^0(\mathbb{CP}^n, \mathcal{O}(km))$ given by $\boldsymbol{z} \mapsto (\boldsymbol{z}, \boldsymbol{z}^{\boldsymbol{\alpha}})$ on $\mathcal{U}_0$. We have

$$\begin{aligned}
\|s'_{\boldsymbol{\alpha},m,k}\|^2 &= \int_{\mathbb{CP}^n} \frac{|\boldsymbol{z}^{\boldsymbol{\alpha}}|^2}{\left(1 + \sum_{i=1}^n |z_i|^2\right)^{mk}} \frac{(m\omega_{FS})^n}{n!} \\
&= (4\pi m)^n \int_{\mathbb{R}^n_{\geq 0}} \frac{r_1^{2\alpha_1+1} \cdots r_n^{2\alpha_n+1}}{(1 + \sum_{i=1}^n r_i^2)^{mk+n+1}} d\boldsymbol{r} \\
&= \frac{(2\pi m)^n (mk)!}{(mk+n)!} \binom{mk}{mk - |\boldsymbol{\alpha}|, \boldsymbol{\alpha}}^{-1}.
\end{aligned}$$

The sections $s'_{\boldsymbol{\alpha},m,k}$, for fixed $m, k \in \mathbb{N}$ and for $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ such that $|\boldsymbol{\alpha}| \leq mk$, form an orthogonal basis of $H^0(\mathbb{CP}^n, \mathcal{O}(mk))$. Normalizing by their $\mathcal{L}^2$ norms yields the result. $\square$

4.2 Symplectic coordinates

Let us quickly discuss the toric and symplectic potentials for the polarization of $\mathbb{CP}^n$ in question. Observe that $\omega \stackrel{\text{def}}{=} m\omega_{FS} = 2i\partial\overline{\partial}\phi(\boldsymbol{t})$ for the toric potential $\phi : \mathbb{R}^n \to \mathbb{R}$, $\phi : \boldsymbol{t} \mapsto \frac{m}{2} \log\left(1 + \sum_{i=1}^n e^{2t_i}\right)$ for $\boldsymbol{t} = (t_1, \cdots, t_n) \in \mathbb{R}^n$. We use the coordinates $(z_1, \cdots, z_n) = \left(e^{t_1 + i\theta_1}, \cdots, e^{t_n + i\theta_n}\right)$ on $(\mathbb{C}^*)^n \subset \mathbb{CP}^n$. The moment map $\mu : \mathbb{CP}^n \mapsto \mathbb{R}^n \cong \mathfrak{t}^*$ is invariant under the real torus $\mathbb{T}^n \subset (\mathbb{C}^*)^n$ and takes the form

$$\mu(\boldsymbol{t}) = \nabla\phi|_{\boldsymbol{t}} = \frac{m}{1 + \sum_{i=1}^n e^{2t_i}} (e^{2t_1}, \cdots, e^{2t_n}).$$

There exists a symplectic potential $u : Simp_n(m) \to \mathbb{R}$ such that

$$u(\boldsymbol{\alpha}) + \phi(\boldsymbol{t}) = \langle \boldsymbol{\alpha}, \boldsymbol{t} \rangle, \qquad \text{for } \boldsymbol{\alpha} = \mu(\boldsymbol{t}),$$

and where $\boldsymbol{\alpha} \in \mathrm{Int}(Simp_n(m))$ and $\boldsymbol{t} \in \mathbb{R}^n$. It is not hard to compute explicitly that

$$u(\boldsymbol{\alpha}) = \frac{1}{2} \left(\sum_{i=1}^n \alpha_i \log \alpha_i + \left(m - \sum_{i=1}^n \alpha_i\right) \log \left(m - \sum_{i=1}^n \alpha_i\right) - m \log m \right),$$

for $\boldsymbol{\alpha} \in Simp_n(m)$, and

$$\nabla u|_{\boldsymbol{\alpha}} = \mu^{-1}(\boldsymbol{\alpha}) = \frac{1}{2}\left(\log\alpha_1 - \log\left(m - \sum_{i=1}^{n}\alpha_i\right), \cdots, \log\alpha_n - \log\left(m - \sum_{i=1}^{n}\alpha_i\right)\right)$$

for $\boldsymbol{\alpha} \in \operatorname{Int}(Simp_n(m))$. As we can see in figure 4.1, the graph of u, for $m = 10$, $n = 1$ and $n = 2$ respectively, is of the expected form.

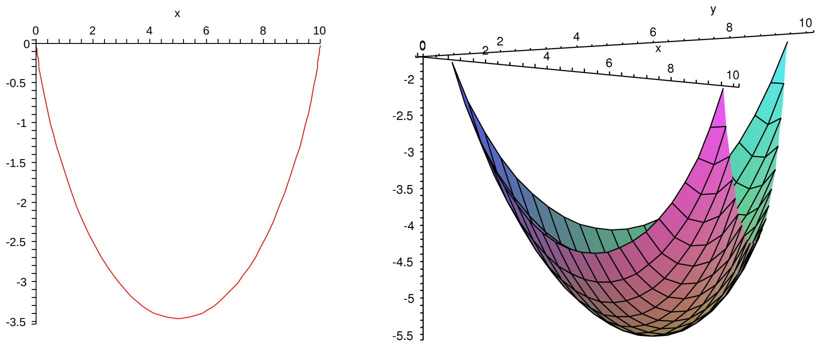

Figure 4.1: Graphs of $x \mapsto u(x)$ for $m = 10$, $n = 1$ and $(x, y) \mapsto u(x, y)$ for $m = 10$, $n = 2$ respectively.

4.3 Density functions

For the polarization $(\mathcal{O}(m), h_{FS}^m) \to (\mathbb{CP}^n, m\omega_{FS})$, we are now investigating the *partial density function* $\rho_{l,m,k}$ corresponding to the subspace $\mathcal{J}_{[1:0:\ldots:0]}^{lk} \subset H^0(\mathbb{CP}^n, \mathcal{O}(mk))$ of sections vanishing to order at least lk at $[1 : 0 : \ldots : 0]$ for $l < m$, $l, m, k \in \mathbb{N}$. For any orthonormal basis $\{s_{1,k}, \ldots, s_{M_k,k}\}$ of $\mathcal{J}_{[1:0:\ldots:0]}^{lk}$, $\rho_{l,m,k} : \mathbb{CP}^n \to \mathbb{R}$ is defined as

$$\rho_{l,m,k}(p) \stackrel{\text{def}}{=} \sum_{j=1}^{M_k} |s_{j,k}(p)|^2_{h_{FS}^{mk}} \quad \text{for } p \in \mathbb{CP}^n.$$

Similarly, we recall that, for any orthonormal basis $\{s_{1,k}, \ldots, s_{N_k,k}\}$ of $H^0(\mathbb{CP}^n, \mathcal{O}(mk))$, the density function $\rho_{m,k} : \mathbb{CP}^n \to \mathbb{R}$ is defined as

$$\rho_{m,k}(p) \stackrel{\text{def}}{=} \sum_{j=1}^{N_k} |s_{j,k}(p)|^2_{h_{FS}^{mk}} \quad \text{for } p \in \mathbb{CP}^n.$$

Lemma 4.3.1. *The density function for the polarization $(\mathcal{O}(m), h_{FS}^m) \to (\mathbb{CP}^n, m\omega_{FS})$ is constant for fixed $k, m \in \mathbb{N}$ and given by*

$$\begin{aligned}\rho_{m,k} &= \frac{1}{(2\pi)^n}\frac{(mk+n)!}{(mk)!m^n} \\ &= \frac{1}{(2\pi)^n}\left(k^n + \frac{1}{2}\frac{n(n+1)}{m}k^{n-1} + \mathcal{O}\left(k^{n-2}\right)\right).\end{aligned}$$

Furthermore, the partial density function $\rho_{l,m,k}$ corresponding to the subspace $\mathcal{J}_{[1:0:\ldots:0]}^{lk} \subset H^0(\mathbb{CP}^n, \mathcal{O}(mk))$ of sections vanishing to order at least lk at $[1 : 0 : \ldots : 0]$, for $l < m$

and $l, m, k \in \mathbb{N}$*, is given by*

$$\rho_{l,m,k}(\boldsymbol{z}) = \frac{1}{(2\pi)^n}\frac{(mk+n)!}{(mk)!m^n}\frac{\sum_{j=lk}^{mk}\binom{mk}{j}\left(\sum_{i=1}^n |z_i|^2\right)^j}{\left(1+\sum_{i=1}^n |z_i|^2\right)^{mk}} \qquad \text{for } \boldsymbol{z} = (z_1, \cdots, z_n) \in \mathbb{C}^n$$

in the coordinate $\psi_0 : \mathbb{C}^n \to \mathcal{U}_0$*. In symplectic coordinates, we have*

$$\rho_{l,m,k}(\boldsymbol{\alpha}) = \frac{1}{(2\pi)^n}\frac{(mk+n)!}{(mk)!m^n} f_{l,m,k}\left(\frac{\sum_{i=1}^n \alpha_i}{m}\right) \qquad \text{for } \boldsymbol{\alpha} \in Simp_n(m),$$

where $f_{l,m,k} : [0,1] \to \mathbb{R}$ *is defined as*

$$f_{l,m,k}(s) \stackrel{\text{def}}{=} \sum_{j=lk}^{mk} \binom{mk}{j} s^j (1-s)^{mk-j} \qquad \text{for } s \in [0,1].$$

Proof. By definition, we have on $\mathcal{U}_0$ that

$$\begin{aligned}
\rho_{m,k}(\boldsymbol{z}) &= \sum_{|\boldsymbol{\alpha}|\le mk} a^2_{\boldsymbol{\alpha},m,k} \left|\boldsymbol{z}^{\boldsymbol{\alpha}}\right|^2_{h_{FS}^{mk}} \\
&= \frac{(mk+n)!}{(2\pi)^n (mk)! m^n} \sum_{|\boldsymbol{\alpha}|\le mk} \binom{mk}{mk-|\boldsymbol{\alpha}|, \boldsymbol{\alpha}} \frac{\left|\boldsymbol{z}^{\boldsymbol{\alpha}}\right|^2}{\left(1+\sum_{i=1}^n |z_i|^2\right)^{mk}} \\
&= \frac{1}{(2\pi)^n}\frac{(mk+n)!}{(mk)!m^n}.
\end{aligned}$$

Similarly, we have on $\mathcal{U}_0$ that

$$\begin{aligned}
\rho_{l,m,k}(\boldsymbol{z}) &= \sum_{lk\le|\boldsymbol{\alpha}|\le mk} a^2_{\boldsymbol{\alpha},m,k} \left|\boldsymbol{z}^{\boldsymbol{\alpha}}\right|^2_{h_{FS}^{mk}} \\
&= \frac{1}{(2\pi)^n}\frac{(mk+n)!}{(mk)!m^n} \sum_{j=lk}^{mk} \binom{mk}{j} \frac{\left(\sum_{i=1}^n |z_i|^2\right)^j}{\left(1+\sum_{i=1}^n |z_i|^2\right)^{mk}}.
\end{aligned}$$

Computing this expression in symplectic coordinates yields the second part of the lemma. □

Remark 4.3.2. *Figure 4.2 indicates that* $f_{l,m,k}$ *converges to a step function with the transitioning behaviour occurring at* $\frac{l}{m}$ *as* $k \to \infty$*.*

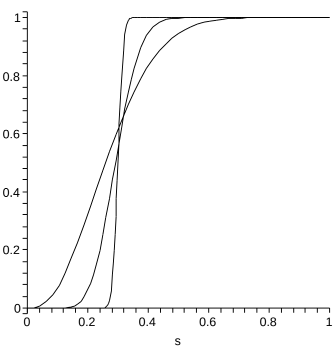

Figure 4.2: Graphs of $f_{l,m,k}$ for $l = 3$, $m = 10$ and $k \in \{1, 3, 100\}$.

In chapter 7, we will investigate the decay behaviour of the analogues of $\rho_{l,m,k}$ for general polarized toric manifolds. The asymptotic expansion of $\rho_{l,m,k}$ on the transitioning region (e.g. at $s = \frac{l}{m}$ in this example) will turn out to be particularly interesting.

Chapter 5

Toric Localization

We provide a proof of a toric localization of sums theorem that enables us to compute the asymptotics of toric Bergman kernels using an orthonormal basis of sections of a small subspace of $H^0(X, L^k)$. Our approach also gives a simple proof of an off-diagonal exponential decay estimate for toric Bergman kernels and extends a localization of sums result by Song and Zelditch (see [SZ10, lemma 1.2]).

5.1 Some toric estimates

Lemma 5.1.1. *Suppose that $\mathbb{T}^n$ acts on $\mathbb{C}^n$ by the standard action*

$$(e^{i\theta_1}, \cdots, e^{i\theta_n}).(z_1, \cdots, z_n) = (e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n) \quad \text{for } \theta_j \in [0, 2\pi), z_j \in \mathbb{C}$$

and that ω is a $\mathbb{T}^n$-invariant Kähler form on $\mathbb{C}^n$.

a) There exists a $\mathbb{T}^n$-invariant smooth Kähler potential $\phi : \mathbb{C}^n \to \mathbb{R}$ such that

$$\omega = 2i\partial\overline{\partial}\phi.$$

b) Up to a constant, the restriction of the moment map $\mu : \mathbb{C}^n \to \mathfrak{t}^* \cong \mathbb{R}^n$ to the open subset $\mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$, for $k \le n$, is given by

$$\begin{aligned} &\mu(x_1, y_1, \cdots, x_k, y_k, t_{k+1}, \cdots, t_n) \\ &\qquad = \left(x_1 \frac{\partial \phi}{\partial x_1} + y_1 \frac{\partial \phi}{\partial y_1}, \cdots, x_k \frac{\partial \phi}{\partial x_k} + y_k \frac{\partial \phi}{\partial y_k}, \frac{\partial \phi}{\partial t_{k+1}}, \cdots, \frac{\partial \phi}{\partial t_n} \right), \end{aligned}$$

for $k \le n$, where we have used coordinates $z = x + iy$ on the $\mathbb{C}$ factors and $z = e^{t+i\theta}$ on the $\mathbb{C}^*$ factors of $\mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$.

Proof. a) By the standard $\partial\overline{\partial}$-lemma, there exists a smooth Kähler potential $\psi : \mathbb{C}^n \to \mathbb{R}$ such that $\omega = 2i\partial\overline{\partial}\psi$ on $\mathbb{C}^n$. We define

$$\phi(\boldsymbol{z}) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \psi\left(\left(e^{i\theta_1}, \cdots, e^{i\theta_n}\right).\boldsymbol{z}\right) d\theta_1 \cdots d\theta_n \quad \text{ for } \boldsymbol{z} \in \mathbb{C}^n.$$

It is now easy to check that
$$2i\partial\overline{\partial}\phi = \omega.$$

b) This is a straightforward calculation. We observe that the vector field $\frac{\partial}{\partial\theta_j}^{\#}$ generated by $\frac{\partial}{\partial\theta_j} \in \mathfrak{t}$ is given by
$$\frac{\partial}{\partial\theta_j}^{\#} = i\left(z_j\frac{\partial}{\partial z_j} - \overline{z}_j\frac{\partial}{\partial\overline{z}_j}\right).$$
Let us use the shorthand notation $\phi_{z_i\overline{z}_j} = \frac{\partial^2\phi}{\partial z_i\partial\overline{z}_j}$. We have
$$i_{\frac{\partial}{\partial\theta_j}^{\#}}\omega = -2\left(z_j\phi_{z_j\overline{z}_k}d\overline{z}_k + \overline{z}_j\phi_{z_k\overline{z}_j}dz_k\right),$$
while
$$\begin{aligned} d\left(x_j\phi_{x_j} + y_j\phi_{y_j}\right) &= d\left(z_j\phi_{z_j} + \overline{z}_j\phi_{\overline{z}_j}\right) \\ &= z_j\phi_{z_jz_k}dz_k + z_j\phi_{z_j\overline{z}_k}d\overline{z}_k + \phi_{z_k}dz_k + \overline{z}_j\phi_{\overline{z}_jz_k}dz_k + \overline{z}_j\phi_{\overline{z}_j\overline{z}_k}d\overline{z}_k + \phi_{\overline{z}_k}d\overline{z}_k. \end{aligned} \tag{5.1.1}$$

By the $\mathbb{T}^n$-invariance of ϕ,
$$z_j\phi_{z_j} - \overline{z}_j\phi_{\overline{z}_j} = 0 \quad \text{for } j \in \{1,\cdots,n\}.$$

Differentiating the above identity with respect to z_j and $\overline{z}_j$, for $j \in \{1,\cdots,n\}$, and substituting the resulting expressions into equation 5.1.1 gives the required equality
$$d\left(x_j\phi_{x_j} + y_j\phi_{y_j}\right) = -i_{\frac{\partial}{\partial\theta_j}^{\#}}\omega \quad \text{for } j \in \{1,\cdots,n\}.$$

A final change of coordinates $z_j = e^{t_j+i\theta_j}$, for $j \in \{k+1,\cdots,n\}$, gives the last few components of the moment map. □

Remark 5.1.2. *The above lemma extends the expressions for the moment map and potential defined on $(\mathbb{C}^*)^n$ as used by Abreu [Abr98, Abr03] to a $\mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ chart.*

Let us denote by $s_{\boldsymbol{\alpha},k}$, for $\boldsymbol{\alpha} \in P \cap \frac{1}{k}\mathbb{Z}^n$, the holomorphic section in $H^0(X_P, L_P^k)$ corresponding to the integral point $k\boldsymbol{\alpha} \in kP \cap \mathbb{Z}^n$ that we discussed in chapter 3.

Lemma 5.1.3. *Let $(L_P, h) \to (X_P, \omega)$ be a polarized toric Kähler manifold and let $\mu : X_P \to \mathfrak{t}^*$ denote a choice of moment map that has P as its image. There exists a continuous function $n : P \times P \twoheadrightarrow [0,1]$ with the following properties:*

a) We have $n(\boldsymbol{\alpha},\boldsymbol{\beta}) = e^{-h(\boldsymbol{\alpha},\boldsymbol{\beta})}$, for all $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathrm{Int}(P)$, where
$$h(\boldsymbol{\alpha},\boldsymbol{\beta}) \stackrel{\text{def}}{=} 2\Big(u(\boldsymbol{\alpha}) - u(\boldsymbol{\beta}) + \left\langle \nabla u|_{\boldsymbol{\beta}}, \boldsymbol{\beta} - \boldsymbol{\alpha}\right\rangle\Big) \geq 0,$$
and $u : P \to \mathbb{R}$ denotes a symplectic potential. $h(\boldsymbol{\alpha},\boldsymbol{\beta}) = 0$ if and only if $\boldsymbol{\alpha} = \boldsymbol{\beta}$, and the only critical points of $h(\boldsymbol{\alpha}, .) : \mathrm{Int}(P) \to \mathbb{R}$ and $h(., \boldsymbol{\alpha}) : \mathrm{Int}(P) \to \mathbb{R}$, for $\boldsymbol{\alpha} \in \mathrm{Int}(P)$, occur at $\boldsymbol{\alpha}$. Furthermore, $\mathrm{Hess}\, h(\boldsymbol{\alpha}, .)|_{\boldsymbol{\alpha}}$ and $\mathrm{Hess}\, h(., \boldsymbol{\alpha})|_{\boldsymbol{\alpha}}$ are positive definite for $\boldsymbol{\alpha} \in \mathrm{Int}(P)$.

b)
$$n(\boldsymbol{\alpha},\boldsymbol{\beta})^k = \frac{\left|s_{\boldsymbol{\alpha},k}\left(\mu^{-1}(\boldsymbol{\beta})\right)\right|^2_{h^k}}{\left|s_{\boldsymbol{\alpha},k}\left(\mu^{-1}(\boldsymbol{\alpha})\right)\right|^2_{h^k}}$$

for $k \in \mathbb{N}$, $\boldsymbol{\alpha} \in P \cap \frac{1}{k}\mathbb{Z}^n$ *and* $\boldsymbol{\beta} \in P$.

c) $n(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 0$ *if and only if* $\boldsymbol{\beta} \in \bigcup_{\{F : F < P, \boldsymbol{\alpha} \notin F\}} F$.

Proof. For $\boldsymbol{v} \in \text{vertices}(P)$, we define $m_{\boldsymbol{v}} : P \times \mathcal{U}_{\sigma_{\boldsymbol{v}}} \to \mathbb{R}_{\geq 0}$ by

$$m_{\boldsymbol{v}}(\boldsymbol{\alpha}, \boldsymbol{z}) = \left|\phi_{[\boldsymbol{\alpha}-\boldsymbol{v}]_{\boldsymbol{v}}}(\boldsymbol{z})\right|^2 h_{\boldsymbol{v}}(\boldsymbol{z}),$$

for $\boldsymbol{z} \in \mathbb{C}^n \cong \mathcal{U}_{\sigma_{\boldsymbol{v}}}$, where $h_{\boldsymbol{v}}$ is the local expression of the Hermitian metric with respect to the standard trivialization of L_P over $\mathcal{U}_{\sigma_{\boldsymbol{v}}}$, and we choose coordinates on $\mathcal{U}_{\sigma_{\boldsymbol{v}}}$ by mapping $u \in \mathcal{U}_{\sigma_{\boldsymbol{v}}}$ to $(u(m(v)_1), \cdots, u(m(v)_n)$ as described in chapter 3. Let us check that $\{m_{\boldsymbol{v}} : \boldsymbol{v} \in \text{vertices}(P)\}$ glue to give a global function $m : P \times X_P \to \mathbb{R}$. Let $\boldsymbol{v}, \boldsymbol{v'}$ be two vertices of P. We check that

$$\begin{aligned} m_{\boldsymbol{v'}}(\boldsymbol{\alpha}, \phi_{A_{\boldsymbol{v'}\boldsymbol{v}}}(\boldsymbol{z})) &= \left|\phi_{[\boldsymbol{\alpha}-\boldsymbol{v'}]_{\boldsymbol{v'}}}\left(\phi_{A_{\boldsymbol{v'}\boldsymbol{v}}}(\boldsymbol{z})\right)\right|^2 h_{\boldsymbol{v'}}\left(\phi_{A_{\boldsymbol{v'}\boldsymbol{v}}}(\boldsymbol{z})\right) \\ &= \left|\phi_{[\boldsymbol{\alpha}-\boldsymbol{v'}]_{\boldsymbol{v}}}(\boldsymbol{z})\right|^2 \left|\phi_{[\boldsymbol{v'}-\boldsymbol{v}]_{\boldsymbol{v}}}(\boldsymbol{z})\right|^2 h_{\boldsymbol{v}}(\boldsymbol{z}) \\ &= \left|\phi_{[\boldsymbol{\alpha}-\boldsymbol{v}]_{\boldsymbol{v}}}(\boldsymbol{z})\right|^2 h_{\boldsymbol{v}}(\boldsymbol{z}) \\ &= m_{\boldsymbol{v}}(\boldsymbol{\alpha}, \boldsymbol{z}) \end{aligned}$$

for $\boldsymbol{z} \in \mathcal{U}_{\sigma_{\boldsymbol{v}} \cap \sigma_{\boldsymbol{v'}}}$ and $\boldsymbol{\alpha} \in P$.

Let us now show that $m(\boldsymbol{\alpha}, \boldsymbol{z}) \neq 0$ for $\boldsymbol{z} \in \mu^{-1}(\boldsymbol{\alpha})$. Let $\boldsymbol{\alpha} \in P$. Pick a vertex $\boldsymbol{v}$ of P such that $\mu^{-1}(\boldsymbol{\alpha}) \subset \mathcal{U}_{\sigma_{\boldsymbol{v}}}$. Our coordinates on $\mathcal{U}_{\sigma_{\boldsymbol{v}}}$ also induce coordinates on $T_N \subset \mathcal{U}_{\sigma_{\boldsymbol{v}}}$, and the action of T_N on $\mathcal{U}_{\sigma_{\boldsymbol{v}}}$ is just the standard diagonal action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$ in these coordinates. Note that our choice of coordinates also corresponds to a choice of basis $(m(v)_1, \cdots, m(v)_n)$ of $M = \mathfrak{t}^* \cong \mathbb{R}^n$. Let $\psi : \mathbb{C}^n \to \mathbb{R}$ denote a $\mathbb{T}^n$-invariant Kähler potential so that $\omega = 2i\partial\overline{\partial}\psi$. By lemma 5.1.1, we have

$$\mu(x_1, y_1, \cdots, x_n, y_n) = \left(x_1 \frac{\partial \psi}{\partial x_1} + y_1 \frac{\partial \psi}{\partial y_1}, \cdots, x_n \frac{\partial \psi}{\partial x_n} + y_n \frac{\partial \psi}{\partial y_n}\right) + [\boldsymbol{v}]_{\boldsymbol{v}},$$

for $(x_1 + iy_1, \cdots, x_n + iy_n) \in \mathbb{C}^n$, in these coordinates on $\mathcal{U}_{\sigma_{\boldsymbol{v}}}$ and $\mathfrak{t}^*$.

If $m_{\boldsymbol{v}}(\boldsymbol{\alpha}, \boldsymbol{z}) = 0$, there exists a $j \in \{1, \cdots, n\}$ such that $|z_j|^{\left([\boldsymbol{\alpha}-\boldsymbol{v}]_{\boldsymbol{v}}\right)_j} = 0$. Hence $z_j = x_j + iy_j = 0$ and $\left([\boldsymbol{\alpha} - \boldsymbol{v}]_{\boldsymbol{v}}\right)_j > 0$. But $(\mu(\boldsymbol{z}) - [\boldsymbol{v}]_{\boldsymbol{v}})_j = x_j \frac{\partial \psi}{\partial x_j} + y_j \frac{\partial \psi}{\partial y_j} = \left([\boldsymbol{\alpha} - \boldsymbol{v}]_{\boldsymbol{v}}\right)_j \neq 0$. This is a contradiction.

We recall that $\mu^{-1}(\boldsymbol{\alpha}) = \text{Orb}_p(\mathbb{T}^n)$ for any $p \in \mu^{-1}(\boldsymbol{\alpha})$. Note that $m(\boldsymbol{\alpha}, .)$ is invariant under the $\mathbb{T}^n$ action for $\boldsymbol{\alpha} \in P$. Because m and μ are continuous, $P \times P$ is Hausdorff and X_P is compact, the function $n : P \times P \to \mathbb{R}$ given by

$$n(\boldsymbol{\alpha}, \boldsymbol{\beta}) \stackrel{\text{def}}{=} \frac{m\left(\boldsymbol{\alpha}, \mu^{-1}(\boldsymbol{\beta})\right)}{m\left(\boldsymbol{\alpha}, \mu^{-1}(\boldsymbol{\alpha})\right)},$$

for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in P$, is continuous.

Note that $\mu^{-1}(\text{Int}(P)) = (\mathbb{C}^*)^n$. Pick local coordinates on $T_N \cong (\mathbb{C}^*)^n$ and a trivialization of L_P over T_N such that $s_{\boldsymbol{\alpha},1}(\boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{\alpha}}$ for $\boldsymbol{z} \in (\mathbb{C}^*)^n$ and $\boldsymbol{\alpha} \in P \cap \mathbb{Z}^n$. Following Abreu, we pick a $\mathbb{T}^n$-invariant Kähler potential $\psi : \mathbb{R}^n \to \mathbb{R}$ (so that $\omega = 2i\partial\overline{\partial}\psi$) such that the Hermitian metric is locally given by $e^{-2\psi}$ in our trivialization and the moment map is $\nabla\psi$. We let $z_j = e^{t_j + i\theta_j}$ for $j \in \{1, \cdots, n\}$ and have

$$m(\boldsymbol{\alpha}, \boldsymbol{z}) = \left|\boldsymbol{z}^{\boldsymbol{\alpha}}\right|^2 e^{-2\psi(\boldsymbol{z})} = e^{-2(\psi(\boldsymbol{t}) - \langle \boldsymbol{\alpha}, \boldsymbol{t} \rangle)}.$$

Finally, applying Legendre duality with $\boldsymbol{t} = \nabla u|_{\boldsymbol{\beta}}$ and $\psi\left(\nabla u|_{\boldsymbol{\beta}}\right) = \langle \boldsymbol{\beta}, \nabla u|_{\boldsymbol{\beta}} \rangle - u(\boldsymbol{\beta})$, we find that

$$n(\boldsymbol{\alpha}, \boldsymbol{\beta}) = e^{-h(\boldsymbol{\alpha}, \boldsymbol{\beta})}, \quad \text{for } \boldsymbol{\alpha}, \boldsymbol{\beta} \in \operatorname{Int}(P),$$

as claimed. Smoothness on $\operatorname{Int}(P) \times \operatorname{Int}(P)$ now follows from the smoothness of u on $\operatorname{Int}(P)$. Positivity and the claim about the zero set of h follows from Young's inequality for Legendre duality. Noting that $\operatorname{Hess}(u)$ is positive definite on $\operatorname{Int}(P)$ and a simple calculation verifies the claim about the critical points of $h(\boldsymbol{\alpha}, .) : \operatorname{Int}(P) \to \mathbb{R}$ and $h(., \boldsymbol{\alpha}) : \operatorname{Int}(P) \to \mathbb{R}$ for $\boldsymbol{\alpha} \in \operatorname{Int}(P)$.

We now prove *c)*. We will work on the coordinate charts $\mathcal{U}_{\sigma_{\boldsymbol{v}}}$ for $\boldsymbol{v} \in \operatorname{vertices}(P)$. We want to show that

$$\bigcup_{\boldsymbol{v} \in \operatorname{vertices}(P)} \mu\left(\mathcal{Z}\left(m_{\boldsymbol{v}}\left(\boldsymbol{\alpha}, .\right)\right)\right) = \bigcup_{\{F : F < P, \boldsymbol{\alpha} \notin F\}} F.$$

Suppose that $\boldsymbol{z} \in \mathcal{Z}\left(m_{\boldsymbol{v}}(\boldsymbol{\alpha}, .)\right) \subset \mathcal{U}_{\sigma_{\boldsymbol{v}}}$ for some $\boldsymbol{v} \in \operatorname{vertices}(P)$. We have $m_{\boldsymbol{v}}(\boldsymbol{\alpha}, \boldsymbol{z}) = \left|\boldsymbol{z}^{[\boldsymbol{\alpha} - \boldsymbol{v}]_{\boldsymbol{v}}}\right|^2 h_{\boldsymbol{v}}(\boldsymbol{z}) = 0$ and

$$\mu(x_1, y_1, \cdots, x_n, y_n) = \left(x_1 \frac{\partial \psi}{\partial x_1} + y_1 \frac{\partial \psi}{\partial y_1}, \cdots, x_n \frac{\partial \psi}{\partial x_n} + y_n \frac{\partial \psi}{\partial y_n}\right) + [\boldsymbol{v}]_{\boldsymbol{v}}$$

for $(x_1 + iy_1, \cdots, x_n + iy_n) = \boldsymbol{z} \in \mathbb{C}^n$. Hence there exists $j \in \{1, \cdots, n\}$ such that $z_j = 0$ and $\left([\boldsymbol{\alpha} - \boldsymbol{v}]_{\boldsymbol{v}}\right)_j \neq 0$. But then $\left(\mu(\boldsymbol{z}) - [\boldsymbol{v}]_{\boldsymbol{v}}\right)_j = 0$. Let $F_j = \left([\boldsymbol{v}]_{\boldsymbol{v}} + \mathcal{Z}(z_j)\right) \cap P$. Then $\mu(\boldsymbol{z}) \in F_j$, $F_j < P$ and $[\boldsymbol{\alpha}]_{\boldsymbol{v}} \notin F_j$.

Now let $F < P$ and $\boldsymbol{\alpha} \in P$ such that $\boldsymbol{\alpha} \notin F$. For any $f \in F$, there exists $\boldsymbol{v} \in \operatorname{vertices}(P)$ and $\boldsymbol{z} \in \mathcal{U}_{\sigma_{\boldsymbol{v}}}$ such that $\mu(\boldsymbol{z}) = f$. We now show that $\boldsymbol{z} \in \mathcal{Z}\left(m_{\boldsymbol{v}}(\boldsymbol{\alpha}, .)\right)$. In our standard chart $\mathcal{U}_{\sigma_{\boldsymbol{v}}}$ and after a reordering of indices, we have

$$\mu\left(\mathcal{U}_{\sigma_{\boldsymbol{v}}}\right) \cap F - [\boldsymbol{v}]_{\boldsymbol{v}} = \left(\mu\left(\mathcal{U}_{\sigma_{\boldsymbol{v}}}\right) - [\boldsymbol{v}]_{\boldsymbol{v}}\right) \cap \mathcal{Z}(z_1) \cap \cdots \cap \mathcal{Z}(z_k)$$

for some $k \in \mathbb{N}$ such that $1 \leq k \leq n$. Then there exists $j \in \{1, \cdots, k\}$ such that $\left([\boldsymbol{\alpha} - \boldsymbol{v}]_{\boldsymbol{v}}\right)_j \neq 0$ and $\left(\mu(\boldsymbol{z}) - [\boldsymbol{v}]_{\boldsymbol{v}}\right)_j = 0$. This implies that $z_j = 0$ and hence $m_{\boldsymbol{v}}(\boldsymbol{\alpha}, \boldsymbol{z}) = 0$.

□

Remark 5.1.4. *Let us remark here that several authors have previously used and observed the convexity properties of h over the interior of the polytope (see e.g. [SZ10, SD10, BGU10]).*

We will make use of the following elementary lemma:

Lemma 5.1.5. *Let $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ and suppose that $f(\boldsymbol{a}) = 0$ and $Df|_{\boldsymbol{a}} = \mathbf{0}$ for some $\boldsymbol{a} \in \mathbb{R}^n$. Then, for any $\boldsymbol{x} \in \mathbb{R}^n$,*

$$f(\boldsymbol{x}) \geq \frac{1}{6} \inf_{\boldsymbol{y} \in line(\boldsymbol{a}, \boldsymbol{x})} \langle \operatorname{Hess} f|_{\boldsymbol{y}} \frac{\boldsymbol{x} - \boldsymbol{a}}{\|\boldsymbol{x} - \boldsymbol{a}\|}, \frac{\boldsymbol{x} - \boldsymbol{a}}{\|\boldsymbol{x} - \boldsymbol{a}\|} \rangle \|\boldsymbol{a} - \boldsymbol{x}\|^2.$$

Proof. We recall the following elementary result:

$$f(\boldsymbol{x}) = \sum_{j=0}^{n} \frac{1}{j!} \left.\frac{\mathrm{d}}{\mathrm{d}\,t}^{j} f\left(\boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a})\right)\right|_0 + \int_0^1 \frac{(1-t)^n}{n!} \left.\frac{\mathrm{d}}{\mathrm{d}\,t}^{j} f\left(\boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a})\right)\right|_t dt.$$

In particular, letting $n = 2$, we find

$$f(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}\|^2 \int_0^1 \frac{(1-t)^2}{2} \langle \operatorname{Hess} f|_{\boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a})} \frac{\boldsymbol{x} - \boldsymbol{a}}{\|\boldsymbol{x} - \boldsymbol{a}\|}, \frac{\boldsymbol{x} - \boldsymbol{a}}{\|\boldsymbol{x} - \boldsymbol{a}\|} \rangle dt.$$

The estimate now follows. $\square$

The following lemma is a standard toric geometry result which comes from the fact that "a symplectic potential restricted to a face F of the polytope P yields a symplectic potential for the corresponding toric subvariety Y_F" (*cf.* [SD10, Lemma 3.5]).

Lemma 5.1.6. *Let (X_P, ω) be a toric Kähler manifold and let $u : P \to \mathbb{R}$ be a symplectic potential for ω. P is given as an intersection of affine halfspaces $P = \cap_{i=1}^{d} H^+_{\boldsymbol{n}_i,\lambda_i}$ with $H^+_{\boldsymbol{n}_i,\lambda_i} \stackrel{\text{def}}{=} \{\boldsymbol{\alpha} \in \mathbb{R}^n : l_i(\boldsymbol{\alpha}) \geq 0\}$ and $l_i(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \langle \boldsymbol{\alpha}, \boldsymbol{n}_i \rangle - \lambda_i$ for some $\lambda_i \in \mathbb{R}$ and primitive $\boldsymbol{n}_i \in \mathbb{Z}^n$. Let $\mathcal{Z}(l_i) \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathbb{R}^n : l_i(\boldsymbol{x}) = 0\}$ and $\boldsymbol{\alpha} \in P$. Without loss of generality, let $\mathcal{Z}(l_1), \ldots, \mathcal{Z}(l_s)$ denote those affine hyperplanes among $\mathcal{Z}(l_1), \cdots, \mathcal{Z}(l_d)$ containing $\boldsymbol{\alpha}$. Let $0 \neq \boldsymbol{m} \in \mathbb{R}^n$ such that $\langle \boldsymbol{m}, \boldsymbol{n}_i \rangle = 0$ for $i \in \{1, \cdots, s\}$. Then*

$$\langle \operatorname{Hess} u|_{\boldsymbol{\alpha}}\, \boldsymbol{m}, \boldsymbol{m} \rangle > 0.$$

Proof. Let $F \stackrel{\text{def}}{=} \mathcal{Z}(l_1, \cdots, l_s)$ denote the face containing $\boldsymbol{\alpha}$ in its relative interior. Pick a $\mathbb{C}^n$ chart $\mathcal{U}_{\sigma_{\boldsymbol{v}}}$ corresponding to a vertex $\boldsymbol{v} = (v_1, \cdots, v_n) \in F$ and coordinates $\mathcal{U}_{\sigma_{\boldsymbol{v}}} \cong \mathbb{C}^n$ such that $\mu^{-1}(F) \cap \mathcal{U}_{\sigma_{\boldsymbol{v}}} \cong \{\boldsymbol{0}\} \times \mathbb{C}^{n-s}$. In these coordinates $l_j(\boldsymbol{\alpha}) = \alpha_j - ([\boldsymbol{v}]_{\boldsymbol{v}})_j$ for $j \in \{1, \cdots, s\}$. By lemma 5.1.1, there exists a torus-invariant Kähler potential $\widehat{\phi}$ on that $\mathbb{C}^n$ chart such that $\omega = 2i\partial\overline{\partial}\widehat{\phi}$. We have

$$\mu(x_1, y_1, \cdots, x_n, y_n) = \left(x_1 \frac{\partial \widehat{\phi}}{\partial x_1} + y_1 \frac{\partial \widehat{\phi}}{\partial y_1}, \cdots, x_n \frac{\partial \widehat{\phi}}{\partial x_n} + y_n \frac{\partial \widehat{\phi}}{\partial y_n} \right) + [\boldsymbol{v}]_{\boldsymbol{v}}$$

for $(x_1+iy_1, \cdots, x_n+iy_n) = \boldsymbol{z} \in \mathbb{C}^n$. In the standard holomorphic trivialization $L_{\mathcal{U}_{\sigma_{\boldsymbol{v}}}} \cong \mathbb{C}^n \times \mathbb{C}$, we have $s_{\boldsymbol{\alpha},1}(\boldsymbol{z}) = \boldsymbol{z}^{[\boldsymbol{\alpha}-\boldsymbol{v}]_{\boldsymbol{v}}}$ and $|s_{\boldsymbol{\alpha},1}(\boldsymbol{z})|_h^2 = \left|\boldsymbol{z}^{[\boldsymbol{\alpha}-\boldsymbol{v}]_{\boldsymbol{v}}}\right|^2 e^{-2\widehat{\phi}(\boldsymbol{z})}$ for $\boldsymbol{z} \in \mathbb{C}^n$ and $\boldsymbol{\alpha} \in \mathbb{Z}^n \cap P$. Over $T_N \cong (\mathbb{C}^*)^n$, the retrivialization $(\boldsymbol{z}, \lambda) \mapsto (\boldsymbol{z}, \boldsymbol{z}^{[\boldsymbol{v}]_{\boldsymbol{v}}}\lambda)$, for $\boldsymbol{z} \in (\mathbb{C}^*)^n$ and $\lambda \in \mathbb{C}$, yields the familiar standard setup where $s_{\boldsymbol{\alpha},1}(\boldsymbol{z}) = \boldsymbol{z}^{[\boldsymbol{\alpha}]_{\boldsymbol{v}}}$ and $|s_{\boldsymbol{\alpha},1}(\boldsymbol{t})|_h^2 = e^{-2(\psi(\boldsymbol{t}) - \langle \boldsymbol{t}, [\boldsymbol{\alpha}]_{\boldsymbol{v}} \rangle)}$ for $\boldsymbol{z} = (e^{t_1+i\theta_1}, \cdots, e^{t_n+i\theta_n}) \in (\mathbb{C}^*)^n$. We have $\omega = 2i\partial\overline{\partial}\psi$, $\mu = \nabla\psi$ and $\psi(\boldsymbol{t}) = \widehat{\phi}(e^{t_1}, \cdots, e^{t_n}) + \langle [\boldsymbol{v}]_{\boldsymbol{v}}, \boldsymbol{t} \rangle$. We denote by u the symplectic potential dual to ψ. We now have

$$u\left(\mu(\boldsymbol{z})\right) = \sum_{i=1}^{n} \log(|z|_i)\left(\mu(\boldsymbol{z}) - [\boldsymbol{v}]_{\boldsymbol{v}}\right)_i - \widehat{\phi}(\boldsymbol{z})$$

for all $\boldsymbol{z} \in (\mathbb{C}^*)^n$. In fact, we note that, for $\boldsymbol{z} \in \{\boldsymbol{0}\} \times (\mathbb{C}^*)^{n-s} \subset \mathbb{C}^n$, we have

$$\log(|z|_i)\left(\mu(\boldsymbol{z}) - [\boldsymbol{v}]_{\boldsymbol{v}}\right)_i = 0, \qquad \text{for } i \in \{1, \cdots, s\},$$

since then $\left(\mu(\boldsymbol{z}) - [\boldsymbol{v}]_{\boldsymbol{v}}\right)_i = 0$ for $i \in \{1, \cdots, s\}$. It follows that the function $\phi : \mathbb{R}^{n-s} \to \mathbb{R}$ given by $\phi(t_{n-s+1}, \cdots, t_n) = \widehat{\phi}(0, \cdots, 0, e^{t_{n-s+1}}, \cdots, e^{t_n})$ is (up to addition of an affine factor) Legendre dual to the restriction of u to $\operatorname{RelInt}(F)$. Since the restriction of ω to $\{\boldsymbol{0}\} \times (\mathbb{C}^*)^{n-s}$ is non-degenerate, it follows that ϕ is strictly convex. The nondegeneracy of $\operatorname{Hess}\left(u|_{\operatorname{RelInt}(F)}\right)$ now follows by Legendre duality. $\square$

Proposition 5.1.7. *Let $(L, h) \to (X_P, \omega)$ be a toric polarization. There exists a constant $c > 0$ such that*

$$n(\boldsymbol{\alpha}, \boldsymbol{\beta}) \leq e^{-c\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|^2}$$

for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in P$.

Proof. Recall that $n(\boldsymbol{\alpha}, \boldsymbol{\beta}) = e^{-h(\boldsymbol{\alpha}, \boldsymbol{\beta})}$ for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{Int}(P)$. We will prove that there exists a constant $c > 0$ such that $h(\boldsymbol{\alpha}, \boldsymbol{\beta}) \geq c\|\boldsymbol{\alpha} - \boldsymbol{\beta}\|^2$ for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{Int}(P)$. Let $u : P \to \mathbb{R}$ denote a symplectic potential for (X_P, ω). We recall that

$$h(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 2\left(u(\boldsymbol{\alpha}) - u(\boldsymbol{\beta}) + \langle \nabla u|_{\boldsymbol{\beta}}, \boldsymbol{\beta} - \boldsymbol{\alpha}\rangle\right).$$

In particular, $\frac{\partial}{\partial \alpha_i} h\Big|_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} = 2\left(\frac{\partial u}{\partial \alpha_i}\Big|_{\boldsymbol{\alpha}} - \frac{\partial u}{\partial \alpha_i}\Big|_{\boldsymbol{\beta}}\right)$ and $\frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h\Big|_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} = 2\, \frac{\partial^2 u}{\partial \alpha_i \partial \alpha_j}\Big|_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{Int}(P)$. Define $f_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} h(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{Int}(P)$. Then

$$h(\boldsymbol{\alpha}, \boldsymbol{\beta}) = f_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) \geq \frac{1}{3}\|\boldsymbol{\alpha} - \boldsymbol{\beta}\|^2 \inf_{\boldsymbol{y} \in \mathrm{line}(\boldsymbol{\alpha}, \boldsymbol{\beta})} \inf_{\{\boldsymbol{v} \in \mathbb{R}^n : \|\boldsymbol{v}\| = 1\}} \langle \mathrm{Hess}\, u|_{\boldsymbol{y}}\, \boldsymbol{v}, \boldsymbol{v}\rangle, \quad \text{for } \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{Int}(P),$$

by lemma 5.1.5. We now have to prove that

$$\inf_{\boldsymbol{y} \in \mathrm{Int}(P)} \inf_{\{\boldsymbol{v} \in \mathbb{R}^n : \|\boldsymbol{v}\| = 1\}} \langle \mathrm{Hess}\, u|_{\boldsymbol{y}}\, \boldsymbol{v}, \boldsymbol{v}\rangle > 0.$$

By the continuity of n, this inequality then gives the required result on the whole of P. For any compact subset $C \subset \mathrm{Int}(P)$ the equality for h on $C \times C$ is obvious since $\mathrm{Hess}\, u$ is positive definite on $\mathrm{Int}(P)$. We just need to show that the above infimum is not equal to zero.

Suppose that there exists a sequence of vectors $\{\boldsymbol{\alpha}_k\}_{k=1}^{\infty} \subset \mathrm{Int}(P)$ and $\{\boldsymbol{v}_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ such that $\|\boldsymbol{v}_k\| = 1$ for all $k \in \mathbb{N}$, and

$$\langle \mathrm{Hess}\, u|_{\boldsymbol{\alpha}_k}\, \boldsymbol{v}_k, \boldsymbol{v}_k\rangle \to 0 \quad \text{as } k \to \infty.$$

A subsequence of $\boldsymbol{\alpha}_k$ and $\boldsymbol{v}_k$ now tends to some $\boldsymbol{\alpha} \in P$ and $\boldsymbol{v} \in \mathbb{R}^n$ such that $\|\boldsymbol{v}\| = 1$ respectively. If $\boldsymbol{\alpha} \in \mathrm{Int}(P)$ we have a contradiction, so we exclude that case from our consideration. P is given as an intersection of affine halfspaces $P = \cap_{i=1}^{d} H^{+}_{\boldsymbol{n}_i, \lambda_i}$ with $H^{+}_{\boldsymbol{n}_i, \lambda_i} \stackrel{\text{def}}{=} \{\boldsymbol{\alpha} \in \mathbb{R}^n : l_i(\boldsymbol{\alpha}) \geq 0\}$ and $l_i(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \langle \boldsymbol{\alpha}, \boldsymbol{n}_i\rangle - \lambda_i$, where $\boldsymbol{n}_i \in \mathbb{Z}^n$ is primitive and $\lambda_i \in \mathbb{R}$ for $i \in \{1, \cdots, d\}$. Without loss of generality, let $\mathcal{Z}(l_1), \ldots, \mathcal{Z}(l_j)$ denote those hyperplanes among $\mathcal{Z}(l_1), \cdots, \mathcal{Z}(l_d)$ containing $\boldsymbol{\alpha}$. By a result of Abreu [Abr03], we have a smooth function $v \in \mathcal{C}^{\infty}(P)$ such that

$$u(\boldsymbol{\alpha}) = \frac{1}{2}\sum_{i=1}^{d} l_i(\boldsymbol{\alpha}) \log(l_i(\boldsymbol{\alpha})) + v(\boldsymbol{\alpha}).$$

We compute that

$$\frac{\partial^2 u}{\partial \alpha_j \partial \alpha_k}\Big|_{\boldsymbol{\alpha}} = \frac{1}{2}\sum_{i=1}^{d} \frac{(\boldsymbol{n}_i)_j (\boldsymbol{n}_i)_k}{l_i(\boldsymbol{\alpha})} + \frac{\partial^2 v}{\partial \alpha_j \partial \alpha_k}\Big|_{\boldsymbol{\alpha}},$$

so that

$$\langle \mathrm{Hess}\, u|_{\boldsymbol{y}}\, \boldsymbol{x}, \boldsymbol{x}\rangle = \frac{1}{2}\sum_{i=1}^{d} \frac{\langle \boldsymbol{n}_i, \boldsymbol{x}\rangle^2}{l_i(\boldsymbol{y})} + \langle \mathrm{Hess}\, v|_{\boldsymbol{y}}\, \boldsymbol{x}, \boldsymbol{x}\rangle.$$

We observe that the $\mathrm{Hess}\, v$ term is bounded on P. As $k \to \infty$, $l_i(\boldsymbol{\alpha}_k) \to 0$ for $i \in \{1, \cdots, j\}$. Now $\frac{\langle \boldsymbol{n}_i, \boldsymbol{v}_k\rangle^2}{l_i(\boldsymbol{\alpha}_k)} \to +\infty$ as $k \to \infty$, for some $i \in \{1, \ldots, j\}$, in which case we are done, or $\langle \boldsymbol{n}_i, \boldsymbol{v}_k\rangle \to 0$ as $k \to \infty$ for all $i \in \{1, \ldots, j\}$. Then $\langle \boldsymbol{n}_i, \boldsymbol{v}\rangle = 0$ for all $i \in \{1, \cdots, j\}$. But then $\langle \mathrm{Hess}\, u|_{\boldsymbol{\alpha}}\, \boldsymbol{v}, \boldsymbol{v}\rangle > 0$ by lemma 5.1.6. □

Example 5.1.8. Let us consider the toric polarization $(\mathcal{O}(1), h_{FS}) \to (\mathbb{CP}^1, \omega_{FS})$, where $\omega_{FS} = i\partial\overline{\partial}\log\left(1 + \|z\|^2\right)$ denotes the Fubini-Study metric which has the symplectic potential $u : [0,1] \to \mathbb{R}$ given by $u(x) = \frac{1}{2}\left(x\log(x) + (1-x)\log(1-x)\right)$. Figure 5.1 shows the decay behaviour of the function $e^{-kh(x,y)}$ for $k = 1, 3, 10$ for this potential.

