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BERGMAN NORM ESTIMATES OF POISSON INTEGRALS

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Abstract. On the half space $\mathbf{R}^n \times \mathbf{R}_+$, it has been known that harmonic Bergman space b^p can contain a positive function only if $p > 1 + \frac{1}{n}$. Thus, for $1 \le p \le 1 + \frac{1}{n}$, Poisson integrals can be b^p-functions only by means of their boundary cancellation properties. In this paper, we describe what those cancellation properties explicitly are. Also, given such cancellation properties, we obtain weighted norm inequalities for Poisson integrals. As a consequence, under weighted integrability condition given by our weighted norm inequalities, we show that our cancellation properties are equivalent to the b^p-containment of Poisson integrals for p under consideration. Our results are sharp in the sense that orders of our weights cannot be improved.

§1. Introduction

For a fixed positive integer n, let $\mathbf{H} = \mathbf{R}^n \times \mathbf{R}_+ \subset \mathbf{R}^{n+1}$ be the upper half space where $\mathbf{R}_+$ denotes the set of all positive real numbers. As is well known, the Poisson kernel $P_t(x)$ for $\mathbf{H}$ is given by

$$P_t(x) = \frac{t}{(|x|^2 + t^2)^m} \qquad (x \in \mathbf{R}^n,\ t > 0)$$

where $m = \frac{n+1}{2}$. For $1 \le p < \infty$, let $L^p = L^p(\mathbf{R}^n)$ be the Lebesgue space on $\mathbf{R}^n$. For $f \in L^p$, the Poisson integral $P[f]$ on $\mathbf{H}$ is defined as the convolution $P_t * f$ of f and P_t. More explicitly,

$$P[f](x,t) = \int_{\mathbf{R}^n} P_t(x-y) f(y)\, dy$$

for $(x,t) \in \mathbf{H}$. For a complex Borel measure μ on $\mathbf{R}^n$, its Poisson integral $P[\mu]$ is defined in a similar way.

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It is well known that the Poisson integral transform is a linear isometry (modulo normalizing constant) of L^p into the harmonic L^p-Hardy space (see, for example, [1]):

$$\sup_{t>0} \int_{\mathbf{R}^n} |P_t * f(x)|^p \, dx = c_n^p \int_{\mathbf{R}^n} |f(x)|^p \, dx.$$

It follows from this type of results on bounded domains (like balls) that the Poisson integral transform takes L^p into the harmonic L^p-Bergman space. The unboundedness of our domain $\mathbf{H}$ makes the situation quite different. In this paper we investigate such phenomena related to harmonic L^p-Bergman spaces caused by the unboundedness of $\mathbf{H}$.

To be more precise, let b^p $(1 \le p < \infty)$ denote the harmonic L^p-Bergman space consisting of all harmonic functions in $L^p(\mathbf{H})$. It is known that Poisson integrals of L^p- functions are not always b^p-functions in general. This follows from the fact [2] that b^p contains a positive function if and only if $p > 1 + \frac{1}{n}$. This means that harmonic functions on $\mathbf{H}$ must have certain types of cancellations to be members of b^p for $1 \le p \le 1 + \frac{1}{n}$ and the same is expected on the boundary if they are represented by Poisson integrals. Such an example is the so-called b^1-cancellation property noticed in [2]: the horizontal zero moments of any b^1-function u are all 0, which means

$$\int_{\mathbf{R}^n} u(x, \delta) \, dx = 0$$

for each $\delta > 0$, and the same is necessarily true for its boundary function in case u is represented by a Poisson integral of some L^1-function. However, this zero moment vanishing property is far from being sufficient, since one may find many odd integrable functions whose Poisson integrals are not contained in b^1. Also, even for $p > 1 + \frac{1}{n}$ where cancellation does not have any effect, it is not hard to find examples of L^p-functions whose Poisson integrals do not belong to b^p. We are led to two questions by these simple observations. First, while it might not be possible to characterize b^p-functions in terms of cancellation properties in general, what types of cancellation properties do they have (if they must)? Secondly, if they already have such cancellation properties, what kinds of norm inequalities hold for Poisson integrals? In this paper we settle these two questions. These problems were originally suggested by Wade Ramey to the third author. We thank him for his suggestion.

First, we have the following cancellation results.

THEOREM 1.1. *Let μ be a complex Borel measure on $\mathbf{R}^n$. If $P[\mu] \in b^p$ for some $1 \le p \le 1 + \frac{1}{n}$, then*

$$\int_{\mathbf{R}^n} d\mu = 0. \tag{1.1}$$

If, in addition, $|x| \in L^1(|\mu|)$ and if $P[\mu] \in b^1$, then the first moments of μ are all 0, or more explicitly,

$$\int_{\mathbf{R}^n} x_j \, d\mu(x) = 0 \tag{1.2}$$

for all j.

Note that (1.1) and (1.2) above are simply $\widehat{u}(0) = 0$ and $\nabla\widehat{u}(0) = 0$, respectively, where $\widehat{\mu}$ denotes the Fourier transform of μ. While we do not use any significant Fourier transform arguments in this paper, we remark that there is a close relation between b^2-norms of Poisson integrals and L^2-norms of Fourier transforms of their boundary functions. See Lemma 3.9. This seems natural by the Plancherel identity, since Poisson integrals are defined in terms of convolution.

Next, given all relevant moment vanishing properties, we consider the question of when $P[f] \in b^p$ holds. In considering such a problem, it might be necessary to derive certain types of norm inequalities. What we have are the following weighted norm inequalities. Here and elsewhere, we use the notation

$$\omega_p(x) = \begin{cases} |x|(\log^+|x|)^p & \text{for} \quad p = 1 \text{ or } p = 1 + \frac{1}{n} \\ |x| & \text{otherwise} \end{cases} \tag{1.3}$$

for simplicity.

THEOREM 1.2. *Let $p \ge 1$ and f be a measurable function on $\mathbf{R}^n$ such that $P[f]$ is well defined. For $1 \le p \le 1 + \frac{1}{n}$, assume $f \in L^1$ and its zero moment is 0. For $p = 1$, we also assume $f \in L^1(|x|dx)$ and its first moments are all 0.*

(1) *For $1 < p < 1 + \frac{1}{n}$ or $p > 1 + \frac{1}{n}$, we have*

$$\int_0^\infty \int_{\mathbf{R}^n} |P_t * f(x)|^p \, dx \, dt \le C_p \int_{\mathbf{R}^n} |f(x)|^p |x| \, dx.$$

(2) *For* $p = 1$ *or* $p = 1 + \frac{1}{n}$, *we have*

$$\int_0^\infty \int_{\mathbf{R}^n} |P_t * f(x)|^p \, dx \, dt \le C_p \int_{\mathbf{R}^n} |f(x)|^p \left[1 + \omega_p(x)\right] \, dx.$$

Having these weighted norm inequalities, one finds that our cancellation results are sharp in the sense that any additional cancellation properties cannot be expected in the conclusion of Theorem 1.1. In fact, given weighted integrability conditions suggested by the above theorem, we have a complete description of Poisson integrals in b^p in terms of cancellation properties.

THEOREM 1.3. *Let* $p \ge 1$ *and* $f \in L^p(\omega_p)$. *Also assume* $f \in L^p$ *for* $1 \le p \le 1 + \frac{1}{n}$.

(1) $P[f] \in b^1$ *if and only if the zero and first moments of* f *are all* 0.

(2) *For* $1 < p \le 1 + \frac{1}{n}$, $P[f] \in b^p$ *if and only if the zero moment of* f *is* 0.

(3) *For* $p > 1 + \frac{1}{n}$, *we always have* $P[f] \in b^p$.

The above theorem recovers some results in [5] where Yi obtained the same for continuous functions with compact support for $1 < p \le 1 + \frac{1}{n}$. Our results Theorem 1.2 and Theorem 1.3 are also sharp in the sense that orders of weights at infinity cannot be reduced.

We will first prove the weighted norm inequalities and then the cancellation properties. Also we will provide various examples related to our results. We divide the proof of the weighted norm inequalities into two sections. In Section 2, all preliminary inequalities we need for the proof of Theorem 1.2 are collected. In Section 3, we prove Theorem 1.2. As consequences of our weighted norm inequalities, we show that functions without necessary cancellation properties can be modified by continuous functions with compact support so that the same type of weighted norm inequalities hold. See Corollary 3.5 and Corollary 3.7. At the end of the section, some observations related to Fourier transforms are included. In section 4, we prove Theorem 1.1 and, as a consequence, we derive Theorem 1.3. In Section 5, we give various examples of functions related to our results. We construct examples for the purpose of showing (i) the moment vanishing properties are not sufficient for the b^p-containment in general, (ii) our weighted integrability condition in Theorem 1.3 is sharp (hence so are the weighted norm inequalities) in the sense that orders at infinity cannot be reduced, and nevertheless (iii) such a weighted integrability condition is not necessary for the b^p-containment in case relevant moment vanishing conditions are already given.

§2. Auxiliary inequalities

In this section we collect inequalities which we need in the proof of weighted norm inequalities in the next section. Our starting point is the following well-known Hardy's inequality. See, for example, [4].

Hardy's Inequality. *Let $p \geq 1$, $r > 0$. Then we have*

$$\int_0^\infty \left(\int_s^\infty \varphi(t)\, dt \right)^p s^{r-1}\, ds \leq \left(\frac{p}{r} \right)^p \int_0^\infty [t\varphi(t)]^p t^{r-1}\, dt \tag{1}$$

$$\int_0^\infty \left(\int_0^s \varphi(t)\, dt \right)^p s^{-r-1}\, ds \leq \left(\frac{p}{r} \right)^p \int_0^\infty [t\varphi(t)]^p t^{-r-1}\, dt \tag{2}$$

for all measurable functions $\varphi \geq 0$ on $(0, \infty)$.

Note. We use the notation $A \lesssim B$ for positive quantities A and B if the ratio A/B has a positive upper bound. Also, we write $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$. Constants involved there may often depend on the dimension and some other parameters, but they will be always independent of particular functions, measures, or points, etc. Sometimes such constants will be explicitly denoted by the same letter C often with subscripts indicating dependency.

Consider operators T_1 and T_2 defined by

$$T_1 h(t) = \int_{|y|>t} h(y)|y|^{-n}\, dy, \quad T_2 h(t) = \int_{|y|<t} h(y)|y|^{-n}\, dy$$

for measurable functions $h \geq 0$ on $\mathbf{R}^n$ and $t > 0$. We need L^p boundedness for these operators, which one may view as a higher dimensional version of Hardy's inequality.

Lemma 2.1. *For $p \geq 1$ and $r > 0$, we have*

$$\int_0^\infty |T_1 h(t)|^p t^{r-1}\, dt \leq C_{p,r} \int_{\mathbf{R}^n} |h(y)|^p |y|^{r-n}\, dy, \tag{1}$$

$$\int_0^\infty |T_2 h(t)|^p t^{-r-1}\, dt \leq C_{p,r} \int_{\mathbf{R}^n} |h(y)|^p |y|^{-r-n}\, dy. \tag{2}$$

In what follows Σ denotes the unit sphere in $\mathbf{R}^n$ centered at the origin.

Proof. Let σ be the surface area measure on Σ. Then it follows from Hardy's inequality and Jensen's inequality that

$$\begin{aligned}\int_0^\infty |T_1 h(t)|^p t^{r-1}\,dt &= \int_0^\infty \left(\int_t^\infty \int_\Sigma h(u\zeta)\,d\sigma(\zeta) u^{-1}\,du\right)^p t^{r-1}\,dt \\ &\le C_{p,r}\int_0^\infty \left(\int_\Sigma h(t\zeta)\,d\sigma(\zeta)\right)^p t^{r-1}\,dt \\ &\le C_{p,r}\int_0^\infty \int_\Sigma |h(t\zeta)|^p\,d\sigma(\zeta)\; t^{r-1}\,dt \\ &= C_{p,r}\int_{\mathbf{R}^n} |h(y)|^p |y|^{r-n}\,dy.\end{aligned}$$

This proves (1). One can see (2) by exactly the same way. The proof is complete. ∎

We also need a logarithmic version of Lemma 2.1. So, consider an operator T_3 defined by

$$T_3 h(t) = \int_{|y|>t} h(y)|y|^{-n}(\log|y|)^{-1}\,dy$$

for measurable functions $h \ge 0$ on $\mathbf{R}^n$ and $t > 1$. For this operator we have the following L^p boundedness.

LEMMA 2.2. *For $p \ge 1$ and $r > 0$, we have*

$$\int_1^\infty |T_3 h(t)|^p t^{-1}(\log t)^{r-1}\,dt \le C_{p,r}\int_{|y|>1} |h(y)|^p |y|^{-n}(\log|y|)^{r-1}\,dy.$$

Proof. By change of variables (after representing the integral in polar coordinates), one can check $T_3 h(t) = T_1\tilde{h}(\log t)$ where $\tilde{h}(y) = h(e^{|y|}|y|^{-1}y)$. Thus, by Lemma 2.1, we have

$$\begin{aligned}&\int_1^\infty |T_3 h(t)|^p t^{-1}(\log t)^{r-1}\,dt \\ &\qquad = \int_1^\infty |T_1\tilde{h}(\log t)|^p t^{-1}(\log t)^{r-1}\,dt \\ &\qquad = \int_0^\infty |T_1\tilde{h}(t)|^p t^{r-1}\,dt\end{aligned}$$

$$\leq C_{p,r} \int_{\mathbf{R}^n} |\tilde{h}(y)|^p |y|^{r-n}\, dy$$
$$= C_{p,r} \int_{|y|>1} |h(y)|^p |y|^{-n} (\log |y|)^{r-1}\, dy.$$

The proof is complete. □

Remark. While it is not needed for our purpose, we remark that the complementary operator

$$T_4 h(t) = \int_{1<|y|<t} h(y)|y|^{-n} (\log |y|)^{-1}\, dy$$

has similar L^p boundedness:

$$\int_1^\infty |T_4 h(t)|^p t^{-1} (\log t)^{-r-1}\, dt \leq C_{p,r} \int_{|y|>1} |h(y)|^p |y|^{-n} (\log |y|)^{-r-1}\, dy$$

for $p \geq 1$ and $r > 0$.

For $t > 0$, let ν_t be the volume measure, normalized to have total mass 1, on the ball in $\mathbf{R}^n$ of radius t centered at the origin. Also, let σ_t be the surface area measure, normalized to have total mass 1, on the sphere in $\mathbf{R}^n$ of radius t centered at the origin. The following L^p boundedness of convolutions with these measures are useful for our purpose.

LEMMA 2.3. *For $p \geq 1$, we have*

$$\text{(1)} \qquad \int_0^\infty \int_{|x|>2t} |h * \nu_t(x)|^p\, dx\, dt \leq \int_{\mathbf{R}^n} |h(x)|^p |x|\, dx$$

$$\text{(2)} \qquad \int_0^\infty \int_{|x|>2t} |h * \sigma_t(x)|^p\, dx\, dt \leq \int_{\mathbf{R}^n} |h(x)|^p |x|\, dx$$

for measurable functions $h \geq 0$ on $\mathbf{R}^n$.

Proof. Since

$$h * \nu_t(x) = \omega_n^{-1} t^{-n} \int_{|x-y|<t} h(y)\, dy$$

where ω_n denotes the volume of the unit ball in $\mathbf{R}^n$, we have by Jensen's inequality

$$|h * \nu_t(x)|^p \leq \omega_n^{-1} t^{-n} \int_{|x-y|<t} |h(y)|^p\, dy.$$

Note that if $|x| > 2t$ and $|x-y| < t$, then $t < |y|$. Thus, it follows from the above that

$$\begin{aligned}
\int_0^\infty \int_{|x|>2t} &|h*\nu_t(x)|^p\,dx\,dt \\
&\le \omega_n^{-1}\int_{\mathbf{R}^n}\int_0^{|x|/2}\int_{|x-y|<t}|h(y)|^p\,dy\,t^{-n}\,dt\,dx \\
&= \omega_n^{-1}\int_{\mathbf{R}^n}\int_0^{\infty}\int_{|x-y|<t<|x|/2}|h(y)|^p t^{-n}\,dx\,dt\,dy \\
&\le \omega_n^{-1}\int_{\mathbf{R}^n}\int_0^{|y|}\int_{|x-y|<t}|h(y)|^p\,dx\,t^{-n}\,dt\,dy \\
&= \int_{\mathbf{R}^n}\int_0^{|y|}|h(y)|^p\,dt\,dy \\
&= \int_{\mathbf{R}^n}|h(y)|^p|y|\,dy,
\end{aligned}$$

which shows (1).

Next, note that

$$h*\sigma_t(x) = \int_\Sigma h(x-t\zeta)\,d\sigma_1(\zeta)$$

and thus Jensen's inequality yields

$$|h*\sigma_t(x)|^p \le \int_\Sigma |h(x-t\zeta)|^p\,d\sigma_1(\zeta).$$

Letting λ_n denote the surface area of Σ, we obtain from the above

$$\begin{aligned}
\int_0^\infty \int_{|x|>2t} &|h*\sigma_t(x)|^p\,dx\,dt \\
&\le \int_{\mathbf{R}^n}\int_0^{|x|/2}\int_\Sigma |h(x-t\zeta)|^p\,d\sigma_1(\zeta)\,dt\,dx \\
&= \lambda_n^{-1}\int_{\mathbf{R}^n}\int_{2|y|<|x|}|h(x-y)|^p|y|^{1-n}\,dy\,dx \\
&= \lambda_n^{-1}\int_{\mathbf{R}^n}\int_{2|x-z|<|x|}|h(z)|^p|x-z|^{1-n}\,dz\,dx \\
&= \lambda_n^{-1}\int_{\mathbf{R}^n}\int_{2|y|<|y+z|}|y|^{1-n}\,dy\,|h(z)|^p\,dz
\end{aligned}$$

$$\leq \lambda_n^{-1} \int_{\mathbf{R}^n} \int_{|y|<|z|} |y|^{1-n}\, dy\, |h(z)|^p\, dz$$
$$= \int_{\mathbf{R}^n} |h(z)|^p |z|\, dz,$$

so that (2) holds. The proof is complete. ∎

As mentioned in the introduction, the following is well known.

LEMMA 2.4. *For $p \geq 1$, we have*

$$\sup_{t>0} \int_{\mathbf{R}^n} |P_t * f(x)|^p\, dx \leq c_n^p \int_{\mathbf{R}^n} |f(x)|^p\, dx$$

for measurable functions $f \geq 0$ on $\mathbf{R}^n$.

When we derive weighted norm inequalities in the next section, we will decompose $\mathbf{R}^n$ into three pieces. We collect here some basic information on those pieces. Here and elsewhere, we let

$$\eta = \eta(x, y, t) = \frac{|y|^2 - 2x \cdot y}{|x|^2 + t^2} \qquad (x,\ y \in \mathbf{R}^n,\ t > 0)$$

for simplicity.

LEMMA 2.5. *If $\eta > \frac{4}{5}$, then the following hold:*

$$|x|^2 + t^2 < 2|x - y|^2, \qquad |x| + t < 6|y|,$$
$$|P_t(x - y) - P_t(x)| \leq P_t(x),$$
$$|P_t(x - y) - P_t(x) + \nabla P_t(x) \cdot y| \leq P_t(x) + |y||\nabla P_t(x)|.$$

Proof. Note that our assumption is $5(|x - y|^2 + t^2) > 9(|x|^2 + t^2)$. The proof is therefore straightforward. ∎

LEMMA 2.6. *If $|\eta| \leq \frac{4}{5}$, then the following hold:*

$$|y| < 3(|x| + t),$$
$$|P_t(x - y) - P_t(x)| \lesssim |y|(1 + t|x|^{-1})|\nabla P_t(x)|,$$
$$|P_t(x - y) - P_t(x) + \nabla P_t(x) \cdot y| \lesssim |y|^2|x|^{-1}|\nabla P_t(x)|.$$

Proof. We have $|x|^2+t^2 \le 5(|x-y|^2+t^2) \le 9(|x|^2+t^2)$. The inequality $|y| < 3(|x|+t)$ is therefore straightforward. Note

$$\begin{aligned}|P_t(x-y) - P_t(x)| &= P_t(x)|(1+\eta)^{-m} - 1| \\ &\lesssim P_t(x)|\eta| \\ &\lesssim |y|(1+|y||x|^{-1})|\nabla P_t(x)| \\ &\lesssim |y|(1+t|x|^{-1})|\nabla P_t(x)|.\end{aligned}$$