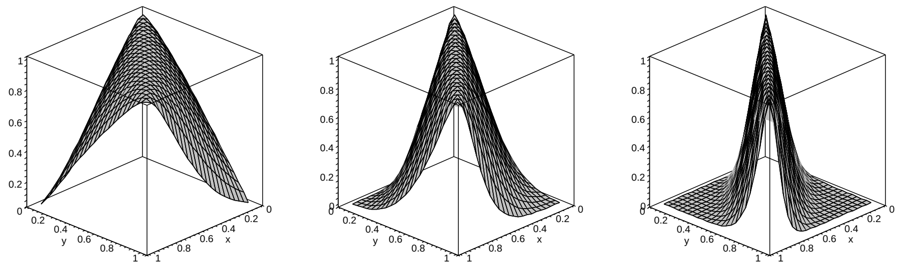

Figure 5.1: Graph of $e^{-kh(x,y)}$ for $k = 1, 3, 10$ on $(\mathbb{CP}^1, \omega_{FS})$.

We are now ready to use our estimate to prove a localization theorem for the toric Bergman kernel.

Theorem 5.1.9 (Generalized toric Bergman kernel localization)**.** *Let P be an integral Delzant polytope in $\mathbb{R}^n$ with the standard lattice $\mathbb{Z}^n$. Let $f_k : P^2 \times \left(P \cap \frac{1}{k}\mathbb{Z}^n\right) \to \mathbb{C}$ be a sequence of functions such that there exists constants $C, M > 0$ such that $|f_k(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})| \leq Ck^M$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in P, \boldsymbol{\gamma} \in P \cap \frac{1}{k}\mathbb{Z}^n$ and all $k \in \mathbb{N}$. Consider*

$$B_k(\boldsymbol{\alpha}, \boldsymbol{\beta}) \stackrel{\text{def}}{=} \sum_{\boldsymbol{\gamma} \in P \cap \frac{1}{k}\mathbb{Z}^n} f_k(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) e_{\boldsymbol{\gamma},k}(\boldsymbol{\alpha}) \otimes \overline{e}_{\boldsymbol{\gamma},k}(\boldsymbol{\beta}),$$

where $e_{\boldsymbol{\gamma},k} = \frac{s_{\boldsymbol{\gamma},k}}{\|s_{\boldsymbol{\gamma},k}\|_{h^k}} \in H^0(X_P, L^k)$ denotes the standard unit norm section corresponding to $\boldsymbol{\gamma} \in P \cap \frac{1}{k}\mathbb{Z}^n$.

a) Then, for any $l > 0$, there exist $E, b > 0$ such that

$$\left| B_k(\boldsymbol{\alpha}, \boldsymbol{\beta}) - \sum_{\boldsymbol{\gamma} \in \mathbb{B}_{\boldsymbol{\alpha}}\left(\sqrt{\frac{b\log k}{k}}\right) \cap \mathbb{B}_{\boldsymbol{\beta}}\left(\sqrt{\frac{b\log k}{k}}\right) \cap P \cap \frac{1}{k}\mathbb{Z}^n} f_k(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) e_{\boldsymbol{\gamma},k}(\boldsymbol{\alpha}) \otimes \overline{e}_{\boldsymbol{\gamma},k}(\boldsymbol{\beta}) \right|_{h^k} \leq Ek^{-l},$$

for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in P$ and $k \in \mathbb{N}$, and where $\mathbb{B}_{\boldsymbol{\alpha}}(r) \stackrel{\text{def}}{=} \{\boldsymbol{\beta} \in \mathbb{R}^n : \|\boldsymbol{\alpha} - \boldsymbol{\beta}\| \leq r\}$.

b) There exist constants $D, c > 0$ such that

$$|B_k(\boldsymbol{\alpha}, \boldsymbol{\beta})|_{h^k} \leq Dk^{M+2n} e^{-ck\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|^2}$$

for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in P$ and $k \in \mathbb{N}$.

Proof. Let us prove b) here. a) is proved similarly.

$$\begin{aligned}
|B_k(\boldsymbol{\alpha},\boldsymbol{\beta})|^2_{h^k} &= \sum_{\boldsymbol{\gamma},\boldsymbol{\delta}\in P\cap\frac{1}{k}\mathbb{Z}^n} (f_k(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})e_{\boldsymbol{\gamma},k}(\boldsymbol{\alpha})\otimes\overline{e}_{\boldsymbol{\gamma},k}(\boldsymbol{\beta}), f_k(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta})e_{\boldsymbol{\delta},k}(\boldsymbol{\alpha})\otimes\overline{e}_{\boldsymbol{\delta},k}(\boldsymbol{\beta}))\\
&= \sum_{\boldsymbol{\gamma},\boldsymbol{\delta}\in P\cap\frac{1}{k}\mathbb{Z}^n} f_k(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})\overline{f_k(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta})}(e_{\boldsymbol{\gamma},k}(\boldsymbol{\alpha}),e_{\boldsymbol{\delta},k}(\boldsymbol{\alpha}))(\overline{e}_{\boldsymbol{\gamma},k}(\boldsymbol{\beta}),\overline{e}_{\boldsymbol{\delta},k}(\boldsymbol{\beta}))\\
&\le C^2k^{2M}\sum_{\boldsymbol{\gamma},\boldsymbol{\delta}\in P\cap\frac{1}{k}\mathbb{Z}^n} |(e_{\boldsymbol{\gamma},k}(\boldsymbol{\alpha}),e_{\boldsymbol{\delta},k}(\boldsymbol{\alpha}))(\overline{e}_{\boldsymbol{\gamma},k}(\boldsymbol{\beta}),\overline{e}_{\boldsymbol{\delta},k}(\boldsymbol{\beta}))|\\
&\le C^2k^{2M}\sum_{\boldsymbol{\gamma},\boldsymbol{\delta}\in P\cap\frac{1}{k}\mathbb{Z}^n} |e_{\boldsymbol{\gamma},k}(\boldsymbol{\alpha})|_{h^k}\,|e_{\boldsymbol{\delta},k}(\boldsymbol{\alpha})|_{h^k}\,|\overline{e}_{\boldsymbol{\gamma},k}(\boldsymbol{\beta})|_{h^k}\,|\overline{e}_{\boldsymbol{\delta},k}(\boldsymbol{\beta})|_{h^k}\\
&= C^2k^{2M}\left(\sum_{\boldsymbol{\gamma}\in P\cap\frac{1}{k}\mathbb{Z}^n} |e_{\boldsymbol{\gamma},k}(\boldsymbol{\alpha})|_{h^k}\,|e_{\boldsymbol{\gamma},k}(\boldsymbol{\beta})|_{h^k}\right)^2\\
&= C^2k^{2M}\left(\sum_{\boldsymbol{\gamma}\in P\cap\frac{1}{k}\mathbb{Z}^n} n(\boldsymbol{\gamma},\boldsymbol{\alpha})^{\frac{k}{2}}n(\boldsymbol{\gamma},\boldsymbol{\beta})^{\frac{k}{2}}\frac{|s_{\boldsymbol{\gamma},k}(\boldsymbol{\gamma})|^2_{h^k}}{\|s_{\boldsymbol{\gamma},k}\|^2_{h^k}}\right)^2\\
&\le C^2k^{2M}\left(\sum_{\boldsymbol{\gamma}\in P\cap\frac{1}{k}\mathbb{Z}^n} e^{-\frac{ck}{2}\left(\|\boldsymbol{\gamma}-\boldsymbol{\alpha}\|^2+\|\boldsymbol{\gamma}-\boldsymbol{\beta}\|^2\right)}\frac{|s_{\boldsymbol{\gamma},k}(\boldsymbol{\gamma})|^2_{h^k}}{\|s_{\boldsymbol{\gamma},k}\|^2_{h^k}}\right)^2\\
&\le C^2k^{2M}e^{-\frac{ck}{2}\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|^2}\left(\sum_{\boldsymbol{\gamma}\in P\cap\frac{1}{k}\mathbb{Z}^n} \frac{|s_{\boldsymbol{\gamma},k}(\boldsymbol{\gamma})|^2_{h^k}}{\|s_{\boldsymbol{\gamma},k}\|^2_{h^k}}\right)^2\\
&\le C^2D^2k^{2M+2n}e^{-\frac{ck}{2}\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|^2}\#\left(P\cap\frac{1}{k}\mathbb{Z}^n\right)^2,
\end{aligned}$$

where c comes from proposition 5.1.7, and we have used the fact that there exists a constant $D \ge 0$ such that

$$\frac{|s_{\boldsymbol{\gamma},k}(\boldsymbol{\alpha})|^2_{h^k}}{\|s_{\boldsymbol{\gamma},k}\|^2_{h^k}} \le \rho_k(\boldsymbol{\alpha}) \le Dk^n$$

for all $\boldsymbol{\alpha}\in P$, $\boldsymbol{\gamma}\in P\cap\frac{1}{k}\mathbb{Z}^n$ and $k\in\mathbb{N}$. The result follows since there are only order k^n elements in the set $P\cap\frac{1}{k}\mathbb{Z}^n$. In the final step we also use the inequality

$$\|\boldsymbol{\alpha}-\boldsymbol{\gamma}\|^2+\|\boldsymbol{\beta}-\boldsymbol{\gamma}\|^2 \ge \frac{1}{2}\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|^2,$$

for $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathbb{R}^n$, which can be proved by observing that, for fixed $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{R}^n$, the critical point $\frac{1}{2}(\boldsymbol{\alpha}+\boldsymbol{\beta})$ of the functional $\xi(\boldsymbol{\gamma}) = \|\boldsymbol{\alpha}-\boldsymbol{\gamma}\|^2+\|\boldsymbol{\beta}-\boldsymbol{\gamma}\|^2$ is the absolute minimum of ξ. □

Similarly, we get the following estimate:

Corollary 5.1.10. *Let B_k be defined as in the previous theorem. For $\boldsymbol{\alpha},\boldsymbol{\beta}\in P$, we have:*

a) For any $\delta > 0$ and $\boldsymbol{\alpha},\boldsymbol{\beta}\in P$,

$$\left|B_k(\boldsymbol{\alpha},\boldsymbol{\beta}) - \sum_{\boldsymbol{\gamma}\in\mathbb{B}_{\boldsymbol{\alpha}}\left(k^{-\frac{1}{2}+\delta}\right)\cap\mathbb{B}_{\boldsymbol{\beta}}\left(k^{-\frac{1}{2}+\delta}\right)\cap P\cap\frac{1}{k}\mathbb{Z}^n} f_k(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})e_{\boldsymbol{\gamma},k}(\boldsymbol{\alpha})\otimes\overline{e}_{\boldsymbol{\gamma},k}(\boldsymbol{\beta})\right|_{h^k} = \mathcal{O}\left(k^{-\infty}\right).$$

b) For any $\delta > 0, C > 0$ and all $\{\boldsymbol{\alpha}_k\}_{k=1}^{\infty}, \{\boldsymbol{\beta}_k\}_{k=1}^{\infty} \subset P$ such that $\|\boldsymbol{\alpha}_k - \boldsymbol{\beta}_k\| \geq Ck^{-\frac{1}{2}+\delta}$ for all $k \geq 0$, we have

$$|B_k(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k)|_{h^k} = \mathcal{O}\left(k^{-\infty}\right).$$

5.2 Comparison with previous results

If we apply part a) of the above corollary to B_k evaluated on the diagonal in $P \times P$, we recover a version of Song and Zelditch's localization lemma [SZ10, lemma 1.2] and of [SZ10, Prop 5.1] which were originally proved using a more complicated argument.

In the special case where $f_k \equiv 1$ for all $k \in \mathbb{N}$, B_k is the Bergman kernel, and we obtain the new localization of sums formula:

$$B_k(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{\boldsymbol{\gamma} \in \mathbb{B}_{\boldsymbol{\alpha}}\left(\sqrt{\frac{b \log k}{k}}\right) \cap \mathbb{B}_{\boldsymbol{\beta}}\left(\sqrt{\frac{b \log k}{k}}\right) \cap P \cap \frac{1}{k}\mathbb{Z}^n} e_{\boldsymbol{\gamma},k}(\boldsymbol{\alpha}) \otimes \overline{e}_{\boldsymbol{\gamma},k}(\boldsymbol{\beta}) + \mathcal{O}\left(k^{-\infty}\right)$$

for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in P$ and all $k \in \mathbb{N}$. Similarly, the density function, which is just the norm of the diagonal of the Bergman kernel, can be localized in this sense. Additionally, we now have an off-diagonal vanishing result in the sense that there exists $c > 0$ and $D \geq 0$ such that

$$|B_k(\boldsymbol{\alpha}, \boldsymbol{\beta})|_{h^k} \leq Dk^{2n} e^{-ck\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|^2}$$

for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in P$ and $k \in \mathbb{N}$. While this is not the sharpest possible estimate, it does illuminate the exponential decay of the Bergman kernel away from the diagonal which is very explicit in the toric case. For general polarized Kähler manifolds $(L, h) \to (X, \omega)$, we know that $B_k(x, y) = \mathcal{O}(k^{-\infty})$ for $d(x, y) > \epsilon > 0$ and $x, y \in X$ [DLM06, proposition 4.1] and have exponential decay and an asymptotic expansion of B_k in a neighbourhood of the diagonal of $X \times X$ due to Dai *et al.* [DLM06, MM07]. One advantage of the simple estimate that we obtain here is that it gives a globaly valid exponential decay estimate for the Bergman kernel in the toric case.

Chapter 6

Euler-Maclaurin Sums

In order to develop a technique for computing the asymptotics of (partial) density functions on a toric polarized manifold, we require an Euler-Maclaurin summation formula dependent on a parameter $k \in \mathbb{N}$. For the convenience of the reader, we will recall some results in this direction. After this, we adapt these results for our purposes.

6.1 The classical Euler-Maclaurin formula

The classical Euler-Maclaurin summation formula provides a comparison between the integral of a function $f \in \mathcal{C}^\infty(\mathbb{R})$ over an interval $[a, b]$, $a < b$, $a, b \in \mathbb{Z}$ and a sum of f and its derivatives over $[a, b] \cap \mathbb{Z}$. Let us set up our notation. We define (see [KSW05])

$$\begin{aligned}
\mathbb{L}(x) &\stackrel{\text{def}}{=} \frac{\frac{x}{2}}{\tanh\left(\frac{x}{2}\right)} \\
&= \sum_{j=0}^{\infty} \frac{B_{2j}}{(2j)!} x^{2j} \\
&= 1 + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \mathcal{O}\left(x^6\right), \qquad \text{for } x \in \mathbb{R},
\end{aligned}$$

and, for $p \in \mathbb{N}$,

$$\mathbb{L}^{2p}(x) \stackrel{\text{def}}{=} 1 + \sum_{j=1}^{p} \frac{B_{2j}}{(2j)!} x^{2j}, \qquad \text{for } x \in \mathbb{R},$$

where B_j, for $j \in \mathbb{N}$, denotes the j^{th} Bernoulli number. For any polytope $P \subset \mathbb{R}^n$ and $\boldsymbol{x} \in P$, let $c(\boldsymbol{x})$ denote the largest codimension of any face containing $\boldsymbol{x}$. We define the *weighted characteristic function* $1_P^w : \mathbb{R}^n \to \mathbb{R}$ by

$$1_P^w(\boldsymbol{x}) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \boldsymbol{x} \notin P \\ 2^{-c(\boldsymbol{x})} & \text{otherwise.} \end{cases}$$

For $f : \mathbb{R}^n \to \mathbb{R}$, we define

$$\sum_{\boldsymbol{x} \in P \cap \mathbb{Z}^n}{}' f(\boldsymbol{x}) \stackrel{\text{def}}{=} \sum_{\boldsymbol{x} \in P \cap \mathbb{Z}^n} 1_P^w(\boldsymbol{x}) f(\boldsymbol{x}),$$

the simplest example of which is $P = [a, b]$, where $a, b \in \mathbb{Z}$ and $a < b$. Then

$$\sum_{x\in[a,b]\cap\mathbb{Z}}{}' f(x) = \frac{1}{2}f(a) + f(a+1) + \cdots + f(b-1) + \frac{1}{2}f(b).$$

The following is a modern formulation of the classical Euler-Maclaurin summation formula for intervals.

Theorem 6.1.1 ([KSW03, Proposition 10]). *Let $f \in \mathcal{C}^\infty(\mathbb{R})$, $a, b \in \mathbb{Z}$, $a < b$ and let $p \in \mathbb{N}_0$. Then*

$$\sum_{x\in[a,b]\cap\mathbb{Z}}{}' f(x) = \mathbb{L}^{2p}\left(\frac{\partial}{\partial\lambda_1}\right)\mathbb{L}^{2p}\left(\frac{\partial}{\partial\lambda_2}\right)\int_{a-\lambda_1}^{b+\lambda_2} f(x)dx\Bigg|_{\lambda_1=\lambda_2=0} + \int_a^b f^{(2p+1)}(x)P_{2p+1}(x)dx,$$

where

$$P_{2p+1}(x) \stackrel{\text{def}}{=} \frac{b_{2p+1}(x - \lfloor x \rfloor)}{(2p+1)!},$$

and $b_{2p+1}(x)$ is the $(2p+1)^{th}$ Bernoulli polynomial for $p \in \mathbb{N}$.

Let us recall the proof of this result here since it provides us with some intuition for the kind of results that we are interested in.

Proof. Recall that the Bernoulli polynomials $b_j : \mathbb{R} \to \mathbb{R}$ are defined recursively by $b_0(x) = 1$, $b'_{n+1}(x) = (n+1)b_n(x)$ and $\int_0^1 b_n(x)dx = 0$. We have $b_1(x) = x - \frac{1}{2}$, and the Bernoulli polynomials satisfy $b_j(1) = (-1)^j b_j(0)$ for all $j \in \mathbb{N}_0$. Furthermore, the Bernoulli numbers $B_j \stackrel{\text{def}}{=} b_j(0)$ satisfy $B_{2j+1} = 0$ for $j \in \mathbb{N}$. The proof is now a simple integration by parts argument. We have

$$\begin{aligned}
&\frac{1}{(2p+1)!}\int_0^1 f^{(2p+1)}(x)b_{2p+1}(x)dx \\
&\qquad = \frac{1}{(2p+1)!}f^{(2p)}(x)b_{2p+1}(x)\Big]_0^1 - \frac{1}{(2p)!}\int_0^1 f^{(2p)}(x)b_{2p}(x)dx \\
&\qquad = -\frac{1}{(2p)!}f^{(2p-1)}(x)b_{2p}(x)\Big]_0^1 + \frac{1}{(2p-1)!}\int_0^1 f^{(2p-1)}(x)b_{2p-1}(x)dx \\
&\qquad = -\sum_{j=1}^{p}\frac{1}{(2j)!}f^{(2j-1)}(x)b_{2j}(x)\Big]_0^1 + f(x)(x-\frac{1}{2})\Big]_0^1 - \int_0^1 f(x)dx \\
&\qquad = -\sum_{j=1}^{p}\frac{B_{2j}}{(2j)!}f^{(2j-1)}(x)\Big]_0^1 + \frac{f(1)+f(0)}{2} - \int_0^1 f(x)dx.
\end{aligned}$$

If we replace $f(x)$ by $f(x+s)$, for some $s \in \mathbb{R}$, we obtain

$$\frac{f(s+1)+f(s)}{2} = \int_s^{s+1} f(x)dx + \sum_{j=1}^{p}\frac{B_{2j}}{(2j)!}f^{(2j-1)}(x)\Big]_s^{s+1} + \int_s^{s+1} f^{(2p+1)}(x)P_{2p+1}(x)dx. \quad (\dagger)$$

Summing over $s \in \{a, \cdots, b-1\}$ gives the result. $\square$

Remark 6.1.2. *Note that*

$$\mathbb{L}^{2p}\left(\frac{\partial}{\partial\lambda_1}\right)\mathbb{L}^{2p}\left(\frac{\partial}{\partial\lambda_2}\right)\int_{a-\lambda_1}^{b+\lambda_2} f(x)dx\Bigg|_{\lambda_1=\lambda_2=0} = \int_a^b f(x)dx + \sum_{j=1}^{p}\frac{B_{2j}}{(2j)!}f^{(2j-1)}(x)\Bigg]_a^b.$$

6.2 The Euler-Maclaurin formula for the standard half-space

We now use equation (†) to derive two results about Euler-Maclaurin sums over half-spaces.

Lemma 6.2.1. *Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $p \in \mathbb{N}_0$. For $k \in \mathbb{N}$, we have*

$$\begin{aligned}
\sum_{\boldsymbol{\alpha}\in\frac{1}{k}(\mathbb{N}_0\times\mathbb{Z}^{n-1})} f(\boldsymbol{\alpha}) &= k^n\mathbb{B}^{2p}\left(\frac{1}{k}\frac{\partial}{\partial\lambda}\right)\int_{\{\boldsymbol{x}\in\mathbb{R}^n:x_1\geq-\lambda\}} f(\boldsymbol{x})d\boldsymbol{x}\Bigg|_{\lambda=0} + \mathcal{O}\left(k^{n-(2p+1)}\right)\\
&= k^n\int_{\mathbb{R}_{\geq0}\times\mathbb{R}^{n-1}} f(\boldsymbol{x})d\boldsymbol{x} + \frac{k^{n-1}}{2}\int_{\mathbb{R}^{n-1}} f(0,x_2,\cdots,x_n)dx_2\cdots dx_n\\
&\quad-\sum_{j=1}^{p}k^{n-2j}\frac{B_{2j}}{(2j)!}\int_{\mathbb{R}^{n-1}}\frac{\partial^{2j-1}f}{\partial x_1^{2j-1}}(0,x_2,\cdots,x_n)dx_2\cdots dx_n\\
&\quad+\mathcal{O}\left(k^{n-(2p+1)}\right),
\end{aligned}$$

where $\mathbb{B}^{2p}(x) \stackrel{\text{def}}{=} \mathbb{L}^{2p}(x) + \frac{x}{2} = 1 + \frac{x}{2} + \sum_{j=1}^{p}\frac{B_{2j}}{(2j)!}x^{2j}$, for $p \in \mathbb{N}$, and $\mathbb{B}^0(x) \stackrel{\text{def}}{=} 1$ for $x \in \mathbb{R}$.

Proof. If $n = 1$, we sum over (†) to obtain

$$\begin{aligned}
\sum_{\alpha\in\frac{1}{k}\mathbb{N}_0} f(\alpha) = \sum_{\alpha\in\mathbb{N}_0} f\left(\frac{\alpha}{k}\right) &= \frac{1}{2}f(0) + \int_0^\infty f\left(\frac{x}{k}\right)dx - \sum_{j=1}^{p}\frac{B_{2j}}{(2j)!}f^{(2j-1)}(0)k^{-(2j-1)}\\
&\quad+k^{-(2p+1)}\int_0^\infty f^{(2p+1)}\left(\frac{x}{k}\right)P_{2p+1}(x)dx\\
&= k\int_0^\infty f(x)\,dx + \frac{1}{2}f(0) - \sum_{j=1}^{p}\frac{B_{2j}}{(2j)!}f^{(2j-1)}(0)k^{-(2j-1)}\\
&\quad+k^{-2p}\int_0^\infty f^{(2p+1)}(x)P_{2p+1}(kx)dx,
\end{aligned}$$

so that the result holds. For $n \geq 2$, we have

$$\begin{aligned}
\sum_{\boldsymbol{\alpha}\in\frac{1}{k}(\mathbb{N}_0\times\mathbb{Z}^{n-1})} f(\boldsymbol{\alpha}) &= \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0\times\mathbb{Z}^{n-1}} f\left(\frac{1}{k}\boldsymbol{\alpha}\right)\\
&= \sum_{(\alpha_1,\cdots,\alpha_{n-1})\in\mathbb{N}_0\times\mathbb{Z}^{n-2}}\Bigg(k\int_{-\infty}^{\infty} f\left(\frac{\alpha_1}{k},\cdots,\frac{\alpha_{n-1}}{k},x_n\right)dx_n\\
&\quad+k^{-2p}\int_{-\infty}^{\infty}\frac{\partial^{2p+1}f}{\partial x_n^{2p+1}}\left(\frac{\alpha_1}{k},\cdots,\frac{\alpha_{n-1}}{k},x_n\right)P_{2p+1}(kx_n)dx_n\Bigg).
\end{aligned}$$

Now observe that there exists $C \geq 0$ such that

$$\sum_{(\alpha_1,\cdots,\alpha_{n-1})\in\mathbb{N}_0\times\mathbb{Z}^{n-2}}\int_{-\infty}^{\infty}\frac{\partial^{2p+1}f}{\partial x_n^{2p+1}}\left(\frac{\alpha_1}{k},\cdots,\frac{\alpha_{n-1}}{k},x_n\right)P_{2p+1}(kx_n)dx_n \leq Ck^{n-1}$$

since there are only $\mathcal{O}\left(k^{n-1}\right)$ terms in the sum which are nonzero (supp f is compact), and the terms of the sum are bounded. Picking $p \in \mathbb{N}_0$ large enough and iterating this procedure yields

$$\sum_{\boldsymbol{\alpha}\in\frac{1}{k}(\mathbb{N}_0\times\mathbb{Z}^{n-1})} f(\boldsymbol{\alpha}) = k^{n-1}\sum_{\alpha_1\in\mathbb{N}_0}\int_{\mathbb{R}^{n-1}} f\left(\frac{\alpha_1}{k}, x_2, \cdots, x_n\right) dx_2\cdots dx_n + \mathcal{O}\left(k^{-\infty}\right).$$

We can now apply the same arguments as in the $n = 1$ case to get

$$\begin{aligned}\sum_{\boldsymbol{\alpha}\in\frac{1}{k}(\mathbb{N}_0\times\mathbb{Z}^{n-1})} f(\boldsymbol{\alpha}) = k^n &\int_{\mathbb{R}_{\geq 0}\times\mathbb{R}^{n-1}} f(\boldsymbol{x})d\boldsymbol{x} \\ &- \sum_{j=1}^{p} k^{n-2j}\frac{B_{2j}}{(2j)!}\int_{\mathbb{R}^{n-1}}\frac{\partial^{2j-1} f}{\partial x_1^{2j-1}}(0, x_2, \cdots, x_n)dx_2\cdots dx_n \\ &+ \frac{k^{n-1}}{2}\int_{\mathbb{R}^{n-1}} f(0, x_2, \cdots, x_n)dx_2\cdots dx_n + \mathcal{O}\left(k^{n-(2p+1)}\right).\end{aligned}$$

□

While the above lemma is entertaining in its own right, let us now focus on Euler-Maclaurin sums over the types of functions that we require for our asymptotic analysis of partial density functions.

Lemma 6.2.2. *Let $\boldsymbol{x}_0 \in \mathbb{R}^n$, $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $h \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\mathcal{U} \subset \mathbb{R}^n$. Suppose that h satisfies $h(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$ with $h(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = \boldsymbol{x}_0$. Suppose furthermore that there exists a constant $c > 0$ such that $h(\boldsymbol{x}) > c > 0$ for all $\boldsymbol{x}$ outside a compact subset of $\mathcal{U}$ and that $\operatorname{Hess} h|_{\boldsymbol{x}_0}$ is positive definite. Suppose that $f(\boldsymbol{x}) = \mathcal{O}\left(|\boldsymbol{x}-\boldsymbol{x}_0|^l\right)$ as $|\boldsymbol{x}-\boldsymbol{x}_0| \to 0$, for some $l \geq 0$, and let $\boldsymbol{\alpha} \in \mathbb{N}_0^n$. Then*

$$\int_{\mathcal{U}} \frac{\partial}{\partial \boldsymbol{x}}^{\boldsymbol{\alpha}}\left(f(\boldsymbol{x})e^{-kh(\boldsymbol{x})}\right) d\boldsymbol{x} = \mathcal{O}\left(k^{\frac{|\boldsymbol{\alpha}|-l-n}{2}}\right).$$

Proof. If $|\boldsymbol{\alpha}| = 0$, this follows from the proof of Laplace's method. Suppose that the result holds for $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ such that $|\boldsymbol{\alpha}| = s$. Suppose $|\boldsymbol{\alpha}| = s+1$ and $\boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{e}_j$ for some $\boldsymbol{\beta} \in \mathbb{N}_0^n$ and some $j \in \{1, \cdots, n\}$. Then

$$\begin{aligned}\int_{\mathcal{U}} \frac{\partial}{\partial \boldsymbol{x}}^{\boldsymbol{\alpha}}\left(f(\boldsymbol{x})e^{-kh(\boldsymbol{x})}\right) d\boldsymbol{x} &= \int_{\mathcal{U}} \frac{\partial}{\partial \boldsymbol{x}}^{\boldsymbol{\beta}}\left(\frac{\partial}{\partial x_j} f(\boldsymbol{x}) - kf(\boldsymbol{x})\frac{\partial}{\partial x_j} h(\boldsymbol{x})\right) e^{-kh(\boldsymbol{x})} d\boldsymbol{x} \\ &= \mathcal{O}\left(k^{\frac{|\boldsymbol{\beta}|-\max(l-1,0)-n}{2}} + k^{\frac{2+|\boldsymbol{\beta}|-(l+1)-n}{2}}\right) \\ &= \mathcal{O}\left(k^{\frac{|\boldsymbol{\alpha}|-l-n}{2}}\right).\end{aligned}$$

The result follows by induction. □

Proposition 6.2.3. *Let $\boldsymbol{x}_0 \in \mathbb{R}^n$, $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $h \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $h(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and $h(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = \boldsymbol{x}_0$. We assume that $\operatorname{Hess} h|_{\boldsymbol{x}_0} > 0$. Suppose furthermore that there exists a constant $c > 0$ such that $h(\boldsymbol{x}) > c > 0$ for all $\boldsymbol{x}$ outside a compact subset of $\mathbb{R}^n$. Then*

$$\sum_{\boldsymbol{\alpha}\in\frac{1}{k}(\mathbb{N}_0\times\mathbb{Z}^{n-1})} f(\boldsymbol{\alpha})e^{-kh(\boldsymbol{\alpha})} = k^n\mathbb{B}^{2p}\left(\frac{1}{k}\frac{\partial}{\partial\lambda}\right)\int_{\{\boldsymbol{x}\in\mathbb{R}^n : x_1\geq-\lambda\}} f(\boldsymbol{x})e^{-kh(\boldsymbol{x})}d\boldsymbol{x}\Bigg|_{\lambda=0} + \mathcal{O}\left(k^{\frac{n-1}{2}-p}\right).$$

Proof. The proof is very similar to the proof of lemma 6.2.1. If $n = 1$, we have

$$\begin{aligned}\sum_{\alpha\in\frac{1}{k}\mathbb{N}_0} f(\alpha)e^{-kh(\boldsymbol{\alpha})} &= \frac{1}{2}f(0)e^{-kh(0)} + k\int_0^\infty f(x)e^{-kh(x)}dx \\ &\quad - \sum_{j=1}^{p}\frac{B_{2j}}{(2j)!}\left(\frac{\partial}{\partial x}\right)^{2j-1}\left(f(x)e^{-kh(x)}\right)\Big|_{x=0} k^{-(2j-1)} \\ &\quad + k^{-2p}\int_0^\infty \left(\frac{\partial}{\partial x}\right)^{2p+1}\left(f(x)e^{-kh(x)}\right)\Big|_{x=0} P_{2p+1}(kx)dx.\end{aligned}$$

Lemma 6.2.2 now implies that the last term above is $\mathcal{O}\left(k^{-p}\right)$ as $k\to\infty$. For $n \geq 2$,

$$\begin{aligned}&\sum_{\boldsymbol{\alpha}\in\frac{1}{k}(\mathbb{N}_0\times\mathbb{Z}^{n-1})} f(\boldsymbol{\alpha})e^{-kh(\boldsymbol{\alpha})} \\ =& \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0\times\mathbb{Z}^{n-2}}\Bigg(k\int_{-\infty}^{\infty} f\left(\frac{\alpha_1}{k},\cdots,\frac{\alpha_{n-1}}{k},x_n\right)e^{-kh\left(\frac{\alpha_1}{k},\cdots,\frac{\alpha_{n-1}}{k},x_n\right)}dx_n \\ &+k^{-2p}\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x_n}\right)^{(2p+1)}\left\{f\left(\frac{\alpha_1}{k},\cdots,\frac{\alpha_{n-1}}{k},x_n\right)e^{-kh\left(\frac{\alpha_1}{k},\cdots,\frac{\alpha_{n-1}}{k},x_n\right)}\right\}P_{2p+1}(kx_n)dx_n\Bigg).\end{aligned}$$

Now observe that, by lemma 6.2.2, there exists $C \geq 0$ such that

$$\begin{aligned}\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0\times\mathbb{Z}^{n-2}}\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x_n}\right)^{(2p+1)}\left\{f\left(\frac{\alpha_1}{k},\cdots,\frac{\alpha_{n-1}}{k},x_n\right)e^{-kh\left(\frac{\alpha_1}{k},\cdots,\frac{\alpha_{n-1}}{k},x_n\right)}\right\}P_{2p+1}(kx_n)dx_n \\ \leq Ck^{p+n-1}.\end{aligned}$$

By choosing p sufficiently large and iterating this procedure, we obtain

$$\begin{aligned}\sum_{\boldsymbol{\alpha}\in\frac{1}{k}(\mathbb{N}_0\times\mathbb{Z}^{n-1})} f(\boldsymbol{\alpha})e^{-kh(\boldsymbol{\alpha})} &= k^{n-1}\sum_{\alpha_1\in\mathbb{N}_0}\int_{\mathbb{R}_{\geq 0}\times\mathbb{R}^{n-1}} f\left(\frac{\alpha_1}{k},x_2,\cdots,x_n\right)e^{-kh\left(\frac{\alpha_1}{k},x_2,\cdots,x_n\right)}d\boldsymbol{x} \\ &\quad +\mathcal{O}\left(k^{-\infty}\right).\end{aligned}$$

Finally, we have

$$\begin{aligned}&\sum_{\boldsymbol{\alpha}\in\frac{1}{k}(\mathbb{N}_0\times\mathbb{Z}^{n-1})} f(\boldsymbol{\alpha})e^{-kh(\boldsymbol{\alpha})} \\ &\qquad = k^n\int_{\mathbb{R}_{\geq 0}\times\mathbb{R}^{n-1}} f(\boldsymbol{x})e^{-kh(\boldsymbol{x})}d\boldsymbol{x} \\ &\qquad + \frac{k^{n-1}}{2}\int_{\mathbb{R}^{n-1}} f(0,x_2,\cdots,x_n)e^{-kh(0,x_2,\cdots,x_n)}dx_2\cdots dx_n \\ &\qquad - \sum_{j=1}^{p}\frac{B_{2j}}{(2j)!}k^{n-2j}\int_{\mathbb{R}^{n-1}}\left(\frac{\partial}{\partial x_1}\right)^{2j-1}\left(f(\boldsymbol{x})e^{-kh(\boldsymbol{x})}\right)\Bigg|_{x_1=0} dx_2\cdots dx_n \\ &\qquad + k^{n-1-2p}\int_{\mathbb{R}_{\geq 0}\times\mathbb{R}^{n-1}}\left(\frac{\partial}{\partial x_1}\right)^{2p+1}\left\{f(\boldsymbol{x})e^{-kh(\boldsymbol{x})}\right\}P_{2p+1}(kx_1)d\boldsymbol{x} \\ &\qquad + \mathcal{O}\left(k^{-\infty}\right).\end{aligned}$$

Since P_{2p+1} is bounded, and due to lemma 6.2.2, the last integral above is $\mathcal{O}\left(k^{p-\frac{n-1}{2}}\right)$ and the result follows. □

6.3 An Euler-Maclaurin formula for integral Delzant polytopes

In this section, we prove some Euler-Maclaurin summation formulas that we will not require in the remainder of this thesis. These results are collected here since these ideas might be helpful for further investigations into the asymptotics of toric partial density functions.

In [KSW05], theorem 6.1.1 was generalized to the positive orthant $\mathcal{O} \stackrel{\text{def}}{=} (\mathbb{R}_{\geq 0})^n$ as follows:

Proposition 6.3.1 (Euler-Maclaurin formula for the standard orthant)**.** *Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $p \in \mathbb{N}$. Then*

$$\sum_{\boldsymbol{x}\in\mathcal{O}\cap\mathbb{Z}^n}{}' f(\boldsymbol{x}) = \prod_{i=1}^n \mathbb{L}^{2p}\left(\frac{\partial}{\partial\lambda_i}\right)\int_{\mathcal{O}(\pm\lambda_1,\ldots,\pm\lambda_n)} f(\boldsymbol{x})d\boldsymbol{x}\Bigg|_{\boldsymbol{\lambda}=\mathbf{0}} + R_{2p+1}(f),$$

where the right hand side is independent of the choice of $\pm$. For $\boldsymbol{\lambda}\in\mathbb{R}^n$, we have

$$\mathcal{O}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \{\boldsymbol{x}\in\mathbb{R}^n : x_i - \lambda_i \geq 0 \textit{ for all } i\in\{1,\ldots,n\}\}$$

and

$$R_{2p+1}(f) \stackrel{\text{def}}{=} \sum_{I\subsetneq\{1,\ldots,n\}}\prod_{i\in I}\mathbb{L}^{2p}\left(\frac{\partial}{\partial\lambda_i}\right)\int_{\mathcal{O}(\pm\lambda_1,\ldots,\pm\lambda_n)}\prod_{i\notin I}P_{2p+1}(x_i)\prod_{i\notin I}\left(\frac{\partial}{\partial x_i}\right)^{2p+1} f(\boldsymbol{x})d\boldsymbol{x}\Bigg|_{\boldsymbol{\lambda}=\mathbf{0}},$$

where

$$P_{2p+1}(x) \stackrel{\text{def}}{=} \frac{b_{2p+1}(x-\lfloor x\rfloor)}{(2p+1)!},$$

and $b_{2p+1}(x)$ is the $(2p+1)^{th}$ Bernoulli polynomial.

A regular integral orthant $\mathcal{C}$ is the image of $\mathcal{O}$ under an affine transformation of the form

$$\boldsymbol{x}\mapsto A_{\mathcal{C}}(\boldsymbol{x}) = B\boldsymbol{x}+\boldsymbol{v} \qquad \boldsymbol{v}\in\mathbb{Z}^n, B\in\mathrm{SL}(n,\mathbb{Z}).$$

For such an orthant $\mathcal{C}$ (see [KSW05]),

$$\sum_{\boldsymbol{x}\in\mathcal{C}\cap\mathbb{Z}^n}{}' f(\boldsymbol{x}) = \prod_{i=1}^n \mathbb{L}^{2p}\left(\frac{\partial}{\partial\lambda_i}\right)\int_{\mathcal{C}(\pm\lambda_1,\ldots,\pm\lambda_n)} f(\boldsymbol{x})d\boldsymbol{x}\Bigg|_{\boldsymbol{\lambda}=\mathbf{0}} + R_{2p+1}(f\circ A_{\mathcal{C}}),$$

where $\mathcal{C}(\lambda_1,\ldots,\lambda_n)$ is the image of $\mathcal{O}(\lambda_1,\ldots,\lambda_n)$ under $A_{\mathcal{C}}$. Let $P\subset\mathbb{R}^n$ be an integral Delzant polytope. For each vertex $\boldsymbol{v}\in P$, we define the *tangent cone at $\boldsymbol{v}$* by

$$\mathcal{C}_{\boldsymbol{v}} \stackrel{\text{def}}{=} \{\boldsymbol{v}+r(\boldsymbol{x}-\boldsymbol{v}) : r\geq 0, \boldsymbol{x}\in P\}.$$

We can pick n *edge vectors* $\alpha_1(\boldsymbol{v}),\ldots,\alpha_n(\boldsymbol{v})$ for $\boldsymbol{v}$ such that

$$\mathcal{C}_{\boldsymbol{v}} = \boldsymbol{v}+\sum_{j=1}^n \mathbb{R}_{\geq 0}\alpha_j(\boldsymbol{v}).$$

A *polarizing vector* is a vector $\boldsymbol{\eta} \in (\mathbb{R}^n)^*$ such that $\langle \boldsymbol{\eta}, \alpha_j(\boldsymbol{v})\rangle \neq 0$ for all vertices $\boldsymbol{v} \in P$ and all $j \in \{1, \dots, n\}$. The *polarized edge vectors* are defined to be

$$\alpha_j^{\#}(\boldsymbol{v}) \stackrel{\text{def}}{=} \begin{cases} \alpha_j(\boldsymbol{v}) & \text{if } \langle \boldsymbol{\eta}, \alpha_j(\boldsymbol{v})\rangle < 0 \\ -\alpha_j(\boldsymbol{v}) & \text{if } \langle \boldsymbol{\eta}, \alpha_j(\boldsymbol{v})\rangle > 0. \end{cases}$$

We define $\#\boldsymbol{v}(\boldsymbol{\eta}) = \#\boldsymbol{v}$ to be the number of edge vectors such $\alpha_j^{\#}(\boldsymbol{v}) \neq \alpha_j(\boldsymbol{v})$. The *polarized tangent cone* is defined by

$$\mathcal{C}_{\boldsymbol{v}}^{\#} = \boldsymbol{v} + \sum_{j=1}^{n} \mathbb{R}_{\geq 0}\alpha_j^{\#}(\boldsymbol{v}).$$

For $f : \mathbb{R}^n \to \mathbb{R}$, we have

$$\sum_{P\cap\mathbb{Z}^n}{}' f = \sum_{\boldsymbol{v}\in vertices(P)} (-1)^{\#\boldsymbol{v}} \sum_{\mathcal{C}_{\boldsymbol{v}}^{\#}\cap\mathbb{Z}^n}{}' f,$$

which enables us to write sums over integral points of an integral Delzant polytope as a sum over cones. If the polytope is given by $P = \{\boldsymbol{x} : l_i(\boldsymbol{x}) \stackrel{\text{def}}{=} \langle \boldsymbol{u}_i, \boldsymbol{x}\rangle + \mu_i \geq 0, i = 1, \dots, d\}$, where $\boldsymbol{u}_i$, for $i \in \{1, \cdots, d\}$, are the primitive *inwards pointing* normal vectors to the $n-1$-dimensional faces of P then the *modified polytope* $P(\boldsymbol{\lambda})$ is defined by $P(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \{\boldsymbol{x} : l_i(\boldsymbol{x}) + \lambda_i \geq 0, i = 1, \dots, d\}$, where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$. A version of the main result in [KSW03], [KSW05] is:

Theorem 6.3.2 ([KSW03, Theorem 1])**.** *Let $P \subset \mathbb{R}^n$ be an integral Delzant polytope, and let $p \in \mathbb{N}_0$ and $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Choose a polarizing vector for P. Then*

$$\sum_{\boldsymbol{x}\in P\cap\mathbb{Z}^n}{}' f(\boldsymbol{x}) = \prod_{i=1}^{d} \mathbb{L}^{2p}\left(\frac{\partial}{\partial\lambda_i}\right) \int_{P(\lambda_1,\dots,\lambda_d)} f(\boldsymbol{x})d\boldsymbol{x}\Bigg|_{\boldsymbol{\lambda}=\boldsymbol{0}} + S_{2p+1}(f),$$

where

$$S_{2p+1}(f) = \sum_{\boldsymbol{v}\in \text{vertices}(P)} (-1)^{\#\boldsymbol{v}} R_{2p+1}(f \circ A_{\mathcal{C}_{\boldsymbol{v}}^{\#}}).$$

We now study a modification of the above result that might be of use for future research investigating the asymptotics of partial toric density functions.

6.4 A parameter dependent Euler-Maclaurin formula for integral Delzant polytopes

Theorem 6.4.1. *Let $\boldsymbol{a} \in \mathbb{R}^n$, $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $h \in \mathcal{C}^\infty(\mathbb{R}^n)$. Suppose that h satisfies $h(\boldsymbol{x}) \geq 0$, for all $\boldsymbol{x} \in \mathbb{R}^n$, and assume that $h(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = \boldsymbol{a}$. Suppose furthermore that* $\operatorname{Hess} h|_{\boldsymbol{a}}$ *is positive definite and that there exists a constant $c > 0$ such that $h(\boldsymbol{x}) - h(\boldsymbol{a}) \geq c$ outside a compact subset of $\mathbb{R}^n$. Let P be an integral Delzant polytope in $\mathbb{R}^n$. Then, for $p \in \mathbb{N}$,*

$$\frac{1}{k^n} \sum_{\boldsymbol{x}\in P\cap\frac{1}{k}\mathbb{Z}^n}{}' f(\boldsymbol{x})\, e^{-kh(\boldsymbol{x})} = \prod_{i=1}^{d} \mathbb{L}^{2p}\left(\frac{1}{k}\frac{\partial}{\partial\lambda_i}\right) \int_{P(\boldsymbol{\lambda})} f(\boldsymbol{x})e^{-kh(\boldsymbol{x})}d\boldsymbol{x}\Bigg|_{\boldsymbol{\lambda}=\boldsymbol{0}} + R_{2p+1}(k),$$

and there exists $C \geq 0$ such that

$$|R_{2p+1}(k)| \leq Ck^{-\left(p+\frac{n+1}{2}\right)} \qquad \textit{for all } k \in \mathbb{N}.$$

Proof. We apply theorem 6.3.2 to the function $\boldsymbol{x} \mapsto f\left(\frac{1}{k}\boldsymbol{x}\right) e^{-kh\left(\frac{1}{k}\boldsymbol{x}\right)}$ on kP. Now $\boldsymbol{x} \in (kP)(\boldsymbol{\lambda})$ if and only if $\frac{1}{k}\boldsymbol{x} \in P\left(\frac{1}{k}\boldsymbol{\lambda}\right)$, so that, for $p \in \mathbb{N}$,

$$\begin{aligned}\sum_{\boldsymbol{x}\in kP\cap\mathbb{Z}^n}' f\left(\frac{1}{k}\boldsymbol{x}\right) e^{-kh\left(\frac{1}{k}\boldsymbol{x}\right)} &= k^n \prod_{i=1}^{d} \mathbb{L}^{2p}\left(\frac{\partial}{\partial\lambda_i}\right) \int_{P\left(\frac{1}{k}\boldsymbol{\lambda}\right)} f(\boldsymbol{x})e^{-kh(\boldsymbol{x})}d\boldsymbol{x}\Bigg|_{\boldsymbol{\lambda}=\boldsymbol{0}} + S_{2p+1}(k)\\ &= k^n \prod_{i=1}^{d} \mathbb{L}^{2p}\left(\frac{1}{k}\frac{\partial}{\partial\lambda_i}\right) \int_{P(\boldsymbol{\lambda})} f(\boldsymbol{x})e^{-kh(\boldsymbol{x})}d\boldsymbol{x}\Bigg|_{\boldsymbol{\lambda}=\boldsymbol{0}} + S_{2p+1}(k),\end{aligned}$$

where

$$S_{2p+1}(k) = \sum_{\boldsymbol{v}\in\text{vertices}(P)} (-1)^{\#\boldsymbol{v}} R_{2p+1}\left(f\left(\frac{1}{k}A_{k\boldsymbol{v}}(\boldsymbol{x})\right) e^{-kh\left(\frac{1}{k}A_{k\boldsymbol{v}}(\boldsymbol{x})\right)}\right),$$

and $A_{k\boldsymbol{v}} : \mathbb{R}^n \to \mathbb{R}^n$ is an affine map. $A_{k\boldsymbol{v}} \stackrel{\text{def}}{=} B_{\boldsymbol{v}}\boldsymbol{x} + k\boldsymbol{v}$ with $B_{\boldsymbol{v}} \in \mathrm{SL}(n,\mathbb{Z})$ maps the positive orthant onto the tangent cone of kP at $k\boldsymbol{v}$. Let us define $f_{\boldsymbol{v}}(\boldsymbol{x}) \stackrel{\text{def}}{=} f(B_{\boldsymbol{v}}\boldsymbol{x} + \boldsymbol{v})$ and $h_{\boldsymbol{v}}(\boldsymbol{x}) \stackrel{\text{def}}{=} h(B_{\boldsymbol{v}}\boldsymbol{x} + \boldsymbol{v})$. We have

$$\begin{aligned}&R_{2p+1}\left(f\left(\frac{1}{k}A_{k\boldsymbol{v}}(\boldsymbol{x})\right) e^{-kh\left(\frac{1}{k}A_{k\boldsymbol{v}}(\boldsymbol{x})\right)}\right)\\ &\quad= R_{2p+1}\left(f_{\boldsymbol{v}}\left(\frac{\boldsymbol{x}}{k}\right) e^{-kh_{\boldsymbol{v}}\left(\frac{\boldsymbol{x}}{k}\right)}\right)\\ &\quad= \sum_{I\subsetneq\{1,\dots,n\}} \prod_{i\in I}\mathbb{L}^{2p}\left(\frac{\partial}{\partial\lambda_i}\right) \int_{\mathcal{O}(\boldsymbol{\lambda})} \prod_{i\notin I} P_{2p+1}(x_i) \prod_{i\notin I}\left(\frac{\partial}{\partial x_i}\right)^{2p+1} f_{\boldsymbol{v}}(\frac{\boldsymbol{x}}{k})e^{-kh_{\boldsymbol{v}}\left(\frac{\boldsymbol{x}}{k}\right)}d\boldsymbol{x}\Bigg|_{\boldsymbol{\lambda}=\boldsymbol{0}}\\ &\quad= \sum_{I\subsetneq\{1,\dots,n\}} \int_{\mathcal{O}} \prod_{i\in I}\mathbb{L}^{2p}\left(-\frac{\partial}{\partial x_i}\right)\left(\prod_{i\notin I} P_{2p+1}(x_i) \prod_{i\notin I}\left(\frac{\partial}{\partial x_i}\right)^{2p+1} f_{\boldsymbol{v}}(\frac{\boldsymbol{x}}{k})e^{-kh_{\boldsymbol{v}}\left(\frac{\boldsymbol{x}}{k}\right)}\right)d\boldsymbol{x},\end{aligned}$$

where $P_{2p+1}(\boldsymbol{x}) = \frac{b_{2p+1}(\boldsymbol{x}-\lfloor\boldsymbol{x}\rfloor)}{m!}$ is smooth on $\mathbb{R}-\mathbb{Z}$. In the integral, we ignore the grid $G \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathbb{R}^n : \text{ there exists } i \in \{1,\dots,n\} \text{ such that } x_i \in \mathbb{Z}\}$. On $(\mathbb{R}_{\geq 0})^n - G$ all derivatives of $P_{2p+1}(x_j), j \in \{1,\dots,n\}$ up to order $2p$ are bounded by some constant $C \geq 0$. Hence $R_{2p+1}(k)$ can be dominated by a linear combination of terms of the form

$$\begin{aligned}Q_{\boldsymbol{\alpha}}(k) &\stackrel{\text{def}}{=} \int_{\mathcal{O}} \frac{\partial}{\partial\boldsymbol{x}}^{\boldsymbol{\alpha}}\left(f_{\boldsymbol{v}}\left(\frac{\boldsymbol{x}}{k}\right)e^{-kh_{\boldsymbol{v}}\left(\frac{\boldsymbol{x}}{k}\right)}\right)d\boldsymbol{x}\\ &= k^{-|\boldsymbol{\alpha}|+n}\int_{\mathcal{O}} \frac{\partial}{\partial\boldsymbol{y}}^{\boldsymbol{\alpha}}\left(f_{\boldsymbol{v}}(\boldsymbol{y})e^{-kh_{\boldsymbol{v}}(\boldsymbol{y})}\right)d\boldsymbol{y},\end{aligned}$$

where $2p+1 \leq |\boldsymbol{\alpha}| \leq (2p+1)n$. We now use lemma 6.2.2 to conclude that

$$k^{-n}Q_{\boldsymbol{\alpha}}(k) = \mathcal{O}\left(k^{-\left(\frac{|\boldsymbol{\alpha}|+n}{2}\right)}\right)$$

and the result follows. □

Chapter 7

Asymptotics

In this chapter, we develop a method for determining the asymptotics of the toric density function which we then adapt to find a new asymptotic expansion for density functions with vanishing along a toric submanifold.

7.1 Introduction

Let $(L_P, h) \to (X_P, \omega)$ be a toric polarization. For $\boldsymbol{\alpha} \in P \cap \frac{1}{k}\mathbb{Z}^n$ and $k \in \mathbb{N}$, let $s_{\boldsymbol{\alpha},k} \in H^0(X, L_P^k)$ denote the standard toric section corresponding to $\boldsymbol{\alpha}$ and k. We have, for $\boldsymbol{\beta} \in \mathrm{Int}(P)$,

$$|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2 = e^{2ku(\boldsymbol{\alpha})}e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})},$$

where $h(\boldsymbol{\alpha}, \boldsymbol{\beta}) \stackrel{\text{def}}{=} 2\Big(u(\boldsymbol{\alpha}) - u(\boldsymbol{\beta}) + \langle \nabla u|_{\boldsymbol{\beta}}, \boldsymbol{\beta} - \boldsymbol{\alpha}\rangle\Big)$, and $u : P \to \mathbb{R}$ denotes a function that is Legendre dual to the toric potential $\phi : \mathbb{R}^n \to \mathbb{R}$ which satisfies $\omega = 2i\partial\overline{\partial}\phi$ on the torus $(\mathbb{C}^*)^n \subset X_P$. Note that

$$\begin{aligned} \|s_{\boldsymbol{\alpha},k}\|^2 &\stackrel{\text{def}}{=} \int_{X_P} |s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2 \frac{\omega^n}{n!} \\ &= (2\pi)^n e^{2ku(\boldsymbol{\alpha})} \int_P e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} d\boldsymbol{\beta}. \end{aligned}$$

The density function $\rho_k : X_P \to \mathbb{R}$ is $\mathbb{T}^n$-invariant and, as a function on P, is given by

$$\rho_k(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \sum_{\boldsymbol{\alpha} \in P \cap \frac{1}{k}\mathbb{Z}^n} \frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} \quad \text{for } \boldsymbol{\beta} \in P \text{ and } k \in \mathbb{N}.$$

We now fix a face $F < P$ and, for $s, k \in \mathbb{N}$, consider the corresponding partial density functions $\rho_{F,s,k}$ of sections of L_P^k vanishing to order at least sk along the toric submanifold $Y_F \subset X_P$ corresponding to F. By invariance under the real torus action, we can think of $\rho_{F,s,k}$ as a function on $P \subset \mathbb{R}^n \cong \mathfrak{t}^*$.

$$\rho_{F,s,k}(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \sum_{\boldsymbol{\alpha} \in P_{F,s} \cap \frac{1}{k}\mathbb{Z}^n} \frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} \quad \text{for } \boldsymbol{\beta} \in P \text{ and } k \in \mathbb{N},$$

where $P_{F,s} \subset P$ denotes the polytope corresponding to $L_P \otimes \mathcal{J}_{Y_F}^s \to X_P$. In this chapter, we discuss a method which enables us to understand the asymptotics of $\rho_{F,s,k}(\boldsymbol{\beta})$ for $\boldsymbol{\beta} \in \mathrm{Int}(P)$ as

$k \to \infty$. We can also use a similar method to determine the asymptotics of $\rho_k(\boldsymbol{\beta})$ for $\boldsymbol{\beta} \in \mathrm{Int}(P)$.

7.2 Overview of our method

To determine the asymptotics of ρ_k and $\rho_{F,s,k}$, we need three essential tools. The first is a localization of sums formula, the second is a version of the Euler-Maclaurin summation formula for integral polytopes which enables us to rewrite sums over integral points of a polytope as sums of certain integrals, and the third is an application of Laplace's method to expand these resulting integrals. Let us first explain this by going through this method in the case of ρ_k before concentrating on $\rho_{F,s,k}$.

7.3 The asymptotics of ρ_k

Recall from chapter 1 that the asymptotics of ρ_k, for a polarized Kähler manifold, have been the focus of recent research efforts. Lu [Lu00] used Tian's holomorphic peak sections to determine the first few coefficients of the asymptotic expansion. In the case of a toric polarization $(L_P, h) \to (X_P, \omega)$, we now determine an explicit formula for all coefficients of the asymptotic expansion of ρ_k using our toric method at points $p \in X_P$ corresponding to interior points of the polytope P under the moment map. Let us note here that Song and Zelditch have also investigated several aspects of toric asymptotics in their work. In [SZ10] (see also references therein), they makes use of the asymptotics of the *toric Szegö kernel* and the *complex stationary phase method* in their considerations. Sena-Dias [SD10] has also developed an integration by parts method to determine the asymptotics of toric density functions building on earlier considerations by Burns, Guillemin and Uribe [BGU10]. The main point of this section is hence to describe our method for expanding ρ_k which will then be extended to partial density functions in the next section. Some of our calculations here will be used for computing the asymptotics of $\rho_{F,s,k}$ later on.

Our first task is to determine the asymptotics of $\frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2}$ for $k \in \mathbb{N}$, $\boldsymbol{\alpha} \in P \cap \frac{1}{k}\mathbb{Z}^n$ and $\boldsymbol{\beta} \in \mathrm{Int}(P)$ as $k \to \infty$. We have

$$\frac{\|s_{\boldsymbol{\alpha},k}\|^2}{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2} = (2\pi)^n\, e^{kh(\boldsymbol{\alpha},\boldsymbol{\beta})} \int_P e^{-kh(\boldsymbol{\alpha},\boldsymbol{\gamma})} d\boldsymbol{\gamma},$$

where $h(\boldsymbol{\alpha},\boldsymbol{\beta}) = 2\left(u(\boldsymbol{\alpha}) - u(\boldsymbol{\beta}) + \langle \nabla u|_{\boldsymbol{\beta}}, \boldsymbol{\beta} - \boldsymbol{\alpha}\rangle\right)$ for $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathrm{Int}(P)$. Furthermore, $h(\boldsymbol{\alpha},\boldsymbol{\beta}) \geq 0$ with equality if and only if $\boldsymbol{\alpha} = \boldsymbol{\beta}$ for $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathrm{Int}(P)$. Note that $\mathrm{Hess}\, h(.,\boldsymbol{\beta})|_{\boldsymbol{\beta}}$ and $\mathrm{Hess}\, h(\boldsymbol{\alpha},.)|_{\boldsymbol{\alpha}}$ are positive definite for $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathrm{Int}(P)$. Due to the canonical singularities of u on ∂P, some care is required when considering the behaviour h near the boundary of P.

Lemma 7.3.1. *Let $\mathcal{K} \subset \mathrm{Int}(P)$ be a compact set. For $j \in \mathbb{N}_0$, there exist smooth functions $a_j : \mathrm{Int}(P) \to \mathbb{R}$ such that, for $p \in \mathbb{N}_0$, $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathrm{Int}(P)$ and all $k \in \mathbb{N}$,*

$$(2\pi)^n \int_P e^{-kh(\boldsymbol{\alpha},\boldsymbol{\gamma})} d\boldsymbol{\gamma} = \sum_{j=0}^{p} a_j(\boldsymbol{\alpha}) k^{-\left(j+\frac{n}{2}\right)} + R_{p,k}(\boldsymbol{\alpha}),$$

and there exists $C_p \geq 0$ such that

$$|R_{p,k}(\boldsymbol{\alpha})| \leq C_p k^{-\left(\frac{n+1}{2}+p\right)} \quad \textit{for all } \boldsymbol{\alpha} \in \mathcal{K}.$$

Furthermore, for $j \in \mathbb{N}_0$, we have

$$
\begin{aligned}
a_j(\boldsymbol{\alpha}) =& (2\pi)^n \sqrt{\frac{\pi^n}{|\operatorname{Hess} u|_{\boldsymbol{\alpha}}|}} \sum_{i=0}^{2j} \frac{(-1)^i}{i!(i+j)!2^{2(i+j)}} \left\langle \operatorname{Hess} u|_{\boldsymbol{\alpha}}^{-1} D_{\boldsymbol{\beta}}, D_{\boldsymbol{\beta}} \right\rangle^{i+j} \\
& \left(h(\boldsymbol{\alpha},\boldsymbol{\beta}) - \langle \operatorname{Hess} u|_{\boldsymbol{\alpha}} (\boldsymbol{\beta}-\boldsymbol{\alpha}), \boldsymbol{\beta}-\boldsymbol{\alpha}\rangle\right)^i \Big|_{\boldsymbol{\alpha}}.
\end{aligned}
$$

Proof. We can apply theorem B.2.2 to h restricted to $\mathcal{U} \times \mathcal{U}$, where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$ such that $\mathcal{K} \subset \mathcal{U} \subset \overline{\mathcal{U}} \subset \operatorname{Int}(P)$. h is then smooth on $\mathcal{U} \times \mathcal{U}$ and all derivatives of h are bounded on $\mathcal{U}$. By B.2.2, we have that, for all $j \in \mathbb{N}_0$ and $p \in \mathbb{N}_0$, there exists $a_j \in \mathcal{C}^\infty(\mathcal{U})$ and $C_p \geq 0$ such that for all $\boldsymbol{\alpha} \in \mathcal{K}$,

$$
\left| \int_{\mathcal{U}} e^{-kh(\boldsymbol{\alpha},\boldsymbol{\gamma})} d\boldsymbol{\gamma} - \sum_{j=0}^{p} k^{-\left(\frac{n}{2}+j\right)} a_j(\boldsymbol{\alpha}) \right| \leq C_p k^{-\left(\frac{n+1}{2}+p\right)},
$$

where

$$
\begin{aligned}
a_j(\boldsymbol{\alpha}) =& \sqrt{\frac{\pi^n}{|\operatorname{Hess} u|_{\boldsymbol{\alpha}}|}} \sum_{i=0}^{2j} \frac{(-1)^i}{i!(i+j)!2^{2(i+j)}} \left\langle \operatorname{Hess} u|_{\boldsymbol{\alpha}}^{-1} D_{\boldsymbol{\beta}}, D_{\boldsymbol{\beta}} \right\rangle^{i+j} \\
& \left(h(\boldsymbol{\alpha},\boldsymbol{\beta}) - \langle \operatorname{Hess} u|_{\boldsymbol{\alpha}} (\boldsymbol{\beta}-\boldsymbol{\alpha}), \boldsymbol{\beta}-\boldsymbol{\alpha}\rangle\right)^i \Big|_{\boldsymbol{\alpha}}.
\end{aligned}
$$

In fact, $a_j \in \mathcal{C}^\infty(\operatorname{Int}(P))$ for all $j \in \mathbb{N}_0$. Finally, observe that there exists $d > 0$ such that $h(\boldsymbol{\alpha},\boldsymbol{\beta}) > d$ for all $\boldsymbol{\alpha} \in \mathcal{K}$ and $\boldsymbol{\gamma} \in P - \mathcal{U}$. Hence

$$
\left| \int_{P-\mathcal{U}} e^{-kh(\boldsymbol{\alpha},\boldsymbol{\gamma})} d\boldsymbol{\gamma} \right| \leq e^{-dk} \operatorname{Vol}(P) = \mathcal{O}\left(k^{-\infty}\right) \quad \text{for all } \boldsymbol{\alpha} \in \mathcal{K}.
$$

The lemma now follows. ☐

Remark 7.3.2. *Define, for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \operatorname{Int}(P)$,*

$$
R(\boldsymbol{\alpha},\boldsymbol{\beta}) \stackrel{\text{def}}{=} h(\boldsymbol{\alpha},\boldsymbol{\beta}) - \langle \operatorname{Hess} u|_{\boldsymbol{\alpha}} (\boldsymbol{\beta}-\boldsymbol{\alpha}), \boldsymbol{\beta}-\boldsymbol{\alpha}\rangle.
$$

The first two terms in the expansion of lemma 7.3.1 are:

$$
\begin{aligned}
a_0(\boldsymbol{\alpha}) =& (2\pi)^n \sqrt{\frac{\pi^n}{|\operatorname{Hess} u|_{\boldsymbol{\alpha}}|}} \\
a_1(\boldsymbol{\alpha}) =& (2\pi)^n \sqrt{\frac{\pi^n}{|\operatorname{Hess} u|_{\boldsymbol{\alpha}}|}} \left(-\frac{1}{2^5} \langle \operatorname{Hess} u|_{\boldsymbol{\alpha}}^{-1} D_{\boldsymbol{\beta}}, D_{\boldsymbol{\beta}} \rangle^2 \, R(\boldsymbol{\alpha},\boldsymbol{\beta})|_{\boldsymbol{\alpha}} \right. \\
& \left. + \frac{1}{2^8 3} \langle \operatorname{Hess} u|_{\boldsymbol{\alpha}}^{-1} D_{\boldsymbol{\beta}}, D_{\boldsymbol{\beta}} \rangle^3 \, R(\boldsymbol{\alpha},\boldsymbol{\beta})^2 \big|_{\boldsymbol{\alpha}} \right) \\
=& \frac{(2\pi)^n}{48} \sqrt{\frac{\pi^n}{|\operatorname{Hess} u|_{\boldsymbol{\alpha}}|}} \left(-9 u_{ij}{}^{ij}(\boldsymbol{\alpha}) + 12 u_{ik}{}^{i}(\boldsymbol{\alpha}) u_j{}^{jk}(\boldsymbol{\alpha}) + 8 u_{ijk}(\boldsymbol{\alpha}) u^{ijk}(\boldsymbol{\alpha}) \right),
\end{aligned}
$$

where we use the Einstein summation convention. Lower indices indicate partial derivatives and indices are raised by the inverse-matrix of $\operatorname{Hess} u|_{\boldsymbol{\alpha}}$*. For example, $u_{ij}{}^{k}(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} u_{ijr}(\boldsymbol{\alpha}) u^{rk}(\boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \in \operatorname{Int}(P)$.*

We can now invert the asymptotic expansion above.

Lemma 7.3.3. *Let $\mathcal{K} \subset \operatorname{Int}(P)$ be a compact set. For $j \in \mathbb{N}_0$, there exists smooth functions $b_j : \operatorname{Int}(P) \to \mathbb{R}$ such that, for $p \in \mathbb{N}_0$, $\boldsymbol{\alpha} \in \mathcal{K} \cap \mathbb{Z}^n$, $\boldsymbol{\beta} \in \mathcal{K}$ and all $k \in \mathbb{N}$,*

$$\left((2\pi)^n \int_P e^{-kh(\boldsymbol{\alpha},\boldsymbol{\gamma})} d\boldsymbol{\gamma} \right)^{-1} = \sum_{j=0}^{p} b_j(\boldsymbol{\alpha}) k^{\frac{n}{2}-j} + S_{p,k}(\boldsymbol{\alpha}),$$

and there exists $C_p \geq 0$ such that

$$|S_{p,k}(\boldsymbol{\alpha})| \leq C_p k^{\frac{n}{2}-(p+\frac{1}{2})}$$

for all $\boldsymbol{\alpha} \in \mathcal{K}$ and $k \gg 0$. Furthermore, for $j \in \mathbb{N}_0$, b_j is determined by the following formal differentiation:

$$b_j(\boldsymbol{\alpha}) = \frac{1}{j!} \frac{\mathrm{d}}{\mathrm{d}\, s} \bigg|_{s=0} \frac{1}{\sum_{l=0}^{\infty} a_j(\boldsymbol{\alpha}) s^l}.$$

Proof. Let $A_k(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} k^{\frac{n}{2}} (2\pi)^n \int_P e^{-kh(\boldsymbol{\alpha},\boldsymbol{\gamma})} d\boldsymbol{\gamma}$. It then follows from lemma 7.3.1 that, for $\boldsymbol{\alpha} \in \operatorname{Int}(P)$,

$$A_k(\boldsymbol{\alpha}) = \sum_{j=0}^{p} a_j(\boldsymbol{\alpha}) k^{-j} + \mathcal{O}\left(k^{-(p+\frac{1}{2})}\right).$$

Note also that $a_0(\boldsymbol{\alpha}) \neq 0$ for all $\boldsymbol{\alpha} \in \operatorname{Int}(P)$. We have

$$\left| \frac{1}{A_k(\boldsymbol{\alpha})} - \frac{1}{\sum_{j=0}^{p} a_j(\boldsymbol{\alpha}) k^{-j}} \right| = \mathcal{O}\left(k^{-(p+\frac{1}{2})}\right) \quad \text{as } k \to \infty,$$

while

$$\left| \frac{1}{\sum_{j=0}^{p} a_j(\boldsymbol{\alpha}) s^j} - \sum_{j=1}^{p} b_j(\boldsymbol{\alpha}) s^j \right| = \mathcal{O}\left(|s|^{p+1}\right) \quad \text{as } |s| \to 0.$$

Replacing s by $\frac{1}{k}$ gives the result. $\square$

Remark 7.3.4. *In particular, we have*

$$\begin{aligned}
b_0(\boldsymbol{\alpha}) &= \frac{1}{a_0(\boldsymbol{\alpha})} \\
&= \frac{1}{(2\pi)^n} \sqrt{\frac{|\operatorname{Hess} u|_{\boldsymbol{\alpha}}|}{\pi^n}} \\
b_1(\boldsymbol{\alpha}) &= -\frac{a_1(\boldsymbol{\alpha})}{a_0(\boldsymbol{\alpha})^2} \\
b_2(\boldsymbol{\alpha}) &= \frac{a_1(\boldsymbol{\alpha})^2 - a_0(\boldsymbol{\alpha}) a_2(\boldsymbol{\alpha})}{a_0(\boldsymbol{\alpha})^3} \\
b_3(\boldsymbol{\alpha}) &= \frac{2a_0(\boldsymbol{\alpha}) a_1(\boldsymbol{\alpha}) a_2(\boldsymbol{\alpha}) - a_3(\boldsymbol{\alpha}) a_0(\boldsymbol{\alpha})^2 - a_1(\boldsymbol{\alpha})^3}{a_0(\boldsymbol{\alpha})^4},
\end{aligned}$$

and a computation shows that e.g.

$$b_1(\boldsymbol{\alpha}) = \frac{1}{48(2\pi)^n} \sqrt{\frac{|\operatorname{Hess} u|_{\boldsymbol{\alpha}}|}{\pi^n}} \left(9 u_{ij}{}^{ij}(\boldsymbol{\alpha}) - 12 u_{ik}{}^{i}(\boldsymbol{\alpha}) u_j{}^{jk}(\boldsymbol{\alpha}) - 8 u_{ijk}(\boldsymbol{\alpha}) u^{ijk}(\boldsymbol{\alpha}) \right).$$

We can now fully determine the asymptotics of ρ_k on $\mathrm{Int}(P)$:

Theorem 7.3.5. *Let $(L_P, h) \to (X_P, \omega)$ be a toric polarization, and let $\mathcal{K} \subset \mathrm{Int}(P)$ be compact. For $j \in \mathbb{N}_0$, there exists smooth functions $d_j : \mathrm{Int}(P) \to \mathbb{R}$ such that, for $\boldsymbol{\beta} \in \mathcal{K}$ and $p \in \mathbb{N}_0$,*

$$\rho_k(\boldsymbol{\beta}) = \sum_{j=0}^{p} d_j(\boldsymbol{\beta}) k^{n-j} + T_{p,k}(\boldsymbol{\beta}),$$

and there exists $C_p \geq 0$ such that

$$|T_{p,k}(\boldsymbol{\beta})| \leq C_p k^{-\left(p+\frac{1}{2}\right)} \quad \textit{for all } \boldsymbol{\beta} \in \mathcal{K} \textit{ and } k \gg 0.$$

We have

$$d_j(\boldsymbol{\beta}) = \sum_{i=0}^{j} c_{i,j-i}(\boldsymbol{\beta}),$$

and

$$c_{i,j}(\boldsymbol{\beta}) = \sqrt{\frac{\pi^n}{\left|\mathrm{Hess}\, u|_{\boldsymbol{\beta}}\right|}} \sum_{l=0}^{2i} \frac{(-1)^l}{l!(l+i)! 2^{2(l+i)}} \left\langle \mathrm{Hess}\, u|_{\boldsymbol{\beta}}^{-1} D_{\boldsymbol{\alpha}}, D_{\boldsymbol{\alpha}} \right\rangle^{l+i}$$
$$b_j(\boldsymbol{\beta}) \left(h(\boldsymbol{\alpha}, \boldsymbol{\beta}) - \langle \mathrm{Hess}\, u|_{\boldsymbol{\beta}} (\boldsymbol{\alpha} - \boldsymbol{\beta}), \boldsymbol{\alpha} - \boldsymbol{\beta} \rangle \right)^l \Big|_{\boldsymbol{\beta}}.$$

Proof. Pick open subsets $\mathcal{U}, \mathcal{V}$ of $\mathbb{R}^n$ such that $\mathcal{K} \subset \mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{V} \subset \overline{\mathcal{V}} \subset \mathrm{Int}(P)$ and a smooth bump function $\psi : \mathbb{R}^n \to [0,1]$ such that $\psi(\boldsymbol{\alpha}) = 1$, for all $\boldsymbol{\alpha} \in \mathcal{U}$, and $\psi(\boldsymbol{\alpha}) = 0$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n - \mathcal{V}$. We have

$$\sum_{\boldsymbol{\alpha} \in P \cap \frac{1}{k}\mathbb{Z}^n} \frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} = \sum_{\boldsymbol{\alpha} \in P \cap \frac{1}{k}\mathbb{Z}^n} \psi(\boldsymbol{\alpha}) \frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} + A_k(\boldsymbol{\beta}),$$

where, for any $j \in \mathbb{N}_0$, there exists $C_j \geq 0$ such that

$$|A_k(\boldsymbol{\beta})| \leq C_j k^{-j} \quad \text{for all } \boldsymbol{\beta} \in \mathcal{K} \text{ and } k \in \mathbb{N}_0.$$

This is easily seen since, by proposition 5.1.7, there exists $c > 0$ such that

$$\frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} = n(\boldsymbol{\alpha}, \boldsymbol{\beta})^k \frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\alpha})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} \leq e^{-ck\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|^2} \rho_k(\boldsymbol{\alpha}).$$

By a standard estimate, there exists $D \geq 0$ such that $\rho_k(\boldsymbol{\alpha}) \leq Dk^n$ for all $\boldsymbol{\alpha} \in P$. Since there are only $\mathcal{O}(k^n)$ elements in $(P - \mathcal{U}) \cap \frac{1}{k}\mathbb{Z}^n$, we have that

$$\sum_{\boldsymbol{\alpha} \in P \cap \frac{1}{k}\mathbb{Z}^n} \frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} (1 - \psi(\boldsymbol{\alpha})) \leq \sum_{\boldsymbol{\alpha} \in (P-\mathcal{U}) \cap \frac{1}{k}\mathbb{Z}^n} \frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} = \mathcal{O}\left(k^{-\infty}\right)$$

for $\boldsymbol{\beta} \in \mathcal{K}$. Note that $\mathrm{supp}\, \psi \subset \mathrm{Int}(P)$ is compact. We apply lemma 7.3.3 to get

$$\sum_{\boldsymbol{\alpha} \in P \cap \frac{1}{k}\mathbb{Z}^n} \psi(\boldsymbol{\alpha}) \frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} = \sum_{j=0}^{p} \left(\sum_{\boldsymbol{\alpha} \in P \cap \frac{1}{k}\mathbb{Z}^n} \psi(\boldsymbol{\alpha}) b_j(\boldsymbol{\alpha}) e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} \right) k^{\frac{n}{2}-j}$$

$$+\sum_{\boldsymbol{\alpha}\in P\cap\frac{1}{k}\mathbb{Z}^n}\psi(\boldsymbol{\alpha})S_{p,k}(\boldsymbol{\alpha})e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})},$$

where $S_{p,k}$ is the remainder term occurring in lemma 7.3.3. There exists $C_p \geq 0$ such that

$$\begin{aligned}\left|\sum_{\boldsymbol{\alpha}\in P\cap\frac{1}{k}\mathbb{Z}^n}\psi(\boldsymbol{\alpha})S_{p,k}(\boldsymbol{\alpha})e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})}\right| &\leq C_p k^{\frac{n}{2}-\left(p+\frac{1}{2}\right)}\sum_{\boldsymbol{\alpha}\in P\cap\frac{1}{k}\mathbb{Z}^n}e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})}\\ &\leq C_p k^{\frac{n}{2}-\left(p+\frac{1}{2}\right)}\sum_{\boldsymbol{\alpha}\in P\cap\frac{1}{k}\mathbb{Z}^n}e^{-ck\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|^2}\\ &=\mathcal{O}\left(k^{n-\left(p+\frac{1}{2}\right)}\right)\quad\text{for all } \boldsymbol{\beta}\in\mathcal{K}.\end{aligned}$$

Note that $\boldsymbol{\alpha}\mapsto h(\boldsymbol{\alpha},\boldsymbol{\beta})$, for $\boldsymbol{\beta}\in\mathcal{K}$, is smooth on $\operatorname{Int}(P)$ and in particular on $\operatorname{supp}\psi$. We can now apply theorem 6.4.1 and, observing that the constants in $\mathcal{O}$ in theorem 6.4.1 can be chosen to vary continuously with the parameter $\boldsymbol{\beta}\in\mathcal{K}$, we conclude that, for $j\in\mathbb{N}_0$,

$$\begin{aligned}\frac{1}{k^n}\sum_{\boldsymbol{\alpha}\in P\cap\frac{1}{k}\mathbb{Z}^n}\psi(\boldsymbol{\alpha})b_j(\boldsymbol{\alpha})e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} &= \prod_{i=1}^{d}\mathbb{L}^{2(p-j)}\left(\frac{1}{k}\frac{\partial}{\partial\lambda_i}\right)\int_{P(\boldsymbol{\lambda})}\psi(\boldsymbol{\alpha})b_j(\boldsymbol{\alpha})e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})}d\boldsymbol{\alpha}\Bigg|_{\boldsymbol{\lambda}=\mathbf{0}} \qquad (7.3.1)\\ &\quad + R_{j,p,k}(\boldsymbol{\beta}),\end{aligned}$$

where there exists $D_j \geq 0$ such that

$$|R_{j,p,k}(\boldsymbol{\beta})| \leq D_j k^{-\left(p-j+\frac{n+1}{2}\right)}\quad\text{for all } \boldsymbol{\beta}\in\mathcal{K}.$$

Since $d(\mathcal{K},\partial P)>0$, equation 7.3.1 simplifies to

$$\sum_{\boldsymbol{\alpha}\in P\cap\frac{1}{k}\mathbb{Z}^n}\psi(\boldsymbol{\alpha})b_j(\boldsymbol{\alpha})e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} = k^n\int_P\psi(\boldsymbol{\alpha})b_j(\boldsymbol{\alpha})e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})}d\boldsymbol{\alpha}+\mathcal{O}\left(k^{\frac{n}{2}-\left(p-j+\frac{1}{2}\right)}\right) \qquad (7.3.2)$$

since all other terms involve integrals and their derivatives over the faces of P. We have

$$\rho_k(\boldsymbol{\beta}) = k^{\frac{3n}{2}}\sum_{j=0}^{p}k^{-j}\int_P\psi(\boldsymbol{\alpha})b_j(\boldsymbol{\alpha})e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})}d\boldsymbol{\alpha}+\mathcal{O}\left(k^{n-\left(p+\frac{1}{2}\right)}\right) \qquad (7.3.3)$$

for $\boldsymbol{\beta}\in\mathcal{K}$. We can now expand these integrals using theorem B.2.2. We have

$$\int_P\psi(\boldsymbol{\alpha})b_j(\boldsymbol{\alpha})e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})}d\boldsymbol{\alpha} = \sum_{i=0}^{p-j}c_{i,j}(\boldsymbol{\beta})k^{-\left(\frac{n}{2}+i\right)}+\mathcal{O}\left(k^{-\left(\frac{n+1}{2}+(p-j)\right)}\right)$$

for $\boldsymbol{\beta}\in\mathcal{K}$, where

$$\begin{aligned}c_{i,j}(\boldsymbol{\beta}) = \sqrt{\frac{\pi^n}{\left|\operatorname{Hess}u|_{\boldsymbol{\beta}}\right|}}\sum_{l=0}^{2i}\frac{(-1)^l}{l!(l+i)!2^{2(l+i)}}\left\langle\operatorname{Hess}u|_{\boldsymbol{\beta}}^{-1}D_{\boldsymbol{\alpha}},D_{\boldsymbol{\alpha}}\right\rangle^{l+i}\\ b_j(\boldsymbol{\beta})\left(h(\boldsymbol{\alpha},\boldsymbol{\beta})-\operatorname{Hess}u|_{\boldsymbol{\beta}}(\boldsymbol{\alpha}-\boldsymbol{\beta}),\boldsymbol{\alpha}-\boldsymbol{\beta}\rangle\right)^l\Big|_{\boldsymbol{\beta}}\end{aligned}$$

is independent of ψ. We conclude that

$$\begin{aligned}\rho_k(\boldsymbol{\beta}) &= k^n \sum_{j=0}^{p} \sum_{i=0}^{p-j} c_{i,j}(\boldsymbol{\beta}) k^{-(i+j)} + \mathcal{O}\left(k^{n-\left(p+\frac{1}{2}\right)}\right) \\ &= \sum_{s=0}^{p} k^{n-s} \sum_{i=0}^{s} c_{i,s-i}(\boldsymbol{\beta}) + \mathcal{O}\left(k^{n-\left(p+\frac{1}{2}\right)}\right)\end{aligned}$$

for all $\boldsymbol{\beta} \in \mathcal{K}$. □

Remark 7.3.6. *In particular, we have*

$$\begin{aligned}d_0(\boldsymbol{\beta}) &= c_{0,0}(\boldsymbol{\beta}) = \frac{1}{(2\pi)^n} \\ d_1(\boldsymbol{\beta}) &= c_{0,1}(\boldsymbol{\beta}) + c_{1,0}(\boldsymbol{\beta}).\end{aligned}$$

Let

$$R(\boldsymbol{\alpha}, \boldsymbol{\beta}) \stackrel{\text{def}}{=} h(\boldsymbol{\alpha}, \boldsymbol{\beta}) - \langle \operatorname{Hess} u|_{\boldsymbol{\beta}} (\boldsymbol{\alpha} - \boldsymbol{\beta}), \boldsymbol{\alpha} - \boldsymbol{\beta} \rangle.$$

We can compute that

$$\begin{aligned}c_{0,1}(\boldsymbol{\beta}) =& -\frac{1}{(2\pi)^n} \frac{a_1(\boldsymbol{\beta})}{a_0(\boldsymbol{\beta})} \\ =& \frac{1}{48(2\pi)^n} \left(9{u_{ij}}^{ij}(\boldsymbol{\beta}) - 12{u_{ik}}^{i}(\boldsymbol{\beta}){u_r}^{kr}(\boldsymbol{\beta}) - 8{u_{ikr}}^{ikr}(\boldsymbol{\beta})\ \right) \\ c_{1,0}(\boldsymbol{\beta}) =& \frac{a_0(\boldsymbol{\beta})}{(2\pi)^n} \left(\frac{1}{2^2} \left\langle \operatorname{Hess} u|_{\boldsymbol{\beta}}^{-1} D_{\boldsymbol{\alpha}}, D_{\boldsymbol{\alpha}} \right\rangle \frac{1}{a_0(\boldsymbol{\alpha})}\bigg|_{\boldsymbol{\beta}} \right. \\ & \left. - \frac{1}{2^5} \left\langle \operatorname{Hess} u|_{\boldsymbol{\beta}}^{-1} D_{\boldsymbol{\alpha}}, D_{\boldsymbol{\alpha}} \right\rangle^2 \frac{R(\boldsymbol{\alpha}, \boldsymbol{\beta})}{a_0(\boldsymbol{\alpha})}\bigg|_{\boldsymbol{\beta}} + \frac{1}{2^8 3} \left\langle \operatorname{Hess} u|_{\boldsymbol{\beta}}^{-1} D_{\boldsymbol{\alpha}}, D_{\boldsymbol{\alpha}} \right\rangle^3 \frac{R(\boldsymbol{\alpha}, \boldsymbol{\beta})^2}{a_0(\boldsymbol{\alpha})}\bigg|_{\boldsymbol{\beta}} \right) \\ =& \frac{1}{48(2\pi)^n} \left(3{u_{ij}}^{ij}(\boldsymbol{\beta}) - 4u_{ijk}(\boldsymbol{\beta}) u^{ijk}(\boldsymbol{\beta})\right).\end{aligned}$$

It is now clear that

$$d_1(\boldsymbol{\beta}) = c_{0,1}(\boldsymbol{\beta}) + c_{1,0}(\boldsymbol{\beta}) = \frac{1}{4(2\pi)^n} \left({u_{ij}}^{ij}(\boldsymbol{\beta}) - {u_{ij}}^{k}(\boldsymbol{\beta}){u_k}^{ij}(\boldsymbol{\beta}) - u_{ijk}(\boldsymbol{\beta}) u^{ijk}(\boldsymbol{\beta})\right).$$

Using Abreu's formula [Abr03, formula 3.3] for the scalar curvature of the metric corresponding to u, we can easily check that

$$d_1(\boldsymbol{\beta}) = \frac{\operatorname{Scal}(\boldsymbol{\beta})}{2(2\pi)^n} \qquad \text{for } \boldsymbol{\beta} \in \operatorname{Int}(P)$$

as expected.

Remark 7.3.7. *Note that the formulas involved in determining the functions $\{d_j\}_{j=0}^{\infty}$ are getting complicated very quickly for large $j \in \mathbb{N}_0$. We know from Tian and Lu's work that these functions should depend only on the geometry of X_P, but matching the explicit formula for d_j in symplectic coordinates to the corresponding geometric quantity becomes non-trivial for large j.*

7.4 The asymptotics of $\rho_{F,s,k}$

7.4.1 Introduction

Fix a face $F < P$ and $s \in \mathbb{N}$ such that $s \leq S(Y_F)$, where $S(Y_F)$ denotes the Seshadri constant with respect to Y_F (see 8.1.2 for a definition). For $k \in \mathbb{N}$, consider the corresponding partial density functions $\rho_{F,s,k}$ of sections of L_P^k vanishing to order at least sk along the toric submanifold $Y_F \subset X_P$ corresponding to F. We have

$$\rho_{F,s,k}(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \sum_{\boldsymbol{\alpha} \in P_{F,s} \cap \frac{1}{k}\mathbb{Z}^n} \frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} \quad \text{for } \boldsymbol{\beta} \in P,$$

where $P_{F,s} \subset P$ denotes the polytope corresponding to $L_P \otimes \mathcal{J}_{Y_F}^s \to X_P$. Let $\mathcal{K} \subset \text{Int}(P)$ be compact and let $\mathcal{U}, \mathcal{V}$ be open subsets of $\mathbb{R}^n$ such that $\mathcal{K} \subset \mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{V} \subset \overline{\mathcal{V}} \subset \text{Int}(P)$. Let $\psi : \mathbb{R}^n \to [0,1]$ be a smooth bump function such that $\psi(\boldsymbol{\alpha}) = 1$, for all $\boldsymbol{\alpha} \in \mathcal{U}$, and $\psi(\boldsymbol{\alpha}) = 0$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n - \mathcal{V}$. Suppose that

$$\begin{aligned} P \stackrel{\text{def}}{=} \Big\{ &\boldsymbol{\alpha} \in \mathbb{R}^n : l_i(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \mu_i - \langle \boldsymbol{\alpha}, \boldsymbol{n}_i \rangle \geq 0 \text{ for some } \mu_i \in \mathbb{R}, \text{ primitive} \\ &\boldsymbol{n}_i \in \mathbb{Z}^n \text{ and } i \in \{1, \cdots, d\}\Big\}, \end{aligned}$$

and assume that $F = \cap_{i=1}^r \mathcal{Z}(l_i)$ for some $r \in \{1, \cdots, n\}$. We define

$$P_{F,s} \stackrel{\text{def}}{=} \left\{ \boldsymbol{\alpha} \in P : \sum_{j=1}^r l_j(\boldsymbol{\alpha}) \geq s \right\}$$
$$F_s \stackrel{\text{def}}{=} \left\{ \boldsymbol{\alpha} \in P : \sum_{j=1}^r l_j(\boldsymbol{\alpha}) = s \right\}$$

for $s \in \mathbb{N}$.

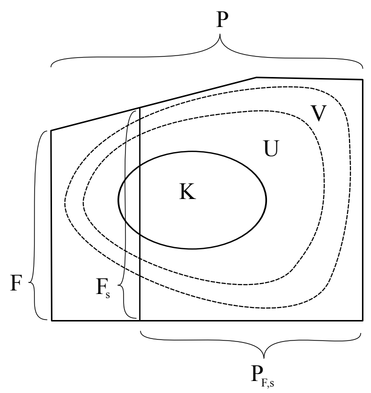

Figure 7.1: Example configuration.

7.4.2 The expansion

In order to simplify our computations, we will from now on assume that we have fixed coordinates such that $P_{F,s} = P \cap \{\boldsymbol{\alpha} \in \mathbb{R}^n : \alpha_1 \geq \nu\}$ and $F_s \stackrel{\text{def}}{=} \{\boldsymbol{\alpha} \in P : \alpha_1 = \nu\}$ for some fixed $\nu \in \mathbb{Z}$ and that the lattice is just $\mathbb{Z}^n \subset \mathbb{R}^n \cong \mathfrak{t}^*$.

Proposition 7.4.1. *Let $(L_P, h) \to (X_P, \omega)$ be a toric polarization and let $\mathcal{K}, F < P, F_s, P_{F,s}, \psi$ be chosen and normalized as described above. We have, for $\boldsymbol{\beta} \in \mathrm{Int}(P)$ and $p, k \in \mathbb{N}_0$:*

$$\rho_{F,s,k}(\boldsymbol{\beta}) = \sum_{j=0}^{p} k^{\frac{3n}{2}-j} \mathbb{B}^{2(p-j)} \left(\frac{1}{k}\frac{\partial}{\partial\lambda}\right) \int_{\alpha_1 \geq \nu-\lambda} \psi(\boldsymbol{\alpha}) b_j(\boldsymbol{\alpha}) e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} d\boldsymbol{\alpha}\Big|_{\lambda=0} + R_{p,k}(\boldsymbol{\beta}),$$

where there exists $C \geq 0$ such that $|R_{p,k}(\boldsymbol{\beta})| \leq Ck^{n-\left(p+\frac{1}{2}\right)}$ for all $\boldsymbol{\beta} \in \mathcal{K}$ and $k \gg 0$.

The following two claims will be useful for our proof of the above proposition.

Claim 1.

$$\rho_{F,s,k}(\boldsymbol{\beta}) = \sum_{\boldsymbol{\alpha} \in P_{F,s} \cap \frac{1}{k}\mathbb{Z}^n} \psi(\boldsymbol{\alpha}) \frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} + R_k(\boldsymbol{\beta}) \quad \text{for all } \boldsymbol{\beta} \in P,$$

and for all $j \in \mathbb{N}_0$, there exists $C_j \geq 0$ such that $|R_k(\boldsymbol{\beta})| \leq C_j k^{-j}$ for all $\boldsymbol{\beta} \in \mathcal{K}$ and all $k \gg 0$.

Proof. By proposition 5.1.7, there exists $c > 0$ such that $e^{-h(\boldsymbol{\alpha},\boldsymbol{\beta})} \leq e^{-c\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|^2}$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in P$. Furthermore, it is a standard result, which is independent of the proof of the asymptotic expansion of ρ_k, that there exists $D \geq 0$ such that $\rho_k(\boldsymbol{\beta}) \leq Dk^n$ for all $\boldsymbol{\beta} \in P$ (see *e.g.* [Bou96, lemma 3.1]). Combining this gives:

$$\begin{aligned}
\sum_{\boldsymbol{\alpha} \in P_{F,s} \cap \frac{1}{k}\mathbb{Z}^n} (1-\psi(\boldsymbol{\alpha})) \frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} &\leq \sum_{\boldsymbol{\alpha} \in (P_{F,s}-\mathcal{U}) \cap \frac{1}{k}\mathbb{Z}^n} \frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} \\
&\leq \sum_{\boldsymbol{\alpha} \in (P_{F,s}-\mathcal{U}) \cap \frac{1}{k}\mathbb{Z}^n} e^{-ck\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|^2} \rho_k(\boldsymbol{\alpha}) \\
&= \mathcal{O}\left(k^{-\infty}\right) \quad \text{for all } \boldsymbol{\beta} \in \mathcal{K} \text{ and } k \gg 0.
\end{aligned}$$

□

Furthermore, we have:

Claim 2. For $p \in \mathbb{N}_0$, $\boldsymbol{\beta} \in \mathrm{Int}(P)$ and $k \in \mathbb{N}_0$, we have

$$\sum_{\boldsymbol{\alpha} \in P_{F,s} \cap \frac{1}{k}\mathbb{Z}^n} \psi(\boldsymbol{\alpha}) \frac{|s_{\boldsymbol{\alpha},k}(\boldsymbol{\beta})|^2}{\|s_{\boldsymbol{\alpha},k}\|^2} = \sum_{j=0}^{p} \left(\sum_{\boldsymbol{\alpha} \in P_{F,s} \cap \frac{1}{k}\mathbb{Z}^n} \psi(\boldsymbol{\alpha}) b_j(\boldsymbol{\alpha}) e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} \right) k^{\frac{n}{2}-j} + Q_{p,k}(\boldsymbol{\beta}),$$

where b_j, for $j \in \mathbb{N}_0$, are the functions appearing in lemma 7.3.3 and where there exists $C_p \geq 0$ such that

$$|Q_{p,k}(\boldsymbol{\beta})| \leq C_p k^{n-\left(p+\frac{1}{2}\right)} \quad \text{for all } \boldsymbol{\beta} \in \mathcal{K} \text{ and all } k \gg 0.$$

Proof. There exists $c > 0$ and, by lemma 7.3.3, there exists $D_p \geq 0$ such that

$$\begin{aligned}
|Q_{p,k}(\boldsymbol{\beta})| &= \left| \sum_{\boldsymbol{\alpha} \in P_{F,s} \cap \frac{1}{k}\mathbb{Z}^n} \psi(\boldsymbol{\alpha}) S_{p,k}(\boldsymbol{\alpha}) e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} \right| \\
&\leq D_p k^{\frac{n}{2}-\left(p+\frac{1}{2}\right)} \sum_{\boldsymbol{\alpha} \in P_{F,s} \cap \frac{1}{k}\mathbb{Z}^n} e^{-ck\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|^2} \\
&= \mathcal{O}\left(k^{n-\left(p+\frac{1}{2}\right)}\right)
\end{aligned}$$

for all $\boldsymbol{\beta} \in \mathcal{K}$. □

Proof of the proposition. By claim 1 and claim 2, we have

$$\rho_{F,s,k}(\boldsymbol{\beta}) = \sum_{j=0}^{p} \left(\sum_{\{\boldsymbol{\alpha} \in \frac{1}{k}\mathbb{Z}^n : \alpha_1 \geq \nu\}} \psi(\boldsymbol{\alpha}) b_j(\boldsymbol{\alpha}) e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} \right) k^{\frac{n}{2}-j} + Q_{p,k}(\boldsymbol{\beta}),$$

where there exists $C \geq 0$ such that $|Q_{p,k}(\boldsymbol{\beta})| \leq Ck^{n-\left(p+\frac{1}{2}\right)}$ for all $\boldsymbol{\beta} \in \mathcal{K}$ and $k \gg 0$. We note that the constants in $\mathcal{O}$ in proposition 6.2.3 depend continuously on our parameter $\boldsymbol{\beta} \in \mathcal{K}$, so that we find

$$\rho_{F,s,k}(\boldsymbol{\beta}) = \sum_{j=0}^{p} \left(k^n \mathbb{B}^{2(p-j)} \left(\frac{1}{k} \frac{\partial}{\partial \lambda} \right) \int_{\alpha_1 \geq \nu - \lambda} \psi(\boldsymbol{\alpha}) b_j(\boldsymbol{\alpha}) e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} d\boldsymbol{\alpha} \bigg|_{\lambda=0} \right) k^{\frac{n}{2}-j} + R_{p,k}(\boldsymbol{\beta}),$$

where there exists $C \geq 0$ such that $|R_{p,k}(\boldsymbol{\beta})| \leq Ck^{n-\left(p+\frac{1}{2}\right)}$ for all $\boldsymbol{\beta} \in \mathcal{K}$ and $k \gg 0$. $\square$

Theorem 7.4.2. *Let $(L_P, h) \to (X_P, \omega)$ be a toric polarization. Fix a nontrivial face $F < P$ and $s \in \mathbb{N}$. Let $\boldsymbol{\beta} \in \operatorname{Int}(P)$. Then*

$$\rho_{F,s,k}(\boldsymbol{\beta}) = \begin{cases} \rho_k(\boldsymbol{\beta}) + \mathcal{O}\left(k^{-\infty}\right) & \text{if } \boldsymbol{\beta} \in \operatorname{Int}(P_{F,s}) \\ \frac{1}{2}\rho_k(\boldsymbol{\beta}) + \sum_{j=0}^{\infty} c_j(\boldsymbol{\beta}) k^{n-\left(j+\frac{1}{2}\right)} + \mathcal{O}\left(k^{-\infty}\right) & \text{if } \boldsymbol{\beta} \in \operatorname{RelInt}(F_s) \\ \mathcal{O}\left(k^{-\infty}\right) & \text{otherwise,} \end{cases}$$

where $c_j \in \mathcal{C}^\infty\left(\operatorname{Int}(P)\right)$ are explicitly computable functions. Now let $\mathcal{K} \subset \operatorname{Int}(P)$ be a compact set. For $p \in \mathbb{N}_0$ and $\boldsymbol{\beta} \in \mathcal{K} \cap \operatorname{RelInt}(F_s)$,

$$\rho_{F,s,k}(\boldsymbol{\beta}) = \frac{1}{2}\rho_k(\boldsymbol{\beta}) + \sum_{j=0}^{p} c_j(\boldsymbol{\beta}) k^{n-\left(j+\frac{1}{2}\right)} + S_{p,k}(\boldsymbol{\beta})$$

for all $k \in \mathbb{N}$, and there exists $D \geq 0$ such that $|S_{p,k}(\boldsymbol{\beta})| \leq Dk^{n-\left(p+\frac{3}{2}\right)}$ for all $\boldsymbol{\beta} \in \mathcal{K} \cap \operatorname{RelInt}(F_s)$ and $k \gg 0$.