Also we have

$$\begin{aligned}&|P_t(x-y) - P_t(x) + \nabla P_t(x)\cdot y| \\ &\quad = P_t(x)\left|(1+\eta)^{-m} - 1 - \frac{2mx\cdot y}{|x|^2+t^2}\right| \\ &\quad = mP_t(x)\left|\eta + O(\eta^2) + \frac{2x\cdot y}{|x|^2+t^2}\right| \\ &\quad \lesssim P_t(x)\left\{|\eta|^2 + \frac{|y|^2}{|x|^2+t^2}\right\} \\ &\quad \lesssim |y|^2|x|^{-1}|\nabla P_t(x)|\left\{1 + \frac{|\eta|^2(|x|^2+t^2)}{|y|^2}\right\} \\ &\quad \lesssim |y|^2|x|^{-1}|\nabla P_t(x)|\left\{1 + \frac{(|y|+|x|)^2}{|x|^2+t^2}\right\} \\ &\quad \lesssim |y|^2|x|^{-1}|\nabla P_t(x)|\end{aligned}$$

where the last inequality follows from the fact that $|y| < 3(|x|+t)$. The proof is complete. ∎

LEMMA 2.7. *If $\eta < -\frac{4}{5}$, then the following hold:*

$$\begin{aligned}&2|x-y| < |x|, \qquad 2t < |x|, \qquad |y| < 2|x|, \\ &|P_t(x-y) - P_t(x)| \lesssim P_t(x-y), \\ &|P_t(x-y) - P_t(x) + \nabla P_t(x)\cdot y| \lesssim P_t(x-y).\end{aligned}$$

Proof. Our assumption is now $5(|x-y|^2+t^2) < |x|^2+t^2$. The proof is therefore straightforward. ∎

§3. Weighted norm inequalities

In this section we obtain weighted norm inequalities for Poisson integrals. Since estimates necessarily depend on good control (by means of cancellation) of dominating terms, it is natural to decompose $\mathbf{R}^n$ into pieces. Here, we consider three pieces. For fixed $x \in \mathbf{R}^n$ and $t > 0$, we let

$$\begin{aligned} K_1 &= \{y \in \mathbf{R}^n : \eta(x,y,t) > 4/5\}, \\ K_2 &= \{y \in \mathbf{R}^n : |\eta(x,y,t)| \le 4/5\}, \\ K_3 &= \{y \in \mathbf{R}^n : \eta(x,y,t) < -4/5\}. \end{aligned}$$

Now, we estimate various types of operators corresponding to these pieces. Why we consider those operators must be clear from Lemma 2.5, Lemma 2.6 and Lemma 2.7. First, for the estimation on K_1, we consider a couple of operators defined by

$$\begin{aligned} \Lambda_1 f(x,t) &= P_t(x) \int_{K_1} f(y)\, dy, \\ \tilde{\Lambda}_1 f(x,t) &= |\nabla P_t(x)| \int_{K_1} f(y)|y|\, dy. \end{aligned}$$

for measurable functions $f \ge 0$ on $\mathbf{R}^n$, $x \in \mathbf{R}^n$ and $t > 0$. For these operators, we have the following L^p boundedness.

LEMMA 3.1. *For measurable functions $f \ge 0$ on $\mathbf{R}^n$, the following hold.*

(1) *For $1 \le p < 1 + \frac{1}{n}$, we have*

$$\int_0^\infty \int_{\mathbf{R}^n} |\Lambda_1 f(x,t)|^p\, dx\, dt \le C_p \int_{\mathbf{R}^n} |f(x)|^p |x|\, dx.$$

(2) *For $p = 1 + \frac{1}{n}$, we have*

$$\int_6^\infty \int_{\mathbf{R}^n} |\Lambda_1 f(x,t)|^p\, dx\, dt \le C_p \int_{\mathbf{R}^n} |f(x)|^p |x| (\log^+ |x|)^p\, dx.$$

(3) *For $p = 1$, we have*

$$\int_6^\infty \int_{\mathbf{R}^n} \tilde{\Lambda}_1 f(x,t)\, dx\, dt \le C \int_{\mathbf{R}^n} f(x) |x| \log^+ |x|\, dx.$$

Proof. First consider the case $1 \le p < 1 + \frac{1}{n}$. By Lemma 2.5, we have

$$\Lambda_1 f(x,t) \le \int_{6|y|>|x|+t} P_t(x) f(y)\, dy \le P_t(x) \int_{6|y|>t} f(y)\, dy.$$

Note

$$\int_{\mathbf{R}^n} |P_t(x)|^p \, dx = t^{n-np} \int_{\mathbf{R}^n} |P_1(x)|^p \, dx \approx t^{n-np}.$$

It follows that

$$\begin{aligned}\int_0^\infty \int_{\mathbf{R}^n} |\Lambda_1 f(x,t)|^p \, dx \, dt &\lesssim \int_0^\infty \left(\int_{6|y|>t} f(y) \, dy \right)^p t^{n-np} \, dt \\ &\lesssim \int_0^\infty \left(\int_{|y|>t} f(y) \, dy \right)^p t^{n-np} \, dt.\end{aligned}$$

Now, apply Lemma 2.1 with $r = n - np + 1 > 0$ to conclude (1).

For the case $p = 1 + \frac{1}{n}$, exactly the same argument yields

$$\begin{aligned}\int_6^\infty \int_{\mathbf{R}^n} |\Lambda_1 f(x,t)|^p \, dx \, dt &\lesssim \int_6^\infty \left(\int_{6|y|>t} f(y) \, dy \right)^p t^{-1} \, dt \\ &= \int_1^\infty \left(\int_{|y|>t} f(y) \, dy \right)^p t^{-1} \, dt,\end{aligned}$$

and therefore (2) follows from Lemma 2.2 with $r = 1$.

For $\tilde{\Lambda}_1$, note

$$\int_{\mathbf{R}^n} |\nabla P_t(x)| \, dx = t^{-1} \int_{\mathbf{R}^n} |\nabla P_1(x)| \, dx \approx t^{-1}$$

and thus a similar argument yields

$$\begin{aligned}\int_6^\infty \int_{\mathbf{R}^n} \tilde{\Lambda}_1 f(x,t) \, dx dt &\lesssim \int_6^\infty t^{-1} \int_{6|y|>t} f(y)|y| \, dy \, dt \\ &= \int_1^\infty t^{-1} \int_{|y|>t} f(y)|y| \, dy \, dt \\ &= \int_{\mathbf{R}^n} f(y)|y| \log^+ |y| \, dy\end{aligned}$$

so that (3) holds. This completes the proof. ∎

Next, for the estimation on K_2, we also consider a couple of operators defined by

$$\begin{aligned}\Lambda_2 f(x,t) &= (1 + t|x|^{-1})|\nabla P_t(x)| \int_{K_2} f(y)|y| \, dy, \\ \tilde{\Lambda}_2 f(x,t) &= |x|^{-1}|\nabla P_t(x)| \int_{K_2} f(y)|y|^2 \, dy\end{aligned}$$

for measurable functions $f \geq 0$ on $\mathbf{R}^n$, $x \in \mathbf{R}^n$ and $t > 0$. For these operators, we have the following L^p boundedness.

LEMMA 3.2. *For measurable functions $f \geq 0$ on $\mathbf{R}^n$, the following hold.*

(1) *For $p > 1$, we have*

$$\int_0^\infty \int_{\mathbf{R}^n} |\Lambda_2 f(x,t)|^p \, dx \, dt \leq C_p \int_{\mathbf{R}^n} |f(x)|^p |x| \, dx.$$

(2) *For $p = 1$, we have*

$$\int_0^\infty \int_{\mathbf{R}^n} \tilde{\Lambda}_2 f(x,t) \, dx \, dt \leq C \int_{\mathbf{R}^n} f(x)|x| \, dx.$$

Proof. First consider the case $p > 1$. By Lemma 2.6, we have

$$\Lambda_2 f(x,t) \leq \int_{|y|<3(|x|+t)} (1 + t|x|^{-1}) f(y)|y||\nabla P_t(x)| \, dy$$

and thus, by integrating in polar coordinates,

$$\begin{aligned}
\int_{\mathbf{R}^n} |\Lambda_2 f(x,t)|^p \, dx &\lesssim \int_0^\infty \left(\int_{|y|<3(u+t)} \frac{t(u+t)f(y)|y|}{(u^2+t^2)^{m+1}} \, dy \right)^p u^{n-1} \, du \\
&\lesssim \int_0^t \left(\int_{|y|<6t} f(y)|y| t^{-n-1} \, dy \right)^p u^{n-1} \, du \\
&\qquad + \int_t^\infty \left(\int_{|y|<6u} f(y)|y| u^{-n-1} \, dy \right)^p u^{n-1} \, du \\
&\lesssim t^{n-np-p} \left(\int_{|y|<6t} f(y)|y| \, dy \right)^p \\
&\qquad + \int_t^\infty \left(\int_{|y|<6u} f(y)|y| \, dy \right)^p u^{n-np-p-1} \, du.
\end{aligned}$$

Note, by interchanging the order of integration,

$$\begin{aligned}
&\int_0^\infty \int_t^\infty \left(\int_{|y|<6u} f(y)|y| \, dy \right)^p u^{n-np-p-1} \, du \, dt \\
&\qquad = \int_0^\infty \left(\int_{|y|<6u} f(y)|y| \, dy \right)^p u^{n-np-p} \, du.
\end{aligned}$$

Consequently,

$$\int_0^\infty \int_{\mathbf{R}^n} |\Lambda_2 f(x,t)|^p \, dx \, dt \lesssim \int_0^\infty \left(\int_{|y|<6u} f(y)|y| \, dy \right)^p u^{n-np-p} \, du.$$

Now, apply Lemma 2.1 with $r = np + p - n - 1 > 0$ to conclude (1).