Proof. Assume that we have normalized coordinates as described. In particular, we have $P_{F,s} = P \cap \{\boldsymbol{\alpha} \in \mathbb{R}^n : \alpha_1 \geq \nu\}$ for some fixed $\nu \in \mathbb{Z}$ and the lattice M is just $\mathbb{Z}^n \subset \mathbb{R}^n$. For the first part, choose a compact set $\mathcal{K} \subset \operatorname{Int}(P)$ such that $\boldsymbol{\beta} \in \mathcal{K}$. We note that by proposition 7.4.1,

$$\begin{aligned} &\rho_{F,s,k}(\boldsymbol{\beta}) = \\ &\quad \sum_{j=0}^{p} k^{\frac{3n}{2}-j} \Bigg(\int_{\alpha_1 \geq \nu} \psi(\boldsymbol{\alpha}) b_j(\boldsymbol{\alpha}) e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} d\boldsymbol{\alpha} \\ &\quad + \delta_{p-j\geq 1} \Bigg\{ \frac{1}{2k} \int_{\mathbb{R}^{n-1}} \psi(\nu, \alpha_2, \cdots, \alpha_n) b_j(\nu, \alpha_2, \cdots, \alpha_n) e^{-kh((\nu,\alpha_2,\cdots,\alpha_n),\boldsymbol{\beta})} d\alpha_2 \cdots d\alpha_n \\ &\quad - \sum_{i=1}^{p-j} k^{-2i} \frac{B_{2i}}{(2i)!} \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial \alpha_1}^{2i-1} \left(\psi(\boldsymbol{\alpha}) b_j(\boldsymbol{\alpha}) e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} \right) \bigg|_{((\nu,\alpha_2,\cdots,\alpha_n),\boldsymbol{\beta})} d\alpha_2 \cdots d\alpha_n \Bigg\} \Bigg) \\ &\quad + R_{p,k}(\boldsymbol{\beta}), \end{aligned}$$

where $R_{p,k}(\boldsymbol{\beta}) = \mathcal{O}\left(k^{n-\left(p+\frac{1}{2}\right)}\right)$ for $\boldsymbol{\beta} \in \mathcal{K}$. If $\beta_1 < \nu$, it is clear that all terms above are $\mathcal{O}\left(k^{-\infty}\right)$ since the point where h achieves its minimum is not in the domains of integration of the summands above. If $\beta_1 > \nu$, we have, for all $j \in \mathbb{N}_0$,

$$\int_{\alpha_1 \geq \nu} \psi(\boldsymbol{\alpha}) b_j(\boldsymbol{\alpha}) e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} d\boldsymbol{\alpha} = \int_{\mathbb{R}^n} \psi(\boldsymbol{\alpha}) b_j(\boldsymbol{\alpha}) e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} d\boldsymbol{\alpha} + \mathcal{O}\left(k^{-\infty}\right),$$

while the other terms are $\mathcal{O}(k^{-\infty})$, and we get back the expansion of $\rho_k(\boldsymbol{\beta})$ (see equation 7.3.3). Suppose now that $\boldsymbol{\beta} \in \mathcal{K} \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_1 = \nu\}$. We have

$$\int_{\alpha_1 \geq \nu} \psi(\boldsymbol{\alpha}) b_j(\boldsymbol{\alpha}) e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} d\boldsymbol{\alpha} = \sum_{l=0}^{2(p-j)} A_{j,l}(\boldsymbol{\beta}) k^{-\frac{n+l}{2}} + \mathcal{O}\left(k^{-\left(\frac{n+1}{2}+p-j\right)}\right),$$

using lemma B.3.6 and theorem B.3.4, where

$$\begin{aligned} A_{j,l}(\boldsymbol{\beta}) = |H(\boldsymbol{\beta})|^{-\frac{1}{2}} \sum_{i=0}^{l} \frac{(-1)^i}{i!} 2^{\frac{n+l}{2}+i} \sum_{|\boldsymbol{\gamma}|=l+2i} \frac{1}{\boldsymbol{\gamma}!} \frac{\partial}{\partial \boldsymbol{\alpha}}^{\boldsymbol{\gamma}} R\left(H^{-\frac{1}{2}}(\boldsymbol{\beta})\boldsymbol{\alpha} + \boldsymbol{\beta}, \boldsymbol{\beta}\right)^i \\ b_j\left(H^{-\frac{1}{2}}(\boldsymbol{\beta})\boldsymbol{\alpha} + \boldsymbol{\beta}\right)\Big|_{\boldsymbol{\alpha}=\mathbf{0}} e\left(\boldsymbol{\gamma}, H^{-\frac{1}{2}}(\boldsymbol{\beta})\boldsymbol{e}_1, 0\right), \end{aligned}$$

for $j, l \in \mathbb{N}_0$, $H(\boldsymbol{\beta}) = \operatorname{Hess} h(.,\boldsymbol{\beta})|_{\boldsymbol{\beta}} = 2 \operatorname{Hess} u|_{\boldsymbol{\beta}}$, $h(\boldsymbol{\alpha},\boldsymbol{\beta}) = 2\left(u(\boldsymbol{\alpha}) - u(\boldsymbol{\beta}) + \langle \nabla u|_{\boldsymbol{\beta}}, \boldsymbol{\beta} - \boldsymbol{\alpha}\rangle\right)$ and

$$R(\boldsymbol{\alpha},\boldsymbol{\beta}) = h(\boldsymbol{\alpha},\boldsymbol{\beta}) - \frac{1}{2}\langle H(\boldsymbol{\beta})(\boldsymbol{\alpha}-\boldsymbol{\beta}), \boldsymbol{\alpha}-\boldsymbol{\beta}\rangle,$$

so that $\frac{\partial}{\partial \boldsymbol{\alpha}}^{\boldsymbol{\gamma}} h(.,\boldsymbol{\beta})\Big|_{\boldsymbol{\beta}} = 2 \frac{\partial}{\partial \boldsymbol{\alpha}}^{\boldsymbol{\gamma}} u\Big|_{\boldsymbol{\beta}}$ for $|\boldsymbol{\gamma}| \geq 2$. By lemma B.3.6, we have, for $\boldsymbol{\beta} \in \mathcal{K} \cap \{\beta_1 = \nu\}$ and $j, l \in \mathbb{N}_0$,

$$A_{j,2l}(\boldsymbol{\beta}) = \frac{1}{2}\sqrt{\frac{(2\pi)^n}{|H(\boldsymbol{\beta})|}} \sum_{i=0}^{2l} \frac{(-1)^i}{i!(i+l)!2^{i+l}} \langle H^{-1}(\boldsymbol{\beta}) D_{\boldsymbol{\alpha}}, D_{\boldsymbol{\alpha}}\rangle^{i+l} b_j(\boldsymbol{\alpha}) R(\boldsymbol{\alpha},\boldsymbol{\beta})^i|_{\boldsymbol{\beta}},$$

where $D_{\boldsymbol{\alpha}} = \left(\frac{\partial}{\partial \alpha_1}, \cdots, \frac{\partial}{\partial \alpha_n}\right)$. We also have, for $p - j \geq 1$,

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^{n-1}} \psi(\nu,\alpha_2,\cdots,\alpha_n) b_j(\nu,\alpha_2,\cdots,\alpha_n) e^{-kh((\nu,\alpha_2,\cdots,\alpha_n),\boldsymbol{\beta})} d\alpha_2 \cdots d\alpha_n \\ &\qquad = \sum_{l=0}^{p-j-1} D_{j,l}(\boldsymbol{\beta}) k^{-\left(\frac{n-1}{2}+l\right)} + \mathcal{O}\left(k^{-\left(\frac{n-1}{2}+p-j\right)}\right), \end{aligned}$$

where, by theorem B.2.2,

$$\begin{aligned} D_{j,l}(\boldsymbol{\beta}) = \frac{1}{2}\sqrt{\frac{\pi^{n-1}}{|G(\boldsymbol{\beta})|}} \\ \sum_{i=0}^{2l} \frac{(-1)^i}{i!(i+l)!2^{2(i+l)}} \langle G^{-1}(\boldsymbol{\beta}) D, D\rangle^{i+l} b_j(\nu,\alpha_2,\cdots,\alpha_n) S((\nu,\alpha_2,\cdots,\alpha_n),\boldsymbol{\beta})^i\Big|_{(\beta_2,\cdots,\beta_n)}, \end{aligned}$$

where now

$$\begin{aligned} G(\boldsymbol{\beta}) &\stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial \alpha_i}\frac{\partial}{\partial \alpha_j} h\right)_{2 \leq i,j \leq n}\Bigg|_{\boldsymbol{\beta}} \\ &= (u_{ij}(\boldsymbol{\beta}))_{2 \leq i,j \leq n} \end{aligned}$$

$$D \overset{\text{def}}{=} (\frac{\partial}{\partial \alpha_2}, \cdots, \frac{\partial}{\partial \alpha_n}),$$

and

$$S((\nu, \alpha_2, \cdots, \alpha_n), \boldsymbol{\beta}) \overset{\text{def}}{=} h\left((\nu, \alpha_2, \cdots, \alpha_n), \boldsymbol{\beta}\right) - \sum_{i,j=2}^{n} u_{ij}(\boldsymbol{\beta})(\alpha_i - \beta_i)(\alpha_j - \beta_j)$$

for $\boldsymbol{\beta} \in \mathcal{K} \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_1 = \nu\}$. Finally, there are explicitly computable functions $C_{i,j,l} \in \mathcal{C}^\infty(\mathrm{Int}(P))$ such that

$$-\frac{B_{2i}}{(2i)!}\int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial \alpha_1}^{2i-1} \left(\psi(\boldsymbol{\alpha}) b_j(\boldsymbol{\alpha}) e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})}\right)\Bigg|_{((\nu,\alpha_2,\cdots,\alpha_n),\boldsymbol{\beta})} d\alpha_2 \cdots d\alpha_n = \sum_{l=0}^{p-(i+j)-1} C_{i,j,l}(\boldsymbol{\beta}) k^{i-\frac{n+1}{2}-l} + \mathcal{O}\left(k^{2i+j-p-\frac{n+1}{2}}\right).$$

To see that the integral above is of order $k^{i-\frac{n+1}{2}}$, we employ reasoning just like in lemma 6.2.2. The full asymptotics can be computed by first expanding the α_1-derivatives and then applying Laplace's expansion for each of the resulting terms. We recall from equation 7.3.3 that, for $p \in \mathbb{N}_0$,

$$\rho_k(\boldsymbol{\beta}) = \sum_{j=0}^{p} k^{\frac{3n}{2}-j} \int_{\mathbb{R}^n} \psi(\boldsymbol{\alpha}) b_j(\boldsymbol{\alpha}) e^{-kh(\boldsymbol{\alpha},\boldsymbol{\beta})} d\boldsymbol{\alpha} + \mathcal{O}\left(k^{n-\left(p+\frac{1}{2}\right)}\right) \tag{7.4.1}$$

for $\boldsymbol{\beta} \in \mathcal{K}$. An application of lemma B.3.6 now yields that

$$\begin{aligned}
\rho_{F,s,k}(\boldsymbol{\beta}) - \frac{1}{2}\rho_k(\boldsymbol{\beta}) = & \sum_{j=0}^{p} k^{\frac{3n}{2}-j} \left(\sum_{l=0}^{p-j-1} A_{j,2l+1}(\boldsymbol{\beta}) k^{-\left(\frac{n+1}{2}+l\right)} \right. \\
& \left. + k^{-1} \sum_{l=0}^{p-j-1} D_{j,l}(\boldsymbol{\beta}) k^{\left(\frac{n-1}{2}-l\right)} + \sum_{i=1}^{p-j} k^{-2i} \sum_{l=0}^{p-(i+j)-1} C_{i,j,l}(\boldsymbol{\beta}) k^{i-l-\frac{n+1}{2}} \right) \\
& + \mathcal{O}\left(k^{n-\left(p+\frac{1}{2}\right)}\right) \\
= k^n & \left(\sum_{j=0}^{p} \sum_{l=0}^{p-j-1} \left(A_{j,2l+1}(\boldsymbol{\beta}) + D_{j,l}(\boldsymbol{\beta})\right) k^{-\left(l+j+\frac{1}{2}\right)} \right. \\
& \left. + \sum_{j=0}^{p} \sum_{i=1}^{p-j} \sum_{l=0}^{p-i-j-1} C_{i,j,l}(\boldsymbol{\beta}) k^{-\left(i+j+l+\frac{1}{2}\right)} \right) + \mathcal{O}\left(k^{n-\left(p+\frac{1}{2}\right)}\right)
\end{aligned}$$

for $\boldsymbol{\beta} \in \mathcal{K} \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_1 = \nu\}$. Here, we use the convention that $\sum_{i=a}^{b} f(i) = 0$ if $a > b$. Note that, in particular,

$$\rho_{F,s,k}(\boldsymbol{\beta}) - \frac{1}{2}\rho_k(\boldsymbol{\beta}) = \left(A_{0,1}(\boldsymbol{\beta}) + D_{0,0}(\boldsymbol{\beta})\right) k^{n-\frac{1}{2}} + \mathcal{O}\left(k^{n-\frac{3}{2}}\right)$$

for $\boldsymbol{\beta} \in \mathcal{K} \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_1 = \nu\}$. □

Remark 7.4.3. *Recall that e.g.* $\rho_{F,s,k}(\boldsymbol{\beta}) = \rho_k(\boldsymbol{\beta}) + \mathcal{O}(k^{-\infty})$ *if* $\boldsymbol{\beta} \in \mathrm{Int}(P_{F,s})$ *in the above theorem means that, for fixed* $\boldsymbol{\beta} \in \mathrm{Int}(P_{F,s})$ *and for any* $n > 0$*, there exists* $C_n \geq 0$ *such that*

$|\rho_{F,s,k}(\boldsymbol{\beta}) - \rho_k(\boldsymbol{\beta})| \leq Ck^{-n}$ for all $k \gg 0$. But it is clear from the proof that for any compact subset $\mathcal{K} \subset \mathrm{Int}(P_{F,s})$ and $n > 0$, we can in fact take a fixed $C_n \geq 0$ for all $\boldsymbol{\beta} \in \mathcal{K}$ and $k \gg 0$ in the estimate. A similar statement holds over $\mathrm{Int}(P - P_{F,s})$.

7.4.3 The $k^{n-\frac{1}{2}}$ term

We will now investigate the $k^{n-\frac{1}{2}}$ term of the asymptotic expansion of $\rho_{F,s,k}$ in more detail. Using the basic integrals that we compute in A.2.1 and A.2.3, we find that, for $\boldsymbol{\beta} \in \mathcal{K} \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_1 = \nu\}$,

$$\begin{aligned} A_{0,1}(\boldsymbol{\beta}) = {} & |H(\boldsymbol{\beta})|^{-\frac{1}{2}} \Bigg(2^{\frac{n+1}{2}} \sum_{s=1}^{n} \frac{\partial}{\partial \alpha_s}\bigg|_{\mathbf{0}} b_0\left(H^{-\frac{1}{2}}(\boldsymbol{\beta})\boldsymbol{\alpha} + \boldsymbol{\beta}\right) e\left(\boldsymbol{e}_s, H^{-\frac{1}{2}}(\boldsymbol{\beta})\boldsymbol{e}_1, 0\right) \\ & - 2^{\frac{n+3}{2}} \sum_{r,s,l=1}^{n} \frac{1}{3!} \frac{\partial^3}{\partial \alpha_r \partial \alpha_s \partial \alpha_l}\bigg|_{\mathbf{0}} R\left(H(\boldsymbol{\beta})^{-\frac{1}{2}}\boldsymbol{\alpha} + \boldsymbol{\beta}\right) b_0\left(H^{-\frac{1}{2}}(\boldsymbol{\beta})\boldsymbol{\alpha} + \boldsymbol{\beta}\right) \\ & e\left(\boldsymbol{e}_r + \boldsymbol{e}_s + \boldsymbol{e}_l, H^{-\frac{1}{2}}(\boldsymbol{\beta})\boldsymbol{e}_1, 0\right)\Bigg) \\ = {} & \frac{1}{12(2\pi)^n\sqrt{\pi}} \frac{u^{111}(\boldsymbol{\beta})}{(u^{11}(\boldsymbol{\beta}))^{\frac{3}{2}}}, \end{aligned}$$

while

$$D_{0,0}(\boldsymbol{\beta}) = \frac{1}{2(2\pi)^n\sqrt{\pi}} \frac{1}{\sqrt{u^{11}(\boldsymbol{\beta})}}$$

for $\boldsymbol{\beta} \in \mathcal{K} \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_1 = \nu\}$. We conclude that

$$\rho_{F,s,k}(\boldsymbol{\beta}) - \frac{1}{2}\rho_k(\boldsymbol{\beta}) = \frac{1}{4(2\pi)^n\sqrt{\pi}} \left(\frac{2}{\sqrt{u^{11}(\boldsymbol{\beta})}} + \frac{1}{3}\frac{u^{111}(\boldsymbol{\beta})}{(u^{11}(\boldsymbol{\beta}))^{\frac{3}{2}}} \right) k^{n-\frac{1}{2}} + \mathcal{O}\left(k^{n-\frac{3}{2}}\right) \tag{7.4.2}$$

for $\boldsymbol{\beta} \in \mathcal{K} \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_1 = \nu\}$.

Proposition 7.4.4. *Let $(L_P, h) \to (X_P, \omega)$ be a toric polarization, let $F < P$ be a non-trivial face and let $s \in \mathbb{N}$. Let $\mathfrak{t}$ denote the Lie algebra of the real torus acting on X_P, and let $\mu : X_P \to \mathfrak{t}^*$ denote a choice of moment map that has P as its image. Suppose that $P_{F,s} = P \cap H_{n,\lambda}$, where $H_{n,\lambda} = \{\boldsymbol{\alpha} \in \mathfrak{t}^* : \langle \boldsymbol{\alpha}, \boldsymbol{n}\rangle - \lambda \geq \nu\}$, and $\boldsymbol{n} \in \mathfrak{t} \cap \mathrm{Ker}(\exp)$ is primitive. Let $K \cong S^1$ denote the circle subgroup generated by $\boldsymbol{n}$, and let N denote the vector field generated by the K-action on X_P, so that $N_p \overset{\mathrm{def}}{=} \frac{\partial}{\partial t}\exp(t\boldsymbol{n}).p\big|_{t=0}$ for $p \in X_P$. Then, for $\boldsymbol{\beta} \in \mathrm{Int}(P) \cap F_s$, we have*

$$\begin{aligned} \rho_{F,s,k}(\boldsymbol{\beta}) - \frac{1}{2}\rho_k(\boldsymbol{\beta}) = {} & \frac{1}{2(2\pi)^n\sqrt{\pi}} \left(\frac{1}{3}\left(\frac{\mathrm{div}\,(JN)_p}{\|JN_p\|} - \mathrm{div}\left(\frac{JN}{\|JN\|}\right)_p \right) + \frac{1}{\|JN_p\|} \right) k^{n-\frac{1}{2}} \\ & + \mathcal{O}\left(k^{n-\frac{3}{2}}\right) \end{aligned}$$

for any $p \in \mu^{-1}(\boldsymbol{\beta})$.

Proof. With our choice of coordinates, we have $N = \frac{\partial}{\partial \theta^1}$. We compute that

$$J\frac{\partial}{\partial \theta^1} = -\sum_{j=1}^{n} u^{1j}\frac{\partial}{\partial \alpha^j},$$

while $\|J\frac{\partial}{\partial\theta^1}\|^2 = u^{11}$. Computing the divergence is now very simple, since the metric has determinant one in symplectic coordinates. We have

$$\operatorname{div}\left(\frac{JN}{\|JN\|}\right) = -\sum_{j=1}^{n}\frac{\partial}{\partial\alpha^j}\left(\frac{u^{1j}}{\sqrt{u^{11}}}\right) = \frac{u^{1j}{}_j}{\sqrt{u^{11}}} - \frac{1}{2}\frac{u^{111}}{(u^{11})^{\frac{3}{2}}}$$

$$\operatorname{div}(JN) = -\sum_{j=1}^{n}\frac{\partial}{\partial\alpha_j}u^{1j} = u^{1j}{}_j.$$

The result now follows from equation 7.4.2. □

Remark 7.4.5. *The higher order asymptotics could be computed explicitly in a similar manner using a computer. Unfortunately, our method of expanding the partial density function is not intrinsically geometric and we only recover the geometric meaning of the $k^{n-\frac{1}{2}}$-coefficient as a last step in the above. We conjecture however that all coefficients in the expansion in theorem 7.4.2 should be geometric.*

Conjecture 7.4.6. *All the functions $c_j : \mathrm{Int}(P) \to \mathbb{R}$, for $j \in \mathbb{N}_0$, appearing in the asymptotic expansion in theorem 7.4.2 are geometric. More precisely, they are determined by the vector field N discussed above and the geometry of (X_P, ω).*

Chapter 8

The Slope Inequality

In this chapter, we discuss the *slope inequality with respect to a complex submanifold* which has important implications for the study of constant scalar curvature Kähler (cscK) metrics within a fixed Kähler class. We then describe an interesting connection between partial density functions and the slope inequality.

8.1 Background

Let $(L, h) \to (X, \omega)$ be a polarized Kähler manifold and let $Y \subset X$ be a complex submanifold of X. Let us now discuss the *slope inequality* for (X, L, Y). For $k \in \mathbb{N}$ and $l \in \mathbb{Q}$ such that $lk \in \mathbb{N}_0$, there exists an asymptotic expansion of the Hilbert-Samuel polynomial $h_{Y,l}$:

$$h_{Y,l}(k) \stackrel{\text{def}}{=} h^0\left(L^k \otimes \mathcal{J}_Y^{lk}\right) = a_0(l)k^n + a_1(l)k^{n-1} + \mathcal{O}\left(k^{n-2}\right) \quad \text{for } k \gg 0, lk \in \mathbb{N}_0.$$

We define $a_0 \stackrel{\text{def}}{=} a_0(0)$, $a_1 \stackrel{\text{def}}{=} a_1(0)$ and recall the Hirzebruch-Riemann-Roch theorem:

Theorem 8.1.1 (Hirzebruch-Riemann-Roch Theorem [Huy05, theorem 5.1.1, p.232])**.** *Let E be a holomorphic vector bundle on a compact complex manifold X. Then its Euler-Poincaré characteristic is given by*

$$\chi(X, E) = \int_X ch(E)td(X),$$

where

$$\chi(X, E) \stackrel{\text{def}}{=} \sum_{i=0}^{\dim(X)} (-1)^i h^i(X, E),$$

and $td(X)$ denotes the Todd class of X.

In the case of an ample holomorphic line bundle L, $\chi(X, L^k) = h^0(X, L^k)$ for all sufficiently large k. We hence obtain, for $k \gg 0$,

$$\begin{aligned} h^0(X, L^k) &= \int_X e^{c_1(L^k)} td(X) \\ &= \int_X \left\{ \left(1 + c_1(L^k) + \frac{c_1(L^k)^2}{2!} + \cdots + \frac{c_1(L^k)^n}{n!}\right) \right. \\ &\quad \left. \left(1 + \frac{1}{2}c_1(X) + \frac{1}{12}\left(c_1(X)^2 + c_2(X)\right) + \dots\right) \right\} \end{aligned}$$

$$= k^n \int_X \frac{c_1(L)^n}{n!} + k^{n-1} \int_X \frac{c_1(L)^{n-1} c_1(X)}{2(n-1)!} + \dots .$$

In particular,

$$\begin{aligned} a_0 &= \int_X \frac{c_1(L)^n}{n!} = \frac{1}{(2\pi)^n} \int_X \frac{\omega^n}{n!} \\ a_1 &= \int_X \frac{c_1(L)^{n-1} c_1(X)}{2(n-1)!} = \frac{1}{2} \frac{1}{(2\pi)^n} \int_X \mathrm{Scal}\, \frac{\omega^n}{n!}. \end{aligned}$$

Let us recall the following from [RT06, Tho06]:

Definition 8.1.2. The *slope* of a polarized manifold (X, L) is given by

$$\mu(X, L) \stackrel{\text{def}}{=} \frac{a_1}{a_0}.$$

For a submanifold $Y \subset X$, the *Seshadri constant* $S(Y)$ is defined as

$$S(Y) \stackrel{\text{def}}{=} \sup_l \left\{ \pi^* L \otimes \mathcal{O}(-lE) \text{ is ample} \right\},$$

where $\pi : Bl_Y(X) \to X$ denotes the blow-up of X along Y and $E = \pi^{-1}(Y)$ the exceptional divisor. The *slope of Y with respect to $c \in \mathbb{R}$* is

$$\mu_c(\mathcal{J}_Y, L) \stackrel{\text{def}}{=} \frac{\int_0^c a_1(l) + \frac{a_0'(l)}{2} dl}{\int_0^c a_0(l) dl},$$

and we say that (X, L) is *slope semi-stable with respect to Y* if

$$\mu_c(\mathcal{J}_Y, L) \leq \mu(X, L) \quad \text{for all } c \in (0, S(Y)]. \tag{8.1.1}$$

We refer to equation 8.1.1 as the *slope inequality* for (X, L, Y). We say that Y *strictly destabilizes* (X, L) if (X, L) is not slope semi-stable with respect to Y.

The following corollary gives us a geometric motivation for studying slope semi-stability:

Corollary 8.1.3 (Ross, Thomas *cf.* [Tho06, Cor 7.4] and [RT06])**.** *Suppose that $Y \subset X$ is a complex submanifold of a polarized Kähler manifold (X, L). If Y strictly destabilizes (X, L), then X does not admit a cscK metric in the class $c_1(L)$.*

Remark 8.1.4. *We only consider slope semi-stability with respect to complex submanifolds, but the notion is well-defined for subschemes [Tho06, RT06].*

8.2 Toric slope stability

We now investigate slope semi-stability for toric submanifolds of a toric polarized manifold $L_P \to X_P$ corresponding to a polytope $P \subset \mathbb{R}^n$. Guillemin and Sternberg's Euler-Maclaurin summation formula for integral Delzant polytopes will be of use to us.

Theorem 8.2.1 (Guillemin-Sternberg, [GS07, theorem 4.1])**.** *For $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and an integral*

Delzant polytope P given by

$$P \stackrel{\text{def}}{=} \Big\{ \boldsymbol{x} \in \mathbb{R}^n : l_i(\boldsymbol{x}) \stackrel{\text{def}}{=} \mu_i - \langle \boldsymbol{x}, \boldsymbol{n}_i \rangle \geq 0 \text{ for some } \mu_i \in \mathbb{R}, \text{ primitive } \boldsymbol{n}_i \in \mathbb{Z}^n \text{ and } i \in \{1, \cdots, d\} \Big\},$$

we have

$$\frac{1}{k^n} \sum_{\boldsymbol{x} \in kP \cap \mathbb{Z}^n} f\left(\frac{1}{k}\boldsymbol{x}\right) \sim \prod_{i=1}^{d} \tau\left(\frac{1}{k}\frac{\partial}{\partial \lambda_i}\right) \int_{P(\boldsymbol{\lambda})} f(\boldsymbol{x}) d\boldsymbol{x} \Bigg|_{\boldsymbol{\lambda}=\boldsymbol{0}},$$

where

$$\begin{aligned}\tau(s) &= \frac{s}{1-e^{-s}} = 1 + \frac{s}{2} + \sum_{j=1}^{\infty} (-1)^{j-1} \frac{B_{2j}}{(2j)!} s^{2j} \\ &= 1 + \frac{s}{2} + \frac{s^2}{12} - \frac{s^4}{720} + \mathcal{O}\left(s^6\right).\end{aligned}$$

Here, B_j denotes the j^{th} Bernoulli number and

$$P(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathbb{R}^n : l_i(\boldsymbol{x}) + \lambda_i \geq 0, i \in \{1, \cdots, d\}\}$$

for $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_d) \in \mathbb{R}^d$.

Remark 8.2.2. *Note that we are now using the outwards pointing primitive normal vectors in the definition of $P(\boldsymbol{\lambda})$ as opposed to 6.3.*

Example 8.2.3. Consider the Delzant polytope given by $(x, y) \in \mathbb{R}^2$ such that

$$\begin{aligned} l_1(x,y) &\stackrel{\text{def}}{=} y \geq 0 \\ l_2(x,y) &\stackrel{\text{def}}{=} 10 - x - y \geq 0 \\ l_3(x,y) &\stackrel{\text{def}}{=} 6 - y \geq 0 \\ l_4(x,y) &\stackrel{\text{def}}{=} x \geq 0. \end{aligned}$$

$$P(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \stackrel{\text{def}}{=} \left\{(x,y) \in \mathbb{R}^2 : l_i(x,y) + \lambda_i \geq 0 \text{ for } i \in \{1,2,3,4\}\right\}.$$

$P(0,0,0,0)$ and $P(0,1,2,3)$ are displayed in figure 8.1. These polytopes correspond to the blow-up of $\mathbb{CP}^2$ in a point together with two different polarizations.

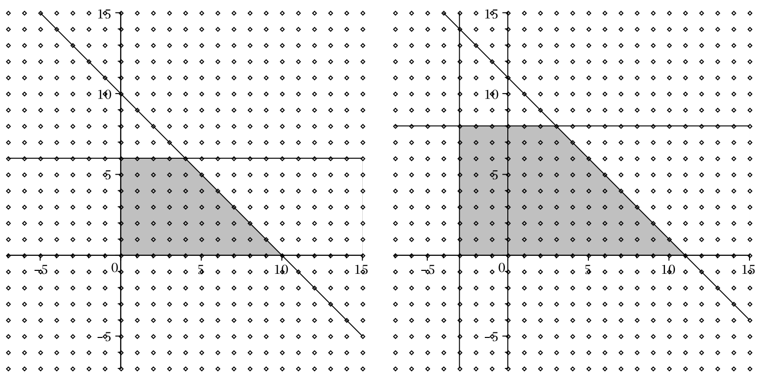

Figure 8.1: $P(0,0,0,0)$ and $P(0,1,2,3)$ from example 8.2.3.

We can expand $h^0(L_P^k)$ in powers of k as follows:

$$h^0(L_P^k) = \#\left(kP \cap \mathbb{Z}^n\right) = \sum_{\boldsymbol{x} \in kP \cap \mathbb{Z}^n} 1 = \mathrm{Vol}(P)k^n + \frac{1}{2}\sum_{i=1}^{d} \frac{\partial}{\partial \lambda_i} \int_{P(\boldsymbol{\lambda})} d\boldsymbol{x} \bigg|_{\boldsymbol{\lambda}=\boldsymbol{0}} k^{n-1} + \mathcal{O}\left(k^{n-2}\right).$$

Following an idea of Donaldson [Don02], we introduce a constant $(n-1)$-form $d\sigma_i$, for $i \in \{1, \cdots, d\}$, such that

$$\mathrm{dVol}_{Eucl} = d\sigma_i \wedge dl_i,$$

where dVol_{Eucl} denotes the Euclidean volume form on $\mathbb{R}^n$. Suppose that the polytope P is given by

$$P \overset{\text{def}}{=} \Big\{ \boldsymbol{x} \in \mathbb{R}^n : l_i(\boldsymbol{x}) \overset{\text{def}}{=} \mu_i - \langle \boldsymbol{x}, \boldsymbol{n}_i \rangle \geq 0 \text{ for some } \mu_i \in \mathbb{R}, \text{ primitive } \boldsymbol{n}_i \in \mathbb{Z}^n \text{ and } i \in \{1, \cdots, d\} \Big\}.$$

Now

$$\begin{aligned} \frac{\partial}{\partial \lambda_i} \int_{P(0,\cdots,0,\underset{i^{th}}{\lambda_i},0,\cdots,0)} \mathrm{dVol}_{Eucl} \Bigg|_{\lambda_i=0} &= \frac{\partial}{\partial \lambda_i} \int_{l_i=-\lambda_i}^{const(P,i)} \left(\int_{\substack{l_i(\boldsymbol{x})=l_i \\ l_j(\boldsymbol{x})\geq 0 \text{ for } j\neq i}} d\sigma_i \right) dl_i \Bigg|_{\lambda_i=0} \\ &= \int_{\substack{l_i(\boldsymbol{x})=0 \\ l_j(\boldsymbol{x})\geq 0 \text{ for } j\neq i}} d\sigma_i \\ &= \mathrm{Vol}_{\sigma_i}(F_i), \end{aligned}$$

where $F_i \overset{\text{def}}{=} \{\boldsymbol{x} \in P : l_i(\boldsymbol{x}) = 0\}$ is a $(n-1)$-dimensional face of P for $i \in \{1, \cdots, d\}$. We recover the well known fact that

$$\mu\left(X_P, L_P\right) = \frac{1}{2} \frac{\sum_{i=1}^{d} \mathrm{Vol}_{\sigma_i}(F_i)}{\mathrm{Vol}(P)} = \frac{1}{2}\frac{\mathrm{Vol}(\partial P)}{\mathrm{Vol}(P)},$$

where $\mathrm{Vol}(P)$ denotes the Euclidean volume of P and $\mathrm{Vol}(\partial P)$ the volume of ∂P taken with respect to the measures $d\sigma_i$ for $i \in \{1, \cdots, d\}$. Let us now apply these ideas to the slope inequality. Let Y_F be a toric submanifold of X_P of complex codimension r corresponding to a face $F < P$ of real codimension r. Without loss of generality, we assume that $F = \{\boldsymbol{x} \in P : l_1(\boldsymbol{x}) = \cdots = l_r(\boldsymbol{x}) = 0\}$. We have

$$H^0(L_P^k \otimes \mathcal{J}_{Y_F}^{lk}) = \mathrm{Span}\left(s_{\boldsymbol{\alpha},k} : \boldsymbol{\alpha} \in P \cap \frac{1}{k}\mathbb{Z}^n \text{ and } \sum_{j=1}^{r} l_j(\boldsymbol{\alpha}) \geq l \right)$$

for $k, l \in \mathbb{N}$. We define $P_l \overset{\text{def}}{=} \{\boldsymbol{x} \in P : \sum_{j=1}^{r} l_j(\boldsymbol{x}) \geq l\}$ and have

$$\begin{aligned} h^0\left(L^k \otimes \mathcal{J}_{Y_F}^{lk}\right) &= \#\left(P_l \cap \frac{1}{k}\mathbb{Z}^n\right) \\ &= \mathrm{Vol}\left(P_l\right) k^n + \frac{1}{2}\mathrm{Vol}\left(\partial P_l\right) k^{n-1} + \mathcal{O}\left(k^{n-2}\right) \\ &= a_0(l)k^n + a_1(l)k^{n-1} + \mathcal{O}\left(k^{n-2}\right), \end{aligned}$$

so that the slope inequality for F takes the form

$$\int_0^c \mathrm{Vol}\,(\partial P_l)\,dl + \mathrm{Vol}\,(P_c) - \mathrm{Vol}\,(P) \leq \frac{\mathrm{Vol}\,(\partial P)}{\mathrm{Vol}(P)} \int_0^c \mathrm{Vol}\,(P_l)\,dl \quad \text{for } c \in (0, S(Y_F)].$$

Let us now rewrite this inequality in a more geometric form. If the codimension r of F is larger than 1, then, for $0 < l < S(Y_F)$, P_l has the following $(n-1)$-dimensional faces:

$$F_0(l) = \left\{ \boldsymbol{x} \in P : l_0(\boldsymbol{x}) \stackrel{\text{def}}{=} \sum_{j=1}^{r} l_j(\boldsymbol{x}) = l \right\}$$

$$F_j(l) = \left\{ \boldsymbol{x} \in P : l_j(\boldsymbol{x}) = 0,\, l_0(\boldsymbol{x}) \stackrel{\text{def}}{=} \sum_{j=1}^{r} l_j(\boldsymbol{x}) \geq l \right\} \quad \text{for } i \in \{1, \cdots, d\}.$$

If $r = 1$, the $n-1$ dimensional faces are given by the above, except that we discard the empty set $F_1(l)$. We note that

$$\begin{aligned} \frac{\partial}{\partial l} a_0(l) &= \frac{\partial}{\partial l} \int_{P_l} \mathrm{dVol}_{Eucl} \\ &= \frac{\partial}{\partial l} \int_{l_0 = l}^{const(P_l)} \left(\int_{\substack{l_0(\boldsymbol{x}) = l_0 \\ l_j(\boldsymbol{x}) \geq 0 \text{ for } j \neq 0}} d\sigma_0 \right) dl_0 \\ &= -\,\mathrm{Vol}_{\sigma_0}\left(F_0(l)\right), \end{aligned}$$

where $\mathrm{dVol}_{Eucl} = d\sigma_0 \wedge dl_0$. The slope inequality now becomes

$$\int_0^c \frac{\mathrm{Vol}\,(\partial P_l - F_0(l))}{\mathrm{Vol}\,(\partial P)} dl \leq \int_0^c \frac{\mathrm{Vol}\,(P_l)}{\mathrm{Vol}(P)} dl.$$

We have proved:

Lemma 8.2.4. *Let (X_P, L_P) be a toric polarization and let Y_F be a toric submanifold of X_P corresponding to a face $F \subset P$ of the polytope $P = \{\boldsymbol{x} \in \mathbb{R}^n : l_i(\boldsymbol{x}) \stackrel{\text{def}}{=} \mu_i - \langle \boldsymbol{n}_i, \boldsymbol{x} \rangle \geq 0$ for $i \in \{1, \cdots, d\}\}$. Suppose that $F = \cap_{i=1}^{r} \mathcal{Z}(l_i) \cap P$ and let $P_l^+ = \{\boldsymbol{x} \in P : \sum_{i=1}^{r} l_i(\boldsymbol{x}) > l\}$. Then (X_P, L_P) is slope semi-stable with respect to Y_F if and only if*

$$\int_0^c \frac{\mathrm{Vol}\left(\partial P_l^+\right)}{\mathrm{Vol}\,(\partial P)} dl \leq \int_0^c \frac{\mathrm{Vol}\left(P_l^+\right)}{\mathrm{Vol}(P)} dl$$

for all $c \in (0, S(Y_F))$.

Proof. The proof is given in the above argument. Note that the slope inequality is continuous in c in the toric case. We can hence discard the case $c = S(Y_F)$. □

8.3 Partial density functions and slope stability

Let $(L, h) \to (X, \omega)$ be a polarized Kähler manifold. We now discuss an idea communicated to us by R. Thomas [KPTS] which shows that sufficient asymptotic information about the partial density function $\rho_{l,k}$ with respect to a complex submanifold $Y \subset X$ might provide a geometric proof of the fact that the existence of a constant scalar curvature metric in the polarization class of (X, L) forces (X, L) to be slope semi-stable with respect to Y (corollary 8.1.3). Fix a complex submanifold $Y \subset X$. We define $q_{l,k} : X \to \mathbb{R}$ by

$$q_{l,k}(p) \stackrel{\text{def}}{=} \rho_{l,k}(p) - \frac{1}{2}\left(\rho_{l,k}(p) - \rho_{l+\frac{1}{k},k}(p)\right),$$

for $p \in X$, $k \in \mathbb{N}$, $l \in (0, S(Y)] \cap \frac{1}{k}\mathbb{Z}$ and where $\rho_{l,k} : X \to \mathbb{R}$ denotes the partial density function with vanishing along Y to order at least lk. Now

$$\begin{aligned}
\int_X \sum_{l=0,\frac{1}{k},\cdots,\frac{ck-1}{k}} q_{l,k}\frac{\omega^n}{n!} &= \sum_{l=0,\frac{1}{k},\cdots,\frac{ck-1}{k}} \Big\{h^0\left(L^k \otimes \mathcal{J}_Y^{lk}\right) \\
&\quad - \frac{1}{2}\left(h^0\left(L^k \otimes \mathcal{J}_Y^{lk}\right) - h^0\left(L^k \otimes \mathcal{J}_Y^{lk+1}\right)\right)\Big\} \\
&= \sum_{l=0,\frac{1}{k},\cdots,\frac{ck-1}{k}} h^0\left(L^k \otimes \mathcal{J}_Y^{lk}\right) - \frac{1}{2}\left(h^0\left(L^k\right) - h^0\left(L^k \otimes \mathcal{J}_Y^{ck}\right)\right) \\
&= \sum_{l=0,\frac{1}{k},\cdots,c} h^0\left(L^k \otimes \mathcal{J}_Y^{lk}\right) - \frac{1}{2}\left(h^0\left(L^k\right) + h^0\left(L^k \otimes \mathcal{J}_Y^{ck}\right)\right) \\
&= \Bigg\{\sum_{l=0,\frac{1}{k},\cdots,c} \left(a_0(l)k^n + a_1(l)k^{n-1}\right) \\
&\quad - \frac{1}{2}\left(\left(a_0 + a_0(c)\right)k^n + \left(a_1 + a_1(c)\right)k^{n-1}\right)\Bigg\} + \mathcal{O}\left(k^{n-1}\right) \\
&= k^{n+1}\int_0^c a_0(l)dl + k^n\frac{a_0(c)+a_0}{2} + k^n\int_0^c a_1(l)dl \\
&\quad - k^n\frac{a_0(c)+a_0}{2} + \mathcal{O}\left(k^{n-1}\right) \\
&= k^{n+1}\int_0^c a_0(l)dl + k^n\int_0^c a_1(l)dl + \mathcal{O}\left(k^{n-1}\right).
\end{aligned} \tag{$\dagger$}$$

Let us now assume that, as distributions over X, we have the following asymptotic expansion

$$q_{l,k} = k^n\frac{1}{(2\pi)^n}\mathbf{1}_{N_l} + k^{n-1}\left(\mathbf{1}_{N_l}\frac{\text{Scal}}{2(2\pi)^n} + f_l\right) + \mathcal{O}\left(k^{n-2}\right) \tag{$\ddagger$}$$

for some function $f_l : X \to \mathbb{R}$ depending on the geometry of X and Y, a subset $N_l \subset X$ and $0 < l \leq S(Y_F)$, and where the constant in $\mathcal{O}$ is independent of $l \in (0, S(Y_F)]$. In particular, we then have

$$\begin{aligned}
\int_X q_{l,k}\frac{\omega^n}{n!} &= k^n\int_X \frac{\mathbf{1}_{N_l}}{(2\pi)^n}\frac{\omega^n}{n!} + k^{n-1}\int_X \left(\mathbf{1}_{N_l}\frac{\text{Scal}(p)}{2(2\pi)^n} + f_l(p)\right)\frac{\omega^n}{n!} + \mathcal{O}\left(k^{n-2}\right) \\
&= k^n\frac{\text{Vol}(N_l)}{(2\pi)^n} + k^{n-1}\int_X \left(\mathbf{1}_{N_l}\frac{\text{Scal}(p)}{2(2\pi)^n} + f_l(p)\right)\frac{\omega^n}{n!} + \mathcal{O}\left(k^{n-2}\right),
\end{aligned}$$

where the constant in $\mathcal{O}$ is again independent of l and $\text{Vol}(N_l) \stackrel{\text{def}}{=} \int_X \mathbf{1}_{N_l}\frac{\omega^n}{n!}$ for our (possibly singular) subset $N_l \subset X$. Then

$$(\dagger) = \sum_{l=0,\frac{1}{k},\cdots,\frac{ck-1}{k}} \left(k^n\frac{\text{Vol}(N_l)}{(2\pi)^n} + k^{n-1}\int_{N_l}\frac{\text{Scal}}{2(2\pi)^n}\frac{\omega^n}{n!} + k^{n-1}\int_X f_l(p)\frac{\omega_p^n}{n!}\right) + \mathcal{O}\left(k^{n-1}\right).$$

Comparing the first term of the asymptotic expansions for $\int_X q_{l,k} \frac{\omega^n}{n!}$ yields $\frac{\mathrm{Vol}(N_l)}{(2\pi)^n} = a_0(l)$. Let us assume now that the scalar curvature is constant. In this case we have $\frac{\mathrm{Scal}(p)}{2} = \frac{a_1}{a_0}$.

$$(\dagger) = k^{n+1} \int_0^c a_0(l)dl + k^n \frac{a_0 - a_0(c)}{2} + k^n \frac{a_1}{a_0} \int_0^c a_0(l)dl + k^{n-1} \sum_{l=0,\frac{1}{k},\cdots,\frac{ck-1}{k}} \int_X f_l(p) \frac{\omega_p^n}{n!}$$
$$+ \mathcal{O}\left(k^{n-1}\right).$$

Subtracting the right hand side above from $(\dagger)$ and dividing by k^n gives

$$\int_0^c a_1(l) + \frac{a_0'(l)}{2} dl - \frac{a_1}{a_0} \int_0^c a_0(l)dl - k^{-1} \sum_{l=0,\frac{1}{k},\cdots,\frac{ck-1}{k}} \int_X f_l(p) \frac{\omega_p^n}{n!} = \mathcal{O}\left(k^{-1}\right).$$

Note that the first part above is the slope inequality which does not depend on k. It follows that *e.g.*

$$k^{-1} \sum_{l=0,\frac{1}{k},\cdots,\frac{ck-1}{k}} \int_X f_l(p) \frac{\omega_p^n}{n!} \leq 0,$$

for $k \gg 0$, would imply the slope inequality with respect to Y.

Remark 8.3.1. *The above calculation reveals some aspects of the deep relationship between the asymptotic expansion of the partial density function $\rho_{l,k}$ and the notion of slope stability with respect to a complex submanifold discussed in this chapter. In particular, we see how sufficient information about the asymptotics of $\rho_{l,k}$ could lead to an alternative proof of corollary 8.1.3. While we have concentrated on the pointwise asymptotics of $\rho_{l,k}$ in the toric case in this thesis, it seems that a future investigation into asymptotic expansions in the sense of equation (‡) might be worthwhile as well.*

Chapter 9

General Polarized Kähler Manifolds

In this chapter, we concentrate on some of the problems that appear when one is trying to understand the asymptotics of the (partial) density function in the case of a general compact polarized Kähler manifold $(L,h) \to (X,\omega)$.

9.1 Special sections

Let $(L,h) \to (X,\omega)$ be a polarization of a compact Kähler manifold (X,ω), so that $\pi : L \to X$ is a holomorphic line bundle with Hermitian metric h such that $iF_h = \omega \in 2\pi c_1(L)$. Let us denote the Hermitian fibre-wise inner product and norm given by h^k on L^k_q by $(s(q), s'(q))_{h^k}$ and $|s(q)|_{h^k}$, respectively, for all $s, s' \in H^0(X, L^k)$ and $q \in X$. We denote the $\mathcal{L}^2$-inner product and norm by $\langle s, s'\rangle_{h^k} \stackrel{\text{def}}{=} \int_X (s,s')_{h^k} \frac{\omega^n}{n!}$ and $\|s\|_{h^k}$, respectively, for all $s, s' \in H^0(X, L^k)$. We will omit the h^k index if it is clear from the context which power of the line bundle we are considering.

9.1.1 Tian's peak sections

The fact that we have a very explicit basis of $H^0(X, L^k)$ is one of the main advantages of the toric case. Recall that each of these basis elements $s_{\boldsymbol{\alpha},k}$ corresponded to an integral point $\boldsymbol{\alpha} \in P \cap \frac{1}{k}\mathbb{Z}^n$ of a polytope P and $s_{\boldsymbol{\alpha},k}$ had the nice property of having "peaked" pointwise norm on the torus $\mu^{-1}(\boldsymbol{\alpha})$ which enabled us to apply Laplace's method and the Euler-Maclaurin summation formulas to extract asymptotic information in chapter 7. In general, there is no such "preferred" basis of $H^0(X, L^k)$, but there are interesting types of sections which we will refer to as *Tian's peak sections* and which do have similar properties. Let us first discuss a type of holomorphic normal coordinate system that is important in this context (see [Boc47, Rua98] for more details).