Similarly, for $p = 1$, we have the following estimate for $\tilde{\Lambda}_2$.

$$\begin{aligned} \int_{\mathbf{R}^n} \tilde{\Lambda}_2 f(x,t) \, dx &\le \int_{\mathbf{R}^n} \int_{|y|<3(|x|+t)} |x|^{-1} f(y)|y|^2 |\nabla P_t(x)| \, dy \, dx \\ &\lesssim \int_0^\infty \int_{|y|<3(u+t)} \frac{t f(y)|y|^2}{(u^2+t^2)^{m+1}} \, dy \, u^{n-1} \, du \\ &\le t^{-n-2} \int_0^t \int_{|y|<6t} f(y)|y|^2 \, dy \, u^{n-1} \, du \\ &\qquad + \int_t^\infty \int_{|y|<6u} f(y)|y|^2 \, dy \, u^{-3} \, du \\ &= n^{-1} t^{-2} \int_{|y|<6t} f(y)|y|^2 \, dy \\ &\qquad + \int_t^\infty \int_{|y|<6u} f(y)|y|^2 \, dy \, u^{-3} \, du. \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^\infty \int_{\mathbf{R}^n} \tilde{\Lambda}_2 f(x,t) \, dx \, dt &\lesssim \int_0^\infty \int_{|y|<6t} f(y)|y|^2 \, dy \, t^{-2} \, dt \\ &\qquad + \int_0^\infty \int_t^\infty \int_{|y|<6u} f(y)|y|^2 \, dy \, u^{-3} \, du \, dt \\ &\lesssim \int_0^\infty \int_{|y|<6t} f(y)|y|^2 \, dy \, t^{-2} \, dt \\ &= 6 \int_{\mathbf{R}^n} f(y)|y| \, dy, \end{aligned}$$

so that (2) holds. The proof is complete. □

Finally, for the estimation on K_3, we consider an operator defined by

$$\Lambda_3 f(x,t) = \int_{K_3} f(y) P_t(x-y) \, dy$$

for measurable functions $f \ge 0$ on $\mathbf{R}^n$, $x \in \mathbf{R}^n$ and $t > 0$. For this operator, we have the following L^p boundedness.

LEMMA 3.3. *For $p \geq 1$ and measurable functions $f \geq 0$ on $\mathbf{R}^n$, we have*

$$\int_0^\infty \int_{\mathbf{R}^n} |\Lambda_3 f(x,t)|^p \, dx \, dt \leq C_p \int_{\mathbf{R}^n} |f(x)|^p |x| \, dx.$$

Proof. By Lemma 2.7, $2t < |x|$ and $2|x-y| < |x|$ on K_3. Thus

$$\begin{aligned}
\Lambda_3 f(x,t) &\leq \int_{2|x-y|<|x|} f(y) P_t(x-y) \, dy \\
&\lesssim t^{-n} \int_{|x-y|<t} f(y) \, dy + \int_{t<|x-y|<|x|/2} t f(y) |x-y|^{-n-1} \, dy \\
&\lesssim f * \nu_t(x) + \int_{t<|x-y|<|x|/2} f(y) |x-y|^{-n} \, dy \\
&= f * \nu_t(x) + \int_{t<|z|<|x|/2} f(x-z) |z|^{-n} \, dz \\
&= f * \nu_t(x) + \int_t^{|x|/2} f * \sigma_s(x) \, s^{-1} \, ds
\end{aligned}$$

Now estimate for the first term of the above follows from Lemma 2.3. For the second term, note that Hardy's inequality with $r = 1$ gives

$$\int_0^{|x|/2} \left(\int_t^{|x|/2} f * \sigma_s(x) \, s^{-1} \, ds \right)^p dt \leq p^p \int_0^{|x|/2} |f * \sigma_t(x)|^p \, dt$$

and therefore desired estimate also follows from Lemma 2.3. The proof is complete. □

We are now ready to prove our weighted norm inequalities. Actually, all the necessary estimates are contained in Lemma 3.1, Lemma 3.2 and Lemma 3.3. What remains is just to combining them together. We begin with the case $p = 1$.

THEOREM 3.4. *Let $f \in L^1 \cap L^1(|x|dx)$. If f satisfies the zero and first moment vanishing conditions*

$$\int_{\mathbf{R}^n} f(x) \, dx = \int_{\mathbf{R}^n} x_j f(x) \, dx = 0 \qquad (1 \leq j \leq n),$$

then we have

$$\int_0^\infty \int_{\mathbf{R}^n} |P_t * f(x)| \, dx \, dt \leq C \int_{\mathbf{R}^n} |f(x)| (1 + |x| \log^+ |x|) \, dx.$$

Proof. Since the zero and first moments are 0 by assumption, we have

$$P_t * f(x) = \int_{\mathbf{R}^n} f(y)\,[P_t(x-y) - P_t(x) + \nabla P_t(x)\cdot y]\;dy$$

and therefore, by Lemma 2.5, Lemma 2.6 and Lemma 2.7,

$$\begin{aligned}|P_t * f(x)| &\le \int_{\mathbf{R}^n} |f(y)|\,|P_t(x-y) - P_t(x) + \nabla P_t(x)\cdot y|\;dy\\ &\lesssim \int_{K_1} |f(y)|\,[P_t(x) + |y||\nabla P_t(x)|]\;dy\\ &\quad + \int_{K_2} |f(y)||y|^2|x|^{-1}|\nabla P_t(x)|\,dy + \int_{K_3} |f(y)|P_t(x-y)\,dy\\ &= \Lambda_1|f|(x,t) + \tilde{\Lambda}_1|f|(x,t) + \tilde{\Lambda}_2|f|(x,t) + \Lambda_3|f|(x,t).\end{aligned}$$

Now, the theorem follows from Lemma 3.1, Lemma 3.2, Lemma 3.3 and Lemma 2.4. The proof is complete. ∎

As a consequence, we have the following modified weighted norm inequalities which might be of some independent interest.

COROLLARY 3.5. *To each $f \in L^1 \cap L^1(|x|dx)$ there corresponds $\tilde{f} \in C_c(\mathbf{R}^n)$ such that*

$$\int_0^\infty \int_{\mathbf{R}^n} |P_t * (f - \tilde{f})(x)|\,dx\,dt \le C \int_{\mathbf{R}^n} |f(x)|(1 + |x|\log^+|x|)\,dx.$$

Proof. Choose $\varphi \in C_c(\mathbf{R}_+)$ and $\psi \in C_c(\mathbf{R}^{n-1})$ such that

$$2\int_0^\infty \varphi(t)\,dt = \int_{\mathbf{R}^{n-1}} \psi(x')\,dx' = 1$$

and define $f_1(x) = [\varphi(x_1) - \varphi(-x_1)]\psi(x_2,\cdots,x_n)$. One can easily check that the zero moment of f_1 is 0, $\int_{\mathbf{R}^n} x_1 f_1(x)\,dx = 1$, and $\int_{\mathbf{R}^n} x_j f_1(x)\,dx = 0$ for $j \ne 1$. Similarly, there exist functions $f_j \in C_c(\mathbf{R}^n)$ with the zero moment 0 such that

$$\int_{\mathbf{R}^n} x_i f_j(x)\,dx = \delta_{ij} \qquad (1 \le i, j \le n)$$

where δ_{ij} is the Kronecker delta. Also let $f_0 \in C_c(\mathbf{R}^n)$ be any even function such that $\int_{\mathbf{R}^n} f_0(x)\,dx = 1$ and define $\tilde{f} = \alpha f_0 + \sum_{j=1}^n \beta_j f_j$ where

$$\alpha = \int_{\mathbf{R}^n} f(x)\,dx, \quad \beta_j = \int_{\mathbf{R}^n} x_j f(x)\,dx$$

for each j. Then we have $\tilde{f} \in C_c(\mathbf{R}^n)$ and it is easily verified that the zero and first moments of the function $f - \tilde{f}$ are all 0. Now, since $f_j \in C_c(\mathbf{R}^n)$ for each j, we have

$$\int_{\mathbf{R}^n} |\tilde{f}(x)|(1 + |x| \log^+ |x|)\, dx \lesssim |\alpha| + \sum_{j=1}^{n} |\beta_j|.$$

On the other hand, we have

$$\begin{aligned} |\alpha| + \sum_{j=1}^{n} |\beta_j| &\lesssim \int_{\mathbf{R}^n} |f(x)|(1 + |x|)\, dx \\ &\lesssim \int_{\mathbf{R}^n} |f(x)|(1 + |x| \log^+ |x|)\, dx. \end{aligned}$$

Therefore the corollary follows from Theorem 3.4. The proof is complete. ∎

For $1 < p \leq 1 + \frac{1}{n}$, we have the following weighted norm inequalities.

THEOREM 3.6. *Let $1 < p \leq 1 + \frac{1}{n}$ and $f \in L^1$. If f satisfies the zero moment vanishing condition*

$$\int_{\mathbf{R}^n} f(x)\, dx = 0,$$

then the following hold.

(1) *For $1 < p < 1 + \frac{1}{n}$, we have*

$$\int_0^\infty \int_{\mathbf{R}^n} |P_t * f(x)|^p\, dx\, dt \leq C_p \int_{\mathbf{R}^n} |f(x)|^p |x|\, dx.$$

(2) *For $p = 1 + \frac{1}{n}$, we have*

$$\int_0^\infty \int_{\mathbf{R}^n} |P_t * f(x)|^p\, dx\, dt \leq C_p \int_{\mathbf{R}^n} |f(x)|^p \left[1 + |x|(\log^+ |x|)^p\right] dx.$$

Proof. As in the proof of Theorem 3.4, we have

$$\begin{aligned} |P_t * f(x)| &\leq \int_{\mathbf{R}^n} |f(y)|\, |P_t(x - y) - P_t(x)|\, dy \\ &\lesssim \int_{K_1} |f(y)| P_t(x)\, dy + (1 + t|x|^{-1}) |\nabla P_t(x)| \int_{K_2} |f(y)||y|\, dy \\ &\quad + \int_{K_3} |f(y)| P_t(x - y)\, dy \\ &= \Lambda_1 |f|(x, t) + \Lambda_2 |f|(x, t) + \Lambda_3 |f|(x, t), \end{aligned}$$

and therefore (1) and (2) are consequences of Lemma 3.1, Lemma 3.2, Lemma 3.3 and Lemma 2.4. The proof is complete. □

As in the case of $p = 1$, we have the following modified weighted norm inequalities. Recall ω_p denotes the weight defined in (1.3).

Corollary 3.7. *Let $1 < p \le 1 + \frac{1}{n}$. Then, to each $f \in L^1$ there corresponds $\tilde{f} \in C_c(\mathbf{R}^n)$ such that*

$$\int_0^\infty \int_{\mathbf{R}^n} |P_t * (f - \tilde{f})(x)|^p \, dx \, dt \le C_p \int_{\mathbf{R}^n} |f(x)|^p (1 + \omega_p(x)) \, dx.$$

Proof. Using notations defined in the proof of Corollary 3.5, put $\tilde{f} = \alpha f_0 \in C_c(\mathbf{R}^n)$. Let q be the conjugate exponent of p. Since $q/p > n$ for $1 < p < 1 + \frac{1}{n}$ and $q/p = n$ for $p = 1 + \frac{1}{n}$, we have

$$\int_{\mathbf{R}^n} (1 + \omega_p(x))^{-q/p} \, dx < \infty.$$

This, together with Hölder's inequality, yields

$$\begin{aligned}\int_{\mathbf{R}^n} |\tilde{f}(x)|^p (1 + \omega_p(x)) \, dx &\lesssim |\alpha|^p \\ &\le \left(\int_{\mathbf{R}^n} |f(x)| \, dx \right)^p \\ &\lesssim \int_{\mathbf{R}^n} |f(x)|^p (1 + \omega_p(x)) \, dx.\end{aligned}$$

Now, since the zero moment of the function $f - \tilde{f}$ is 0, the theorem follows from Theorem 3.6. The proof is complete. □

Finally, for $p > 1 + \frac{1}{n}$, we have the following weighted norm inequalities.