Proposition 9.1.1 (Bochner). *Let X be a compact Kähler manifold and $M \geq 0$. For any $x \in X$, there exists a holomorphic coordinate map $\boldsymbol{z} = \phi_x$ centred at x for which there exist a real-valued Kähler potential K around x such that all the $(0,l), (1,l), (l,1), (l,0)$ terms in the Taylor expansion of K vanish at x for $l \leq M$, except for the $(1,1)$ term which is equal to $\|\boldsymbol{z}\|^2$. These coordinates are called K-*coordinates of order M centred at x.

Note that, in K-coordinates of order $M \geq 4$ centred at x, we have

$$K(\boldsymbol{z}) = \|\boldsymbol{z}\|^2 + R_{i\bar{j}k\bar{l}} z_i \bar{z}_j z_k \bar{z}_l + \mathcal{O}\left(\|\boldsymbol{z}\|^5\right),$$

where $R_{i\bar{j}k\bar{l}}$ denote the components of the curvature tensor at x.

Definition 9.1.2. Let $(L, h) \to (X, \omega)$ be a polarized compact Kähler manifold and let $x_0 \in X$. Choose local K-coordinates of order M centred at x_0. Pick a local holomorphic frame e_L of L at x_0 such that the local function $a : \mathcal{U} \to \mathbb{R}$ representing h satisfies $a(\boldsymbol{z}) = \exp(-K(\boldsymbol{z}))$ for all $\boldsymbol{z} \in \mathcal{U}$. We call such a local frame *adapted* to the K-coordinates.

Theorem 9.1.3 (Tian [Tia90], see also [Rua98, Lemma 3.1])**.** *Let $(L, h) \to (X, \omega)$ be a polarized compact Kähler manifold and let $x_0 \in X$. Choose local K-coordinates of order 4 centred at x_0 and an adapted frame of L.*

For any n-tuple of integers $\boldsymbol{p} = (p_1, \cdots, p_n) \in \mathbb{N}_0^n$ and an integer $p' > |\boldsymbol{p}| \stackrel{\text{def}}{=} p_1 + \cdots + p_n$, there exists $k_0 > 0$ such that, for $k > k_0$, there exists a holomorphic global section $s_{\boldsymbol{p},p',k} \in H^0(X, L^k)$, satisfying $\|s_{\boldsymbol{p},p',k}\|_{h^k}^2 = 1$ and

$$\int_{X - \left\{\|\boldsymbol{z}\| \leq \frac{\log k}{\sqrt{k}}\right\}} |s_{\boldsymbol{p},p',k}|_{h^k}^2 \frac{\omega^n}{n!} = \mathcal{O}\left(k^{-2p'}\right),$$

and, locally at x_0,

$$s_{\boldsymbol{p},p',k}(\boldsymbol{z}) = \lambda_{\boldsymbol{p}} \left(\boldsymbol{z}^{\boldsymbol{p}} + \mathcal{O}\left(\|\boldsymbol{z}\|^{2p'}\right)\right) e_L^{\otimes k} \left(1 + \mathcal{O}\left(k^{-2p'}\right)\right),$$

where the constant in $\mathcal{O}$ depends only on p' and the geometry of X. Moreover,

$$\lambda_{\boldsymbol{p}}^{-2} = \int_{\|\boldsymbol{z}\| \leq \frac{\log k}{\sqrt{k}}} |\boldsymbol{z}^{\boldsymbol{p}}|^2 \, a^k \frac{\omega^n}{n!},$$

where a^k is the function representing the Hermitian metric h^k in the local K-coordinates.

We refer to the sections in theorem 9.1.3 as "peak sections" since their norm is concentrated near a point. Such sections have been utilised to prove several interesting results. Tian originally used them in his paper [Tia90] to prove a version of theorem 10.1.1 which we discuss in chapter 10.

Recall that, on a polarized Kähler manifold $(L, h) \to (X, \omega)$, the density function $\rho_k : X \to \mathbb{R}$ is defined to be the norm of the diagonal of the Bergman kernel B_k on L^k. Let $s_{1,k}, \cdots, s_{N_k,k}$ be a orthonormal basis of $H^0(X, L^k)$ for $k \in \mathbb{N}$. Then

$$\rho_k(x) = \sum_{i=1}^{N_k} |s_{i,k}(x)|_{h^k}^2 \qquad \text{for } x \in X \text{ and } k \in \mathbb{N}.$$

Theorem 9.1.4 (Catlin [Cat99], Tian [Tia90], Yau, Zelditch [Zel98])**.** *Let $(L, h) \to (X, \omega)$ be a polarized Kähler manifold. There is a complete asymptotic expansion*

$$\rho_k(x) \sim \sum_{j=0}^{\infty} a_j(x) k^{n-j} \qquad \textit{for } x \in X \textit{ and as } k \to \infty,$$

for certain smooth functions $\{a_j\}_{j=0}^{\infty}$ on X. More precisely, for any $R, r \in \mathbb{N}$, there exists a *constant $C_{R,r} \geq 0$, depending on R, r and the manifold (X, ω), such that*

$$\left| \rho_k(x) - \sum_{j<R} a_j(x) k^{n-j} \right|_{\mathcal{C}^r(X)} \leq C_{R,r} k^{n-R} \qquad \text{for all } k \in \mathbb{N}_0 \text{ and } x \in X.$$

Peaked sections were subsequently used by Lu to explicitly determine a_0 up to a_3. In particular:

Theorem 9.1.5 (Lu [Lu00]). *We have $a_0(x) = \frac{1}{(2\pi)^n}$ and $a_1(x) = \frac{1}{2} \frac{1}{(2\pi)^n} \operatorname{Scal}(x)$ in the expansion above.*

9.1.2 Subspaces of $H^0(X, L^k)$

Let $(L, h) \to (X, \omega)$ denote a polarized compact Kähler manifold as before. Tian proved the existence of "peak sections" of L^k, for $k \gg 0$, using Hörmander's $\overline{\partial}$-estimates. We will now discuss how to think of these sections as elements of a natural subspace of $H^0(X, L^k)$.

Let us denote the space of global holomorphic sections of L^k vanishing to order at least l at $p \in X$ by $\mathcal{J}_k^l(p)$. Similarly, for a complex submanifold $Y \subset X$, we denote by $\mathcal{J}_k^l(Y)$ the global sections of L^k that vanish to order at least l along Y. Given a polarization, we have a decomposition of the vector space $H^0(X, L^k)$, for any $l \in \mathbb{N}$ and $p \in X$, as a direct sum of mutually orthogonal subspaces.

$$\begin{aligned} H^0(X, L^k) &= \mathcal{J}_k^l(p) \oplus \mathcal{J}_k^l(p)^\perp \\ &= \mathcal{J}_k^l(p) \oplus \left(\mathcal{J}_k^{l-1}(p) \cap \mathcal{J}_k^l(p)^\perp \right) \oplus \mathcal{J}_k^{l-1}(p)^\perp \\ &= \mathcal{J}_k^l(p) \oplus \left\{ \bigoplus_{j=1}^{l} \left(\mathcal{J}_k^{j-1}(p) \cap \mathcal{J}_k^j(p)^\perp \right) \right\}. \end{aligned}$$

Note that, for l large enough, $\mathcal{J}_k^l(p) = \{0\}$ and that $\mathcal{J}_k^0(p) \stackrel{\text{def}}{=} H^0(X, L^k)$.

Definition 9.1.6. Let $(L, h) \to (X, \omega)$ be a polarized compact Kähler manifold. We call a sequence of unit norm sections $\{s_k\}_{k=1}^\infty$, where $s_k \in H^0(X, L^k)$ for $k \in \mathbb{N}$, peaked at $p \in X$ if, in local K-coordinates of order 4 centred at p, we have

$$\lim_{k\to\infty} \left| \int_{X - \{\|\boldsymbol{z}\| \leq \frac{\log k}{\sqrt{k}}\}} |s_k|_{h^k}^2 \frac{\omega^n}{n!} \right| = 0.$$

We would like to argue - and this is already hinted at in Tian's lemma 9.1.7 below - that sequences of unit norm elements $s_k \in \mathcal{J}_k^{j-1}(p) \cap \mathcal{J}_k^j(p)^\perp$ as $k \to \infty$, for some fixed $j \in \mathbb{N}$, are a natural setting for discussing a version of Tian's peak sections at p. The fact that sequences of unit norm generators of the one-dimensional vector spaces $\mathcal{J}_k(p)^\perp$, for $k \gg 0$, are peaked in the above sense follows *e.g.* from [MM07, 5.1.25, p.217]. We will now prove a little generalization of this result which follows this intuition. First, let us recall:

Lemma 9.1.7 (see [Tia90, lemma 3.1]). *Let $(L, h) \to (X, \omega)$ be a polarized Kähler manifold and let $s_{\boldsymbol{p},p',k}$ be one of Tian's peak sections, which is peaked at $x_0 \in X$ for $\boldsymbol{p} \in \mathbb{N}_0^n$, $p', k \in \mathbb{N}$ and $p' > |\boldsymbol{p}|$. Let $s \in H^0(X, L^k)$ with $\|s\| = 1$ such that s does not contain $\boldsymbol{z}^p$ in its Taylor* *expansion in adapted coordinates at x_0. Then*

$$\langle s_{\boldsymbol{p},p',k}, s\rangle = \mathcal{O}\left(k^{-1}\right),$$

where the constant in $\mathcal{O}$ depends only on $\boldsymbol{p}, p'$ and the geometry of (X,ω).

The next result illustrates how the space $\mathcal{J}_k^l(p)^\perp$ can be considered as a vector space of peaked sections following Tian's ideas [Tia90].

Proposition 9.1.8. *Suppose that $(L,h) \to (X,\omega)$ is a polarized Kähler manifold. Let $l \in \mathbb{N}$, $x_0 \in X$ and let $\{s_k\}_{k=1}^\infty$ be a sequence such that $s_k \in \mathcal{J}_k^l(p)^\perp \subset H^0(X,L^k)$ and $\|s_k\|_{h^k} = 1$ for all $k \in \mathbb{N}$. Then there exists a constant $C \geq 0$ such that, in local K-coordinates of order 4 centred x_0, we have*

$$\left| \int_{X - \left\{\|\boldsymbol{z}\| \leq \frac{\log(k)}{\sqrt{k}}\right\}} |s_k|^2_{h^k} \frac{\omega^n}{n!} \right| \leq C k^{-1} \quad \textit{for all } k \in \mathbb{N}.$$

In particular, $\{s_k\}_{k=1}^\infty$ is peaked at x_0.

Proof. Let $p' > l$. Choose $k_0 > 0$ such that, for all $k > k_0$ and $\boldsymbol{p} \in \mathbb{N}_0^n$ such that $|\boldsymbol{p}| < l$, Tian's section $s_{\boldsymbol{p},p',k} \in H^0(X,L^k)$ with parameters $\boldsymbol{p}, p'$ and k and peaked norm at x_0, exists. We define the following vector spaces for $k \geq k_0$:

$$\begin{aligned} H_k &\stackrel{\text{def}}{=} H^0(X,L^k) \\ T_k &\stackrel{\text{def}}{=} \operatorname{Span}\left(s_{\boldsymbol{p},p',k} : \boldsymbol{p} \in \mathbb{N}_0^n, |\boldsymbol{p}| < l, s_{\boldsymbol{p},p',k} \in H^0(X,L^k)\right) \\ J_k &\stackrel{\text{def}}{=} \mathcal{J}_k^l(x_0). \end{aligned}$$

Furthermore, let $n_k \stackrel{\text{def}}{=} \dim H^0(X,L^k)$ and $m \stackrel{\text{def}}{=} \dim T_k$ for $k > k_0$. We observe that

$$H_k = T_k \oplus J_k = J_k^\perp \oplus J_k.$$

We let

$$W_k \stackrel{\text{def}}{=} \left(T_k^\perp \cap J_k\right)^\perp \cap J_k = \left(T_k + J_k^\perp\right) \cap J_k$$

and note that $v_k \stackrel{\text{def}}{=} \dim W_k \leq 2m$ for all $k \in \mathbb{N}$. We have an orthogonal decomposition

$$H_k = J_k^\perp \oplus W_k \oplus \left(T_k^\perp \cap J_k\right)$$

and an alternative decomposition

$$H_k = T_k \oplus W_k \oplus \left(T_k^\perp \cap J_k\right)$$

and observe that $T_k \subset J_k^\perp \oplus W_k = \left(T_k^\perp \cap J_k\right)^\perp$. We pick, for each k, an orthonormal ordered basis $(w_{1,k}, \cdots, w_{v_k,k})$ of W_k, $(t_{1,k}, \cdots, t_{m,k})$ of T_k, $(j_{1,k}, \cdots, j_{n_k-m-v_k,k})$ of $T_k^\perp \cap J_k$ and $(f_{1,k}, \cdots, f_{m,k})$ of $J_k^\perp$. Observe that

$$(e_{1,k}, \cdots, e_{n_k,k}) \stackrel{\text{def}}{=} (f_{1,k}, \cdots, f_{m,k}, w_{1,k}, \cdots, w_{v_k,k}, j_{1,k}, \cdots, j_{n_k-m-v_k,k})$$

forms an ordered orthonormal basis of H_k. We have a second basis of H_k:

$$(e'_{1,k}, \cdots, e'_{n_k,k}) \stackrel{\text{def}}{=} (t_{1,k}, \cdots, t_{m,k}, w_{1,k}, \cdots, w_{v_k,k}, j_{1,k}, \cdots, j_{n_k - v_k - m,k}).$$

Now

$$t_{i,k} = \sum_{j=1}^{m} \langle t_{i,k}, f_{j,k} \rangle f_{j,k} + \sum_{j=1}^{v_k} \langle t_{i,k}, w_{j,k} \rangle w_{j,k}.$$

Collecting the change of basis coefficients $e'_{i,k} = \phi_{ij,k} e_{j,k}$ in a matrix ϕ_k, we have

$$\phi_k = \left(\begin{array}{cc|c} A_k & B_k & \multirow{2}{*}{0} \\ 0 & \mathbb{I}_{v_k} & \\ \hline \multicolumn{2}{c|}{0} & \mathbb{I}_{n_k - m} \end{array} \right),$$

where $(A_k)_{ij} \stackrel{\text{def}}{=} \langle t_{i,k}, f_{j,k} \rangle$ and $(B_k)_{ij} \stackrel{\text{def}}{=} \langle t_{i,k}, w_{j,k} \rangle$ with

$$\phi_k^{-1} = \left(\begin{array}{cc|c} A_k^{-1} & -A_k^{-1} B_k & \multirow{2}{*}{0} \\ 0 & \mathbb{I}_{v_k} & \\ \hline \multicolumn{2}{c|}{0} & \mathbb{I}_{n_k - m} \end{array} \right).$$

Claim. A_k is an asymptotically unitary matrix in the sense that

$$A_k \overline{A}_k^t = \mathbb{I}_m + \mathcal{O}\left(k^{-1}\right),$$

where $\mathcal{O}\left(k^{-1}\right)$ denotes a matrix all of whose entries are $\mathcal{O}\left(k^{-1}\right)$.

Proof of the claim.

$$\begin{aligned} \langle t_{i,k}, t_{j,k} \rangle &= \left\langle \sum_{l=1}^{m} (A_k)_{il} f_{l,k} + \sum_{l=1}^{v_k} (B_k)_{il} w_{l,k}, \sum_{r=1}^{m} (A_k)_{jr} f_{r,k} + \sum_{r=1}^{v_k} (B_k)_{jr} w_{r,k} \right\rangle \\ &= \sum_{l=1}^{m} (A_k)_{il} (\overline{A}_k)_{jl} + \sum_{l=1}^{v_k} (B_k)_{il} (\overline{B}_k)_{jl}, \end{aligned}$$

but $(B_k)_{ij} = \mathcal{O}\left(\frac{1}{k}\right)$ and $\langle t_{i,k}, t_{j,k} \rangle = \delta_{ij} + \mathcal{O}\left(\frac{1}{k}\right)$ by lemma 9.1.7. Noting also that $v_k \leq 2m$, for all $k \in \mathbb{N}$, gives the result. □

The above claim and the fact that $(B_k)_{ij} = \mathcal{O}\left(\frac{1}{k}\right)$ now give that

$$f_{i,k} = \sum_{j=1}^{m} (A_k^{-1})_{ij} t_{j,k} - \sum_{j=1}^{v_k} \underbrace{(A_k^{-1} B_k)_{ij}}_{\mathcal{O}\left(\frac{1}{k}\right)} w_{j,k}.$$

Let us rewrite this as

$$f_{i,k} = \sum_{j=1}^{m} (A_k^{-1})_{ij} t_{j,k} + \eta_{i,k},$$

where $\|\eta_{i,k}\| = \mathcal{O}\left(\frac{1}{k}\right)$. Let $\mathcal{B}_k \stackrel{\text{def}}{=} \{\boldsymbol{z} \in X : \|\boldsymbol{z}\| \leq \frac{\log(k)}{\sqrt{k}}\}$ in local K-coordinates of order 4 centred at x_0. Point-wise, we have, for $q \in X$,

$$
\begin{aligned}
|f_{i,k}(q)|^2 =& (f_{i,k}(q), f_{i,k}(q)) \\
=& \left(\sum_{j=1}^{m} (A_k^{-1})_{ij} t_{j,k}(q) + \eta_{i,k}(q), \sum_{l=1}^{m} (A_k^{-1})_{il} t_{l,k}(q) + \eta_{i,k}(q) \right) \\
=& \sum_{j,l=1}^{m} (A_k^{-1})_{ij} (\overline{A}_k^{-1})_{il} \left(t_{j,k}(q), t_{l,k}(q)\right) + \left(\eta_{i,k}(q), \eta_{i,k}(q)\right) + \sum_{j=1}^{m} \Bigg((A_k^{-1})_{ij} \left(t_{j,k}(q), \eta_{i,k}(q)\right) \\
&+ ((\overline{A}_k)^{-1})_{ij} \left(\eta_{i,k}(q), t_{j,k}(q)\right) \Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\mathcal{B}_k} (t_{j,k}, \eta_{i,k}) \frac{\omega^n}{n!} &\leq \int_{\mathcal{B}_k} |(t_{j,k}, \eta_{i,k})| \frac{\omega^n}{n!} \\
&\leq \int_{\mathcal{B}_k} |t_{j,k}| \, |\eta_{i,k}| \frac{\omega^n}{n!} \\
&\leq \sqrt{\int_{\mathcal{B}_k} |t_{j,k}|^2 \frac{\omega^n}{n!} \int_{\mathcal{B}_k} |\eta_{j,k}|^2 \frac{\omega^n}{n!}} \\
&= \mathcal{O}\left(\frac{1}{k}\right).
\end{aligned}
$$

Finally,

$$
\int_{\mathcal{B}_k} |f_{i,k}|^2 \frac{\omega^n}{n!} = 1 + \mathcal{O}\left(\frac{1}{k}\right) \quad \text{for } i \in \{1, \cdots, m\}.
$$

Without loss of generality, we can assume that $s_k = f_{1,k}$ and the result follows. $\square$

We expect this kind of "peaked" behaviour to extend in the obvious way to sequences $\{s_k\}_{k=1}^{\infty}$, where $s_k \in \mathcal{J}_k^l(Y)$, $l \in \mathbb{N}$ and $Y \subset X$ is a complex submanifold. Also, we conjecture that we have the following:

Conjecture 9.1.9. *Let $(L, h) \to (X, \omega)$ be a polarized Kähler manifold. Let $Y \subset X$ be a complex submanifold. Let $\{r_k\}_{k=1}^{\infty} \subset \mathbb{R}_{\geq 0}$ be a sequence such that $\frac{r_k}{k} \to 0$ as $k \to \infty$. Then, for any sequence $\{s_k\}_{k=1}^{\infty}$, where $s_k \in \mathcal{J}_k^{r_k}(Y)^{\perp}$ and $\|s_k\| = 1$ for all $k \in \mathbb{N}$, we have*

$$
\lim_{k \to \infty} \left| \int_{X - N(Y)} |s_k|^2 \frac{\omega^n}{n!} \right| = 0
$$

for any neighbourhood $N(Y)$ of Y.

In future, it might be interesting to explore in detail how such sections are "peaked" asymptotically. Considering what we have found so far, we make the following definition:

Definition 9.1.10. Let $(L, h) \to (X, \omega)$ be a polarization of the compact Kähler manifold (X, ω). We call a set $S_k = \{p_i : i \in I\}$, where p_i are points (submanifolds) of X a *sampling set of points (submanifolds)* of L^k if

$$
\operatorname{Span}\left(\cup_{i \in I} \mathcal{J}_k(p_i)^{\perp}\right) = H^0(X, L^k),
$$

where $\perp$ is taken with respect to the $\mathcal{L}^2$ inner product on $H^0(X, L^k)$ given by integrating the fibre-wise inner product given by h^k with respect to the volume form $\frac{\omega^n}{n!}$. We call a sampling set of points (submanifolds) *minimal*, if $\#I$ is minimal among all sampling sets of points (submanifolds) of L^k.

In the following, let $(L, h) \to (X, \omega)$ be as in the above definition. We make two simple observations.

Lemma 9.1.11.

$$\operatorname{Span}\left(\cup_{p \in X} \mathcal{J}_k(p)^\perp\right) = H^0(X, L^k).$$

Proof. Suppose $s \in \operatorname{Span}\left(\cup_{p \in X} \mathcal{J}_k(p)^\perp\right)^\perp = \cap_{p \in X} \mathcal{J}_k(p)$. Then clearly $s = 0$. $\square$

Lemma 9.1.12. *Let $N_k = \dim(H^0(X, L^k)$, then there exists $k_0 > 0$ such that for all $k > k_0$ and any $p \in X$, there exists a minimal sampling set of points I_k of L^k with precisely N_k distinct points. Furthermore, we can assume without loss of generality that $p \in I_k$.*

Proof. Choose $k_0 \in \mathbb{N}$ such that L^k is very ample for $k > k_0$. Each $\mathcal{J}_k(p)$ has dimension $N_k - 1$ and $\mathcal{J}_k(p)^\perp$ has dimension 1. Let $p_1 = p$ and define $V_1 = \mathcal{J}_k(p)^\perp$. Using 9.1.11, we observe that either $N_k = 1$ or there exists another point $p_2 \in X$ such that

$$V_2 = \operatorname{Span}\left(V_1, \mathcal{J}_k(p_2)^\perp\right)$$

has dimension $\dim(V_1) + 1$. Continuing by induction and defining

$$V_{n+1} = \operatorname{Span}\left(V_n, \mathcal{J}_k(p_{n+1})^\perp\right),$$

we arrive at the result. $\square$

Proposition 9.1.13. *Let $\{p_1, \ldots, p_{N_k}\} \subset X$ be a minimal sampling set of points for L^k and let L_p, for $p \in X$, denote the fibre of L over p. There exists a (non-canonical) vector space isomorphism*

$$\Phi_k : H^0(X, L^k) \to \bigoplus_{i=1}^{N_k} L_{p_i} \cong \mathbb{C}^{N_k}.$$

defined by evaluation. $\Phi_k(s) \overset{\text{def}}{=} (s(p_1), \ldots, s(p_{N_k}))$ *for* $s \in H^0(X, L^k)$.

Proof. Clearly Φ_k is linear and $s(p_i) = 0$ for all $i \in \{1, \cdots, N_k\}$ implies $s = 0$. By dimension counting, Φ_k is an isomorphism. $\square$

Remark 9.1.14. *Suppose that, for $Y \subset X$ a complex submanifold of X, we are interested in studying the space $\mathcal{J}_k^l(Y)$ for some $l, k \in \mathbb{N}$. Then the simple identities*

$$\mathcal{J}_k^l(Y) = \cap_{p \in Y} \mathcal{J}_k^l(p)$$
$$\mathcal{J}_k^l(Y)^\perp = \operatorname{Span}\left(\cup_{p \in Y} \mathcal{J}_k^l(p)^\perp\right)$$

give us an intuitive understanding in terms of "peaked sections" and suggest that one could consider studying these spaces using $\mathcal{J}_k^l(p)$ for a large enough finite set of points points $p \in Y$. This idea is reminiscent of the method used by Donaldson in [Don96], where a large collection of peaked sections with an evenly distributed "net" of peak points is being utilized.

9.2 Localization of the density function

We now prove a "localization of sums" result for the density function of the Bergman kernel. First, we need a little lemma to simplify the proof.

Lemma 9.2.1. *Let $(L, h) \to (X, \omega)$ be a polarization of a Kähler manifold X of complex dimension n. For each $p \in X$, there exists a holomorphic coordinate chart which is centred at p:*

$$\psi_p : \mathcal{U}_p \to \mathcal{V}_p \subset \mathbb{C}^n \qquad \text{for } p \in \mathcal{U}_p \subset X,$$

such that the following holds:

- *There exists a real valued Kähler potential $\phi_p : \mathcal{V}_p \to \mathbb{R}$ for ω such that $\omega = i\partial\overline{\partial}\phi_p$ and a constant $C_p \geq 0$ such that*
$$\left|\phi_p(\boldsymbol{z}) - \|\boldsymbol{z}\|^2\right| \leq C_p\|\boldsymbol{z}\|^4,$$
for all $\boldsymbol{z} \in \mathcal{V}_p$.
- *Geodesic distance on X and vector space norm on $\mathcal{V}_p$ are related by*
$$\frac{1}{2}\|\boldsymbol{z} - \boldsymbol{w}\| \leq d\left(\psi_p^{-1}(\boldsymbol{z}), \psi_p^{-1}(\boldsymbol{w})\right) \leq 2\|\boldsymbol{z} - \boldsymbol{w}\|$$
for all $\boldsymbol{z}, \boldsymbol{w} \in \mathcal{V}_p$.
- *$L_{\mathcal{U}_p} \cong \mathcal{U}_p \times \mathbb{C}$ is trivialized in such a way that the Hermitian norm of the trivializing section $e_p : \mathcal{U}_p \to \mathbb{C}$ is given by*
$$a_p(\boldsymbol{z}) \stackrel{\text{def}}{=} \|e_p(\boldsymbol{z})\|^2 = e^{-\phi_p(\boldsymbol{z})},$$
and we have the estimate
$$\frac{1}{2} \leq a_p(\boldsymbol{z}) \leq 1$$
for all $\boldsymbol{z} \in \mathcal{V}_p$.

Proof. For every $p \in X$, we can use holomorphic K-coordinates of order 4 at p on a small open set containing p. In particular, we then have

$$\phi_p(\boldsymbol{z}) = \|\boldsymbol{z}\|^2 + R_{i\bar{j}k\bar{l}}(p) z_i \overline{z}_j z_k \overline{z}_l + \mathcal{O}(\|\boldsymbol{z}\|^5),$$

for all $\boldsymbol{z} \in \mathcal{V}_p$, which gives us the first estimate on a sufficiently small neighbourhood of p. We note that, at p, the metric is just the identity matrix. Since the metric varies smoothly, geodesic distance near p is then also approximately equal to the vector space distance. We restrict to a small enough coordinate neighbourhood to obtain the required estimate

$$\frac{1}{2}\|\boldsymbol{z} - \boldsymbol{w}\| \leq d\left(\psi_p^{-1}(\boldsymbol{z}), \psi_p^{-1}(\boldsymbol{w})\right) \leq 2\|\boldsymbol{z} - \boldsymbol{w}\|$$

for all $\boldsymbol{z} \in \mathcal{V}_p$. We can choose a trivialization of L over a neighbourhood of p such that $a_p = e^{-\phi_p}$. The final estimate

$$\frac{1}{2} \leq a_p(\boldsymbol{z}) \leq 1$$

just follows from the corresponding estimate for ϕ_p on a small enough neighbourhood of p.

□

Proposition 9.2.2. *Let $(L,h) \to (X,\omega)$ be a polarization of a Kähler manifold (X,ω) of complex dimension n. Let $Y \subset X$ be an embedded complex submanifold of X of complex dimension j. For any $M > 1$, there exists $r > 0$ and a tubular neighbourhood*

$$T_r(Y) = \{p \in X : d(p,Y) \leq r\},$$

where $d(p,Y)$ denotes the geodesic distance between p and Y such that

$$|s(q)|^2 \leq M^{-k}\|s\|^2$$

for all $s \in \mathcal{J}_k^k(Y)$ and all $q \in T_r(Y)$.

Proof. We work in local coordinates. There exists r, r' such that $r > r' > 0$, and there exists a finite collection of coordinate neighbourhoods $\psi_\alpha : \mathcal{U}_\alpha \to \mathcal{V}_\alpha \subset \mathbb{C}^n$ centred at points $p_\alpha \in Y$, such that $\overline{\mathcal{P}}_r^n(\mathbf{0}) \subset \mathcal{V}_\alpha$ (see appendix A for notation) for all α, and $T_{r'}(Y) \subset \bigcup_\alpha \psi_\alpha^{-1}\left(\mathcal{P}_{\frac{r}{2}}^n(\mathbf{0})\right)$. Let us assume furthermore that these coordinate patches are chosen as in lemma 9.2.1 and that

$$\left\{\psi_\alpha^{-1}(\boldsymbol{z}) : \boldsymbol{z} \in \mathcal{V}_\alpha \text{ and } z_{j+1} = \cdots = z_n = 0\right\} = \mathcal{U}_\alpha \cap Y \qquad \text{for all } \alpha.$$

Note that, for $0 \neq s \in H^0(X, L^k)$, $\frac{s}{\|s\|}$ can be extended to an orthonormal basis of $H^0(X, L^k)$, and by the Catlin-Tian-Yau-Zelditch expansion, there exists a constant $C > 0$ such that the density function ρ_k satisfies $\rho_k(p) \leq Ck^n$, for all $p \in X$ and all $k \in \mathbb{N}$. Including the case $s = 0$, we hence obtain the estimate

$$|s(p)|^2 \leq Ck^n\|s\|^2 \quad \text{for all } p \in X \text{ and all } k \in \mathbb{N}.$$

Let a_α^k denote the local expression of the Hermitian metric h^k on $\mathcal{U}_\alpha$. Suppose that on $\mathcal{U}_\alpha$ $s = f_\alpha e_\alpha^{\otimes k}$ for some holomorphic function f_α on $\mathcal{U}_\alpha$. We have

$$|f_\alpha(\boldsymbol{z})|^2 \leq Ck^n\|s\|^2 a_\alpha^{-k}(\boldsymbol{z}) \leq Ck^n 2^k\|s\|^2 \quad \text{for all } \boldsymbol{z} \in \mathcal{U}_\alpha \text{ and all } \alpha.$$

Now let $s \in \mathcal{J}_k^k(p)$. For all α, f_α satisfies

$$\left.\frac{\partial}{\partial \boldsymbol{z}}^{\boldsymbol{\beta}} f_\alpha\right|_{(z_1,\ldots,z_j,0,\ldots,0)} = 0$$

for all $\boldsymbol{\beta} \in \mathbb{N}_0^n$ with $|\boldsymbol{\beta}| < k$ and all $(z_1, \ldots, z_j, 0, \ldots, 0) \in \mathcal{V}_\alpha$. We can apply corollary A.1.4 to obtain

$$\begin{aligned} |f_\alpha(\boldsymbol{z})| &\leq \|(z_{j+1}, \ldots, z_n)\|^k \frac{2}{r^k} \sup_{\boldsymbol{w} \in \overline{\mathcal{P}}_r^n(\mathbf{0})} |f_\alpha(\boldsymbol{w})| \\ &\leq \|(z_{j+1}, \ldots, z_n)\|^k \frac{2}{r^k}\sqrt{Ck^n 2^k}\|s\| \qquad (\dagger) \end{aligned}$$

for all $(z_1, \ldots, z_n) \in \mathcal{P}_{\frac{r}{2}}^n(\mathbf{0})$ and all α. Now pick $M' > 0$ large enough so that, for $\boldsymbol{z} \in \mathcal{W} \stackrel{\text{def}}{=} \{\boldsymbol{z} : \boldsymbol{z} \in \mathcal{P}_{\frac{r}{2}}^n(\mathbf{0}) \text{ and } \|(z_{j+1}, \ldots, z_n)\| \leq M^{-M'}\}$, $(\dagger)$ is bounded by $M^{-\frac{k}{2}}\|s\|$ for all $k \in \mathbb{N}$ and all α. We then have

$$|s(q)|^2 \leq M^{-k}\|s\|^2 a_\alpha^k(q) \leq M^{-k}\|s\|^2,$$

for $q \in \psi_\alpha^{-1}(\mathcal{W})$, all α and all $k \in \mathbb{N}$. There exists a small tubular neighbourhood $N_{r''}(Y)$ of Y such that $r'' \leq r'$ and

$$N_{r''}(Y) \subset \bigcup_\alpha \psi_\alpha^{-1}(\mathcal{W}).$$

The required inequality now holds on this tubular neighbourhood of Y.

□

We can now use the preceding proposition to prove a "localization of sums" result for the full density function. Recall that, for any fixed $p \in X$, we can choose an orthonormal basis of sections $s_{1,k}, \cdots, s_{N_k,k}$ of $H^0(X, L^k)$ such that $s_{1,k} \in \mathcal{J}_k(p)^\perp$ and $s_{2,k}, \cdots, s_{N_k,k} \in \mathcal{J}_k(p)$ for all $k \in \mathbb{N}$ (Note that $\mathcal{J}_k(p)^\perp$ is one-dimensional).

It follows that, for such a choice of orthonormal basis, $\rho_k(p) = |s_{1,k}(p)|^2_{h^k}$, and we understand the asymptotics of ρ_k at p once we understand the peaked sections $s_{1,k}$ at p for all $k \in \mathbb{N}$. Let us now focus on the asymptotics of ρ_k in a neighbourhood of some point p. In this case, it might still be useful to compute the asymptotics of ρ_k by an orthonormal basis of a small subspace of $H^0(X, L^k)$. The following corollary confirms that this is possible in the case where p is not just a point but any embedded complex submanifold of X.

Corollary 9.2.3 (Localization of the density function on a tubular neighbourhod)**.** *Let $(L, h) \to (X, \omega)$ be a polarization of a Kähler manifold (X, ω). Denote by ρ_k the density function for this polarization and let $Y \subset X$ be an embedded complex submanifold of X. There exists $r > 0$ and, for any $l \in \mathbb{N}$, a constant $C_l \geq 0$ such that*

$$\left| \rho_k(p) - \sum_{j=1}^{N_k} |s_{k,j}(p)|^2 \right| \leq C_l k^{-l}$$

for all $p \in T_r(Y)$ and $k \in \mathbb{N}$. Here, $\{s_{k,j}\}_{j=1}^{N_k}$ denotes any orthonormal basis of the space $\mathcal{J}_k^k(Y)^\perp$ and $|.|$ denotes the fibre-wise norm on L^k. In particular, the asymptotic expansion of $\rho_k(p)$ is equal to the asymptotic expansion of $\sum_{j=1}^{N_k} |s_{k,j}|^2(p)$ for $p \in T_r(Y)$.

Proof. Suppose that X is of complex dimension n. We use the estimate from the previous proposition together with the fact that

$$\dim\left(H^0(X, L^k)\right) = \mathcal{O}(k^n).$$

□

Remark 9.2.4. *It is conceivable that the above idea could be explored to understand the partial density functions with imposed vanishing along submanifolds in this general setting. If one could manage to localize the sum appearing in the definition of the partial density function to a subspace $\mathcal{J}_k^{r_k}(Y)^\perp \subset H^0(X, L^k)$ for a suitably chosen sequence $\{r_k\}_{k=1}^\infty \subset \mathbb{N}$ and to then understand these subspaces in terms of Tian's peak sections, then one might hope to apply methods similar to the toric case in this setting.*

Chapter 10

Induced Metrics on Blow-ups

In this chapter, we study how partial density functions with vanishing along a finite set of points on a polarized Kähler manifold $(L, h) \to (X, \omega)$ are related to certain metrics on blow-ups of (X, ω). We then study a polarization and blow-up of $\mathbb{CP}^n$ and $\mathbb{C}^n$ in detail.

10.1 Pull-back metrics

Let $(L, h) \to (X, \omega)$ be a polarization, so that $\omega \in 2\pi c_1(L)$ is a Kähler form on X. We recall that the evaluation map $ev_k(p)(s) \stackrel{\text{def}}{=} s(p)$ for $p \in X$ and $s \in H^0(X, L^k)$ induces an embedding $i_k : X \hookrightarrow \mathbb{P}\left(H^0(X, L^k)^*\right)$, $i_k(p) = [ev_k(p)]$ for $p \in X$ and for all large enough $k \in \mathbb{N}$. We call such embeddings *Kodaira embeddings*. Any ordered basis $\boldsymbol{b}_k \stackrel{\text{def}}{=} (s_{1,k}, \cdots, s_{N_k,k})$ of $H^0(X, L^k)$ determines an isomorphism $H^0(X, L^k) \cong H^0(X, L^k)^*$ under which this embedding takes the form $i_{\boldsymbol{b}_k} : X \hookrightarrow \mathbb{P}\left(H^0(X, L^k)\right)$

$$i_{\boldsymbol{b}_k} : p \mapsto [f_{1,k}(p) : \cdots : f_{N_k,k}(p)],$$

where we let $\{f_{i,k}\}_{i=1}^{N_k}$ denote the local holomorphic functions representing the sections $\{s_{i,k}\}_{i=1}^{N_k}$ in a fixed trivializing chart $\mathcal{U} \times \mathbb{C}$ of L^k such that $p \in \mathcal{U}$. In future, we will abuse notation and will not distinguish between $f_{i,k}$ and $s_{i,k}$ when this does not cause confusion. Let us denote by $\omega_{FS} \in 2\pi c_1(\mathcal{O}(1))$ the Fubini-Study metric on $\mathbb{CP}^n$. It is locally given by $\omega_{FS} = i\partial\overline{\partial} \log\left(1 + \sum_{j=1}^n |z_j|^2\right)$ on the coordinate patch $U_0 = \{[1 : z_1 : \cdots : z_n] : z_j \in \mathbb{C}\} \subset \mathbb{CP}^n$. Suppose that, for each $k \in \mathbb{N}$, we have picked an ordered basis $\boldsymbol{b}_k = (s_{1,k}, \cdots, s_{N_k,k})$ of $H^0(X, L^k)$. We define

$$\begin{aligned}\omega_{\boldsymbol{b}_k} &\stackrel{\text{def}}{=} \frac{1}{k} i_{\boldsymbol{b}_k}^* \omega_{FS} \\ &= \frac{i}{k} \partial\overline{\partial} \log\left(\sum_{j=1}^{N_k} |s_{j,k}|^2\right).\end{aligned}$$

Since $L^k = i_{\boldsymbol{b}_k}^* \mathcal{O}(k)$ for $k \gg 0$, it is clear that $\omega_{\boldsymbol{b}_k} \in 2\pi c_1(L)$ for all $k \gg 0$, and ω_k is invariant under the action of $U(N_k)$ on the ordered bases of $H^0(X, L^k)$. Using the Hermitian metric h on L, we can in fact define a canonical sequence of such metrics $\omega_k \stackrel{\text{def}}{=} \omega_{\boldsymbol{b}_k}$ by choosing an orthonormal basis $\boldsymbol{b}_k$ of $H^0(X, L^k)$ with respect to the $\mathcal{L}^2$ inner product induced by h^k on $H^0(X, L^k)$ for each $k \in \mathbb{N}$. The sequence $\{\omega_k\}_{k=1}^\infty$ then converges to ω in the following sense:

Theorem 10.1.1 (Ruan [Rua98], Tian [Tia90])**.** *Let $(L,h) \to (X,\omega)$ be a polarization of a Kähler manifold. Denote by g and g_k the Riemannian metrics corresponding to ω and ω_k respectively. Then, for any $l \in \mathbb{N}$, we have*

$$\|g - g_k\|_{\mathcal{C}^l(X)} = \mathcal{O}\left(\frac{1}{k}\right).$$

Remark 10.1.2. *Tian originally proved a weaker version of this result. The version of the theorem above is due to Ruan.*

One of the aims of this chapter is to illustrate why an extension of these ideas to certain metrics on the blow-up of X at finitely many points might be of interest. After introducing their construction, we will study the examples of blow-ups of $\mathbb{CP}^n$ and $\mathbb{C}^n$ in some detail.

10.2 Some facts about blow-ups

Let us now review the standard notion of blowing up a complex manifold X of dimension n in a finite set of points $p_1, \cdots, p_s$. We will take an elementary differential-geometric approach to this, and we describe the coordinate charts of the resulting manifold $\mathrm{Bl}_{p_1,\cdots,p_s} X$ explicitly. The main purpose of this section is to fix our notation and to recall some standard facts about blow-ups.

Recall that, for finitely many distinct points $p_1, \cdots, p_s \in X$, the *blow-up of X at $p_1, \cdots, p_s$*, denoted by $\mathrm{Bl}_{p_1,\cdots,p_s}(X)$, is a complex manifold which can be defined by the following gluing construction: Suppose that, for $i \in \{1, \cdots, s\}$, $\psi_i : \mathcal{U}_i \to \mathcal{V}_i \subset \mathbb{C}^n$ are disjoint holomorphic coordinate charts centred at $p_i \in \mathcal{U}_i \subset X$ respectively. We define

$$\mathcal{W}_i = \left\{(\boldsymbol{a}, [\boldsymbol{t}]) \in \mathcal{V}_i \times \mathbb{CP}^{n-1} : \boldsymbol{a} \in [\boldsymbol{t}]\right\}$$

and consider the holomorphic maps $\pi_i : \mathcal{W}_i \to \mathcal{V}_i$ given by $\pi_i : (\boldsymbol{a}, [\boldsymbol{t}]) \mapsto \boldsymbol{a}$ for $i \in \{1, \cdots, s\}$ and $(\boldsymbol{a}, [\boldsymbol{t}]) \in \mathcal{W}_i$. We define $\mathrm{Bl}_{p_1,\cdots,p_s}(X)$ by holomorphically gluing $\mathcal{W}_1, \cdots, \mathcal{W}_s$ to $X - \bigcup_{i=1}^s \{p_i\}$ via $\pi_1, \cdots, \pi_s$:

$$\mathrm{Bl}_{p_1,\cdots,p_s}(X) \stackrel{\text{def}}{=} \left(X - \bigcup_{i=1}^{s} \{p_i\}\right) \cup_{\pi_1} \mathcal{W}_1 \cup_{\pi_2} \cdots \cup_{\pi_s} \mathcal{W}_s.$$

The maps π_i descend to a well defined holomorphic map $\pi : \mathrm{Bl}_{p_1,\cdots,p_s}(X) \to X$ which is called the *blow-down map* of $\mathrm{Bl}_{p_1,\cdots,p_s}(X)$. The complex hypersurface $E_i \stackrel{\text{def}}{=} \pi_i^{-1}(p_i)$, for $i \in \{1, \cdots, s\}$, is called the *exceptional divisor over p_i* and π is a biholomorphism between $\mathrm{Bl}_{p_1,\cdots,p_s}(X) - \bigcup_{i=1}^s E_i$ and $X - \bigcup_{i=1}^s \{p_i\}$. We call the data $\pi : Bl_{p_1,\cdots,p_s}(X) \to X$ the *blow up of X at $p_1, \cdots, p_s$*.

Suppose that $(L^m, h) \to (X, \omega)$ is a polarization of a compact Kähler manifold. Fix some distinct points $p_1, \cdots, p_s \in X$ and $\boldsymbol{l} = (l_1, \cdots, l_s) \in \mathbb{N}^s$. On the blow up $\pi : \mathrm{Bl}_{p_1,\cdots,p_s}(X) \to X$, we consider the holomorphic line bundle

$$\widehat{L} = \widehat{L}_{p_1,\cdots,p_s,\boldsymbol{l},m} \stackrel{\text{def}}{=} \pi^* L^m \otimes \mathcal{O}(-l_1 E_1) \otimes \cdots \otimes \mathcal{O}(-l_s E_s).$$

The following lemma is a standard result on blow-ups.

Lemma 10.2.1. *Let X be a complex manifold and L a positive line bundle on X. Fix distinct points $p_1, \cdots, p_s \in X$. For any $\boldsymbol{l} = (l_1, \cdots, l_s) \in \mathbb{N}^s$, there exist $m_0 > 0$ such that, for all*

$m \geq m_0$ and $m \in \mathbb{N}$, $\widehat{L} = \widehat{L}_{p_1,\cdots,p_s,\boldsymbol{l},m}$ is positive. Furthermore, there exists an isomorphism of vector spaces

$$\phi_k : H^0(X, \mathcal{J}_{p_1}^{l_1 k} \otimes \cdots \otimes \mathcal{J}_{p_s}^{l_s k} \otimes L^{mk}) \to H^0(\mathrm{Bl}_{p_1,\cdots,p_s} X, \widehat{L}^k), \text{ for each } k \in \mathbb{N},$$

where $\mathcal{J}_{p_i}^{l_i k}$ denotes the ideal sheaf of holomorphic functions on X vanishing to order at least $l_i k$ at $p_i \in X$. ϕ_k is unique up to multiplication by a nonzero complex number.

Proof. The positivity claim is a standard result. A proof can be found *e.g.* in [Huy05, lemma 5.3.2]. Let us sketch the proof of the second part of the lemma. We decompose $\phi_k = b_k \circ a_k \circ \pi^*$, where

$$\pi^* : H^0\left(X, \mathcal{J}_{p_1}^{l_1 k} \otimes \cdots \mathcal{J}_{p_s}^{l_s k} \otimes L^{mk}\right) \to H^0\left(\mathrm{Bl}_{p_1,\cdots,p_s}(X), \mathcal{J}_{E_1}^{l_1 k} \otimes \cdots \otimes \mathcal{J}_{E_s}^{l_s k} \otimes \pi^* L^{mk}\right)$$

denotes the pull-back of sections. π^* is injective since π is surjective and for $\dim(X) \geq 2$ surjective by Hartog's theorem. Also, for $dim(X) = 1$, π is a biholomorphism, so π^* is an isomorphism. We have

$$a_k : H^0\left(\mathrm{Bl}_{p_1,\cdots,p_s}(X), \mathcal{J}_{E_1}^{l_1 k} \otimes \cdots \otimes \mathcal{J}_{E_s}^{l_s k} \otimes \pi^* L^{mk}\right) \to V_k \subset H^0\left(\mathrm{Bl}_{p_1,\cdots,p_s}(X), \pi^* L^{mk}\right)$$

denoting the natural isomorphism to the space V_k of holomorphic sections of $\pi^* L^{mk}$ vanishing to order at least $l_i k$ along E_i for all $i \in \{1, \cdots, s\}$. Finally, there is an isomorphism

$$b_k : V_k \to H^0\left(\mathrm{Bl}_{p_1,\cdots,p_s}(X), \mathcal{O}\left(-l_1 k E_1\right) \otimes \cdots \otimes \mathcal{O}\left(-l_s k E_s\right) \otimes \pi^* L^{mk}\right),$$

where $b_k : s \mapsto d \otimes s$, for $s \in V_k$, $k \in \mathbb{N}$, and d is a meromorphic section of $\mathcal{O}\left(-l_1 k E_1\right) \otimes \cdots \otimes \mathcal{O}\left(-l_s k E_s\right)$ obtained by locally dividing by the defining functions of the divisor $\sum_{i=1}^{s} l_i k E_i$. Given two such sections d, d', we note that $\frac{d}{d'} \in H^0(X, \mathcal{O}) - \{0\} = \mathbb{C}^*$, which implies that ϕ_k is unique up to multiplication by a nonzero complex number. $\square$

10.3 Induced metrics on $\mathrm{Bl}_{p_1,\cdots,p_s}(X)$

10.3.1 Construction

Consider a polarization $(L^m, h) \to (X, \omega)$ and the sequence of pull back metrics $\omega_k \in 2\pi c_1(L^m)$ studied by Tian and Ruan. For some distinct points $p_1, \cdots, p_s \in X$ and $\boldsymbol{l} \in \mathbb{N}^s$, we consider $\mathrm{Bl}_{p_1,\cdots,p_s}(X)$ with the line bundle $\widehat{L} = \widehat{L}_{p_1,\cdots,p_s,\boldsymbol{l},m}$ over it. We assume that m is large enough so that $i_k : \mathrm{Bl}_{p_1,\cdots,p_s}(X) \to \mathbb{P}(H^0(\mathrm{Bl}_{p_1,\cdots,p_s}(X), \widehat{L}^k)^*)$ is an embedding for all $k \in \mathbb{N}$.

Lemma 10.3.1. *There exists a natural sequence of Kähler metrics $\{\widehat{\omega}_k\}_{k=1}^{\infty} \subset 2\pi c_1(\widehat{L})$ on $\mathrm{Bl}_{p_1,\cdots,p_s}(X)$ which is induced by the polarization.*

Proof. For $k \in \mathbb{N}$, we pick an orthonormal basis $\{s_j\}_{j=1}^{M_k}$ of $H^0(X, \mathcal{J}_{p_1}^{l_1 k} \otimes \cdots \otimes \mathcal{J}_{p_s}^{l_s k} \otimes L^{mk})$ thought of as a subspace of $H^0(X, L^k)$. Define

$$\widehat{\omega}_k \overset{\text{def}}{=} \frac{i}{k} \partial \overline{\partial} \log \left(\sum_{j=1}^{M_k} |\widehat{s}_j|^2 \right),$$

where $\widehat{s} = \phi_k(s)$, and ϕ_k denotes a choice of isomorphism

$$\phi_k : H^0(X, \mathcal{J}_{p_1}^{l_1 k} \otimes \cdots \otimes \mathcal{J}_{p_s}^{l_s k} \otimes L^{mk}) \to H^0(\mathrm{Bl}_{p_1,\cdots,p_s} X, \widehat{L}^k) \text{ for each } k \in \mathbb{N}.$$

Since the choice of isomorphism is unique up to multiplication by a nonzero complex number, $\widehat{\omega}_k$ does not depend on it. Due to the obvious $U(M_k)$ invariance of the defining equation, $\widehat{\omega}_k$ is independent of our choice of orthonormal basis as well. Thinking of $k\widehat{\omega}_k$ as the pull-back of the Fubini-Study metric under the Kodaira embedding given by an ordering of our choice of basis, we see that $\widehat{\omega}_k \in 2\pi c_1(\widehat{L})$ and that it is a Kähler metric on $\mathrm{Bl}_{p_1,\cdots,p_s}(X)$. □

10.3.2 Relationship with $\rho_{\boldsymbol{l},k}$

For $(L^m, h) \to (X, \omega)$, $p_1, \cdots, p_s \in X$, $\boldsymbol{l} = (l_1, \cdots, l_s)$ and $\widehat{L} \to \mathrm{Bl}_{p_1,\cdots,p_s}(X)$ as in the previous section, we consider the partial density function $\rho_{\boldsymbol{l},k} : X \to \mathbb{R}$, defined by $\rho_{\boldsymbol{l},k}(p) \stackrel{\text{def}}{=} \sum_{j=1}^{M_k} |s_{j,k}(p)|^2_{h^k}$ for $p \in X$, where $\{s_{j,k}\}_{j=1}^{M_k}$ denotes any orthonormal basis of $H^0(X, \mathcal{J}_{p_1}^{l_1 k} \otimes \cdots \otimes \mathcal{J}_{p_s}^{l_s k} \otimes L^{mk})$.

Lemma 10.3.2. *On $X - \bigcup_{i=1}^s \{p_i\}$, we have*

$$\pi_* \widehat{\omega}_k - \omega = \frac{i}{k} \partial\overline{\partial} \log \rho_{\boldsymbol{l},k}.$$

Proof. This just follows from the definitions. On $\mathrm{Bl}_{p_1,\cdots,p_s}(X) - \bigcup_{i=1}^s \{E_i\}$, $\mathcal{O}(-l_i k E_i)$ is trivial for each $i \in \{1, \cdots, s\}$. In each trivializing chart of the form $\psi : \pi^{-1}(\mathcal{U}) \times \mathbb{C} \to \widehat{L}\Big|_{\pi^{-1}(\mathcal{U})}$ such that $\mathcal{U} \subset X - \bigcup_{i=1}^s \{p_i\}$ is an open set, the lift of a section $s \in J_k$, where

$$J_k \stackrel{\text{def}}{=} \{s \in H^0(X, L^k) : s \text{ vanishes to order at least } l_i k \text{ along } p_i \text{ for all } i \in \{1, \cdots, s\}\},$$

takes the form

$$\widehat{s} : \widehat{u} \to (\widehat{u}, s(\pi(\widehat{u}))), \text{ for } \widehat{u} \in \pi^{-1}(\mathcal{U}),$$

where s denotes the local holomorphic function representing s now. For any basis $\boldsymbol{b}_k = (s_{1,k}, \cdots, s_{M_k,k})$ of J_k, we hence have that

$$\pi_* \widehat{\omega}_k = \frac{i}{k} \partial\overline{\partial} \log \left(\sum_{j=1}^{M_k} |s_{j,k}|^2 \right),$$

while

$$-\omega = \frac{i}{k} \partial\overline{\partial} \log(h^k),$$

where h^k denotes a local function representing the fibre-wise Hermitian metric on L^k. □

It would be interesting to completely understand the decay properties of the derivatives of $\rho_{\boldsymbol{l},k}$, and we conjecture that, away from the exceptional divisors, $\widehat{\omega}_k - \pi^* \omega$ rapidly decays as $k \to \infty$.

Conjecture 10.3.3. *There exists a neighbourhood N of $p_1, \cdots, p_s$ such that, for any $l \in \mathbb{N}$, we have*

$$\|\pi_* \widehat{\omega}_k - \omega\|_{\mathcal{C}^l(X-N)} = \mathcal{O}\left(k^{-1}\right).$$

10.4 cscK metrics and some open questions

In this section we will discuss some open questions as well as some background material related to induced Kähler metrics on blow-ups. We will not aim to answer the questions that we pose here, but hope that the reader might appreciate some additional motivation for studying such metrics. In the following sections, we will then investigate the cases of $\mathrm{Bl}_{[1:0\cdots:0]}(\mathbb{CP}^n)$ and $\mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n)$ in more detail.

10.4.1 Balanced metrics

Balanced metrics have been discussed by Donaldson [Don01] in the context of finding constant scalar curvature Kähler (cscK) metrics in a given Kähler class. These efforts are part of a bigger program of trying to identify "best" Kähler representatives within a given Kähler class. Extremal Kähler (eK) and in particular cscK metrics are natural such representatives and are of particular importance within this approach.

Let us first review the necessary terminology (see [Don01, AL04]). A complex manifold $X \subset \mathbb{CP}^n$ is called *balanced* if there exists a $\lambda > 0$ such that

$$\int_X \frac{z_i \overline{z}_j}{\sum_{l=1}^n |z_l|^2} \, \mathrm{dVol} = \lambda \delta_{ij},$$

where dVol is the volume form on X induced from the Fubini-Study metric on $\mathbb{CP}^n$. For a complex manifold X endowed with a positive line bundle $L \to X$ such that X can be embedded into $\mathbb{P}\left(H^0(X, L)^*\right)$ by a Kodaira embedding, a basis $\boldsymbol{b}$ of $H^0(X, L)$ is called *balanced* if $i_{\boldsymbol{b}}(X)$ is balanced. Finally, the pair (X, L), where X is a complex manifold and L a holomorphic line bundle over X, is called *balanced* if there exists a basis of $H^0(X, L)$ that is balanced. The following lemma gives another characterization of balanced metrics.

Lemma 10.4.1 (see [AL04] for a similar version)**.** *Let X be a compact complex manifold and let $L \to X$ be a positive holomorphic line bundle. (X, L) is balanced if and only if there exists a Kähler form $\omega \in 2\pi c_1(L)$ such that the density function ρ_k associated to ω is constant on X.*

Let us denote by $\mathrm{Aut}(X, L)$ the group of biholomorphisms of X that lift to bundle isomorphisms of L. $\mathrm{Aut}(X, L)/\mathbb{C}^*$ denotes the same group modulo the trivial automorphism group $\mathbb{C}^*$.

Theorem 10.4.2 (Donaldson, (see [Don01, AL04]))**.** *Suppose that* $\mathrm{Aut}(X, L)/\mathbb{C}^*$ *is discrete. Let $\boldsymbol{b}, \boldsymbol{b'}$ denote two balanced bases of $H^0(X, L)$ with $n_k \stackrel{\text{def}}{=} \dim\left(H^0(X, L^k)\right)$. Then there exists $U \in U(n_k)$ and $\lambda > 0$ such that*

$$\boldsymbol{b} = \lambda U.\boldsymbol{b}'.$$

In particular, we can unambiguously define $\omega_k \stackrel{\text{def}}{=} \frac{1}{k} i_{\boldsymbol{b}}^* \omega_{FS} \in 2\pi c_1(L)$ for any balanced basis $\boldsymbol{b}$ of $H^0(X, L^k)$ if $\mathrm{Aut}(X, L)/\mathbb{C}^*$ is discrete. We call such a metric ω_k a *balanced metric*. The following theorem demonstrates the importance of these balanced metrics.

Theorem 10.4.3 (Donaldson, [Don01])**.** *Let $(L, h) \to (X, \omega)$ be a polarization.*

a) Suppose that $\mathrm{Aut}(X, L)/\mathbb{C}^*$ *is discrete and (X, L^k) is balanced for all sufficiently large k. Suppose that the balanced metrics ω_k converge in $\mathcal{C}^\infty$ to some limit ω_∞ as $k \to \infty$. Then ω_∞ has constant scalar curvature.*

b) Suppose that $\mathrm{Aut}(X, L)/\mathbb{C}^*$ *is discrete and that* ω_∞ *is a Kähler metric in the class* $2\pi c_1(L)$ *with constant scalar curvature. Then* (X, L^k) *is balanced for large enough* k *and the sequence of balanced metrics* ω_k *converges in* $\mathcal{C}^\infty$ *to* ω_∞ *as* $k \to \infty$*.*

It might be interesting to study the relationship between these results and our induced metrics on the blow-up of X.

Question 10.4.4. Suppose that $\mathrm{Aut}(X, L)/\mathbb{C}^*$ is discrete and that $\omega \in 2\pi c_1(L)$ is a cscK metric on X. Part b) of the above theorem gives us a sequence of balanced metrics ω_k converging to ω. For finitely many distinct points $p_1, \cdots, p_s \in X$, $\boldsymbol{l} \in \mathbb{N}^s$ and $m, k \in \mathbb{N}$, one could study the blow-up $\mathrm{Bl}_{p_1,\cdots,p_s}(X)$ and the metrics $\widehat{\omega}_{k,j} \in 2\pi c_1\left(L_{p_1,\cdots,p_s,\boldsymbol{l},m}\right)$, induced by the polarization $(L, h_{\omega_k}) \to (X, \omega_k)$ for large enough $m \in \mathbb{N}$ and for $k, j \in \mathbb{N}$. Some questions that one might want to attack in this context are:

a) Does there exists a sequence $\{\widehat{\omega}_{k_l,j_l}\}_{l=1}^{\infty}$ such that $\widehat{\omega}_{k_l,j_l} \to \widehat{\omega}$ for some cscK (or at least geometrically interesting) Kähler form $\widehat{\omega}$ as $l \to \infty$?

b) For small parameters $\boldsymbol{l} = (l_1, \cdots, l_n) \in \mathbb{N}^n$ and large $m \in \mathbb{N}$, does there exists a balanced basis of $L^k_{p_1,\cdots,p_s,\boldsymbol{l},m}$ for large k? Which necessary and/or sufficient conditions can be identified to ensure the existence of a balanced basis on the blow-up?

10.4.2 cscK metrics on blow-ups

Let us now review a positive result which confirms the existence of cscK metrics on the blow-up of a cscK manifold with a small parameter. We refer the interested reader to [Tho06, §5] for a related discussion.

Theorem 10.4.5 (Arezzo, Packard [AP06, AP07])**.** *Assume that* (X, ω) *is a compact cscK manifold without nontrivial vanishing holomorphic vector field. Consider the blow-up* $\pi : \mathrm{Bl}_{p_1,\cdots,p_s}(X) \to X$ *at finitely many distinct distinct* $p_1, \cdots, p_s \in X$*. Then, for all* $l_1, \cdots, l_s > 0$*, there exists* $\lambda_0 > 0$ *such that for all* $\lambda \in (0, \lambda_0)$*,* $\mathrm{Bl}_{p_1,\cdots,p_s}(X)$ *has a constant scalar curvature Kähler form* $\widehat{\omega}_\lambda$ *such that*

$$\widehat{\omega}_\lambda \in \pi^*[\omega] - \lambda^2 \left(l_1 PD[E_1] + \cdots + l_s PD[E_s]\right),$$

where $PD[E_i]$ *denotes the Poincaré dual of* E_i*. In addition, if the scalar curvature of* ω *is not zero, then the scalar curvatures of* ω *and of* $\widehat{\omega}_\lambda$ *have the same signs.*

Remark 10.4.6. *The theorem above has been extended to the case where nontrivial vanishing holomorphic vector fields on* X *do occur. See [AL04, AP07, AP09] for details. The reader interested in the related story of extremal Kähler metrics on blow-ups may consult [APS06, Szé11].*

Question 10.4.7. In the case where we have a polarization, does there exist a natural sequence of balanced metrics on the blow-up tending to a cscK metric such as the ones identified in the above theorem (Note that $\mathrm{Aut}(X, L)/\mathbb{C}^*$ might not be discrete)? Can we understand the relationship between such a sequence of balanced metrics on the blow-up and a corresponding sequence of balanced metrics on the base-manifold?

Remark 10.4.8. *The existence of the cscK metric* $\widehat{\omega}_\lambda$ *in the above theorem is proved using a gluing argument and uses analytical methods. A (possibly misguided) hope might be to try to find a more "algebraic" proof of this theorem involving* e.g. *balanced metrics.*

10.5 Example: Balanced blow-up of $\mathbb{CP}^n$

10.5.1 Introduction

Let us now discuss the induced metrics $\widehat{\omega}_k$ in the case of $\mathbb{CP}^n$ blown up at the point $p = [1 : 0 : \cdots : 0]$.

A polarization of $\mathrm{Bl}_{[1:0:\cdots:0]}(\mathbb{CP}^n)$

Let us pick $m \in \mathbb{N}$ and revisit the polarization

$$(\mathcal{O}(m), h_{FS}^m) \to (\mathbb{CP}^n, \omega = m\omega_{FS})$$

which we discussed previously. We consider the blow-up $\pi : \mathrm{Bl}_{[1:0\cdots:0]}(\mathbb{CP}^n) \to \mathbb{CP}^n$ with exceptional divisor E and, for $l < m$ and $l \in \mathbb{N}$, the line bundle

$$\widehat{L}_{l,m} \stackrel{\text{def}}{=} \pi^*\mathcal{O}(m) \otimes \mathcal{O}(-lE) \to \mathrm{Bl}_{[1:0:\cdots:0]}(\mathbb{CP}^n).$$

As usual, $\mathrm{Bl}_{[1:0:\cdots:0]}(\mathbb{CP}^n)$ is obtained by gluing

$$\mathcal{W} = \left\{(\boldsymbol{z}, [\boldsymbol{t}]) \in \mathcal{U}_0 \times \mathbb{CP}^{n-1} : \boldsymbol{z} \in [\boldsymbol{t}]\right\}$$

and $\mathbb{CP}^n - \{[1 : 0 : \cdots : 0]\}$ via the projection map $\pi : \mathcal{W} \to \mathcal{U}_0$ onto the first factor. As a toric variety, $\mathrm{Bl}_{[1:0:\cdots:0]}(\mathbb{CP}^n)$ and $\widehat{L}_{l,m}$ are determined by the polytope $P_{l,m}$ obtained by intersecting the standard simplex in $\mathbb{R}^n$ of side length m, denoted by $Simp_n(m)$, with the half-space $\{\boldsymbol{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i \geq l\}$. Each vertex of $P_{l,m}$ gives a chart and a trivialization of $\widehat{L}_{l,m}$ over it.

Observe that $\mathcal{W} = \cup_{i=1}^n \mathcal{W}_i$, where $\mathcal{W}_i = \{(\boldsymbol{a}, [t_1 : \cdots : t_n]) \in \mathcal{W} : t_j \neq 0\}$. We have the coordinate maps $\xi_i : \mathbb{C}^n \to \mathcal{W}_i$ given by

$$\xi_i(\boldsymbol{w}) = (w_i(w_1, \ldots, w_{i-1}, 1, w_{i+1}, \cdots, w_n), [w_1 : \cdots : w_{i-1} : 1 : w_{i+1} : \cdots : w_n]).$$

It is then clear that, for $i \in \{1, \cdots, n\}$, $E \cap \mathcal{W}_i = \mathcal{Z}(w_i)$, and one can check that $\mathcal{W}_i$ is the toric chart corresponding to the vertex $(0, \cdots, 0, l, 0, \cdots, 0)$ of $P_{l,m}$ with the non-zero entry appearing in the i^{th} position.

Any section s of $\mathcal{O}(mk)$ vanishing to order at least lk at $p = [1 : 0 : \cdots : 0]$ lifts to a section $\widehat{s}$ of $\widehat{L}_{l,m}^k$. In particular, a holomorphic global section given by $s(\boldsymbol{w}) = (\boldsymbol{w}, p(\boldsymbol{w}))$ in the toric trivialization $\mathcal{O}(mk)|_{U_0} \cong \mathbb{C}^n \times \mathbb{C}$, where p is a polynomial of degree at most mk vanishing to order at least lk at 0, can be lifted to

$$\begin{aligned}\widehat{s}(\boldsymbol{w}) &= \left(\boldsymbol{w}, w_j^{-lk} p(\pi(\boldsymbol{w}))\right) \\ &= \left(\boldsymbol{w}, w_j^{-lk} p(w_j(w_1, \ldots, w_{j-1}, 1, w_{j+1}, \ldots, w_n))\right), \boldsymbol{w} \in \mathbb{C}^n\end{aligned}$$

in the toric trivialization of $\widehat{L}_{l,m}^k$ over $\mathcal{W}_j \cong \mathbb{C}^n$ for $j \in \{1, \cdots, n\}$.

A balanced basis and the induced metrics $\widehat{\omega}_k$

Let us return to the orthonormal basis of $\left(H^0\left(\mathbb{CP}^n, \mathcal{O}(mk)\right), \langle ., .\rangle\right)$ which we discussed in chapter 4 and which is given by

$$\{s_{\boldsymbol{\alpha},m,k} \in H^0\left(\mathbb{CP}^n, \mathcal{O}(mk)\right) : \boldsymbol{\alpha} \in \mathbb{N}_0^n, |\boldsymbol{\alpha}| \leq mk\}.$$

On the toric defining trivialization $\mathcal{O}(mk)|_{\mathcal{U}_0} \cong \mathcal{U}_0 \times \mathbb{C}$, $s_{\boldsymbol{\alpha},m,k}$ takes the form

$$s_{\boldsymbol{\alpha},m,k} : \boldsymbol{z} \mapsto (\boldsymbol{z}, a_{\boldsymbol{\alpha},m,k}\boldsymbol{z}^{\boldsymbol{\alpha}}) \qquad \text{for } \boldsymbol{z} \in \mathbb{C}^n,$$

and

$$a_{\boldsymbol{\alpha},m,k} \stackrel{\text{def}}{=} \sqrt{\frac{(mk+n)!}{(2\pi)^n (mk)! m^n} \binom{mk}{mk - |\boldsymbol{\alpha}|, \boldsymbol{\alpha}}}.$$

Since we have seen that $\rho_{m,k}$ is constant, an application of 10.4.1 yields the following:

Corollary 10.5.1. *The orthonormal basis of $\left(H^0\left(\mathbb{CP}^n, \mathcal{O}(mk)\right), \langle ., .\rangle\right)$ given by*

$$\{s_{\boldsymbol{\alpha},m,k} \in H^0\left(\mathbb{CP}^n, \mathcal{O}(mk)\right) : \boldsymbol{\alpha} \in \mathbb{N}_0^n, |\boldsymbol{\alpha}| \leq mk\}$$

is a balanced basis.

Let us think of $m, l \in \mathbb{N}$ as fixed now. We now consider the induced metrics

$$\widehat{\omega}_k \stackrel{\text{def}}{=} \frac{1}{k}\partial\overline{\partial}\log\left(\sum_{|\boldsymbol{\alpha}|=lk}^{mk} |\widehat{s}_{\boldsymbol{\alpha},m,k}|^2\right).$$

10.5.2 Asymptotic behaviour of $\widehat{\omega}_k$ away from E

Let us investigate the behaviour of $\widehat{\omega}_k$ at points whose image under the moment map lie far enough away from the vertex $\mathbf{0} \in Simp_n(m)$ that we are blowing up. Recall also that $\widehat{\omega}_k$ depends on two parameters $l, m \in \mathbb{N}$, where $l < m$. We have the following result:

Lemma 10.5.2. *Let $\mu : \mathbb{CP}^n \to \mathfrak{t}^* \cong \mathbb{R}^n$ denote the choice of moment map discussed in chapter 4 and let $p \in \mathbb{CP}^n - \{[1 : 0 : \cdots : 0]\}$. Suppose that $\sum_{i=1}^n \mu(p)_i > l$. Then, for any $j \in \mathbb{N}_0$, we have*

$$\| \nabla^j \left(\pi_*\widehat{\omega}_k - \omega\right)\big|_p\| = \mathcal{O}\left(k^{-\infty}\right),$$

where ∇ denotes the connection corresponding to ω and $\|.\|$ the norm induced by ω.

Proof. We have $\pi_*\widehat{\omega}_k - \omega = \frac{i}{k}\partial\overline{\partial}\log\rho_{l,m,k}$, and we recall from chapter 4 that the partial density function $\rho_{l,m,k}$ is given by

$$\rho_{l,m,k}(\boldsymbol{\alpha}) = \frac{1}{(2\pi)^n}\frac{(mk+n)!}{(mk)!m^n} f_{l,m,k}\left(\frac{\sum_{i=1}^n \alpha_i}{m}\right),$$

for $\boldsymbol{\alpha} \in Simp_n(m)$, and

$$f_{l,m,k}(s) \stackrel{\text{def}}{=} \sum_{j=lk}^{mk} \binom{mk}{j} s^j (1-s)^{mk-j} \qquad \text{for } s \in [0, 1].$$

We are done if we can show that $f^{(j)}_{l,m,k}(s) = \mathcal{O}\left(k^{-\infty}\right)$ for $s > \frac{l}{m}$ and $j \in \mathbb{N}$ and $f_{l,m,k}(s) = 1 + \mathcal{O}\left(k^{-\infty}\right)$ for $s > \frac{l}{m}$. For $l = 0$, we have $f_{0,m,k}(s) = (s + 1 - s)^{mk} = 1$, and it is clear that $0 \leq f_{l,m,k} \leq 1$ for $k, l \in \mathbb{N}_0$, $0 \leq l \leq m$ and $s \in [0,1]$. We have

$$s^j(1-s)^{mk-j} = e^{-k\gamma_k(s)},$$

where $\gamma_k(s) \stackrel{\text{def}}{=} -\frac{1}{k}\left(j \log s + (mk-j)\log(1-s)\right)$. We note that $\gamma_k'(s) = \frac{smk-j}{ks(1-s)}$ and $\gamma_k''(s) = \frac{m}{s^2(1-s)^2}\left(s^2 - \frac{2j}{mk}s + \frac{j}{mk}\right)$, so that $\gamma_k'(s) = 0$ if and only if $s = \frac{j}{mk}$ and $\gamma_k''(s) > 0$ for $s \in (0,1)$. We have

$$\binom{mk}{j} e^{-k\gamma_k\left(\frac{j}{mk}\right)} = \frac{j^j(mk)!(mk-j)^{mk-j}}{j!(mk)^{mk}(mk-j)!} \sim \sqrt{\frac{mk}{2\pi j(mk-j)}} = \mathcal{O}(1) \quad \text{as } k \to \infty.$$

We now define $h_k(j,m,s) \stackrel{\text{def}}{=} \gamma_k(s) - \gamma_k\left(\frac{j}{mk}\right) \geq 0$. h_k is strictly convex and has minimum $h_k\left(j, m, \frac{j}{mk}\right) = 0$.

$$\begin{aligned} f_{l,m,k}(s) &= \sum_{j=lk}^{mk} \binom{mk}{j} e^{-k\gamma_k(s)} \\ &= \sum_{j=lk}^{mk} b_{j,m,k} e^{-kh_k(j,m,s)}, \end{aligned}$$

where $b_{j,m,k} = \mathcal{O}(1)$ as $k \to \infty$. If $s > \frac{l}{m}$, it is not hard to see that there exists $c > 0$ such that $h_k(j,m,s) > c$ for all $j < lk$ and all $k \in \mathbb{N}$. For $s > \frac{l}{m}$ we have

$$\sum_{j=0}^{lk-1} b_{j,m,k} e^{-kh_k(j,m,s)} = \mathcal{O}\left(k^{-\infty}\right).$$

Hence, for $j > 0$ and $s > \frac{l}{m}$,

$$\begin{aligned} &f_{l,m,k}(s) = 1 + \mathcal{O}\left(k^{-\infty}\right) \qquad \text{and} \\ &f^{(j)}_{l,m,k}(s) = \mathcal{O}\left(k^{-\infty}\right). \end{aligned}$$

The last statement above follows since the derivatives of $f_{l,m,k}$ are sums of terms which are products of two factors. The first is of polynomial order in k and the second factor is $e^{-kh_k(j,m,s)}$, which is $\mathcal{O}\left(k^{-\infty}\right)$ as $k \to \infty$ for $s > \frac{l}{m}$. The result follows. □

10.5.3 Behaviour of $\widehat{\omega}_k$ on E

Let us work in the chart $\mathcal{W}_1$ now. We have

$$\begin{aligned} \widehat{\omega}_k &= \frac{i}{k}\partial\overline{\partial}\log\left(\sum_{|\boldsymbol{\alpha}|=lk}^{mk} \left|\widehat{s}_{\boldsymbol{\alpha},m,k}(\boldsymbol{w})\right|^2\right) \\ &= \frac{i}{k}\partial\overline{\partial}\log \sum_{lk \leq j \leq mk} \sum_{|\boldsymbol{\alpha}|=j} \binom{mk}{mk-j, \boldsymbol{\alpha}} \left|w_1^{j-lk} w_2^{\alpha_2} \cdots w_n^{\alpha_n}\right|^2 \\ &= \frac{i}{k}\partial\overline{\partial}\log \sum_{lk \leq j \leq mk} \binom{mk}{j} \sum_{|\boldsymbol{\alpha}|=j} \frac{j!}{\boldsymbol{\alpha}!} \left|w_1^{\alpha_1}(w_1w_2)^{\alpha_2} \cdots (w_1w_n)^{\alpha_n}\right|^2 |w_1|^{-2lk} \end{aligned}$$

$$
\begin{aligned}
&= \frac{i}{k}\partial\overline{\partial}\log \sum_{j=0}^{(m-l)k} \binom{mk}{j+lk} |w_1|^{-2lk} \left(|w_1|^2 \left(1+|w_2|^2+\cdots+|w_n|^2\right)\right)^{j+lk} \\
&= il\partial\overline{\partial}\log\left(1+|w_1|^2+\cdots+|w_n|^2\right) \\
&\quad + \frac{i}{k}\partial\overline{\partial}\log \sum_{j=0}^{(m-l)k} \binom{mk}{j+lk} \left(|w_1|^2 \left(1+|w_2|^2+\cdots+|w_n|^2\right)\right)^{j}.
\end{aligned}
$$

We define

$$
\eta_k \stackrel{\text{def}}{=} \frac{i}{k}\partial\overline{\partial}\log \sum_{j=0}^{(m-l)k} \binom{mk}{j+lk} \left(|w_1|^2 \left(1+|w_2|^2+\cdots+|w_n|^2\right)\right)^{j}.
$$

Observe now that on $E \cap \mathcal{W}_1$.

$$
\begin{aligned}
\eta_k &= \frac{i}{k}\partial\overline{\partial}\log \sum_{j=0}^{(m-l)k} \frac{((m-l)k)!(lk)!}{((m-l)k-j)!(j+lk)!} |\pi(\boldsymbol{w})|^{2j} \\
&= \frac{i}{k}\partial\overline{\partial}\log\left(1+\frac{(m-l)k}{lk+1}|\pi(\boldsymbol{w})|^2\left(1+\mathcal{O}\left(|\pi(\boldsymbol{w})|^2\right)\right)\right) \\
&= i\frac{m-l}{lk+1}\partial\overline{\partial}\,|\pi(\boldsymbol{w})|^2 \\
&= i\frac{m-l}{lk+1}(1+|w_2|^2+\cdots+|w_n|^2)dw_1\wedge d\overline{w}_1.
\end{aligned}
$$

The calculation can similarly be carried out on the remaining charts. If we denote by $\sigma : \mathcal{W} \to \mathbb{CP}^{n-1}$ the projection onto $\mathbb{CP}^{n-1}$, we have:

Lemma 10.5.3. *On the exceptional divisor $E \subset \mathrm{Bl}_{[1:0:\cdots:0]}(\mathbb{CP}^n)$, we have*

$$
\widehat{\omega}_k = l\sigma^*\omega_{FS} + \mathcal{O}\left(k^{-1}\right).
$$

10.6 Example: Balanced blow-up of $\mathbb{C}^n$

10.6.1 Introduction

Let us now consider induced metrics on $\mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n)$ in detail. This is the simplest example of a blow-up of a non-compact manifold and therefore of special interest to us.

A balanced basis of sections of $\mathbb{C}^n \times \mathbb{C}$

In analogy with the compact case, we consider the polarization

$$
(L^k \stackrel{\text{def}}{=} \mathbb{C}^n \times \mathbb{C}, h^k) \to (\mathbb{C}^n, \omega),
$$

where $h^k = e^{-k\frac{\|\boldsymbol{z}\|^2}{2}}$ and $\omega = i\partial\overline{\partial}\frac{\|\boldsymbol{z}\|^2}{2}$. We work with the basis of sections

$$
\left\{ s_{\boldsymbol{\alpha},k}(\boldsymbol{z}) = \sqrt{\left(\frac{k}{2\pi}\right)^n \left(\frac{k}{2}\right)^{|\boldsymbol{\alpha}|} \frac{1}{\boldsymbol{\alpha}!}}\,\boldsymbol{z}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathbb{N}_0^n \right\}
$$

of L^k discussed previously (see chapter 2). Despite the fact that the notion of balanced bases was only defined for polarizations of a *compact* manifold, we have seen that the corresponding density function $\rho_k(\boldsymbol{z}) = \left(\frac{k}{2\pi}\right)^n$ on $\mathbb{C}^n$ is constant in the example above. Following [AL04, Section 5], we will therefore regard the basis of sections given above as balanced.

The induced metrics

We now consider the blowup $\pi : \mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n) \to \mathbb{C}^n$. For any $l, k \in \mathbb{N}$, the line bundle $\pi^* L^k \otimes \mathcal{O}(-lkE)$ has a basis of holomorphic sections given by the lift of the basis $\{s_{\boldsymbol{\alpha},k} : |\boldsymbol{\alpha}| \geq lk, \boldsymbol{\alpha} \in \mathbb{N}_0\}$ of

$$\mathcal{J}_{l,k} \stackrel{\text{def}}{=} \left\{ s \in H^0(\mathbb{C}^n, L^k) : s \text{ vanishes to order at least } lk \text{ at } \mathbf{0} \right\}.$$

We pick charts $\mathcal{W}_1, \cdots, \mathcal{W}_n$ of $\mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n) = \{(\boldsymbol{z}, [\boldsymbol{l}]) \in \mathbb{C}^n \times \mathbb{CP}^{n-1} : \boldsymbol{z} \in [\boldsymbol{l}]\}$ defined by $\mathcal{W}_j = \{(\boldsymbol{z}, [\boldsymbol{l}]) \in \mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n) : l_j \neq 0\}$ for $j \in \{1, \cdots, n\}$.

Let us work in the chart $\mathcal{W}_1 \cong \mathbb{C}^n$ and a corresponding trivialization on $\mathcal{W}_1 \times \mathbb{C}$ of $\pi^* L^k \otimes \mathcal{O}(-lk)$ now. On $\mathcal{W}_1$, we have $\pi(\boldsymbol{w}) = w_1(1, w_2, \cdots, w_n)$ for $\boldsymbol{w} = (w_1, \cdots, w_n) \in \mathbb{C}^n$. A global holomorphic section $s : \boldsymbol{z} \mapsto (\boldsymbol{z}, \boldsymbol{z}^{\boldsymbol{\alpha}})$ of L^k vanishing to order at least lk at $\mathbf{0}$ can be lifted to a section $\widehat{s} \in H^0(\mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n), \pi^* L^k \otimes \mathcal{O}(-lk))$ which is given locally on $\mathcal{W}_1 \times \mathbb{C}$ as $\boldsymbol{w} \mapsto (\boldsymbol{w}, w_1^{|\boldsymbol{\alpha}|-lk} w_1^{\alpha_2} \ldots w_n^{\alpha_n})$. We are now interested in the properties of the induced metrics $\widehat{\omega}_k$ on $\mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n)$. We have

$$\widehat{\omega}_k = \frac{i}{k} \partial\overline{\partial} \log \left(\sum_{|\boldsymbol{\alpha}| \geq lk} |\widehat{s}_{\boldsymbol{\alpha},k}|^2 \right).$$

Let us now check that $\widehat{\omega}_k$ really defines a Kähler form for all $k \in \mathbb{N}$ in this non-compact setting.

Proposition 10.6.1. *Let $P \subset \mathbb{R}^n$ be an integral Delzant polytope with finitely many vertices. Consider the associated toric basis $\{s_{\boldsymbol{\alpha}} \in H^0(X_P, L_P) : \boldsymbol{\alpha} \in P \cap \mathbb{Z}^n\}$ of $H^0(X_P, L_P)$. Fix constants $a_{\boldsymbol{\alpha}} \geq 0$, for $\boldsymbol{\alpha} \in P \cap \mathbb{Z}^n$, and, for any vertex $\boldsymbol{v}$ of P, denote by $m(\boldsymbol{v})_1, \cdots, m(\boldsymbol{v})_n$ the primitive edge vectors emanating from $\boldsymbol{v}$. The 2-form*

$$\omega = i\partial\overline{\partial} \log \sum_{\boldsymbol{\alpha} \in P \cap \mathbb{Z}^n} a_{\boldsymbol{\alpha}} |s_{\boldsymbol{\alpha}}|^2$$

defines a non-degenerate Kähler form on X_P if, for all $\boldsymbol{v} \in \mathrm{vertices}(P)$,

$$a_{\boldsymbol{v}}, a_{\boldsymbol{v}+m(\boldsymbol{v})_1}, \cdots, a_{\boldsymbol{v}+m(\boldsymbol{v})_n} > 0.$$

Proof. We just need to check that in each chart $\mathcal{U}_{\sigma_{\boldsymbol{v}}}$, for $\boldsymbol{v} \in \mathit{vertices}(P)$, $g(.,.) = \omega(., J.)$ is positive definite. Pick $\boldsymbol{v} \in \mathit{vertices}(P)$ and assume without loss of generality that we have normalized coordinates on $\mathcal{U}_{\sigma_{\boldsymbol{v}}}$ such that $P \subset \mathbb{R}^n_{\geq 0}$, $\boldsymbol{v} = \mathbf{0}$ and, for $i \in \{1, \cdots, n\}$, $m(\boldsymbol{v})_i = \boldsymbol{e}_i$, where $\boldsymbol{e}_i$ denotes the i^{th} standard basis element of $\mathbb{R}^n$. On $\mathcal{U}_{\sigma_{\boldsymbol{v}}} \cong \mathbb{C}^n$ with complex coordinates $\boldsymbol{z} = (z_1, \cdots, z_n)$, we have

$$\begin{aligned} \omega &= i\partial\overline{\partial} \log \sum_{\boldsymbol{\alpha} \in P \cap \mathbb{Z}^n} a_{\boldsymbol{\alpha}} |\boldsymbol{z}^{\boldsymbol{\alpha}}|^2 \\ &= i g_{i\overline{j}} dz_i \wedge d\overline{z}_j, \end{aligned}$$

where

$$g_{i\bar{j}}(\boldsymbol{u}) = \frac{1}{f(\boldsymbol{u})^2}\Big(\left(f_{ij}(\boldsymbol{u})f(\boldsymbol{u}) - f_i(\boldsymbol{u})f_j(\boldsymbol{u})\right)\bar{z}_i z_j + \delta_{ij} f_i(\boldsymbol{u})f(\boldsymbol{u})\Big),$$

and we define $\boldsymbol{u} \overset{\text{def}}{=} \boldsymbol{u}(\boldsymbol{z}) = (|z_1|^2, \cdots, |z_n|^2)$ and $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$, $f : \boldsymbol{u} \mapsto \sum_{\boldsymbol{\alpha}\in P\cap\mathbb{Z}^n} a_{\boldsymbol{\alpha}} \boldsymbol{u}^{\boldsymbol{\alpha}}$, so that

$$\sum_{\boldsymbol{\alpha}\in P\cap\mathbb{Z}^n} a_{\boldsymbol{\alpha}} \left|s_{\boldsymbol{\alpha}}(\boldsymbol{z})\right|^2 = f\left(|z_1|^1, \cdots, |z_n|^2\right) = f(\boldsymbol{u}).$$

The matrix representation of the metric in $(\boldsymbol{x}, \boldsymbol{y})$-coordinates, where $z_j = x_j + iy_j$ for $j \in \{1, \cdots, n\}$, is

$$g = 2\begin{pmatrix} \Re(g_{i\bar{j}}) & \Im(g_{i\bar{j}}) \\ -\Im(g_{i\bar{j}}) & \Re(g_{i\bar{j}}) \end{pmatrix}.$$

We just have to show that this matrix is positive definite for all $\boldsymbol{u} \in \mathbb{R}^n_{\geq 0}$. Let $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^n$ and $\boldsymbol{u} = (|z_1|^2, \cdots, |z_n|^2) \in \mathbb{R}^n_{\geq 0}$. Define $R_{ij} \overset{\text{def}}{=} \Re(g_{i\bar{j}})$, $I_{ij} \overset{\text{def}}{=} \Im(g_{i\bar{j}})$ for $i, j \in \{1, \cdots, n\}$.

$$\begin{aligned}
S(\boldsymbol{u}) &\overset{\text{def}}{=} \sum_{i,j=1}^{n} \boldsymbol{b}^t R(\boldsymbol{u})\boldsymbol{b} + 2\boldsymbol{b}^t I(\boldsymbol{u})\boldsymbol{c} + \boldsymbol{c}^t R(\boldsymbol{u})\boldsymbol{c} \\
&= \frac{1}{f(\boldsymbol{u})^2} \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in P\cap\mathbb{Z}^n} a_{\boldsymbol{\alpha}} a_{\boldsymbol{\beta}} \boldsymbol{u}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \sum_{i,j=1}^{n} \left(\frac{\alpha_i\alpha_j - \alpha_i\beta_j}{u_i u_j}\right)\Big(b_i\left(x_i x_j + y_i y_j\right) b_j \\
&\quad + 2b_i\left(x_i y_j - x_j y_i\right) c_j + c_i\left(x_i x_j + y_i y_j\right) c_j\Big) \\
&= \frac{1}{2f(\boldsymbol{u})^2} \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in P\cap\mathbb{Z}^n} a_{\boldsymbol{\alpha}} a_{\boldsymbol{\beta}} \boldsymbol{u}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \sum_{i,j=1}^{n} \left(\frac{\alpha_i\alpha_j - \alpha_i\beta_j - \beta_i\alpha_j + \beta_i\beta_j}{u_i u_j}\right)\Big(b_i\left(x_i x_j + y_i y_j\right) b_j \\
&\quad + 2b_i\left(x_i y_j - x_j y_i\right) c_j + c_i\left(x_i x_j + y_i y_j\right) c_j\Big) \\
&= \frac{1}{2f(\boldsymbol{u})^2} \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in P\cap\mathbb{Z}^n} a_{\boldsymbol{\alpha}} a_{\boldsymbol{\beta}} \boldsymbol{u}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \left\{\Big(\sum_{i=1}^{n}(x_i b_i + y_i c_i)\frac{\alpha_i - \beta_i}{u_i}\Big)^2\right. \\
&\quad \left. + \Big(\sum_{i=1}^{n}(y_i b_i - x_i c_i)\frac{\alpha_i - \beta_i}{u_i}\Big)^2\right\} \\
&= \frac{a_{\boldsymbol{0}}}{f(\boldsymbol{u})^2} \sum_{j=1}^{n} a_{\boldsymbol{e}_j}\left(b_j^2 + c_j^2\right) \\
&\quad + \frac{1}{2f(\boldsymbol{u})^2} \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in P\cap\mathbb{Z}^n \\ |\boldsymbol{\alpha}+\boldsymbol{\beta}|>1}} a_{\boldsymbol{\alpha}} a_{\boldsymbol{\beta}} \boldsymbol{u}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \left\{\Big(\sum_{i=1}^{n}(x_i b_i + y_i c_i)\frac{\alpha_i - \beta_i}{u_i}\Big)^2\right. \\
&\quad \left. + \Big(\sum_{i=1}^{n}(y_i b_i - x_i c_i)\frac{\alpha_i - \beta_i}{u_i}\Big)^2\right\}.
\end{aligned}$$

We now see that $S(\boldsymbol{u}) \geq 0$ with equality occurring if and only if $\boldsymbol{b} = \boldsymbol{c} = \boldsymbol{0}$. Hence, the tensor $g = \omega(., J.)$ associated to ω is positive definite on the chart $\mathcal{U}_{\sigma_{\boldsymbol{v}}}$ and g is indeed a Kähler metric. □

Corollary 10.6.2. *The* 2*-form*

$$\widehat{\omega}_k = \frac{i}{k}\partial\overline{\partial}\log\left(\sum_{|\boldsymbol{\alpha}|\geq lk} \left|\widehat{s}_{\boldsymbol{\alpha},k}\right|^2\right).$$

on $\mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n)$*, which we discussed above, is a Kähler form.*

Proof. Consider the non-compact polytope $P \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n x_i \geq lk\}$. Noting that each vertex of P satisfies the conditions listed in the definition of a Delzant polytope, we can associate a non-compact manifold X_P and a line bundle $L_P \to X_P$ to P by gluing $\mathbb{C}^n$-charts corresponding to the vertices of P in the same way as in the compact case (see chapter 3).

Comparing this with the charts of $\mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n)$ introduced above, we realise that the $\mathcal{W}_i$-chart is exactly the chart corresponding to the vertex $(0, \cdots, 0, lk, 0, \cdots, 0)$, where the nonzero entry occurs in the i^{th} position. We have $X_P = \mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n)$ and $L_P = \pi^* L^k \otimes \mathcal{O}(-lkE)$. Similarly, a basis of holomorphic sections of L_P corresponds to integral points of P. We note that the proof of proposition 10.6.1 still applies in this non-compact setting. Also note that we are now working with a converging sum of infinitely many terms here. $\square$

10.6.2 Asymptotic behaviour of $\widehat{\omega}_k$ away from E

Let us recall the notion of an Asymptotically Euclidean metric.

Definition 10.6.3 ([Joy01])**.** Let X be a non-compact manifold of dimension n, and let g be a Riemannian metric on X. We say that (X, g) is an *Asymptotically Euclidean manifold of order* j or an *AE manifold of order* j for short, and we say that g is an *AE metric of order* j, if the following conditions hold. There should exist a compact subset $S \subset X$ and a map $\pi : X - S \to \mathbb{R}^n$ that is a diffeomorphism between $X - S$ and the subset $\{z \in \mathbb{R}^n : \|z\| > R\}$ for some fixed $R > 0$. Under this diffeomorphism, the push-forward metric $\pi_*(g)$ should satisfy

$$\nabla^s \left(\pi_* g - g_{Eucl}\right) = \mathcal{O}\left(r^{-j-s}\right)$$

on $\{z \in \mathbb{R}^n : \|z\| > R\}$ for all $s \in \mathbb{N}_0$, where $r(\boldsymbol{z}) \stackrel{\text{def}}{=} \|\boldsymbol{z}\|$, for $\boldsymbol{z} \in \mathbb{R}^n$, denotes the radius function. Here, ∇ denotes the Levi-Civita connection of the Euclidean metric g_{Eucl}. We shall call the map $\pi : X - S \to \mathbb{R}^n$ an *asymptotic coordinate system* for X.

Lemma 10.6.4. *Suppose that* $\eta \stackrel{\text{def}}{=} i\partial\overline{\partial} f(\|\boldsymbol{z}\|)$ *on* $\mathbb{C}^n - \{\mathbf{0}\}$ *for some smooth function* $f : \mathbb{R}_{>0} \to \mathbb{R}$*. Then the tensor* $\nu = \eta(., J.)$ *satisfies*

$$\|\nabla^s \nu\| = \mathcal{O}\left(r^{-j-s}\right) \text{ as } r \to \infty$$

for some $j \in \mathbb{N}_0$ *and all* $s \in \mathbb{N}_0$ *if*

$$f^{(1+s)}(r) = \mathcal{O}\left(r^{1-(j+s)}\right) \qquad \text{for all } s \in \mathbb{N}_0 \text{ as } r \to \infty.$$

Here, ∇ *denotes the Euclidean connection on* $\mathbb{C}^n \cong \mathbb{R}^{2n}$*, and* $\|.\|$ *is taken with respect to the Euclidean metric.*

Proof. Take coordinates $(z_1, \cdots, z_n) = (x_1 + iy_1, \cdots, x_n + iy_n)$ on $\mathbb{C}^n$. We compute that the representation of ν in $(\boldsymbol{x}, \boldsymbol{y})$ coordinates as a $2n \times 2n$-matrix is:

$$\begin{aligned}\nu = {} & \frac{f''(r)}{4} \begin{pmatrix} (r_{x_i} r_{x_j} + r_{y_i} r_{y_j})_{ij} & (r_{x_j} r_{y_i} - r_{y_j} r_{x_i})_{ij} \\ -(r_{x_j} r_{y_i} - r_{y_j} r_{x_i})_{ij} & (r_{x_i} r_{x_j} + r_{y_i} r_{y_j})_{ij} \end{pmatrix} \\ & + \frac{f'(r)}{4} \begin{pmatrix} (r_{x_i x_j} + r_{y_i y_j})_{ij} & (r_{x_j y_i} - r_{y_j x_i})_{ij} \\ -(r_{x_j y_i} - r_{y_j x_i})_{ij} & (r_{x_i x_j} + r_{y_i y_j})_{ij} \end{pmatrix},\end{aligned}$$

where $1 \leq i, j \leq n$, and we use the notation $(a_{ij})_{ij}$ to denote the matrix with entries a_{ij}. We also use the shorthand $r_{x_i x_j} = \frac{\partial^2}{\partial x_i \partial x_j} r$ etc., where r denotes the radius function on $\mathbb{C}^n$. Observe that, for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^n$, $\frac{\partial}{\partial \boldsymbol{x}}^{\boldsymbol{\alpha}} \frac{\partial}{\partial \boldsymbol{y}}^{\boldsymbol{\beta}} r = \mathcal{O}\left(r^{1-(|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|)}\right)$ as $r \to \infty$. Investigating the asymptotics of $\frac{\partial}{\partial \boldsymbol{x}}^{\boldsymbol{\alpha}} \frac{\partial}{\partial \boldsymbol{y}}^{\boldsymbol{\beta}}$ applied to each of the entries of the above matrices yields the result. □

Example 10.6.5. We have in particular that, for $\eta = i\partial\overline{\partial}\|\boldsymbol{z}\|^{2-j}$, $\nu \overset{\text{def}}{=} \eta(.,J.)$ satisfies

$$\|\nabla^s \nu\| = \mathcal{O}\left(r^{-j-s}\right) \text{ as } r \to \infty.$$

Lemma 10.6.6. *For each fixed $k \in \mathbb{N}$, the Riemannian metric corresponding to $\widehat{\omega}_k$ on $\mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n)$ that we discussed above is Asymptotically Euclidean on $\mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n) - E$ for arbitrarily high order $j \in \mathbb{N}_0$.*

Proof. We recall that on $X - \{\mathbf{0}\}$

$$\pi_* \widehat{\omega}_k - \omega_{Eucl} = \frac{i}{k}\partial\overline{\partial}\log(\rho_{l,k}).$$

Let us denote by g_{Eucl} the Euclidean metric corresponding to $\omega_{Eucl} \overset{\text{def}}{=} \frac{i}{2}\partial\overline{\partial}\|\boldsymbol{z}\|^2$ and by $\pi_*\widehat{g}_k$ the Riemannian metric corresponding to $\pi_*\widehat{\omega}_k$. We have, for $\nu_k(.,.) \overset{\text{def}}{=} \frac{i}{k}\partial\overline{\partial}\log\rho_{l,k}(.,J.)$,

$$\|\nabla^j\left(\pi_*\widehat{g}_k - g_{Eucl}\right)\| = \|\nabla^j \nu_k\|.$$

We have seen in chapter 2 that

$$\rho_{l,k}(\boldsymbol{z}) = \left(\frac{k}{2\pi}\right)^n \left(1 - \frac{\Gamma\left(lk, \frac{k\|\boldsymbol{z}\|^2}{2}\right)}{(kl-1)!}\right).$$

Let $f(r) \overset{\text{def}}{=} \log g(r)$ and $g(r) \overset{\text{def}}{=} 1 - \frac{\Gamma\left(lk, \frac{kr^2}{2}\right)}{(kl-1)!}$ for $r \in \mathbb{R}_{\geq 0}$. We apply lemma 10.6.4 to f. Observe that the r-derivatives of f are finite sums of fractions $\frac{p(r)}{q(r)}$, where $q(r) = g(r)^l$ for some $l > 0$ and p is a polynomial in derivatives of $\frac{\partial}{\partial r}\Gamma\left(lk, \frac{kr^2}{2}\right) = -e^{-\frac{kr^2}{2}}\left(\frac{kr^2}{2}\right)^{lk-1} kr$. We also have $g(r) \to 1$ as $r \to \infty$. It is hence clear that all derivatives of f are $\mathcal{O}\left(r^{-\infty}\right)$ as $r \to \infty$. The result now follows from lemma 10.6.4. □

Finally, we have the following lemma describing the asymptotics in k:

Lemma 10.6.7. *For $\boldsymbol{z} \in \mathbb{C}^n$ such that $\|\boldsymbol{z}\| > \sqrt{2l} > 0$ and for $j \in \mathbb{N}_0$, we have*

$$\|\left.\nabla^j\right|_{\boldsymbol{z}}\left(\pi_*\widehat{\omega}_k - \omega_{Eucl}\right)\| = \mathcal{O}\left(k^{-\infty}\right),$$

where ∇ is the Euclidean connection, and $\|.\|$ is taken with respect to g_{Eucl}.