Theorem 3.8. *Let $p > 1 + \frac{1}{n}$. For measurable functions $f \ge 0$, we have*

$$\int_0^\infty \int_{\mathbf{R}^n} |P_t * f(x)|^p \, dx \, dt \le C_p \int_{\mathbf{R}^n} |f(x)|^p |x| \, dx. \tag{3.1}$$

Proof. Recall that we have

$$\int_{\mathbf{R}^n} P_t(x - y) \, dy = c_n$$

for any $t > 0$ and $x \in \mathbf{R}^n$. Also, recall that $5(|x-y|^2+t^2) > 9(|x|^2+t^2)$ on K_1 by definition. Thus we have $P_t(x-y) \lesssim P_t(x)$ on K_1. Now, Jensen's inequality yields

$$\begin{aligned}\left(\int_{K_1} f(y)P_t(x-y)\,dy\right)^p &\le C_p \int_{K_1} |f(y)|^p P_t(x-y)\,dy\\ &\lesssim P_t(x)\int_{K_1} |f(y)|^p\,dy\\ &= \Lambda_1|f|^p(x,t)\end{aligned}$$

and therefore the estimate on K_1 follows from Lemma 3.1.

Since $|x|^2+t^2 \le 5(|x-y|^2+t^2) \le 9(|x|^2+t^2)$ on K_2 by definition, we also have $P_t(x-y) \lesssim P_t(x)$ on K_2. Thus,

$$\begin{aligned}&\int_0^\infty \int_{\mathbf{R}^n} \left(\int_{K_2} f(y)P_t(x-y)\,dy\right)^p dx\,dt\\ &\quad\lesssim \int_0^\infty \int_{\mathbf{R}^n} \left(\int_{K_2} f(y)P_t(x)\,dy\right)^p dx\,dt\\ &\quad\lesssim \int_0^\infty \left(\int_{|y|<6t} f(y)\,dy\right)^p t^{n-np}\,dt\\ &\quad\lesssim \int_0^\infty \left(\int_{|y|<t} f(y)\,dy\right)^p t^{n-np}\,dt.\end{aligned}$$

where the second inequality can be verified by an easy modification of the estimate for Λ_2 in the proof of Lemma 3.2. Thus, the estimate on K_2 follows from Lemma 2.1 with $r = np-n-1 > 0$. Finally, the estimate on K_3 follows from Lemma 3.3. The proof is complete. □

Remark. Note that $|x|$ is an A_p weight (see [3]) if and only if $p > 1+\frac{1}{n}$. Thus, for the case $p > 1+\frac{1}{n}$, we could derive the weighted norm inequality (3.1) by utilizing the well-known A_p weight theory. On the other hand, for $1 \le p \le 1+\frac{1}{n}$, weights under consideration are not A_p weights, but still appear to be quite natural. This seems to cause the fact that we only have the modified weighted norm inequalities Corollary 3.5 and Corollary 3.7, in case the weights are already fixed.

The case $p = 2$ is something special, because we then have Fourier transform tools at hand. In the rest of this section, we mention some results in that direction. The following Plancherel type theorem is noticed in [5].

LEMMA 3.9. *For $f \in L^2$, we have*

$$\int_0^\infty \int_{\mathbf{R}^n} |P_t * f(x)|^2 \, dx \, dt \ = c_n \int_{\mathbf{R}^n} |\widehat{f}(x)|^2 |x|^{-1} \, dx \tag{3.2}$$

where $\widehat{f}$ denotes the Fourier transform of f.

This yields some immediate consequences. That is, for $f \in L^1 \cap L^2$, one can see $P[f] \in b^2$ only if $\widehat{f}(0) = 0$ (by continuity of $\widehat{f}$) for $n = 1$. On the other hand, we always have $P[f] \in b^2$ for $n \geq 2$. Such b^p containment results will be investigated in the next section. Here, we mention some inequalities which can be derived from (3.2).

PROPOSITION 3.10. $(n = 1)$ *Let $f \in L^1$. If f satisfies the zero moment vanishing condition*

$$\int_{\mathbf{R}} f(x) \, dx = 0,$$

then we have

$$\int_0^\infty \int_{\mathbf{R}} |P_t * f(x)|^2 \, dx \, dt \leq C \left(\int_{\mathbf{R}} |f(x)||x| \, dx \right)^2 + C \int_{\mathbf{R}} |f(x)|^2 \, dx. \tag{3.3}$$

Proof. We may assume $f \in L^2$. Since $\widehat{f}(0) = 0$ by assumption, we have

$$|\widehat{f}(x)| = |\widehat{f}(x) - \widehat{f}(0)| \lesssim |x| \int_{\mathbf{R}} |f(y)||y| \, dy$$

for all $x \in \mathbf{R}$. It follows that

$$\begin{aligned} \int_{\mathbf{R}} |\widehat{f}(x)|^2 |x|^{-1} \, dx &= \int_{|x|<1} |\widehat{f}(x)|^2 |x|^{-1} \, dx + \int_{|\mathbf{x}|>1} |\widehat{f}(x)|^2 |x|^{-1} \, dx \\ &\lesssim \left(\int_{\mathbf{R}} |f(x)||x| \, dx \right)^2 + \int_{\mathbf{R}} |f(x)|^2 \, dx \end{aligned}$$

and therefore (3.3) holds by Lemma 3.9. The proof is complete. □

For $n \geq 2$, note that $|x|^{-1}$ is integrable near the origin. Thus, a similar argument yields the following.

PROPOSITION 3.11. $(n \geq 2)$ *For measurable functions $f \geq 0$ on $\mathbf{R}^n$, we have*

$$\int_0^\infty \int_{\mathbf{R}^n} |P_t * f(x)|^2 \, dx \, dt \leq C \left(\int_{\mathbf{R}^n} f(x) \, dx \right)^2 + C \int_{\mathbf{R}^n} |f(x)|^2 \, dx.$$

For $n \geq 2$ and $p = 2$, one can also see that order of weight in Theorem 3.8 is sharp in the following sense.

PROPOSITION 3.12. $(n \geq 2)$ *We have*

$$\int_0^\infty \int_{\mathbf{R}^n} |P_t * f(x)|^2 \, dx \, dt \leq C_\alpha \int_{\mathbf{R}^n} |f(x)|^2 |x|^\alpha \, dx \tag{3.4}$$

for all measurable functions $f \geq 0$ on $\mathbf{R}^n$ if and only if $\alpha = 1$.

Proof. We only need prove the necessity by Theorem 3.8. So assume (3.4) holds and take $f(x) = e^{-t|x|}$ for $t > 0$. Note $f \in L^2$ for any $t > 0$. We have $\widehat{f}(x) = c_n P_t(x)$ (see [4], page 16) and thus

$$\int_{\mathbf{R}^n} |\widehat{f}(x)|^2 |x|^{-1} \, dx = c_n^2 \, t^{-n-1} \int_{\mathbf{R}^n} |x|^{-1} (1 + |x|^2)^{-n-1} \, dx.$$

On the other hand, we have

$$\int_{\mathbf{R}^n} |f(x)|^2 |x|^\alpha \, dx = t^{-n-\alpha} \int_{\mathbf{R}^n} e^{-|x|} |x|^\alpha \, dx$$

so that $\alpha = 1$ by Lemma 3.9. The proof is complete. □

Having seen the above proposition, one might guess the sharpness of orders of weights considered in our weighted norm inequalities. Our results are indeed sharp in the sense that orders cannot be reduced at infinity. We will see examples in Section 5. See Proposition 5.1, Proposition 5.4, Proposition 5.5 and Proposition 5.6.

§4. Harmonic Bergman functions

We have seen that the zero or first moment vanishing hypothesis played the key role in the proof of Theorem 3.4 and Theorem 3.6. In this section we show that those cancellation properties are indeed necessary, which justifies our hypotheses taken in the weighted norm inequalities of the previous section.

THEOREM 4.1. *Let μ be a complex Borel measure on $\mathbf{R}^n$. If $P[\mu] \in b^p$ for some $1 \leq p \leq 1 + \frac{1}{n}$, then $\mu(\mathbf{R}^n) = 0$.*

Proof. Assume μ is real without loss of generality. Put $F = P[\mu]$ and assume $F \in b^p$ for some $1 \leq p \leq 1 + \frac{1}{n}$. Note that

$$\begin{aligned} F(x,t) &= \int_{\mathbf{R}^n} P_t(x-y)\,d\mu(y) \\ &= t^{-n}\int_{|y|<R} d\mu(y) + \int_{|y|<R} (P_t(x-y) - t^{-n})\,d\mu(y)| \\ &\quad + \int_{|y|\geq R} P_t(x-y)\,d\mu(y) \end{aligned} \tag{4.1}$$

for any $R > 0$. Now, assume $\mu(\mathbf{R}^n) \neq 0$. We may further assume $\mu(\mathbf{R}^n) > 1$. Now choose $R > 1$ sufficiently large so that

$$\int_{|y|<R} d\mu(y) - \int_{|y|\geq R} d|\mu|(y) \geq 1 \tag{4.2}$$

holds. Also, fix $N > 0$ such that

$$mN^{-1}||\mu|| < \frac{1}{2}. \tag{4.3}$$

Also note

$$|P_t(x-y) - t^{-n}| \leq \frac{mt|x-y|^2}{t^{n+1}(|x-y|^2+t^2)} \leq \frac{m|x-y|}{2t^{n+1}} \tag{4.4}$$

where the first inequality can be seen from the elementary inequality

$$a^{-m} - b^{-m} \leq m(b-a)b^{-1}a^{-m} \qquad (0 < a < b). \tag{4.5}$$

Now, it follows from (4.1), (4.2) and (4.4) that

$$\begin{aligned} |F(x,t)| &\geq t^{-n}\int_{|y|<R} d\mu(y) - mt^{-n-1}\int_{|y|<R} \frac{|x-y|}{2}\,d|\mu|(y) \\ &\qquad -t^{-n}\int_{|y|\geq R} d|\mu|(y) \\ &\geq t^{-n} - mt^{-n-1}\int_{|y|<R} \frac{|x-y|}{2}\,d|\mu|(y). \end{aligned}$$

Therefore, for $|x| < R$ and $NR < t < 2NR$, we have by (4.3)

$$|F(x,t)| \geq t^{-n}(1 - mt^{-1}R||\mu||) \gtrsim N^{-n}R^{-n},$$

which yields

$$\int_{NR}^{2NR} \int_{|x|<R} |F(x,t)|^p \, dx \, dt \gtrsim N^{1-np} R^{n-np+1} \gtrsim N^{1-np}$$

where the second inequality holds, because $n - np + 1 \geq 0$. But, this is impossible by the fact

$$\int_{NR}^{2NR} \int_{|x|<R} |F(x,t)|^p \, dx \, dt \leq \int_{NR}^{\infty} \int_{\mathbf{R}^n} |F(x,t)|^p \, dx \, dt \to 0$$

as R tends to infinity. This completes the proof. ∎

As a consequence, we have the following b^p-cancellation property. The case $p = 1$ has been known [2].

COROLLARY 4.2. *Let $1 \leq p \leq 1 + \frac{1}{n}$ and $F \in b^p$. Then*

$$\int_{\mathbf{R}^n} F(x, \delta) \, dx = 0$$

whenever $F(\cdot, \delta) \in L^1$.

Proof. For $t > \delta$, we have $F(x,t) = P[f](x,t)$ where $f = F(\cdot, \delta)$. Hence, the corollary follows from Theorem 4.1. ∎

In addition to the zero moment vanishing property, b^1-functions represented by Poisson integrals must also have the following first moment vanishing property on the boundary.

THEOREM 4.3. *Let μ be a complex Borel measure on $\mathbf{R}^n$ and suppose*

$$\int_{\mathbf{R}^n} |x| \, d|\mu|(x) < \infty.$$

If $P[\mu] \in b^1$, then

$$\int_{\mathbf{R}^n} x_j \, d\mu(x) = 0 \qquad (1 \leq j \leq n).$$

Proof. Assume μ is real without loss of generality. Put $F = P[\mu]$ and assume $F \in b^1$. Note that, since the zero moment of μ is 0 by Theorem 4.1, we have

$$
\begin{aligned}
F(x,t) &= \int_{\mathbf{R}^n} (P_t(x-y) - P_t(x))\, d\mu(y) \\
&= -\int_{\mathbf{R}^n} \nabla P_t(x)\cdot y\, d\mu(y) + \int_{2|y|>t} \nabla P_t(x)\cdot y\, d\mu(y) \\
&\qquad + \int_{2|y|\le t} (P_t(x-y) - P_t(x) + \nabla P_t(x)\cdot y)\, d\mu(y) \\
&\qquad + \int_{2|y|>t} (P_t(x-y) - P_t(x))\, d\mu(y) \\
&=: I + II + III + IV
\end{aligned}
\tag{4.6}
$$

for any $t > 0$. Put

$$\alpha_j = \int_{\mathbf{R}^n} x_j\, d\mu(x) \qquad (1 \le j \le n)$$

and assume $\alpha_j \ge 0$ for each j by making change of variables, if needed. Let $\alpha = \sum \alpha_j$. It remains to show $\alpha = 0$. Suppose not. Then we may further assume $\alpha > 1$. Now we estimate terms in (4.6). Fix $t > 1$ and x such that $2|x| < t$, and $t < 4\sqrt{n}x_j$ for each j. For the first term, we have

$$|I| = \tfrac{2mt}{(|x|^2+t^2)^{m+1}} \sum \alpha_j x_j \ge \tfrac{mt^2}{2\sqrt{n}(|x|^2+t^2)^{m+1}} \gtrsim \tfrac{1}{t^{n+1}}. \tag{4.7}$$

For the second term, we have

$$|II| \le \frac{2m|x|}{t^{n+2}} \int_{2|y|>t} |y|\, d\mu(y) = t^{-n-1} o(1) \tag{4.8}$$

as t tends to infinity. To estimate the third term, note that

$$
\begin{aligned}
\frac{1}{t}\int_{2|y|\le t} |y|^2\, d|\mu|(y) &= \frac{1}{t}\int_{2|y|\le\sqrt{t}} |y|^2\, d|\mu|(y) + \frac{1}{t}\int_{\sqrt{t}<2|y|\le t} |y|^2\, d|\mu|(y) \\
&\le \frac{1}{2\sqrt{t}}\int_{\mathbf{R}^n} |y|\, d|\mu|(y) + \int_{2|y|>\sqrt{t}} |y|\, d|\mu|(y) \\
&= o(1)
\end{aligned}
$$

as t tends to infinity. Also note that $\big||y|^2 - 2x\cdot y\big| \le |y|(|y| + 2|x|) \le 3t^2/4$ for $2|y| \le t$. Hence, by Lemma 2.6, we have

$$|III| \lesssim \frac{1}{t^{n+2}} \int_{2|y|\le t} |y|^2\, d|\mu|(y) = t^{-n-1} o(1) \tag{4.9}$$

as t tends to infinity. For the last term, we have by (4.5)

$$\begin{aligned}|P_t(x-y)-P_t(x)| &\le \frac{m\,t\,\big||x|^2-|x-y|^2\big|}{t^{n-1}(|x-y|^2+t^2)(|x|^2+t^2)}\\ &\le \frac{m\,|y|\,(t|x|+t|x-y|)}{t^{n-1}(|x-y|^2+t^2)(|x|^2+t^2)}\\ &\le \frac{m|y|}{t^{n+1}}\end{aligned}$$

and therefore

$$|IV| \le \frac{m}{t^{n+1}}\int_{2|y|>t}|y|\,d|\mu|(y) = t^{-n-1}o(1). \tag{4.10}$$

Thus, by (4.6), (4.7), (4.8), (4.9) and (4.10), we finally see that there is a positive constant N such that

$$\int_N^\infty\int_{D_t}|F(x,t)|\,dx\,dt \gtrsim \int_N^\infty t^{-1}\,dt = \infty$$

where D_t is the set of all points x such that $2|x| < t$ and $t < 4\sqrt{n}x_j$ for all j. Since $F \in b^1$ by assumption, this is a contradiction. The proof is complete. ∎

As a consequence of Theorem 4.3 and Corollary 4.2, we have the following b^1-cancellation property. The proof is similar to that of Corollary 4.2.

Corollary 4.4. *For $F \in b^1$, we have*

$$\int_{\mathbf{R}^n} F(x,\delta)\,dx = 0$$

for any $\delta > 0$. Also, we have

$$\int_{\mathbf{R}^n} x_jF(x,\delta)\,dx = 0 \qquad (1 \le j \le n)$$

whenever $F(\cdot,\delta) \in L^1(|x|dx)$.

The first moment vanishing property sometimes forces functions in b^1 of certain type to be identically 0. For example, consider a positive finite Borel measure μ on $\mathbf{R}^k_+$ $(1 \le k \le n)$ and let $\tilde\mu$ be its reflection with respect to the origin. Also, let λ be a positive finite Borel measure on $\mathbf{R}^{n-k}$. Assume first moments of μ and λ are all well defined.

COROLLARY 4.5. *Let μ and λ be as above. If $P[(\mu-\tilde{\mu})\times\lambda]\in b^1$, then either $\mu=0$ or $\lambda=0$.*

Proof. By Theorem 4.3, the first moments of $(\mu-\tilde{\mu})\times\lambda$ are all 0. In particular, for $1\le j\le k$, we have

$$\begin{aligned} 0 &= ||\lambda|| \int_{\mathbf{R}^k} x_j \, d(\mu-\tilde{\mu}) \\ &= 2||\lambda|| \int_{\mathbf{R}^k_+} x_j \, d\mu. \end{aligned}$$

Summing up all these together, we obtain

$$||\lambda|| \int_{\mathbf{R}^k_+} |x'| \, d\mu(x') = 0$$

and therefore $\mu=0$ or $\lambda=0$ as desired. The proof is complete. $\square$

Remark. Consider any closed cone $E\subset\mathbf{R}^{n+1}$ with vertex at the origin whose radial projection to the unit sphere is properly contained in $\mathbf{R}^{n+1}_+$. For measures μ and λ considered in Corollary 4.5, one can actually obtain a direct estimate on E:

$$P[(\mu-\tilde{\mu})\times\lambda](w)\ge C|w|^{-n-1} \qquad (w\in E,\ |w|\ge 1)$$

for some $C>0$ if $\mu\times\lambda\ne 0$, which implies $P[(\mu-\tilde{\mu})\times\lambda]\notin b^1$.

In the next section we will see examples (Proposition 5.1, Proposition 5.4, Proposition 5.5) showing that the above cancellation properties do not characterize the b^p-containment of Poisson integrals. However, such cancellation properties and b^p-containment of Poisson integrals are equivalent under restrictions which are obviously suggested by weighted norm inequalities of the previous section. For the case $p=1$, we have the following.

THEOREM 4.6. *Suppose $f\in L^1$ satisfies*

$$\int_{\mathbf{R}^n} |f(x)||x| \log^+ |x| \, dx < \infty. \tag{4.11}$$

Then $P[f]\in b^1$ if and only if

$$\int_{\mathbf{R}^n} f(x)\,dx = \int_{\mathbf{R}^n} x_j f(x)\,dx = 0 \qquad (1\le j\le n).$$

Proof. This follows from Theorem 3.4, Theorem 4.1 and Theorem 4.3. ∎

Similarly, for the case $1 < p \leq 1 + \frac{1}{n}$, the following holds. Recall ω_p denotes the weight defined in (1.3). Also, note that the condition (4.12) below, together with $f \in L^p$, implies $f \in L^1$ (see the proof of Corollary 3.7).