Proof. Following similar reasoning as in lemma 10.5.2, we observe that, for $\boldsymbol{z} \in \mathbb{C}^n - \{\mathbf{0}\}$,

$$\begin{aligned}\pi_*\widehat{\omega}_k - \left.\omega_{Eucl}\right|_{\boldsymbol{z}} &= \frac{i}{k}\partial\overline{\partial}\log\rho_{l,k}(\boldsymbol{z}) \\ &= \frac{i}{k}\partial\overline{\partial}\log\left(f_{l,k}(\|\boldsymbol{z}\|)\right),\end{aligned}$$

where we use equation 2.2.2, recall the strictly convex function $h(y) = y - l\log(y+l) + l\log(l)$, for $y \in (-l, \infty)$, from chapter 2 and define

$$f_{l,k}(r) \stackrel{\text{def}}{=} \int_{-l}^{\frac{r^2}{2}-l} \frac{1}{y+l} e^{-kh(y)} dy \quad \text{for } r \in \mathbb{R}_{\geq 0}.$$

In chapter 2, we noted that h achieves its absolute minimum at 0 and $h(0) = 0$. It is not hard to see that, for $r > \sqrt{2l}$, $f_{l,k}(r) = ak^{-\frac{1}{2}} + \mathcal{O}\left(k^{-1}\right)$ for some $a \neq 0$ and that $f_{l,k}^{(j)}(r) = \mathcal{O}\left(k^{-\infty}\right)$ for all $j \in \mathbb{N}$. The lemma now follows by expanding $\nabla^j \left(\frac{i}{k}\partial\overline{\partial}\log f_{l,k}(\|\boldsymbol{z}\|)\right)$ in terms of $f_{l,k}$ and its derivatives. $\square$

10.6.3 Behaviour of $\widehat{\omega}_k$ on E

Let us now analyse the behaviour of $\widehat{\omega}_k$ near the exceptional divisor E. We work again in the coordinate patch $\mathcal{W}_1$.

$$\begin{aligned}
\widehat{\omega}_k &= \frac{i}{k}\partial\overline{\partial}\log \sum_{|\boldsymbol{\alpha}|\geq lk} \left|\widehat{s}_{\boldsymbol{\alpha},k}\right|^2 \\
&= \frac{i}{k}\partial\overline{\partial}\log \left(\frac{k}{2\pi}\right)^n \sum_{|\boldsymbol{\alpha}|\geq lk} \left(\frac{k}{2}\right)^{|\boldsymbol{\alpha}|} \frac{1}{\boldsymbol{\alpha}!} \left|w_1\right|^{2(|\boldsymbol{\alpha}|-lk)} \left|w_2\right|^{2\alpha_2} \dots \left|w_n\right|^{2\alpha_n} \\
&= \frac{i}{k}\partial\overline{\partial}\log \sum_{j=lk}^{\infty} \left(\frac{k}{2}\right)^j \frac{1}{j!} \left|w_1\right|^{2(j-lk)} \sum_{|\boldsymbol{\alpha}|=j} \frac{j!}{\boldsymbol{\alpha}!} 1^{2\alpha_1} \left|w_2\right|^{2\alpha_2} \dots \left|w_n\right|^{2\alpha_n} \\
&= \frac{i}{k}\partial\overline{\partial}\log \sum_{j=lk}^{\infty} \left(\frac{k}{2}\right)^j \frac{1}{j!} \left|w_1\right|^{2(j-lk)} \left(1 + \left|w_2\right|^2 + \dots + \left|w_n\right|^2\right)^j.
\end{aligned}$$

Let us denote

$$\begin{aligned}
\|\boldsymbol{z}\|^2 &= \left|w_1\right|^2 \left(1 + \left|w_2\right|^2 + \dots + \left|w_n\right|^2\right) \\
\|\boldsymbol{u}\|^2 &= \left|w_2\right|^2 + \dots + \left|w_n\right|^2,
\end{aligned}$$

where $\|\boldsymbol{z}\| = \|\pi(\boldsymbol{w})\|$. We have

$$\begin{aligned}
\widehat{w}_k &= \frac{i}{k}\partial\overline{\partial}\log \sum_{j=0}^{\infty} \left(\frac{k}{2}\right)^{j+lk} \frac{1}{(j+lk)!} \|\boldsymbol{z}\|^{2j} (1 + \|\boldsymbol{u}\|^2)^{lk} \\
&= li\partial\overline{\partial}\log\left(1 + \|\boldsymbol{u}\|^2\right) + \frac{i}{k}\partial\overline{\partial}\log \sum_{j=0}^{\infty} \frac{(lk)!}{(j+lk)!} \left(\frac{k\|\boldsymbol{z}\|^2}{2}\right)^j \\
&= li\partial\overline{\partial}\log\left(1 + \|\boldsymbol{u}\|^2\right) + \frac{i}{k}\partial\overline{\partial}\log \left(1 + \frac{k}{2(lk+1)}\|\boldsymbol{z}\|^2 \left(1 + \mathcal{O}\left(\|\boldsymbol{z}\|^2\right)\right)\right).
\end{aligned}$$

We note that $i\partial\overline{\partial}\log(1 + \|\boldsymbol{u}\|^2)$ is the Fubini-Study Kähler form on $E \cong \mathbb{CP}^{n-1}$. Let us denote the projection $\mathrm{Bl}_{\mathbf{0}}\left(\mathbb{C}^n\right) \to \mathbb{CP}^{n-1}$ by σ. It is given by the restriction of the projection $\sigma : \mathbb{C}^n \times \mathbb{CP}^{n-1} \to \mathbb{CP}^{n-1}$ to $\mathrm{Bl}_{\mathbf{0}}\left(\mathbb{C}^n\right)$. We have on $\mathcal{W}_1 \cap E = \mathcal{Z}(w_1)$ that

$$\widehat{\omega}_k = l\sigma^*\omega_{FS} + \frac{i}{2(lk+1)} \left(1 + \left|w_2\right|^2 + \dots + \left|w_2\right|^2\right) dw_1 \wedge d\overline{w}_1.$$

More invariantly, we have:

Lemma 10.6.8. *On the exceptional divisor $E \subset \mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n)$,*

$$\widehat{\omega}_k = l\sigma^*\omega_{FS} + i\frac{\partial\overline{\partial}\|\pi\|^2}{2(lk+1)}.$$

Proof. The above calculation on $\mathcal{W}_1$ equally carries through on $\mathcal{W}_2, \cdots, \mathcal{W}_n$. □

Remark 10.6.9. *Note the similarities between lemma 10.6.8 and lemma 10.5.3 and also between lemma 10.6.7 and lemma 10.5.2. The Fubini-Study metric on the exceptional divisor E is in a sense "glued" to a metric which converges to the original metric away from E.*

10.6.4 Discussion of $\mathrm{Scal}(\widehat{g}_k)$

Since $\widehat{\omega}_k = i\partial\overline{\partial}f(\|\boldsymbol{z}\|^2)$, where

$$f(u) = \frac{1}{k}\left(\frac{ku}{2} - \log\left(1 - \frac{\Gamma\left(lk, \frac{ku}{2}\right)}{(kl-1)!}\right)\right) \quad \text{for } u \in \mathbb{R}_{\geq 0},$$

the metric $\pi_*\widehat{g}_k$ on $\mathbb{C}^n - \{\mathbf{0}\}$ is $U(n)$ invariant, and the formula for the scalar curvature simplifies. We find that $\mathrm{Scal}(\pi_*\widehat{g})$ is given by

$$\mathrm{Scal}(\pi_*\widehat{g})(u) = \frac{f'(u)\left(g'(u) + ug''(u)\right) + (n-1)g'(u)\left(f'(u) + uf''(u)\right)}{nf'(u)\left(f'(u) + uf''(u)\right)},$$

where $g(u) = \log\left(n!\left(f'(u)\right)^{n-1}\left(f'(u) + uf''(u)\right)\right)$ for $u \in \mathbb{R}_{\geq 0}$. This formula and its derivation can be found in Simanca's paper [Sim91]. It is also worth mentioning that, in this paper, Simanca proves the existence of a cscK metric on $\mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^n)$ thought of as the total space of the line bundle $\mathcal{O}(-1) \to \mathbb{CP}^{n-1}$. More precisely,

Theorem 10.6.10 (Simanca [Sim91])**.** *Let δ be a real constant. Then, blowing up a sufficiently small symmetric neighborhood of the origin in $\mathbb{C}^n$, we obtain a disk bundle $\pi : D \to \mathbb{CP}^{n-1}$ whose total space carries a complete Kähler metric of constant scalar curvature δ with radially symmetric Kähler potential. If $\delta \leq 0$ the bundle D can be taken to be the universal line bundle $\mathcal{O}(-1) \to \mathbb{CP}^{n-1}$ of Chern class -1, while in the case where $\delta > 0$, the bundle D is properly contained in L.*

Remark 10.6.11. *Simanca's theorem generalizes the well known Burns metric on $\mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^2)$ obtained by restricting the product of the Fubini-Study and Euclidean metric on $\mathbb{C}^2 \times \mathbb{CP}^1$ to $\mathrm{Bl}_{\mathbf{0}}(\mathbb{C}^2) \subset \mathbb{C}^2 \times \mathbb{CP}^1$.*

Figure 10.1 illustrates how $\mathrm{Scal}(\pi_*\widehat{g})$, thought of as a function of $u = \|\boldsymbol{z}\|^2$ changes with $k \in \{1, 2, 3, 4\}$ for fixed $l = 10$. Here, $n = 2$ and the graph for $k = 1$ is the one having the lowest limiting value near 0. These values then increase with k. We can clearly spot the transitioning behaviour at $u = 2l = 20$ and the rapid decay of $\mathrm{Scal}(u)$ as $u \to \infty$.

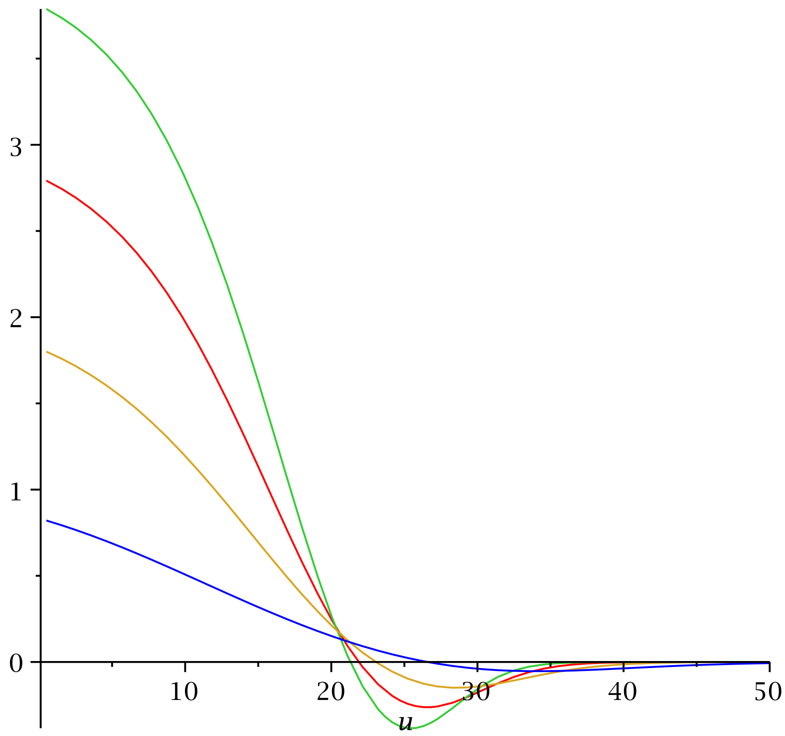

Figure 10.1: $\mathrm{Scal}\,(\pi_* \widehat{g})$ for $l = 10$ and $k \in \{1, 2, 3, 4\}$ in the variable $u = \|\boldsymbol{z}\|^2$ in dimension $n = 2$.

Appendix A

Technical Results

After reviewing some specific error term estimates in the smooth and holomorphic versions of Taylor's theorem, we prove an estimate for holomorphic functions with vanishing in A.1.4. In A.2, we then discuss a type of integral that will play a role similar to the *generalized error functions* which we defined in chapter 2.

A.1 Taylor's error term estimates

Let us first recall an estimate for the remainder in Taylor's theorem. We define $\overline{\mathcal{B}}_r(\boldsymbol{a}) \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathbb{R}^m : \|\boldsymbol{x} - \boldsymbol{a}\| \leq r\}$ for $\boldsymbol{a} \in \mathbb{R}^m$ and $r \geq 0$.

Theorem A.1.1 (Taylor's theorem in several variables)**.** *Let $f : \mathcal{U} \to \mathbb{R}^m$ be a smooth function, where $\mathcal{U} \subset \mathbb{R}^m$ is an open set and assume that $\overline{\mathcal{B}}_r(\boldsymbol{a}) \subset \mathcal{U}$, for some $r > 0$ and $\boldsymbol{a} \in \mathbb{R}^m$. Then, for any $n \in \mathbb{N}_0$ and $\boldsymbol{x} \in \overline{\mathcal{B}}_r(\boldsymbol{a})$, we have*

$$f(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|=0}^{n} \frac{1}{\boldsymbol{\alpha}!} \left.\frac{\partial}{\partial \boldsymbol{x}}^{\boldsymbol{\alpha}} f\right|_{\boldsymbol{a}} (\boldsymbol{x} - \boldsymbol{a})^{\boldsymbol{\alpha}} + \sum_{|\boldsymbol{\alpha}|=n+1} R_{\boldsymbol{\alpha}}(\boldsymbol{x})(\boldsymbol{x} - \boldsymbol{a})^{\boldsymbol{\alpha}},$$

for some functions $R_{\boldsymbol{\alpha}} : \mathcal{U} \to \mathbb{R}$ satisfying

$$|R_{\boldsymbol{\alpha}}(\boldsymbol{x})| \leq \frac{1}{\boldsymbol{\alpha}!} \sup_{\boldsymbol{s} \in \overline{\mathcal{B}}_r(\boldsymbol{a})} \left.\frac{\partial}{\partial \boldsymbol{x}}^{\boldsymbol{\alpha}} f\right|_{\boldsymbol{s}},$$

for $\boldsymbol{x} \in \overline{\mathcal{B}}_r(\boldsymbol{a})$, and where we use multi-index notation for $\boldsymbol{\alpha} \in \mathbb{N}_0^m$.

We now review the analogue of Taylor's theorem for a holomorphic function. Let $\boldsymbol{a} \in \mathbb{C}^m$. For $r > 0$, we define the polydisc

$$\mathcal{P}_r^m(\boldsymbol{a}) = \{\boldsymbol{z} \in \mathbb{C}^m : |z_i - a_i| < r \text{ for all } i \in \{1, \cdots, m\}\}$$

and denote by $\overline{\mathcal{P}}_r^m(\boldsymbol{a})$ its closure. Recall that

$$\frac{1}{1-z} = \sum_{j=0}^{n} z^j + z^{n+1} \frac{1}{1-z},$$

for $z \in \mathbb{C} - \{1\}$. Similarly, for $s, z \in \mathbb{C}$ with $s \neq z$ and $s \neq 0$, we can deduce from the above that

$$\frac{1}{s-z}=\sum_{j=0}^{n}\frac{z^j}{s^{j+1}}+z^{n+1}\frac{1}{(s-z)s^{n+1}}.$$

Now suppose that $f:\mathcal{U}\to\mathbb{C}$ is holomorphic on $\mathcal{U}$, where $\mathcal{U}$ is an open subset of $\mathbb{C}$. Suppose that $\overline{\mathcal{P}}^1_r(0)\subset\mathcal{U}$ for some $r>0$. Then, by Cauchy's integral formula, we have

$$\begin{aligned}
f(z)&=\frac{1}{2\pi i}\int_{|s|=r}\frac{f(s)}{s-z}ds\\
&=\frac{1}{2\pi i}\left(\sum_{j=1}^{n}z^j\int_{|s|=r}\frac{f(s)}{s^{j+1}}ds+z^{n+1}\int_{|s|=r}\frac{f(s)}{(s-z)s^{n+1}}ds\right)\\
&=\sum_{j=0}^{n}\frac{1}{j!}\left.\frac{\partial}{\partial z}^{j}f\right|_0 z^j+R_{n+1}(z)z^{n+1},
\end{aligned}$$

for $|z|<r$, and where we define

$$R_{n+1}(z)\overset{\text{def}}{=}\frac{1}{2\pi i}\int_{|s|=r}\frac{f(s)}{(s-z)s^{n+1}}ds.$$

If $|z|\leq\frac{r}{2}$, we have

$$|R_{n+1}(z)|\leq\frac{2}{r^{n+1}}\sup_{s\in\mathcal{P}^1_r(0)}|f(s)|.$$

From this, the holomorphic Taylor theorem in one variable follows.

Theorem A.1.2 (A version of the holomorphic Taylor theorem in one variable)**.** *Let $f:\mathcal{U}\to\mathbb{C}$ be a holomorphic function on $\mathcal{U}$, where $\mathcal{U}$ is an open subset of $\mathbb{C}$, and assume that $\mathcal{P}^1_r(a)\subset\mathcal{U}$ for some $r>0$ and $a\in\mathcal{U}$. Then, for any $n\in\mathbb{N}_0$ and $z\in\overline{\mathcal{P}}^1_{\frac{r}{2}}(a)$, we have*

$$f(z)=\sum_{j=0}^{n}\frac{1}{j!}\left.\frac{\partial}{\partial z}^{j}f\right|_a(z-a)^j+R_{n+1}(z)(z-a)^{n+1},$$

where $R_{n+1}:\mathcal{P}^1_{\frac{r}{2}}(a)\to\mathbb{C}$ is a function satisfying

$$|R_{n+1}(z)|\leq\frac{2}{r^{n+1}}\sup_{s\in\overline{\mathcal{P}}^1_r(a)}|f(s)|.$$

We can now generalize this to m complex dimensions. Suppose $f:\mathcal{U}\to\mathbb{C}$ is holomorphic on an open set $\mathcal{U}\subset\mathbb{C}^m$, and suppose that $\overline{\mathcal{P}}^m_r(\mathbf{0})\subset\mathcal{U}$. Let $\boldsymbol{z}\in\overline{\mathcal{P}}^m_{\frac{r}{2}}(\mathbf{0})$. $r'\overset{\text{def}}{=}\max\{|z_j|:j\in\{1,\ldots,m\}\}$. We assume that $r'>0$ since the result is trivial otherwise. Let $u(s)=s\boldsymbol{z}$ for $s\in\mathbb{C}$. We have

$$\begin{aligned}
f(\boldsymbol{z})=f(u(1))&=\frac{1}{2\pi i}\int_{|s|=\frac{r}{r'}}\frac{f(u(s))}{s-1}ds\\
&=\frac{1}{2\pi i}\left(\sum_{j=0}^{n}\int_{|s|=\frac{r}{r'}}\frac{f(u(s))}{s^{j+1}}ds+\int_{|s|=\frac{r}{r'}}\frac{f(u(s))}{(s-1)s^{n+1}}ds\right).
\end{aligned}$$

We note that
$$\left.\frac{\partial}{\partial s}^{j} f\circ u\right|_{0} = \sum_{|\boldsymbol{\alpha}|=j}\binom{j}{\boldsymbol{\alpha}}\left.\frac{\partial}{\partial \boldsymbol{z}}^{\boldsymbol{\alpha}} f\right|_{\boldsymbol{0}} \boldsymbol{z}^{\boldsymbol{\alpha}},$$

so that

$$f(\boldsymbol{z}) = \sum_{|\boldsymbol{\alpha}|=0}^{n} \frac{1}{\boldsymbol{\alpha}!}\left.\frac{\partial}{\partial \boldsymbol{z}}^{\boldsymbol{\alpha}} f\right|_{\boldsymbol{0}} \boldsymbol{z}^{\boldsymbol{\alpha}} + R_{n+1}(\boldsymbol{z}),$$

where
$$R_{n+1}(\boldsymbol{z}) = \frac{1}{2\pi i}\int_{|s|=\frac{r}{r'}} \frac{f(u(s))}{(s-1)s^{n+1}}ds.$$

For $\|\boldsymbol{z}\| \leq \frac{r}{2}$ we then have $r' \leq \frac{r}{2}$, so that

$$\begin{aligned}|R_{n+1}(\boldsymbol{z})| &\leq \frac{1}{\frac{r}{r'}-1}\left(\frac{r'}{r}\right)^{n} \sup_{|s|=\frac{r}{r'}} |f(u(s))| \\ &\leq \|\boldsymbol{z}\|^{n+1}\frac{2}{r^{n+1}} \sup_{\boldsymbol{w}\in\overline{\mathcal{P}}^{m}_{r}(\boldsymbol{0})} |f(\boldsymbol{w})|\,.\end{aligned}$$

We have proved the holomorphic version of Taylor's theorem in several variables.

Theorem A.1.3 (A version of the holomorphic Taylor theorem in several variables)**.** *Let $f : \mathcal{U} \to \mathbb{C}$ be a holomorphic function on $\mathcal{U}$, where $\mathcal{U}$ is an open subset of $\mathbb{C}^m$, and assume that $\overline{\mathcal{P}}^{m}_{r}(\boldsymbol{a}) \subset \mathcal{U}$ for some $r > 0$ and $\boldsymbol{a} \in \mathcal{U}$. Then, for any $n \in \mathbb{N}_0$ and $z \in \overline{\mathcal{P}}^{m}_{\frac{r}{2}}(\boldsymbol{a})$, we have*

$$f(\boldsymbol{z}) = \sum_{|\boldsymbol{\alpha}|=0}^{n} \frac{1}{\boldsymbol{\alpha}!}\left.\frac{\partial}{\partial \boldsymbol{z}}^{\boldsymbol{\alpha}} f\right|_{\boldsymbol{a}} (\boldsymbol{z}-\boldsymbol{a})^{\boldsymbol{\alpha}} + R_{n+1}(\boldsymbol{z}),$$

where

$$|R_{n+1}(\boldsymbol{z})| \leq \|\boldsymbol{z}-\boldsymbol{a}\|^{n+1}\frac{2}{r^{n+1}} \sup_{\boldsymbol{w}\in\overline{\mathcal{P}}^{m}_{r}(\boldsymbol{a})} |f(\boldsymbol{w})|\,.$$

Let us now derive a little corollary from this.

Corollary A.1.4 (Holomorphic Taylor theorem with vanishing)**.** *Suppose that $f : \mathcal{U} \to \mathbb{C}$ is holomorphic on $\mathcal{U}$, where $\mathcal{U}$ is an open subset of $\mathbb{C}^m$. Suppose that $\overline{\mathcal{P}}^{m}_{r}(\boldsymbol{0}) \subset \mathcal{U}$ for some $r > 0$. Assume that, for fixed $j \leq m$, f vanishes to order at least n along $\{z_{j+1} = \cdots = z_m = 0\} \cap \mathcal{U}$, i.e.*
$$\left.\frac{\partial}{\partial \boldsymbol{z}}^{\boldsymbol{\alpha}} f\right|_{(z_1,\ldots,z_j,0,\ldots,0)} = 0$$
for all $\boldsymbol{\alpha} \in \mathbb{N}_0^m$ such that $|\boldsymbol{\alpha}| < n$ and all $(z_1, \ldots, z_j, 0, \ldots, 0) \in \mathcal{U}$. Then

$$|f(\boldsymbol{z})| \leq \|(z_{j+1}, \ldots, z_m)\|^{n}\frac{2}{r^{n}} \sup_{\boldsymbol{u}\in\overline{\mathcal{P}}^{m}_{r}(\boldsymbol{0})} |f(\boldsymbol{u})|$$

for all $\boldsymbol{z} \stackrel{\text{def}}{=} (z_1, \ldots, z_m) \in \overline{\mathcal{P}}^{j}_{r}(\boldsymbol{0}) \times \overline{\mathcal{P}}^{m-j}_{\frac{r}{2}}(\boldsymbol{0})$.

Proof. Fix $\boldsymbol{z} = (z_1, \ldots, z_j) \in \overline{\mathcal{P}}^{j}_{r}(\boldsymbol{0})$ and consider the holomorphic restriction $\boldsymbol{w} \mapsto f(\boldsymbol{z},\boldsymbol{w})$ for $\boldsymbol{w} \in \overline{\mathcal{P}}^{m-j}_{r}(\boldsymbol{0})$. Applying the holomorphic Taylor theorem to the restriction yields, for

$\boldsymbol{w} \in \overline{\mathcal{P}}^{m-j}_{\frac{r}{2}}(\mathbf{0})$,

$$\begin{aligned} |f(\boldsymbol{z},\boldsymbol{w})| &\leq \|\boldsymbol{w}\|^n \frac{2}{r^n} \sup_{\boldsymbol{w}\in\overline{\mathcal{P}}^{m-j}_r(\mathbf{0})} |f(\boldsymbol{z},\boldsymbol{w})| \\ &\leq \|\boldsymbol{w}\|^n \frac{2}{r^n} \sup_{(\boldsymbol{v},\boldsymbol{w})\in\overline{\mathcal{P}}^{m}_r(\mathbf{0})} |f(\boldsymbol{v},\boldsymbol{w})| . \end{aligned}$$

Since the second estimate holds for all $\boldsymbol{z} \in \overline{\mathcal{P}}^j_r(\mathbf{0})$, the result follows. □

A.2 Basic integrals

Lemma A.2.1. *Let $k \in \mathbb{N}$, $\boldsymbol{\beta} \in \mathbb{N}_0^n$ and assume that $|\boldsymbol{\beta}| = j$ for some $j \in \mathbb{N}_0$. Then*

$$\begin{aligned} \int_{\mathbb{R}^n} \boldsymbol{x}^{2\boldsymbol{\beta}} e^{-\frac{k}{2}\langle \boldsymbol{x},\boldsymbol{x}\rangle} d\boldsymbol{x} &= \sqrt{\frac{(2\pi)^n}{k^n}} \left(\frac{2}{k}\right)^j \langle D, D\rangle^j \frac{\boldsymbol{x}^{2\boldsymbol{\beta}}}{j!}\bigg|_{\mathbf{0}} \\ &= \sqrt{\frac{(2\pi)^n}{k^n}} \left(\frac{2}{k}\right)^j \frac{(2\boldsymbol{\beta})!}{\boldsymbol{\beta}!} , \end{aligned}$$

where $D \overset{\text{def}}{=} \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$.

Proof. We note that $\langle D, D\rangle^{|\boldsymbol{\beta}|} \frac{\boldsymbol{x}^{2\boldsymbol{\beta}}}{|\boldsymbol{\beta}|!}\Big|_{\mathbf{0}} = \frac{(2\boldsymbol{\beta})!}{\boldsymbol{\beta}!}$, so that

$$\begin{aligned} \int_{\mathbb{R}^n} \boldsymbol{x}^{2\boldsymbol{\beta}} e^{-\langle \boldsymbol{x},\boldsymbol{x}\rangle} d\boldsymbol{x} &= \prod_{j=1}^n \int_{\mathbb{R}} x^{2\beta_j} e^{-x^2} dx \\ &= \prod_{j=1}^n \Gamma(\beta_j + \frac{1}{2}) \\ &= \prod_{j=1}^n \sqrt{\pi} 2^{1-2\beta_j} \frac{\Gamma(2\beta_j)}{\Gamma(\beta_j)} \\ &= \sqrt{\pi^n} 2^{-2j} \frac{(2\boldsymbol{\beta})!}{\boldsymbol{\beta}!} \\ &= \sqrt{\pi^n} 2^{-2j} \langle D, D\rangle^j \frac{\boldsymbol{x}^{2\boldsymbol{\beta}}}{j!}\bigg|_{\mathbf{0}} , \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} \boldsymbol{x}^{2\boldsymbol{\beta}} e^{-\frac{k}{2}\langle \boldsymbol{x},\boldsymbol{x}\rangle} d\boldsymbol{x} &= \left(\frac{k}{2}\right)^{-\frac{n}{2}-j} \int_{\mathbb{R}^n} \boldsymbol{y}^{2\boldsymbol{\beta}} e^{\langle \boldsymbol{y},\boldsymbol{y}\rangle} d\boldsymbol{y} \\ &= \sqrt{\frac{(2\pi)^n}{k^n}} \left(\frac{2}{k}\right)^j \frac{(2\boldsymbol{\beta})!}{\boldsymbol{\beta}!}. \end{aligned}$$

□

Recall the definition of the error function: $\operatorname{erf}(c) \overset{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^c e^{-x^2} dx$ for $c \in \mathbb{R}$.

Lemma A.2.2. *Let $B \in M_{n\times n}(\mathbb{R})$ be symmetric and positive definite, $\lambda \in \mathbb{R}$, $\boldsymbol{n} \in \mathbb{R}^n - \{\mathbf{0}\}$ and $\boldsymbol{\beta} \in \mathbb{N}_0^n$.*

If $|\boldsymbol{\beta}| = 2r$, for some $r \in \mathbb{N}_0$, pick $i_k, j_k \in \{1, \ldots, n\}$ such that $i_k \leq j_k$, for $k \in \{1, \ldots, r\}$, *and* $\sum_{k=1}^{r}(\boldsymbol{e}_{i_k}+\boldsymbol{e}_{j_k})=\boldsymbol{\beta}$. *Then*

$$\int_{\langle \boldsymbol{x},\boldsymbol{n}\rangle\geq\lambda}\boldsymbol{x}^{\boldsymbol{\beta}}e^{-\langle \boldsymbol{x},B\boldsymbol{x}\rangle}d\boldsymbol{x}=\left(\prod_{k=1}^{r}\frac{1}{\delta_{i_kj_k}-2}\frac{\partial}{\partial a_{i_kj_k}}\right)\sqrt{\frac{\pi^n}{|A|}}\frac{1-\operatorname{erf}\left(\frac{\lambda}{\|\boldsymbol{n}\|_{A^{-1}}}\right)}{2}\Bigg|_{A=B}.$$

If $|\boldsymbol{\beta}|=2r+1$, *for some* $r\in\mathbb{N}_0$, *pick* $i_k,j_k,s\in\{1,\dots,n\}$ *such that* $i_k\leq j_k$ *for* $k\in\{1,\dots,r\}$ *and* $\boldsymbol{e}_s+\sum_{k=1}^{r}(\boldsymbol{e}_{i_k}+\boldsymbol{e}_{j_k})=\boldsymbol{\beta}$. *Then*

$$\int_{\langle \boldsymbol{x},\boldsymbol{n}\rangle\geq\lambda}\boldsymbol{x}^{\boldsymbol{\beta}}e^{-\langle \boldsymbol{x},B\boldsymbol{x}\rangle}d\boldsymbol{x}=\left(\prod_{k=1}^{r}\frac{1}{\delta_{i_kj_k}-2}\frac{\partial}{\partial a_{i_kj_k}}\right)\left(\sqrt{\frac{\pi^{n-1}}{|A|}}\frac{\left(A^{-1}\boldsymbol{n}\right)_s}{2\|\boldsymbol{n}\|_{A^{-1}}}e^{-\left(\frac{\lambda^2}{\|\boldsymbol{n}\|^2_{A^{-1}}}\right)}\right)\Bigg|_{A=B},$$

where $\|\boldsymbol{n}\|^2_{A^{-1}}=\sum_{i,j=1}^{n}n_iA^{-1}_{ij}n_j$. $\boldsymbol{e}_i$ *denotes the* i^{th} *standard basis vector of* $\mathbb{R}^n$, *and we think of the symmetric matrix* $A=(a_{ij})_{1\leq i,j\leq n}$ *as a function in the* $\frac{n(n+1)}{2}$ *variables* a_{ij} *for* $1\leq i\leq j\leq n$.

Proof. For $A\in M_{n\times n}(\mathbb{R})$ positive definite and symmetric, we have

$$\begin{aligned}\int_{\langle \boldsymbol{x},\boldsymbol{n}\rangle\geq\lambda}e^{-\langle \boldsymbol{x},A\boldsymbol{x}\rangle}d\boldsymbol{x}&=\int_{\langle \boldsymbol{y},(A^{-\frac{1}{2}})^t\boldsymbol{n}\rangle\geq\lambda}e^{-\langle \boldsymbol{x},\boldsymbol{x}\rangle}\left|A^{\frac{1}{2}}\right|^{-1}d\boldsymbol{y} && (\boldsymbol{x}=A^{-\frac{1}{2}}\boldsymbol{y})\\&=\int_{\langle \boldsymbol{x},S^t(A^{-\frac{1}{2}})^t\boldsymbol{n}\rangle\geq\lambda}e^{-\langle \boldsymbol{x},\boldsymbol{x}\rangle}|A|^{-\frac{1}{2}}d\boldsymbol{x} && (\boldsymbol{y}=S\boldsymbol{x}),\end{aligned}$$

where we pick $S\in SO(n)$ such that $S^t(A^{-\frac{1}{2}})^t\boldsymbol{n}=\mu\boldsymbol{e}_1$ with $\mu\geq 0$. In fact, $\mu=\|\boldsymbol{n}\|_{A^{-1}}$ since $\|S^t(A^{-\frac{1}{2}})^t\boldsymbol{n}\|^2=\|(A^{-\frac{1}{2}})^t\boldsymbol{n}\|^2=\boldsymbol{n}^tA^{-\frac{1}{2}}(A^{-\frac{1}{2}})^t\boldsymbol{n}=\boldsymbol{n}^tA^{-1}\boldsymbol{n}$.

$$\begin{aligned}\int_{\langle \boldsymbol{x},\boldsymbol{n}\rangle\geq\lambda}e^{-\langle \boldsymbol{x},A\boldsymbol{x}\rangle}d\boldsymbol{x}&=\int_{\substack{x_1\geq\frac{\lambda}{\|\boldsymbol{n}\|_{A^{-1}}}\\ x_2,\dots,x_n\in\mathbb{R}}}e^{-\langle \boldsymbol{x},\boldsymbol{x}\rangle}|A|^{-\frac{1}{2}}d\boldsymbol{x}\\&=|A|^{-\frac{1}{2}}\pi^{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{\pi}\left(1-\operatorname{erf}\left(\frac{\lambda}{\|\boldsymbol{n}\|_{A^{-1}}}\right)\right)\right)\\&=\sqrt{\frac{\pi^n}{|A|}}\frac{1-\operatorname{erf}\left(\frac{\lambda}{\|\boldsymbol{n}_{A^{-1}}\|}\right)}{2},\end{aligned}$$

where we have used $\int_{\mathbb{R}}e^{-x^2}dx=\sqrt{\pi}$, $\int_{x\geq c}e^{-x^2}dx=\frac{\sqrt{\pi}}{2}(1-\operatorname{erf}(c))$. We now differentiate for $1\leq i\leq j\leq n$.

$$\frac{\partial}{\partial a_{ij}}e^{-\langle \boldsymbol{x},A\boldsymbol{x}\rangle}=(\delta_{ij}-2)x_ix_je^{-\langle \boldsymbol{x},A\boldsymbol{x}\rangle},$$

where we think of A as a function in the matrix entries a_{ij} for $1\leq i\leq j\leq n$. With $i_1,j_1,\dots,i_r,j_r$ chosen as described, we get the first part of the result. Similarly, we have

$$\begin{aligned}\int_{\langle \boldsymbol{x},\boldsymbol{n}\rangle\geq\lambda}\boldsymbol{x}_se^{-\langle \boldsymbol{x},A\boldsymbol{x}\rangle}d\boldsymbol{x}&=\int_{\langle \boldsymbol{x},\left(A^{-\frac{1}{2}}\right)^t\boldsymbol{n}\rangle\geq\lambda}\left(A^{-\frac{1}{2}}\boldsymbol{y}\right)_se^{-\langle \boldsymbol{y},\boldsymbol{y}\rangle}|A|^{-\frac{1}{2}}d\boldsymbol{y} && (\boldsymbol{x}=A^{-\frac{1}{2}}\boldsymbol{y})\\&=\int_{\langle \boldsymbol{x},S^t\left(A^{-\frac{1}{2}}\right)^t\boldsymbol{n}\rangle\geq\lambda}\left(A^{-\frac{1}{2}}S\boldsymbol{x}\right)_se^{-\langle \boldsymbol{x},\boldsymbol{x}\rangle}\left|A^{-\frac{1}{2}}\right|d\boldsymbol{x} && (\boldsymbol{y}=S\boldsymbol{x}),\end{aligned}$$

with $S \in SO(n)$ chosen such that $S^t \left(A^{-\frac{1}{2}}\right)^t \boldsymbol{n} = \|\boldsymbol{n}\|_{A^{-1}} \boldsymbol{e}_s$. We have

$$\begin{aligned}
\int_{\langle \boldsymbol{x},\boldsymbol{n}\rangle \geq \lambda} \boldsymbol{x}_s e^{-\langle \boldsymbol{x}, A\boldsymbol{x}\rangle} d\boldsymbol{x} &= |A|^{-\frac{1}{2}} \int_{\substack{x_s \geq \frac{\lambda}{\|\boldsymbol{n}\|_{A^{-1}}} \\ x_1,\dots,\hat{x}_s,\dots,x_n \in \mathbb{R}}} \sum_{l=1}^{n} \left(A^{-\frac{1}{2}} S\right)_{sl} x_l e^{-\langle \boldsymbol{x},\boldsymbol{x}\rangle} d\boldsymbol{x} \\
&= \left|A^{-\frac{1}{2}}\right| \int_{\substack{x_s \geq \frac{\lambda}{\|\boldsymbol{n}\|_{A^{-1}}} \\ x_s,\dots,\hat{x}_s,\dots,x_n \in \mathbb{R}}} \left(A^{-\frac{1}{2}} S\right)_{ss} x_s e^{-\langle \boldsymbol{x},\boldsymbol{x}\rangle} d\boldsymbol{x} \qquad \left(\int_{\mathbb{R}} x e^{-x^2} = 0\right) \\
&= |A|^{-\frac{1}{2}} (\sqrt{\pi})^{n-1} \left(A^{-\frac{1}{2}} S\right)_{ss} \int_{x \geq \frac{\lambda}{\|\boldsymbol{n}\|_{A^{-1}}}} x e^{-x^2} dx \\
&= |A|^{-\frac{1}{2}} (\sqrt{\pi})^{n-1} \left(A^{-\frac{1}{2}} S\right)_{ss} \frac{1}{2} e^{-\left(\frac{\lambda}{\|\boldsymbol{n}\|_{A^{-1}}}\right)^2}.
\end{aligned}$$

But $S^t(A^{-\frac{1}{2}})^t \boldsymbol{n} = \|\boldsymbol{n}\|_{A^{-1}} \boldsymbol{e}_s$, so that $\|\boldsymbol{n}\|_{A^{-1}} (A^{-\frac{1}{2}})^t S \boldsymbol{e}_s = \|\boldsymbol{n}\|_{A^{-1}} A^{-\frac{1}{2}} S \boldsymbol{e}_s = A^{-1}\boldsymbol{n}$. Hence $A^{-\frac{1}{2}} S \boldsymbol{e}_s = \frac{A^{-1}\boldsymbol{n}}{\|\boldsymbol{n}\|_{A^{-1}}}$ and we have

$$\int_{\langle \boldsymbol{x},\boldsymbol{n}\rangle \geq \lambda} \boldsymbol{x}_s e^{-\langle \boldsymbol{x}, A\boldsymbol{x}\rangle} d\boldsymbol{x} = \sqrt{\frac{\pi^{n-1}}{|A|}} \frac{\left(A^{-1}\boldsymbol{n}\right)_s}{\|\boldsymbol{n}\|_{A^{-1}}} \frac{\exp\left(-\left(\frac{\lambda}{\|\boldsymbol{n}\|_{A^{-1}}}\right)^2\right)}{2}.$$

The general result follows by differentiation. □

Let us now look at the integrals

$$e(\boldsymbol{\gamma}, \boldsymbol{n}, \lambda) \stackrel{\text{def}}{=} \int_{\langle \boldsymbol{x},\boldsymbol{n}\rangle \geq \lambda} \boldsymbol{x}^{\boldsymbol{\gamma}} e^{-\langle \boldsymbol{x},\boldsymbol{x}\rangle} d\boldsymbol{x}$$

for $\boldsymbol{\gamma} \in \mathbb{N}_0^n$, $\boldsymbol{n} \in \mathbb{R}^n - \{\boldsymbol{0}\}$ and $\lambda \in \mathbb{R}$. We are interested in the case where $\lambda = 0$. We note that

$$e(\boldsymbol{\gamma}, \boldsymbol{n}, 0) = \int_{\langle \boldsymbol{x},\boldsymbol{n}\rangle \geq 0} \boldsymbol{x}^{\boldsymbol{\gamma}} e^{-\langle \boldsymbol{x},\boldsymbol{x}\rangle} d\boldsymbol{x} = (-1)^{|\gamma|} \int_{\langle \boldsymbol{x},\boldsymbol{n}\rangle \leq 0} \boldsymbol{x}^{\boldsymbol{\gamma}} e^{-\langle \boldsymbol{x},\boldsymbol{x}\rangle} d\boldsymbol{x}. \tag{A.2.1}$$

If $|\boldsymbol{\gamma}| = 2r$, for some $r \in \mathbb{N}_0$,

$$\begin{aligned}
e(\boldsymbol{\gamma}, \boldsymbol{n}, 0) &= \frac{1}{2} \int_{\mathbb{R}^n} \boldsymbol{x}^{\boldsymbol{\gamma}} e^{-\langle \boldsymbol{x},\boldsymbol{x},\rangle} d\boldsymbol{x} \\
&= \frac{1}{2} \prod_{j=1}^{n} \int_{\mathbb{R}} x^{\gamma_i} e^{-x^2} dx \\
&= \begin{cases} \sqrt{\pi^n} 2^{-(|\gamma|+1)} \frac{(2\boldsymbol{\beta})!}{\boldsymbol{\beta}!} & \text{it } \boldsymbol{\gamma} = 2\boldsymbol{\beta} \text{ for some } \boldsymbol{\beta} \in \mathbb{N}_0^n \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{A.2.2}$$

If $|\boldsymbol{\gamma}| = 1$, so that $\boldsymbol{\gamma} = \boldsymbol{e}_s$, for some $s \in \{1, \cdots, n\}$, we apply the previous lemma to find

$$e(\boldsymbol{e}_s, \boldsymbol{n}, 0) = \frac{\sqrt{\pi^{n-1}}}{2} \frac{n_s}{\|\boldsymbol{n}\|}.$$

If $|\boldsymbol{\gamma}| = 3$, so that $\boldsymbol{\gamma} = \boldsymbol{e}_r + \boldsymbol{e}_l + \boldsymbol{e}_s$, for some $r, l, s \in \{1, \cdots, n\}$ with $r \leq l$, we have

$$e(\boldsymbol{\gamma}, \boldsymbol{n}, 0) = \frac{1}{2} \frac{\sqrt{\pi^{n-1}}}{\delta_{rl} - 2} \frac{\partial}{\partial a_{rl}} \left. \frac{\left(A^{-1}\boldsymbol{n}\right)_s}{\|\boldsymbol{n}\|_{A^{-1}} \sqrt{|A|}} \right|_{A=I_n}.$$

We compute that

$$\frac{\partial}{\partial a_{rl}}\left(A^{-1}\boldsymbol{n},\boldsymbol{e}_s\right)=\left(a^{sl}a^{lk}\delta_{lr}-a^{sl}a^{rj}-a^{sr}a^{lj}\right)n_j$$

$$\frac{\partial}{\partial a_{rl}}|A|=\left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0}|A+\lambda\left(E_{rl}+E_{lr}-\delta_{lr}E_{ll}\right)|=\left(2-\delta_{lr}\right)|A|\,a^{rl},$$

where E_{rl} denotes the $n\times n$ matrix with zeroes everywhere except in the (r,l)-entry. Finally,

$$\frac{\partial}{\partial a_{rl}}\|n\|_{A^{-1}}=\frac{n_i n_j}{2\|n\|_{A^{-1}}}\left(a^{il}a^{lj}\delta_{lr}-a^{li}a^{rj}-a^{lj}a^{ri}\right).$$

A short computation then shows that

$$e(\boldsymbol{e}_r+\boldsymbol{e}_s+\boldsymbol{e}_l,\boldsymbol{n},0)=\frac{\sqrt{\pi^{n-1}}}{4}\left(\frac{\delta_{sl}n_r+\delta_{rs}n_l+\delta_{rl}n_s}{\|\boldsymbol{n}\|}-\frac{n_r n_s n_l}{\|\boldsymbol{n}\|^3}\right).\tag{A.2.3}$$

Remark A.2.3. *These functions $\lambda\mapsto e(\boldsymbol{\gamma},\boldsymbol{n},\lambda)$, for $\lambda\in\mathbb{R}$, serve a role similar to the* generalized error functions *that we defined in chapter 2 with the difference being that we have not normalized them in the same way here.*

Appendix B

Laplace's Method

After proving a version of Laplace's theorem in n dimensions, we investigate how Laplace's method can be applied to certain integrals over half-spaces. These integrals are of importance in chapter 7 where we use them to investigate the asymptotic expansion of partial density functions on toric polarized Kähler manifolds.

B.1 Background

Laplace's method (see [dB81, BH75]) provides a means of determining the asymptotics of an integral of the type

$$\mathrm{I}_k \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-kh(\boldsymbol{x})} d\boldsymbol{x}$$

as $k \in \mathbb{N}$ tends to infinity, where $f, h \in \mathcal{C}^\infty(\mathbb{R}^n)$ and h has an absolute minimum which it attains only at $\boldsymbol{x}_0 \in \mathbb{R}^n$. Suppose that $\left.\mathrm{Hess}(h)\right|_{\boldsymbol{x}_0}$ is positive definite, that I_k is finite for $k \in \mathbb{N}$ and that there exists $c > 0$ such that $h(\boldsymbol{x}) - h(\boldsymbol{x}_0) \geq c$ outside a compact subset of $\mathbb{R}^n$. Laplace's method then gives an asymptotic expansion in k,

$$e^{kh(\boldsymbol{x}_0)}\, \mathrm{I}_k = \sum_{j=0}^{\infty} a_j k^{-\left(\frac{n}{2}+j\right)} + \mathcal{O}\left(k^{-\infty}\right) \qquad \text{as } k \to \infty,$$

where the coefficients a_j are determined by various derivatives of f and h at $\boldsymbol{x}_0$. We will derive a special version of Laplace's method in this chapter, and we apply this method to study integrals similar to the one above. For a treatment of the related method of stationary phase, the reader may consult Hörmander's book [Hör90].

B.2 A version of Laplace's method

Let us now provide a proof of a parameter dependent version of Laplace's method which will be useful to us.

Definition B.2.1. Let $\mathcal{U} \subset \mathbb{R}^n$ be an open bounded set and let $f, h \in \mathcal{C}^\infty(\mathcal{U} \times \mathcal{U})$ such that, for $\boldsymbol{x} \in \mathcal{U}$, $\left.\mathrm{Hess}\, h(\boldsymbol{x}, .)\right|_{\boldsymbol{x}} > 0$. Suppose that $h(\boldsymbol{x}, \boldsymbol{y}) \geq 0$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{U}$ and that $h(\boldsymbol{x}, \boldsymbol{y}) = 0$ if and only if $\boldsymbol{x} = \boldsymbol{y}$. Suppose furthermore that f, h and their derivatives are bounded in $\mathcal{U} \times \mathcal{U}$

and that, for any $\boldsymbol{x} \in \mathcal{U}$ and $\boldsymbol{y}_0 \in \partial\mathcal{U}$, we have $\lim_{\boldsymbol{y}\to\boldsymbol{y_0}} h(\boldsymbol{x},\boldsymbol{y}) > 0$. We define

$$\mathrm{I}_k(\boldsymbol{x}) \stackrel{\text{def}}{=} \int_{\mathcal{U}} f(\boldsymbol{x},\boldsymbol{y})e^{-kh(\boldsymbol{x},\boldsymbol{y})}d\boldsymbol{y} \quad \text{for } k \in \mathbb{N} \text{ and } \boldsymbol{x} \in \mathcal{U}.$$

Let $\mathrm{H}(\boldsymbol{x}) \stackrel{\text{def}}{=} \mathrm{Hess}\left(h\left(\boldsymbol{x},.\right)\right)|_{\boldsymbol{x}}$. We use the notation $\mathrm{H}(\boldsymbol{x}) > 0$ to denote that $\mathrm{H}(\boldsymbol{x})$ is positive definite, and we define

$$\begin{aligned} R(\boldsymbol{x},\boldsymbol{y}) &\stackrel{\text{def}}{=} h(\boldsymbol{x},\boldsymbol{y}) - \frac{1}{2}\langle \mathrm{H}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}), \boldsymbol{y}-\boldsymbol{x}\rangle \\ \mathrm{D}_{\boldsymbol{y}} &\stackrel{\text{def}}{=} (\partial_{y_1}, \cdots, \partial_{y_n}) \end{aligned}$$

for $\boldsymbol{x},\boldsymbol{y} \in \mathcal{U}$.