THEOREM 4.7. *Let $1 < p \leq 1 + \frac{1}{n}$. Suppose $f \in L^p$ satisfies*

$$\int_{\mathbf{R}^n} |f(x)|^p \omega_p(x)\, dx < \infty. \tag{4.12}$$

Then $P[f] \in b^p$ if and only if

$$\int_{\mathbf{R}^n} f(x)\, dx = 0.$$

Proof. This follows from Theorem 3.6 and Theorem 4.1. ∎

Remark. We will see examples (Proposition 5.1, Proposition 5.4, Proposition 5.5) showing that the converses of the moment vanishing properties do not hold. Consequently, Theorem 4.6 and Theorem 4.7 do not hold without the weighted integrability conditions (4.11) and (4.12). Nevertheless, one may consider some other aspects of those weighted integrability conditions. Namely, one may ask whether their orders are optimal. Also, one may ask whether they are necessary for the b^p-containment of Poisson integrals in case relevant moment vanishing conditions are already given. Answers are *yes* for the first one and *no* for the second one. We will see examples in the next section. See Proposition 5.1, Proposition 5.4, Proposition 5.5 for the first one and Proposition 5.7, Proposition 5.8, Proposition 5.9 for the second one.

We now turn to the case $p > 1 + \frac{1}{n}$. In this case one can immediately see $P[f] \in b^p$ from Theorem 3.8 whenever f satisfies the condition (4.13) below and $P[f]$ defines an actual harmonic function.

THEOREM 4.8. *Let $p > 1 + \frac{1}{n}$. If f is a measurable function on $\mathbf{R}^n$ such that*

$$\int_{\mathbf{R}^n} |f(x)|^p |x|\, dx < \infty, \tag{4.13}$$

then $P[f] \in b^p$.

Proof. As mentioned above, we only need to check that $P[f]$ is well defined. First note that

$$\int_{|x|>1} |f(x)|^p \, dx \le \int_{|x|>1} |f(x)|^p |x| \, dx < \infty.$$

Let q be the conjugate exponent of p. Since $q/p < n$, we have

$$\int_{|x|<1} |x|^{-q/p} \, dx < \infty$$

and thus

$$\int_{|x|<1} |f(x)| \, dx \lesssim \left(\int_{|x|<1} |f(x)|^p |x| \, dx \right)^{1/p} < \infty$$

by Hölder's inequality. Accordingly, $f \in L^1 + L^p$ and $P[f]$ is indeed a harmonic function. The proof is complete. ∎

Remark. For $1 \le p \le 1 + \frac{1}{n}$, we have seen that the b^p-containment of Poisson integrals, under restrictions (4.11) or (4.12), is characterized by the zero or first moment vanishing condition. As mentioned before, such characterizations fail to hold without any extra conditions. On the other hand, for $p > 1 + \frac{1}{n}$, one cannot expect any cancellation property by Theorem 4.8 and, in fact, there are many positive functions in b^p (consider Poisson integrals of positive functions with compact support). As far as such positive functions are concerned, the bound $p = 1 + \frac{1}{n}$ is known to be precise as mentioned in the introduction.

We now close this section with the following for the case $p = 2$. Note that there is no implication between this and our results above.

PROPOSITION 4.9. *For functions $f \in L^1 \cap L^2$, the following hold.*

(1) $(n = 1)$ *For $f \in L^1(|x|dx)$, we have $P[f] \in b^2$ if and only if*

$$\int_{\mathbf{R}^n} f(x) \, dx = 0.$$

(2) $(n \ge 2)$ *We always have $P[f] \in b^2$.*

Proof. We have (1) by Proposition 3.10, Theorem 4.1 and (2) by Proposition 3.11. ∎

§5. Examples

In this section we give various examples related to theorems obtained in the previous sections. We will assume $n = 1$ for simplicity and thus $1+\frac{1}{n} = 2$. Similar arguments will produce examples for $n > 1$. What we are concerned here are (i) the failure of the converses of the moment vanishing properties, (ii) the sharpness of the orders of weights in the weighted integrability conditions for the b^p-containment (and the same for the weighted norm inequalities), and (iii) the failure of b^p-containment characterizations by means of our weighted integrability conditions in case relevant moment vanishing conditions are given.

We first construct examples simultaneously concerning (i) and (ii) for $1 \le p \le 2$. Our examples show that the moment vanishing properties are not sufficient for the b^p-containment. In other words, the sufficiencies of Theorem 4.6 and Theorem 4.7 do not hold without weighted integrability conditions (4.11) and (4.12). However, as far as orders of weights (in those weighted integrability conditions) are concerned, our examples show that our results are sharp (hence so are the weighted norm inequalities) in the sense that orders at infinity cannot be reduced. We begin with the easiest case.

Proposition 5.1. *For $1 < p < 2$, there exists a function $f \in L^1 \cap L^p$ such that*

$$\int_{-\infty}^{\infty} f(x)\,dx = 0, \qquad \int_{-\infty}^{\infty} |f(x)|^p |x|^{1-\varepsilon}\,dx < \infty \tag{5.1}$$

for any $\varepsilon > 0$, but $P[f] \notin b^p$.

Proof. Let $\mathcal{X}$ be the characteristic function of the interval $[1,\infty)$ and define

$$f(x) = |x|^{-\alpha}(\mathcal{X}(x) - \mathcal{X}(-x)) \tag{5.2}$$

where $\alpha = \frac{2}{p}$. Then, clearly $f \in L^1 \cap L^p$ and (5.1) holds. We will show that $P[f] \notin b^p$. Now, for $0 < x < t$ and $t > 1$, we have

$$\begin{aligned} P_t * f(x) &= \int_1^{\infty} \left(\frac{t}{|x-y|^2+t^2} - \frac{t}{|x+y|^2+t^2} \right) \frac{dy}{y^\alpha} \\ &= \int_1^{\infty} \frac{4txy}{(|x-y|^2+t^2)(|x+y|^2+t^2)} \frac{dy}{y^\alpha} \\ &\ge \int_t^{\infty} \frac{4txy}{(|x+y|^2+t^2)^2} \frac{dy}{y^\alpha} \end{aligned}$$

$$\begin{aligned} &\gtrsim tx \int_t^\infty y^{-3-\alpha}\, dy \\ &\gtrsim t^{-1-\alpha} x \end{aligned}$$

and thus

$$\begin{aligned} \int_1^\infty \int_0^t |P_t * f(x)|^p \, dx\, dt &\gtrsim \int_1^\infty \int_0^t t^{-p-2} x^p \, dx\, dt \\ &\gtrsim \int_1^\infty t^{-1}\, dt \\ &= \infty \end{aligned}$$

as desired. The proof is complete. ∎

The cases $p = 1$ and $p = 2$ are more subtle. To construct examples for those cases, we first prove a couple of lemmas.

LEMMA 5.2. *Let $\alpha > 1$ and $\varepsilon > 0$. Then, for $t > 2e$, we have*

(1) $$\int_e^t \frac{dy}{(\log y)^\varepsilon} \approx \frac{t}{(\log t)^\varepsilon},$$

(2) $$\int_t^\infty \frac{dy}{y^\alpha (\log y)^{1+\varepsilon}} \approx \frac{1}{t^{\alpha-1}(\log t)^{1+\varepsilon}}.$$

Proof. For $2e < t \le e^{\varepsilon+1}$ (if there is any such t), we have

$$\int_e^{2e} \frac{dy}{(\log y)^\varepsilon} + \int_{2e}^t \frac{dy}{(\log y)^\varepsilon} \lesssim 1 \lesssim \frac{t}{(\log t)^\varepsilon}.$$

For $t > e^{\varepsilon+1}$, we have by integration by parts

$$\begin{aligned} \int_{e^{\varepsilon+1}}^t \frac{dy}{(\log y)^\varepsilon} &= \frac{t}{(\log t)^\varepsilon} - \frac{e^{\varepsilon+1}}{(\varepsilon+1)^\varepsilon} + \varepsilon \int_{e^{\varepsilon+1}}^t \frac{dy}{(\log y)^{\varepsilon+1}} \\ &\le \frac{t}{(\log t)^\varepsilon} + \frac{\varepsilon}{\varepsilon+1} \int_{e^{\varepsilon+1}}^t \frac{dy}{(\log y)^\varepsilon}. \end{aligned}$$

Therefore, for $t > 2e$, we have

$$\int_e^t \frac{dy}{(\log y)^\varepsilon} \lesssim \frac{t}{(\log t)^\varepsilon}.$$

The other direction of the above inequality is clear and thus (1) holds. A similar estimate yields (2). The proof is complete. ∎

LEMMA 5.3. *For $\varepsilon > 0$, consider functions f_1 and f_2 defined by*

$$f_j(x) = \alpha_j x^{-j}(\log x)^{-1-\varepsilon}\mathcal{X}(x) \qquad (j = 1, 2)$$

where $\mathcal{X}$ is the characteristic function of the interval $[e, \infty)$ and each $\alpha_j > 0$ is chosen so that $\int f_j(x)dx = 1$. Then, we have

(1) $\quad P_t(x) - P_t * f_1(x) \gtrsim t^{-1}(\log t)^{-\varepsilon}$

(2) $\quad P_t(x) - \varepsilon\alpha_2 P_t'(x) - P_t * f_2(x) \gtrsim t^{-2}(\log t)^{-\varepsilon}$

for all x, t with $t < x < 2t$ and t sufficiently large.