We would now like to determine the asymptotics of I_k as k tends to infinity. The following theorem is closely related to the method of stationary phase [Hör90, p.220, theorem 7.7.5 and lemma 7.7.3]. For our case, we will provide a basic proof rather than following Hörmander's Fourier transform approach.

Theorem B.2.2. *For $j \in \mathbb{N}$, there exists $a_j \in \mathcal{C}^\infty(\mathcal{U})$, and, for $p \in \mathbb{N}_0$, there exists a continuous function $C_p : \mathcal{U} \to \mathbb{R}_{\geq 0}$ such that*

$$\left| \mathrm{I}_k(\boldsymbol{x}) - \sum_{j=0}^{p} k^{-\left(\frac{n}{2}+j\right)} a_j(\boldsymbol{x}) \right| \leq C_p(\boldsymbol{x}) k^{-\left(\frac{n+1}{2}+p\right)}, \quad \text{for } \boldsymbol{x} \in \mathcal{U},$$

and where

$$a_j(\boldsymbol{x}) = \sqrt{\frac{(2\pi)^n}{|\mathrm{H}(\boldsymbol{x})|}} \sum_{i=0}^{2j} \frac{(-1)^i}{i!(i+j)!2^{i+j}} \left.\langle \mathrm{H}^{-1}(\boldsymbol{x})\,\mathrm{D}_{\boldsymbol{y}}, \mathrm{D}_{\boldsymbol{y}}\rangle^{i+j} f(\boldsymbol{x},\boldsymbol{y}) R(\boldsymbol{x},\boldsymbol{y})^i \right|_{\boldsymbol{x}}.$$

In order to prove this result, we first need a few lemmas that will simplify the argument.

Lemma B.2.3. *Let h,$\mathcal{U}$ be as before and let $A > 0$, $\delta \in \left(0, \frac{1}{2}\right)$, $\boldsymbol{x} \in \mathcal{U}$ and $\left\{\boldsymbol{y}_j\right\}_{j=1}^{\infty} \subset \mathcal{U}$ such that*

$$\|\boldsymbol{y}_k - \boldsymbol{x}\| > Ak^{-\frac{1}{2}+\delta} \quad \text{for all } k \in \mathbb{N}.$$

Then there exists a continuous function $C : \mathcal{U} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$kh(\boldsymbol{x},\boldsymbol{y}_k) \geq C(\boldsymbol{x},A)k^{2\delta} \quad \text{for all } k \in \mathbb{N}.$$

In particular, for $l \in \mathbb{N}$, there exists a continuous function $D_l : \mathcal{U} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ such that

$$e^{-kh(\boldsymbol{x},\boldsymbol{y_k})} \leq D_l(\boldsymbol{x},A)k^{-l} \quad \text{for all } k \in \mathbb{N}.$$

Proof. For $\boldsymbol{x} \in \mathcal{U}$ fixed, there exists an open convex neighbourhood $\mathcal{V}_{\boldsymbol{x}} \subset \mathcal{U}$ of $\boldsymbol{x}$ on which $h(\boldsymbol{x},.)$ is strictly convex. We have

$$h\left(\boldsymbol{x},(1-s)\boldsymbol{x}+s\boldsymbol{y}\right) < (1-s)h\left(\boldsymbol{x},\boldsymbol{x}\right) + sh\left(\boldsymbol{x},\boldsymbol{y}\right) \leq h(\boldsymbol{x},\boldsymbol{y}) \quad \text{for } s \in (0,1), \boldsymbol{x} \in \mathcal{U}, \boldsymbol{y} \in \mathcal{V}_{\boldsymbol{x}}.$$

If $\boldsymbol{y}_k \in \mathcal{U} - \mathcal{V}_{\boldsymbol{x}}$, there exists $\lambda(\boldsymbol{x}) > 0$, depending continuously on $\boldsymbol{x} \in \mathcal{U}$, such that $h(\boldsymbol{x},\boldsymbol{y}) \geq$

$\lambda(\boldsymbol{x})$. In this case, we have the obvious estimate

$$kh(\boldsymbol{x},\boldsymbol{y}_k) \geq k\lambda(\boldsymbol{x}),$$

which is even stronger than what we want to prove. We now assume without loss of generality that $\boldsymbol{y}_k \in \mathcal{V}_{\boldsymbol{x}}$. Let $s_k = \frac{Ak^{-\frac{1}{2}+\delta}}{\|\boldsymbol{y}_k-\boldsymbol{x}\|}$. Then

$$\begin{aligned} h\left(\boldsymbol{x},\boldsymbol{x}+s_k\left(\boldsymbol{y}_k-\boldsymbol{x}\right)\right) &= h\left(\boldsymbol{x},\boldsymbol{x}+Ak^{-\frac{1}{2}+\delta}\frac{\boldsymbol{y}_k-\boldsymbol{x}}{\|\boldsymbol{y}_k-\boldsymbol{x}\|}\right) \\ &< h\left(\boldsymbol{x},\boldsymbol{y}_k\right). \end{aligned}$$

We have

$$h(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}\langle \mathrm{H}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}),\boldsymbol{y}-\boldsymbol{x}\rangle + R(\boldsymbol{x},\boldsymbol{y}),$$

and

$$R(\boldsymbol{x},\boldsymbol{y}) = \sum_{\left\{\boldsymbol{\gamma}\in\mathbb{N}_0^n:|\boldsymbol{\gamma}|=3\right\}} R_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{y}-\boldsymbol{x})^{\boldsymbol{\gamma}},$$

where by Taylor's standard estimate (theorem A.1.1),

$$|R_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y})| \leq D_{\boldsymbol{\gamma}}(\boldsymbol{x}) = \sup_{\boldsymbol{y}\in\overline{B_{\boldsymbol{x}}(A)}} \left| \frac{1}{\gamma!}\,\frac{\partial}{\partial \boldsymbol{y}}^{\gamma} h(\boldsymbol{x},\boldsymbol{y})\bigg|_{\boldsymbol{y}} \right|$$

for $\boldsymbol{y} \in \overline{B_{\boldsymbol{x}}(A)} \stackrel{\text{def}}{=} \{\boldsymbol{y}\in\mathcal{U} : \|\boldsymbol{y}-\boldsymbol{x}\| \leq A\}$. Defining $E(\boldsymbol{x}) \stackrel{\text{def}}{=} \sum_{|\boldsymbol{\gamma}|=3} D_{\boldsymbol{\gamma}}(\boldsymbol{x})$, we have

$$|R(\boldsymbol{x},\boldsymbol{y})| \leq E(\boldsymbol{x})\|\boldsymbol{y}-\boldsymbol{x}\|^3 \qquad \text{for } \boldsymbol{y}\in\overline{B_{\boldsymbol{x}}(A)}.$$

For $k \in \mathbb{N}$ and $\boldsymbol{y}_k \in \mathcal{V}_{\boldsymbol{x}}$ such that $\|\boldsymbol{y}_k - \boldsymbol{x}\| > Ak^{-\frac{1}{2}+\delta}$, we have

$$\begin{aligned} kh(\boldsymbol{x},\boldsymbol{y}_k) &> kh\left(\boldsymbol{x},\boldsymbol{x}+Ak^{-\frac{1}{2}+\delta}\frac{\boldsymbol{y}_k-\boldsymbol{x}}{\|\boldsymbol{y}_k-\boldsymbol{x}\|}\right) \\ &\geq k\left|A^2k^{-1+2\delta}\left|\frac{1}{2}\langle\mathrm{H}(\boldsymbol{x})\frac{\boldsymbol{y}_k-\boldsymbol{x}}{\|\boldsymbol{y}_k-\boldsymbol{x}\|},\frac{\boldsymbol{y}_k-\boldsymbol{x}}{\|\boldsymbol{y}_k-\boldsymbol{x}\|}\rangle\right|\right. \\ &\quad \left. -\left|R\left(\boldsymbol{x},\boldsymbol{x}+Ak^{-\frac{1}{2}+\delta}\frac{\boldsymbol{y}_k-\boldsymbol{x}}{\|\boldsymbol{y}_k-\boldsymbol{x}\|}\right)\right|\right| \\ &= k^{2\delta}\left|A^2\left|\frac{1}{2}\langle\mathrm{H}(\boldsymbol{x})\frac{\boldsymbol{y}_k-\boldsymbol{x}}{\|\boldsymbol{y}_k-\boldsymbol{x}\|},\frac{\boldsymbol{y}_k-\boldsymbol{x}}{\|\boldsymbol{y}_k-\boldsymbol{x}\|}\rangle\right|\right. \\ &\quad \left. -\left|k^{1-2\delta}R\left(\boldsymbol{x},\boldsymbol{x}+Ak^{-\frac{1}{2}+\delta}\frac{\boldsymbol{y}_k-\boldsymbol{x}}{\|\boldsymbol{y}_k-\boldsymbol{x}\|}\right)\right|\right|. \end{aligned}$$

Note that

$$\left|k^{1-2\delta}R\left(\boldsymbol{x},\boldsymbol{x}+Ak^{-\frac{1}{2}+\delta}\frac{\boldsymbol{y}_k-\boldsymbol{x}}{\|\boldsymbol{y}_k-\boldsymbol{x}\|}\right)\right| \leq E(\boldsymbol{x})A^3k^{-\frac{1}{2}+\delta}$$

tends to 0 as $k\to\infty$, while

$$\frac{A^2}{2}\left|\langle\mathrm{H}(\boldsymbol{x})\frac{\boldsymbol{y}_k-\boldsymbol{x}}{\|\boldsymbol{y}_k-\boldsymbol{x}\|},\frac{\boldsymbol{y}_k-\boldsymbol{x}}{\|\boldsymbol{y}_k-\boldsymbol{x}\|}\rangle\right| \geq F(\boldsymbol{x}) \stackrel{\text{def}}{=} \frac{A^2}{2}\inf_{\|\boldsymbol{u}\|=1}\langle\mathrm{H}(\boldsymbol{x})\boldsymbol{u},\boldsymbol{u}\rangle > 0.$$

We have

$$kh(\boldsymbol{x}, \boldsymbol{y}_k) \geq k^{2\delta} \left(F(\boldsymbol{x}) - E(\boldsymbol{x}) A^3 k^{-\frac{1}{2}+\delta} \right) > 0$$

if $k \geq K(\boldsymbol{x}) \stackrel{\text{def}}{=} \left(\frac{E(\boldsymbol{x})A^3}{F(\boldsymbol{x})} \right)^{\frac{1}{\frac{1}{2}-\delta}} + 1$, and if $k \leq K(\boldsymbol{x})$, we have

$$kh(\boldsymbol{x}, \boldsymbol{y}_k) \geq I(\boldsymbol{x}) = \inf_{\left\{ \boldsymbol{u} \in \mathcal{U} : \|\boldsymbol{u} - \boldsymbol{x}\| = AK(\boldsymbol{x})^{-\frac{1}{2}+\delta} \right\}} h(\boldsymbol{x}, \boldsymbol{u}) > 0.$$

Finally,

$$kh(\boldsymbol{x}, \boldsymbol{y}_k) \geq C(\boldsymbol{x}) k^{2\delta},$$

where $C(\boldsymbol{x}) = \min \left(\frac{I(\boldsymbol{x})}{K(\boldsymbol{x})^{2\delta}}, F(\boldsymbol{x}) - E(\boldsymbol{x}) A^3 K(\boldsymbol{x})^{-\frac{1}{2}+\delta} \right)$. Tracing through the proof, it is obvious that C is continuous with respect to both $\boldsymbol{x} \in \mathcal{U}$ and the parameter $A > 0$.

For the last part of the result, we observe that, for $l \in \mathbb{R}_{\geq 0}$ and $\boldsymbol{x} \in \mathcal{U}$, there exists $K(\boldsymbol{x}) \geq 1$ such that

$$e^{-C(\boldsymbol{x})k^{2\delta}} < k^{-l} \text{ for all } k > K(\boldsymbol{x}),$$

and $K : \mathcal{U} \to \mathbb{R}_{>0}$ is continuous. If $1 \leq k \leq K(\boldsymbol{x})$,

$$e^{-C(\boldsymbol{x})k^{2\delta}} \leq 1 \leq K(\boldsymbol{x})^l k^{-l},$$

so that

$$e^{-kh(\boldsymbol{x}, \boldsymbol{y}_k)} \leq e^{-C(\boldsymbol{x})k^{2\delta}} \leq D_l(\boldsymbol{x}) k^{-l} \qquad \text{for all } k \in \mathbb{N},$$

where $D_l(\boldsymbol{x}) \stackrel{\text{def}}{=} \max\left(1, K(\boldsymbol{x})^l\right)$. Furthermore, D_l depends continuously on the parameter $A > 0$. □

Corollary B.2.4. *Let δ, f, h and $\mathcal{U}$ be defined as before. Then, for $l \in \mathbb{N}_0$, there exists a continuous function $D_l : \mathcal{U} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ such that*

$$\left| \int_{\left\{ \boldsymbol{y} \in \mathcal{U} : \|\boldsymbol{y} - \boldsymbol{x}\| \geq Ak^{-\frac{1}{2}+\delta} \right\}} f(\boldsymbol{x}, \boldsymbol{y}) e^{-kh(\boldsymbol{x}, \boldsymbol{y})} d\boldsymbol{y} \right| \leq D_l(\boldsymbol{x}, A) k^{-l} \quad \textit{for } \boldsymbol{x} \in \mathcal{U} \textit{ and } A > 0.$$

Proof. We have seen that there exists a continuous function $C : \mathcal{U} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $kh(\boldsymbol{x}, \boldsymbol{y}) \geq C(\boldsymbol{x}, A) k^{2\delta}$ if $\|\boldsymbol{y} - \boldsymbol{x}\| \geq Ak^{-\frac{1}{2}+\delta}$. We have

$$\left| \int_{\left\{ \boldsymbol{y} \in \mathcal{U} : \|\boldsymbol{y} - \boldsymbol{x}\| \geq Ak^{-\frac{1}{2}+\delta} \right\}} f(\boldsymbol{x}, \boldsymbol{y}) e^{-kh(\boldsymbol{x}, \boldsymbol{y})} d\boldsymbol{y} \right| \leq e^{-(k-1)k^{2\delta-1}C(\boldsymbol{x}, A)} \int_{\mathcal{U}} |f(\boldsymbol{x}, \boldsymbol{y})| \, e^{-h(\boldsymbol{x}, \boldsymbol{y})} d\boldsymbol{y}.$$

For fixed $l \in \mathbb{N}_0$, there exists a continuous function $K : \mathcal{U} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$e^{-(k-1)k^{2\delta-1}C(\boldsymbol{x}, A)} \leq k^{-l} \qquad \text{for } k > K(\boldsymbol{x}, A).$$

If $1 \leq k \leq K(\boldsymbol{x}, A)$,

$$e^{-(k-1)k^{2\delta-1}C(\boldsymbol{x}, A)} \leq 1 \leq K(\boldsymbol{x}, A)^l k^{-l},$$

so that, for $k \in \mathbb{N}$, we have

$$e^{-(k-1)k^{2\delta-1}C(\boldsymbol{x}, A)} \leq D_l(\boldsymbol{x}, A) k^{-l},$$

where $D_l(\boldsymbol{x}, A) \stackrel{\text{def}}{=} \max(1, K(\boldsymbol{x}, A)^l)$. The result follows. □

Lemma B.2.5. *Let $h, f, \mathcal{U}$ and R be defined as before. For $p \in \mathbb{N}_0$, there exists a continuous function $C_p : \mathcal{U} \to \mathbb{R}_{\geq 0}$ such that*

$$\left| \mathrm{I}_k(\boldsymbol{x}) - \sum_{i=0}^{p} \int_{\mathcal{U}} f(\boldsymbol{x}, \boldsymbol{y}) \frac{(-kR(\boldsymbol{x}, \boldsymbol{y}))^i}{i!} e^{-\frac{k}{2}\langle \mathrm{H}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}), \boldsymbol{y}-\boldsymbol{x} \rangle} d\boldsymbol{y} \right| \leq C_p(\boldsymbol{x}) k^{-\frac{n+p+1}{2}}$$

for $\boldsymbol{x} \in \mathcal{U}$.

Proof. For $\boldsymbol{x} \in \mathcal{U}$, let $r(\boldsymbol{x}) = \min\left(1, \frac{d(\boldsymbol{x}, \partial\overline{\mathcal{U}})}{2}\right)$. There exists a continuous function $C : \mathcal{U} \to \mathbb{R}_{\geq 0}$ (see the proof of lemma B.2.3) such that

$$|R(\boldsymbol{x}, \boldsymbol{y})| \leq C(\boldsymbol{x}) \|\boldsymbol{y} - \boldsymbol{x}\|^3 \qquad \text{for } \boldsymbol{y} \in \overline{B_{\boldsymbol{x}}(r(\boldsymbol{x}))},$$

where, for $r \in \mathbb{R}_{\geq 0}$, $\overline{B_{\boldsymbol{x}}(r)} \stackrel{\text{def}}{=} \{\boldsymbol{y} \in \mathcal{U} : \|\boldsymbol{y} - \boldsymbol{x}\| \leq r\}$. Suppose that $\|\boldsymbol{y} - \boldsymbol{x}\| \leq k^{-\frac{1}{3}} r(\boldsymbol{x})$. Then $|kR(\boldsymbol{x}, \boldsymbol{y})| \leq C(\boldsymbol{x})$. There exists a continuous function $D : \mathcal{U} \to \mathbb{R}_{\geq 0}$ such that

$$\left| e^t - \sum_{j=0}^{p} \frac{t^j}{j!} \right| \leq D(\boldsymbol{x}) \, |t|^{p+1} \qquad \text{for } t \in \mathbb{R} \text{ such that } |t| \leq C(\boldsymbol{x}).$$

Hence

$$\left| e^{-kR(\boldsymbol{x}, \boldsymbol{y})} - \sum_{j=0}^{p} \frac{(-kR(\boldsymbol{x}, \boldsymbol{y}))^j}{j!} \right| \leq D(\boldsymbol{x}) \, |kR(\boldsymbol{x}, \boldsymbol{y})|^{p+1} \leq C(\boldsymbol{x})^{p+1} D(\boldsymbol{x}) k^{p+1} \|\boldsymbol{y} - \boldsymbol{x}\|^{3(p+1)}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{U}$ such that $\|\boldsymbol{y} - \boldsymbol{x}\| \leq k^{-\frac{1}{3}} r(\boldsymbol{x})$. We have

$$\begin{aligned} A_k(\boldsymbol{x}) &\stackrel{\text{def}}{=} \left| \mathrm{I}_k(\boldsymbol{x}) - \int_{\overline{B_{k^{-\frac{1}{3}} r(\boldsymbol{x})}(\boldsymbol{x})}} f(\boldsymbol{x}, \boldsymbol{y}) \sum_{i=0}^{p} \frac{(-kR(\boldsymbol{x}, \boldsymbol{y}))^i}{i!} e^{-\frac{k}{2}\langle H(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}), \boldsymbol{y}-\boldsymbol{x} \rangle} d\boldsymbol{y} \right| \\ &\leq \left| \mathrm{I}_k(\boldsymbol{x}) - \int_{\overline{B_{k^{-\frac{1}{3}} r(\boldsymbol{x})}(\boldsymbol{x})}} f(\boldsymbol{x}, \boldsymbol{y}) e^{-kh(\boldsymbol{x}, \boldsymbol{y})} d\boldsymbol{y} \right| \\ &\quad + \int_{\overline{B_{k^{-\frac{1}{3}} r(\boldsymbol{x})}(\boldsymbol{x})}} |f(\boldsymbol{x}, \boldsymbol{y})| \left| e^{-kR(\boldsymbol{x}, \boldsymbol{y})} - \sum_{i=0}^{p} \frac{(-kR(\boldsymbol{x}, \boldsymbol{y}))^i}{i!} \right| e^{-\frac{k}{2}\langle H(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}), \boldsymbol{y}-\boldsymbol{x} \rangle} d\boldsymbol{y}. \end{aligned}$$

We have $|f(\boldsymbol{x}, \boldsymbol{y})| \leq E$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{U}$ and some constant $E > 0$. Corollary B.2.4 gives, for any $l \geq 0$, a continuous function $C_l : \mathcal{U} \to \mathbb{R}_{\geq 0}$ such that

$$\left| \mathrm{I}_k(\boldsymbol{x}) - \int_{\overline{B_{k^{-\frac{1}{3}} r(\boldsymbol{x})}(\boldsymbol{x})}} f(\boldsymbol{y}) e^{-kh(\boldsymbol{x}, \boldsymbol{y})} d\boldsymbol{y} \right| \leq C_l(\boldsymbol{x}) k^{-l}.$$

We find that, for $l \geq \frac{n+p+1}{2}$,

$$\begin{aligned} A_k(\boldsymbol{x}) &\leq C_l(\boldsymbol{x}) k^{-l} \\ &\quad + \int_{\overline{B_{k^{-\frac{1}{3}} r(\boldsymbol{x})}(\boldsymbol{x})}} E C(\boldsymbol{x})^{p+1} D(\boldsymbol{x}) k^{p+1} \|\boldsymbol{y} - \boldsymbol{x}\|^{3(p+1)} e^{-\frac{k}{2}\langle H(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}, \boldsymbol{y}-\boldsymbol{x} \rangle} d\boldsymbol{y} \\ &\leq C_l(\boldsymbol{x}) k^{-l} + E C(\boldsymbol{x})^{p+1} D(\boldsymbol{x}) k^{p+1} \int_{\mathcal{U}} \|\boldsymbol{y}\|^{3(p+1)} e^{-\frac{k}{2}\langle H(\boldsymbol{x})\boldsymbol{y}, \boldsymbol{y} \rangle} d\boldsymbol{y} \end{aligned}$$

$$\leq F(\boldsymbol{x}) k^{-\frac{n+p+1}{2}},$$

where $F : \mathcal{U} \to \mathbb{R}_{\geq 0}$ is a continuous function. ☐

Consider $\mathrm{D}_{\boldsymbol{y}} \stackrel{\text{def}}{=} (\partial_{y_1}, \dots, \partial_{y_n})$. We have, using multi-index notation,

$$\begin{aligned}\langle \mathrm{D}_{\boldsymbol{y}}, \mathrm{D}_{\boldsymbol{y}} \rangle^j &= \left(\partial_{y_1}^2 + \cdots + \partial_{y_n}^2\right)^j \\ &= \sum_{\boldsymbol{\beta} \in \mathbb{N}_0^n : |\boldsymbol{\beta}| = j} \binom{j}{\boldsymbol{\beta}} \left(\partial_{y_1}^2\right)^{\beta_1} \dots \left(\partial_{y_n}^2\right)^{\beta_n}.\end{aligned}$$

Suppose that $\boldsymbol{\beta} \in \mathbb{N}_0^n$ and $|\boldsymbol{\beta}| = j$. Then

$$\langle \mathrm{D}_{\boldsymbol{y}}, \mathrm{D}_{\boldsymbol{y}} \rangle^j \boldsymbol{y}^{2\boldsymbol{\beta}}\big|_{\mathbf{0}} = \binom{j}{\boldsymbol{\beta}} (2\boldsymbol{\beta})!\,.$$

Lemma B.2.6. *Let $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, and let H be a positive definite symmetric matrix. Then*

$$\sum_{|\boldsymbol{\beta}|=j} \binom{j}{\boldsymbol{\beta}} \frac{\partial}{\partial \boldsymbol{y}}^{2\boldsymbol{\beta}} f\left(H^{-\frac{1}{2}} \boldsymbol{y} + \boldsymbol{x}\right)\bigg|_{\mathbf{0}} = \langle H^{-1} \mathrm{D}_{\boldsymbol{y}}, \mathrm{D}_{\boldsymbol{y}} \rangle^j f\big|_{\boldsymbol{x}}.$$

Proof. For $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$\begin{aligned}\langle H^{-1} \mathrm{D}_{\boldsymbol{y}}, \mathrm{D}_{\boldsymbol{y}} \rangle^j f\big|_{\boldsymbol{x}} &= \langle H^{-\frac{1}{2}} \mathrm{D}_{\boldsymbol{y}}, H^{-\frac{1}{2}} \mathrm{D}_{\boldsymbol{y}} \rangle^j f\Big|_{\boldsymbol{x}} \\ &= \left(\sum_{j=1}^n \left(\sum_{k=1}^n H_{jk}^{-\frac{1}{2}} \partial_{y_k}\right)^2\right)^j f\Bigg|_{\boldsymbol{x}} \\ &= \sum_{|\boldsymbol{\beta}|=j} \binom{j}{\boldsymbol{\beta}} \prod_{l=1}^n \left(\sum_{k=1}^n H_{lk}^{-\frac{1}{2}} \partial_{y_l}\right)^{2\beta_l} f\Bigg|_{\boldsymbol{x}},\end{aligned}$$

while *e.g.*

$$\frac{\partial}{\partial y_k} f\left(H^{-\frac{1}{2}} \boldsymbol{y} + \boldsymbol{x}\right)\bigg|_{\mathbf{0}} = \left(\sum_{j=1}^n H_{kj}^{-\frac{1}{2}} \partial_{y_j}\right) f\Bigg|_{\mathbf{0}}.$$

Iteration yields the lemma. ☐

Lemma B.2.7. *Let $\mathcal{U}$, H and f be defined as before. For every $p \in \mathbb{N}_0$, there exists a continuous function $C_p : \mathcal{U} \to \mathbb{R}_{\geq 0}$ such that*

$$\left| \int_{\mathcal{U}} f(\boldsymbol{x}, \boldsymbol{y}) e^{-\frac{k}{2} \langle \mathrm{H}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}), \boldsymbol{y}-\boldsymbol{x} \rangle} d\boldsymbol{y} - \sum_{j=0}^p a_j(\boldsymbol{x}) k^{-(\frac{n}{2}+j)} \right| \leq C_p(\boldsymbol{x}) k^{-\frac{n+2p+1}{2}} \quad \text{for all } \boldsymbol{x} \in \mathcal{U},$$

where

$$a_j(\boldsymbol{x}) = \frac{1}{2^j j!} \sqrt{\frac{(2\pi)^n}{|H(\boldsymbol{x})|}} \langle \mathrm{H}^{-1}(\boldsymbol{x}) \mathrm{D}_{\boldsymbol{y}}, \mathrm{D}_{\boldsymbol{y}} \rangle^j f(\boldsymbol{x}, \boldsymbol{y})\big|_{\boldsymbol{x}}.$$

Proof. We Taylor-expand

$$f\left(\boldsymbol{x}, \mathrm{H}^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x}\right) = \sum_{s=0}^{2p} \sum_{|\boldsymbol{\gamma}|=s} \frac{1}{\boldsymbol{\gamma}!} \frac{\partial}{\partial \boldsymbol{y}}^{\boldsymbol{\gamma}} f\left(\boldsymbol{x}, \mathrm{H}(\boldsymbol{x})^{-\frac{1}{2}} \boldsymbol{y} + \boldsymbol{x}\right)\bigg|_{\mathbf{0}} \boldsymbol{y}^{\boldsymbol{\gamma}} + R_{2p+1}(\boldsymbol{x}, \boldsymbol{y}),$$

where

$$|R_{2p+1}(\boldsymbol{x}, \boldsymbol{y})| \leq C(\boldsymbol{x})\|\boldsymbol{y}\|^{2p+1} \qquad \text{for } \boldsymbol{y} \in \overline{B_{r(\boldsymbol{x})}(\mathbf{0})},$$

and where we let $r(\boldsymbol{x}) \stackrel{\text{def}}{=} \min\left(1, \frac{d(\boldsymbol{x}, \partial\overline{\mathcal{U}})}{2}\right)$ and note that $C : \mathcal{U} \to \mathbb{R}_{\geq 0}$ is continuous. Hence

$$\begin{aligned}
&\int_{\overline{B_{r(\boldsymbol{x})}(\boldsymbol{x})}} f(\boldsymbol{x}, \boldsymbol{y}) e^{-\frac{k}{2}\langle \mathrm{H}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}), \boldsymbol{y}-\boldsymbol{x}\rangle} d\boldsymbol{y} \\
&= |\mathrm{H}(\boldsymbol{x})|^{-\frac{1}{2}} \int_{\mathcal{V}(\boldsymbol{x})} f\left(\boldsymbol{x}, H^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x}\right) e^{-\frac{k}{2}\langle \boldsymbol{y}, \boldsymbol{y}\rangle} d\boldsymbol{y} + \mathcal{O}\left(k^{-\infty}\right) \\
&= |\mathrm{H}(\boldsymbol{x})|^{-\frac{1}{2}} \sum_{s=0}^{p} \sum_{|\boldsymbol{\beta}|=s} \frac{1}{(2\boldsymbol{\beta})!} \frac{\partial}{\partial \boldsymbol{y}}^{2\boldsymbol{\beta}} f\left(\boldsymbol{x}, \mathrm{H}^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x}\right)\Bigg|_{\mathbf{0}} \int_{\mathbb{R}^n} \boldsymbol{y}^{2\boldsymbol{\beta}} e^{-\frac{k}{2}\langle \boldsymbol{y}, \boldsymbol{y}\rangle} d\boldsymbol{y} + \mathcal{O}\left(k^{-\frac{n+2p+1}{2}}\right),
\end{aligned}$$

where $\mathcal{V}(\boldsymbol{x}) \stackrel{\text{def}}{=} \left\{\boldsymbol{y} \in \mathbb{R}^n : H^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x} \in \overline{B_{r(\boldsymbol{x})}(\boldsymbol{x})}\right\}$. If we now apply lemma A.2.1 and the localization result of corollary B.2.4 together with lemma B.2.6, we arrive at the result. □

Now we can provide a proof of theorem B.2.2.

Proof. Combining B.2.5 and B.2.7 gives

$$\begin{aligned}
\mathrm{I}_k(\boldsymbol{x}) = \sqrt{\frac{(2\pi)^n}{|\mathrm{H}(\boldsymbol{x})|}} \sum_{i=0}^{2p} k^i \frac{(-1)^i}{i!} \sum_{\substack{s=0 \\ 2s \geq 3i}}^{p+i} k^{-(\frac{n}{2}+s)} \frac{1}{2^s s!} \\
\langle \mathrm{H}^{-1}(\boldsymbol{x})\, \mathrm{D}_{\boldsymbol{y}}, \mathrm{D}_{\boldsymbol{y}}\rangle^s f(\boldsymbol{x}, \boldsymbol{y}) R(\boldsymbol{x}, \boldsymbol{y})^i\big|_{\boldsymbol{x}} + \mathcal{O}\left(k^{-\frac{n+2p+1}{2}}\right),
\end{aligned}$$

and the constant in $\mathcal{O}$ depends continuously on $\boldsymbol{x} \in \mathcal{U}$. Changing the summation variables yields the theorem. □

B.3 Laplace's method over halfspaces

Definition B.3.1. Let f and h be defined as before. We define

$$\mathrm{J}_k(\boldsymbol{x}) = \int_{\mathcal{U} \cap H_+(\boldsymbol{n}, \lambda)} f(\boldsymbol{x}, \boldsymbol{y}) e^{-kh(\boldsymbol{x}, \boldsymbol{y})} d\boldsymbol{y},$$

where $\boldsymbol{n} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $H_+(\boldsymbol{n}, \lambda) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{n}, \boldsymbol{x}\rangle - \lambda \geq 0\}$.

Lemma B.3.2. *Let $h, f, \mathcal{U}$ and R be defined as before. For $p \in \mathbb{N}_0$, there exists a continuous function $C_p : \mathcal{U} \to \mathbb{R}_{\geq 0}$ such that*

$$\left| \mathrm{J}_k(\boldsymbol{x}) - \sum_{i=0}^{p} \int_{\mathcal{U}} f(\boldsymbol{x}, \boldsymbol{y}) \frac{(-kR(\boldsymbol{x}, \boldsymbol{y}))^i}{i!} e^{-\frac{k}{2}\langle \mathrm{H}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}), \boldsymbol{y}-\boldsymbol{x}\rangle} d\boldsymbol{y} \right| \leq C_p(\boldsymbol{x}) k^{-\frac{n+p+1}{2}}$$

for $\boldsymbol{x} \in \mathcal{U}$.

Proof. The same arguments as in B.2.5 apply. □

Lemma B.3.3. *Let f and h be defined as before. For $p \in \mathbb{N}$, there exists a continuous function $C_p : \mathcal{U} \to \mathbb{R}_{\geq 0}$ such that*

$$\left| \int_{\mathcal{U} \cap H_+(\boldsymbol{n},\lambda)} f(\boldsymbol{x},\boldsymbol{y}) e^{-\frac{k}{2}\langle \mathrm{H}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}),\boldsymbol{y}-\boldsymbol{x}\rangle} d\boldsymbol{y} - \sum_{j=0}^{p} k^{-\frac{n+j}{2}} a_j \left(\boldsymbol{x}, \mathrm{H}^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{n}, \sqrt{\frac{k}{2}}(\lambda - \langle \boldsymbol{x},\boldsymbol{n}\rangle) \right) \right| \leq C_p(\boldsymbol{x}) k^{-\frac{n+p+1}{2}},$$

where

$$a_j(\boldsymbol{x},\boldsymbol{n},\lambda) \stackrel{\text{def}}{=} 2^{\frac{n+j}{2}} \left|\mathrm{H}(\boldsymbol{x})\right|^{-\frac{1}{2}} \sum_{|\boldsymbol{\gamma}|=j} \frac{1}{\boldsymbol{\gamma}!} \frac{\partial}{\partial \boldsymbol{y}}^{\boldsymbol{\gamma}} f\left(\boldsymbol{x}, \mathrm{H}^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x}\right)\Big|_{\boldsymbol{0}} e(\boldsymbol{\gamma},\boldsymbol{n},\lambda)$$

and

$$e(\boldsymbol{\gamma},\boldsymbol{n},\lambda) \stackrel{\text{def}}{=} \int_{H_+(\boldsymbol{n},\lambda)} \boldsymbol{x}^{\boldsymbol{\gamma}} e^{-\langle \boldsymbol{x},\boldsymbol{x}\rangle} d\boldsymbol{x}.$$

Proof.

$$\int_{\mathcal{U} \cap H_+(\boldsymbol{n},\lambda)} f(\boldsymbol{x},\boldsymbol{y}) e^{-\frac{k}{2}\langle \mathrm{H}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}),\boldsymbol{y}-\boldsymbol{x}\rangle} d\boldsymbol{y} = \left|\mathrm{H}(\boldsymbol{x})\right|^{-\frac{1}{2}} \int_{\mathcal{V}(\boldsymbol{x}) \cap H_+(\mathrm{H}^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{n},\lambda - \langle \boldsymbol{x},\boldsymbol{n}\rangle)} f(\boldsymbol{x}, \mathrm{H}^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y}+\boldsymbol{x}) e^{-\frac{k}{2}\langle \boldsymbol{y},\boldsymbol{y}\rangle} d\boldsymbol{y},$$

where $\mathcal{V}(\boldsymbol{x}) \stackrel{\text{def}}{=} \left\{\boldsymbol{y} \in \mathbb{R}^n : \mathrm{H}^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x} \in \mathcal{U}\right\}$. We Taylor-expand

$$f\left(\boldsymbol{x}, \mathrm{H}^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x}\right) = \sum_{s=0}^{p} \sum_{|\boldsymbol{\gamma}|=s} \frac{1}{\boldsymbol{\gamma}!} \frac{\partial}{\partial \boldsymbol{y}}^{\boldsymbol{\gamma}} f\left(\boldsymbol{x}, \mathrm{H}(\boldsymbol{x})^{-\frac{1}{2}}\boldsymbol{y} + \boldsymbol{x}\right)\Big|_{\boldsymbol{0}} \boldsymbol{y}^{\boldsymbol{\gamma}} + R_{p+1}(\boldsymbol{x},\boldsymbol{y}),$$

where

$$|R_{p+1}(\boldsymbol{x},\boldsymbol{y})| \leq C(\boldsymbol{x})\|\boldsymbol{y}\|^{p+1} \qquad \text{for } \boldsymbol{y} \in \overline{B_{r(\boldsymbol{x})}(\boldsymbol{0})},$$

and where we define $r(\boldsymbol{x}) \stackrel{\text{def}}{=} \min\left(1, \frac{d(\boldsymbol{x},\partial\overline{\mathcal{U}})}{2}\right)$ and note that $C : \mathcal{U} \to \mathbb{R}_{\geq 0}$ is continuous. Hence

$$\begin{aligned}
&\int_{B_{r(\boldsymbol{x})}(\boldsymbol{x}) \cap H_+(\boldsymbol{n},\lambda)} f(\boldsymbol{x},\boldsymbol{y}) e^{-\frac{k}{2}\langle \mathrm{H}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}),\boldsymbol{y}-\boldsymbol{x}\rangle} d\boldsymbol{y} \\
&= \left|\mathrm{H}(\boldsymbol{x})\right|^{-\frac{1}{2}} \int_{\mathcal{W}(\boldsymbol{x})} f\left(\boldsymbol{x}, H^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x}\right) e^{-\frac{k}{2}\langle \boldsymbol{y},\boldsymbol{y}\rangle} d\boldsymbol{y} + \mathcal{O}\left(k^{-\infty}\right) \\
&= \left|\mathrm{H}(\boldsymbol{x})\right|^{-\frac{1}{2}} \sum_{s=0}^{p} \sum_{|\boldsymbol{\beta}|=s} \frac{1}{\boldsymbol{\beta}!} \frac{\partial}{\partial \boldsymbol{y}}^{\boldsymbol{\beta}} f\left(\boldsymbol{x}, \mathrm{H}^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x}\right)\Big|_{\boldsymbol{0}} \int_{H_+(\mathrm{H}^{-1}(\boldsymbol{x})\boldsymbol{n},\lambda - \langle \boldsymbol{x},\boldsymbol{n}\rangle)} \boldsymbol{y}^{\boldsymbol{\beta}} e^{-\frac{k}{2}\langle \boldsymbol{y},\boldsymbol{y}\rangle} d\boldsymbol{y} \\
&\quad + \mathcal{O}\left(k^{-\frac{n+p+1}{2}}\right) \\
&= \left|\mathrm{H}(\boldsymbol{x})\right|^{-\frac{1}{2}} \sum_{s=0}^{p} k^{-\frac{n+s}{2}} \left(2^{\frac{n+s}{2}} \sum_{|\boldsymbol{\beta}|=s} \frac{1}{\boldsymbol{\beta}!} \frac{\partial}{\partial \boldsymbol{y}}^{\boldsymbol{\beta}} f\left(\boldsymbol{x}, \mathrm{H}^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x}\right)\Big|_{\boldsymbol{0}} \right. \\
&\quad \left. e\left(\boldsymbol{\beta}, \mathrm{H}^{-1}(\boldsymbol{x})\boldsymbol{n}, \sqrt{\frac{k}{2}}\left(\lambda - \langle \boldsymbol{x},\boldsymbol{n}\rangle\right)\right)\right) + \mathcal{O}\left(k^{-\frac{n+p+1}{2}}\right),
\end{aligned}$$

where $\mathcal{W}(\boldsymbol{x}) \stackrel{\text{def}}{=} \left\{\boldsymbol{y} \in \mathbb{R}^n : H^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x} \in B_{r(\boldsymbol{x})}(\boldsymbol{x}) \cap H_+(\boldsymbol{n}, \lambda)\right\}$. If we now apply lemma A.2.1 and the localization result of corollary B.2.4 together with lemma B.2.6, we arrive at the result. □

Theorem B.3.4. *Let f, h, R, H and $\mathcal{U}$ be defined before. For $j \in \mathbb{N}$, there exists $b_j \in \mathcal{C}^\infty(\mathcal{U})$ and, for $p \in \mathbb{N}_0$, there exists a continuous function $C_p : \mathcal{U} \to \mathbb{R}_{\geq 0}$ such that*

$$\left| \mathrm{J}_k(\boldsymbol{x}) - \sum_{j=0}^{p} k^{-\left(\frac{n+j}{2}\right)} b_j(\boldsymbol{x}, k) \right| \leq C_p(\boldsymbol{x}) k^{-\left(\frac{n+p+1}{2}\right)},$$

for $\boldsymbol{x} \in \mathcal{U}$, and where

$$\begin{aligned} b_j(\boldsymbol{x}, k) \stackrel{\text{def}}{=} |\mathrm{H}(\boldsymbol{x})|^{-\frac{1}{2}} \sum_{i=0}^{j} \frac{(-1)^i}{i!} 2^{\frac{n+j+2i}{2}} & \\ \sum_{|\gamma|=j+2i} \frac{1}{\gamma!} \frac{\partial}{\partial \boldsymbol{y}}^{\gamma} R\left(\boldsymbol{x}, \mathrm{H}^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x}\right)^i f\left(\boldsymbol{x}, \mathrm{H}^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x}\right)\Big|_{\boldsymbol{0}} & \\ e\left(\boldsymbol{\gamma}, \mathrm{H}^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{n}, \sqrt{\frac{k}{2}}\left(\lambda - \langle \boldsymbol{x}, \boldsymbol{n}\rangle\right)\right) &, \end{aligned}$$

and

$$e\left(\boldsymbol{\gamma}, \lambda, \boldsymbol{n}\right) \stackrel{\text{def}}{=} \int_{H_+(\boldsymbol{n},\lambda)} \boldsymbol{x}^{\boldsymbol{\gamma}} e^{-\langle \boldsymbol{x}, \boldsymbol{x}\rangle} d\boldsymbol{x}$$

can be explicitly evaluated using lemma A.2.2.

Proof. The proof is a simple Taylor expansion analogous to some of the preceding results. We observe that, for $p \in \mathbb{N}_0$, there exists a continuous function $C_p : \mathcal{U} \to \mathbb{R}_{\geq 0}$ such that

$$\left| \mathrm{J}_k(\boldsymbol{x}) - \sum_{i=0}^{p} \frac{(-1)^i}{i!} k^i \int_{H_+(\boldsymbol{n},\lambda)} R(\boldsymbol{x}, \boldsymbol{y})^i f(\boldsymbol{x}, \boldsymbol{y}) e^{-\frac{k}{2}\langle \mathrm{H}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}), \boldsymbol{y}-\boldsymbol{x}\rangle} d\boldsymbol{y} \right| \leq C_p(\boldsymbol{x}) k^{-\frac{n+p+1}{2}}.$$

If we now expand $R^i f$ in the above integral up to order $p + 2i$ according to lemma B.3.3, we arrive at the result. □

Corollary B.3.5. *Let f, h, R, H and $\mathcal{U}$ be defined as before, and let $\{\boldsymbol{x}_k\}_{k=0}^{\infty}$ be a sequence in $\mathcal{U}$, $\delta \in (0, \frac{1}{2})$, $\lambda \in \mathbb{R}$ and $M \geq 0$. Suppose that $\langle \boldsymbol{x}_k, \boldsymbol{n}\rangle - \lambda \geq M k^{-\frac{1}{2}+\delta}$ for all $k \in \mathbb{N}_0$. Then $\mathrm{J}_k(\boldsymbol{x}) = \mathrm{I}_k(\boldsymbol{x}_k) + \mathcal{O}\left(k^{-\infty}\right)$. If $\langle \boldsymbol{x}_k, \boldsymbol{n}\rangle - \lambda \leq -M k^{-\frac{1}{2}+\delta}$ for all $k \in \mathbb{N}_0$, then $\mathrm{J}_k(\boldsymbol{x}) = \mathcal{O}\left(k^{-\infty}\right)$.*

Proof. If $\langle \boldsymbol{x}_k, \boldsymbol{n}\rangle - \lambda \geq M k^{-\frac{1}{2}+\delta}$ for all $k \in \mathbb{N}_0$, then

$$\begin{aligned} e\left(\boldsymbol{\gamma}, \boldsymbol{n}, \sqrt{\frac{k}{2}}\left(\lambda - \langle \boldsymbol{x}_k, \boldsymbol{n}\rangle\right)\right) - \int_{\mathbb{R}^n} \boldsymbol{y}^{\boldsymbol{\gamma}} e^{-\langle \boldsymbol{y}, \boldsymbol{y}\rangle} d\boldsymbol{y} &= \int_{\sqrt{\frac{k}{2}}(\lambda - \langle \boldsymbol{x}_k, \boldsymbol{n}\rangle) > \langle \boldsymbol{y}, \boldsymbol{n}\rangle} \boldsymbol{y}^{\boldsymbol{\gamma}} e^{-\langle \boldsymbol{y}, \boldsymbol{y}\rangle} d\boldsymbol{y} \\ &= \mathcal{O}\left(k^{-\infty}\right), \end{aligned}$$

which can be easily seen by changing coordinates in the integral to $\boldsymbol{x}$ such that $x_1 = \langle \boldsymbol{y}, \boldsymbol{n}\rangle$ and by observing that $\sqrt{\frac{k}{2}}\left(\lambda - \langle \boldsymbol{x}_k, \boldsymbol{n}\rangle\right) \leq -M k^{\delta}$. Similarly, if $\langle \boldsymbol{x}_k, \boldsymbol{n}\rangle - \lambda \leq -M k^{-\frac{1}{2}+\delta}$ for all

$k \in \mathbb{N}_0$,

$$e\left(\boldsymbol{\gamma}, \boldsymbol{n}, \sqrt{\frac{k}{2}}\,(\lambda - \langle \boldsymbol{x}_k, \boldsymbol{n}\rangle)\right) = \int_{\sqrt{\frac{k}{2}}(\lambda - \langle \boldsymbol{x}_k, \boldsymbol{n}\rangle) \le \langle \boldsymbol{y}, \boldsymbol{n}\rangle} \boldsymbol{y}^{\boldsymbol{\gamma}} e^{-\langle \boldsymbol{y}, \boldsymbol{y}\rangle} d\boldsymbol{y},$$

and the result follows since $\sqrt{\frac{k}{2}}\,(\lambda - \langle \boldsymbol{x}_k, \boldsymbol{n}\rangle) \ge M k^{\delta}$. □

Lemma B.3.6. *Consider a setup as in theorem B.3.4 and suppose that $\boldsymbol{x} \in H(\boldsymbol{n}, \lambda)$. Then $b_j(\boldsymbol{x}, k)$ is independent of k, for $j \in \mathbb{N}_0$, and we have $b_{2j}(\boldsymbol{x}, k) = \frac{1}{2} a_j(\boldsymbol{x})$ for all $j \in \mathbb{N}_0$ and $k \in \mathbb{N}$, where a_j, for $j \in \mathbb{N}_0$, are the functions appearing in theorem B.2.2.*

Proof. We have $\langle \boldsymbol{x}, \boldsymbol{n}\rangle - \lambda = 0$, so that it is obvious that $b_j(\boldsymbol{x}, k)$ is independent of k. Recall from equation A.2.2 that

$$e(\boldsymbol{\gamma}, \boldsymbol{n}, 0) = \begin{cases} \sqrt{\pi^n}\, 2^{-(|\boldsymbol{\gamma}|+1)} \frac{(2\boldsymbol{\beta})!}{\boldsymbol{\beta}!} & \text{if } \boldsymbol{\gamma} = 2\boldsymbol{\beta} \text{ for some } \boldsymbol{\beta} \in \mathbb{N}_0^n \\ 0 & \text{otherwise.} \end{cases}$$

We hence have

$$\begin{aligned} b_{2j}(\boldsymbol{x}, k) = {} & |H(\boldsymbol{x})|^{-\frac{1}{2}} \sum_{i=0}^{2j} \frac{(-1)^i}{i!} 2^{\frac{n}{2}+i+j} \sum_{|\boldsymbol{\gamma}|=2(i+j)} \frac{1}{\boldsymbol{\gamma}!} \frac{\partial}{\partial \boldsymbol{y}}^{\boldsymbol{\gamma}} R(\boldsymbol{x}, H^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x})^i \\ & f(\boldsymbol{x}, H^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x})\Big|_{\boldsymbol{0}} e(\boldsymbol{\gamma}, H^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{n}, 0) \\ = {} & \frac{1}{2}\sqrt{\frac{(2\pi)^n}{|H(\boldsymbol{x})|}} \sum_{i=0}^{2j} \frac{(-1)^i}{i!(i+j)!2^{i+j}} \sum_{|\boldsymbol{\beta}|=i+j} \binom{i+j}{\boldsymbol{\beta}} \frac{\partial}{\partial \boldsymbol{y}}^{2\boldsymbol{\beta}} R(\boldsymbol{x}, H^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x})^i \\ & f(\boldsymbol{x}, H^{-\frac{1}{2}}(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{x})\Big|_{\boldsymbol{0}} \\ = {} & \frac{1}{2} a_j(\boldsymbol{x}). \end{aligned}$$

□

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