Proof. Assume $t < x < 2t$ and t is sufficiently large. By Lemma 5.2, we have

$$\begin{aligned}
&\int_e^\infty \frac{1}{(x-y)^2 + t^2}\,\frac{dy}{(\log y)^{1+\varepsilon}} \\
&\approx \frac{1}{t^2}\int_e^t \frac{dy}{(\log y)^{1+\varepsilon}} + \int_t^\infty \frac{1}{y^2}\,\frac{dy}{(\log y)^{1+\varepsilon}} \\
&\approx t^{-1}(\log t)^{-1-\varepsilon} \\
&= t^{-1}(\log t)^{-\varepsilon} o(1).
\end{aligned} \tag{5.3}$$

Also, note that

$$\begin{aligned}
&\int_e^\infty \frac{y}{(x-y)^2 + t^2}\,\frac{dy}{(\log y)^{1+\varepsilon}} \\
&\geq \int_t^\infty \frac{y}{(x-y)^2 + t^2}\,\frac{dy}{(\log y)^{1+\varepsilon}} \\
&\gtrsim \int_t^\infty \frac{dy}{y(\log y)^{1+\varepsilon}} \\
&\gtrsim (\log t)^{-\varepsilon}
\end{aligned} \tag{5.4}$$

and therefore, by (5.3) and (5.4),

$$\begin{aligned}
P_t(x) - P_t * f_1(x) &= \alpha_1 \int_e^\infty [P_t(x) - P_t(x-y)] y^{-1}(\log y)^{-1-\varepsilon}\,dy \\
&= \alpha_1 P_t(x) \int_e^\infty \frac{y - 2x}{(x-y)^2 + t^2}\,\frac{dy}{(\log y)^{1+\varepsilon}} \\
&\gtrsim \alpha_1 P_t(x)(\log t)^{-\varepsilon}[1 + o(1)] \\
&\gtrsim t^{-1}(\log t)^{-\varepsilon}.
\end{aligned}$$

This proves (1). A similar argument yields (note $\alpha_1 = \varepsilon^{-1}$)

$$\begin{aligned} P_t(x) - P_t * f_2(x) &= \alpha_2 P_t(x) \int_e^\infty \frac{y-2x}{(x-y)^2+t^2} \frac{dy}{y(\log y)^{1+\varepsilon}} \\ &= \alpha_2 P_t(x)[t^{-1}(\log t)^{-\varepsilon} o(1) - 2\varepsilon x t^{-1} P_t * f_1(x)] \\ &= -2\varepsilon\alpha_2 x t^{-1} P_t(x)\, P_t * f_1(x) + t^{-2}(\log t)^{-\varepsilon} o(1) \end{aligned}$$

where the second equality holds by (5.3). Since $P_t'(x) = -2xt^{-1}[P_t(x)]^2$, it follows that

$$\begin{aligned} &P_t(x) - \varepsilon\alpha_2 P_t'(x) - P_t * f_2(x) \\ &\quad = 2\varepsilon\alpha_2 x t^{-1} P_t(x)\,[P_t(x) - P_t * f_1(x)] + t^{-2}(\log t)^{-\varepsilon} o(1) \\ &\quad \approx t^{-1}[P_t(x) - P_t * f_1(x)] + t^{-2}(\log t)^{-\varepsilon} o(1) \\ &\quad \gtrsim t^{-2}(\log t)^{-\varepsilon}. \end{aligned}$$

The last inequality holds by (1). The proof is complete. □

We are now ready to construct examples for $p = 1$ and $p = 2$. We first consider the simpler case $p = 2$.

Proposition 5.4. *There exists a function $f \in L^1 \cap L^2$ such that*

$$\int_{-\infty}^\infty f(x)\,dx = 0, \int_{-\infty}^\infty |f(x)|^2 |x| (\log^+ |x|)^{2-\varepsilon}\,dx < \infty \tag{5.5}$$

for any $\varepsilon > 0$, but $P[f] \notin b^2$.

Proof. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be the characteristic functions of the interval $[-\frac{1}{2}, \frac{1}{2}]$ and $[e, \infty)$, respectively. Consider a function f defined by

$$f(x) = \mathcal{X}_1(x) - 2x^{-1}(\log x)^{-3/2}\mathcal{X}_2(x) =: f_1 + f_2.$$

Clearly, $f \in L^1 \cap L^2$ and (5.5) is easily seen. We claim

$$|P_t * f(x)| \gtrsim t^{-1}(\log t)^{-1/2} \tag{5.6}$$

for all x, t with $t < x < 2t$ and t sufficiently large. This yields

$$\int_N^\infty \int_t^{2t} |P_t * f(x)|^2\,dx\,dt \gtrsim \int_N^\infty (t \log t)^{-1}\,dt = \infty$$

for N large and hence $P[f] \notin b^2$. Now, assume $t < x < 2t$ and t is sufficiently large. We first estimate f_1. Since

$$\begin{aligned}\int_{-1/2}^{1/2} \frac{y}{(x-y)^2+t^2}\,dy &= 2\int_0^{1/2} \frac{y}{(x-y)^2+t^2} - \frac{y}{(x+y)^2+t^2}\,dy \\ &= 8x\int_0^{1/2} \frac{y^2}{((x-y)^2+t^2)((x+y)^2+t^2)}\,dy \\ &\lesssim t^{-3},\end{aligned}$$

we have

$$\begin{aligned}P_t * f_1(x) &= \int_{-1/2}^{1/2} P_t(x-y)\,dy \\ &= P_t(x) + \int_{-1/2}^{1/2} [P_t(x-y) - P_t(x)]\,dy \\ &= P_t(x) + P_t(x)\int_{-1/2}^{1/2} \frac{2xy - y^2}{(x-y)^2+t^2}\,dy \\ &= P_t(x) + O(t^{-3}).\end{aligned}$$

For f_2, we obtain from Lemma 5.3

$$P_t * f_2(x) + P_t(x) \gtrsim t^{-1}(\log t)^{-1/2}.$$

It follows that

$$P_t * f(x) \gtrsim t^{-1}(\log t)^{-1/2} + O(t^{-3}) \gtrsim t^{-1}(\log t)^{-1/2}$$

so that (5.6) holds. The proof is complete.

PROPOSITION 5.5. *There exists a function $f \in L^1 \cap L^1(|x|dx)$ such that*

$$\begin{gathered}\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} xf(x)\,dx = 0, \\ \int_{-\infty}^{\infty} |f(x)||x|(\log^+|x|)^{1-\varepsilon}\,dx < \infty\end{gathered} \tag{5.7}$$

for any $\varepsilon > 0$, but $P[f] \notin b^1$.

Proof. Let $\mathcal{X}_1, \mathcal{X}_2$ be the characteristic functions of the intervals $[-\frac{1}{2}, \frac{1}{2}]$ and $[e, \infty)$, respectively. Also, for a constant $\alpha > 0$ chosen so that

$$\alpha^2 \int_e^\infty \frac{dy}{y^2(\log y)^2} = 1,$$

let $\mathcal{X}_3$ be the characteristic function of the interval $[0, \alpha]$ and consider a function f defined by

$$\begin{aligned} f(x) &= \mathcal{X}_1(x) - \alpha^2 (x \log x)^{-2} \mathcal{X}_2(x) + (\mathcal{X}_3(x) - \mathcal{X}_3(-x)) \\ &=: f_1 + f_2 + f_3. \end{aligned}$$

Clearly, $f \in L^1 \cap L^1(|x|dx)$ and (5.7) is easily verified. As in the proof of Proposition 5.4, it suffice to prove

$$|P_t * f(x)| \gtrsim t^{-2}(\log t)^{-1} \tag{5.8}$$

for all x, t with $t < x < 2t$ and t sufficiently large. So, assume $t < x < 2t$ and t is sufficiently large. For f_1, as is seen in the proof of Proposition 5.4, we have

$$P_t * f_1(x) = P_t(x) + O(t^{-3}).$$

For f_2, we obtain from lemma 5.3

$$P_t * f_2(x) + P_t(x) - \alpha^2 P_t'(x) \gtrsim t^{-2}(\log t)^{-1}.$$

For f_3, we have

$$\begin{aligned} P_t * f_3(x) &= \int_0^\alpha [P_t(x+y) - P_t(x-y)]\, dy \\ &= -\alpha^2 P_t'(x) + \int_0^\alpha [P_t(x+y) - P_t(x-y) + 2P_t'(x)y]\, dy \\ &= -\alpha^2 P_t'(x) + 2P_t'(x) \int_0^\alpha \frac{y^3(y^2 - 2x^2 + 2t^2)}{((x-y)^2+t^2)((x+y)^2+t^2)}\, dy \\ &= -\alpha^2 P_t'(x) + O(t^{-4}). \end{aligned}$$

Combining these estimates all together, we have

$$P_t * f(x) \gtrsim t^{-2}(\log t)^{-1} + O(t^{-3}) \gtrsim t^{-2}(\log t)^{-1}$$

so that (5.8) holds. The proof is complete. ∎

Orders of weights in the weighted norm inequalities for $p > 2$ is also sharp in the following sense.

PROPOSITION 5.6. *For $p > 2$, there exists a measurable function $f \geq 0$ such that*

$$\int_{-\infty}^{\infty} |f(x)|^p |x|^{1-\varepsilon}\, dx < \infty \tag{5.9}$$

for any $\varepsilon > 0$, but $P[f] \notin b^p$.

Proof. Let $\mathcal{X}$ be the characteristic function of the interval $[1, \infty)$ and consider a function f defined by $f(x) = |x|^{-\alpha}\mathcal{X}(x)$ where $\alpha = \frac{2}{p}$. Clearly we have (5.9). Now, for $0 < x < t$ and $1 < t < y$, we have $P_t(x - y) \gtrsim ty^{-2}$ and thus

$$P_t * f(x) \geq \int_t^{\infty} P_t(x-y) y^{-\alpha}\, dy \gtrsim t^{-\alpha}.$$

Therefore

$$\int_1^{\infty} \int_0^t |P_t * f(x)|^p\, dx\, dt \gtrsim \int_1^{\infty} t^{-1}\, dt = \infty,$$

which shows $P[f] \notin b^p$. The proof is complete. □

We now construct examples concerning (iii). For $1 \leq p \leq 2$, we have seen above that the weighted integrability conditions (4.11) and (4.12) play essential roles in Theorem 4.6 and Theorem 4.7. However, once relevant moment vanishing conditions are given, they are not necessary for the b^p-containment, as the following examples show. We first consider the case $1 < p < 2$.

PROPOSITION 5.7. *For $1 < p < 2$, there exists a function $f \in L^1 \cap L^p$ such that*

$$\int_{-\infty}^{\infty} f(x)\, dx = 0, \qquad \int_{-\infty}^{\infty} |f(x)|^p |x|\, dx = \infty \tag{5.10}$$

and yet $P[f] \in b^p$.

Proof. Fix α with $1 < \alpha \leq \frac{2}{p}$ and let

$$f = \sum_{k=1}^{\infty} k^{-\alpha} (\mathcal{X}_{2k} - \mathcal{X}_{2k-1}) \tag{5.11}$$

where $\mathcal{X}_k$ denotes the characteristic function of the interval $[k, k+1]$. It is easy to see that f satisfies (5.10) and $f \in L^1 \cap L^p$. Put $F = P[f]$. We claim

$$|F(x,t)| \lesssim t^{-1}(1+|x|)^{-1} \qquad (t>1,\ x \in \mathbf{R}), \tag{5.12}$$

which implies

$$\int_1^\infty \int_{-\infty}^\infty |F(x,t)|^p \, dx\, dt \lesssim \int_1^\infty \int_0^\infty \frac{1}{t^p(1+x)^p}\, dx\, dt < \infty$$

and thus $F \in b^p$ by Lemma 2.4 as desired. Assume $t > 1$ in the rest of the proof. Note

$$\begin{aligned}
&|P[\mathcal{X}_{2k} - \mathcal{X}_{2k-1}](x,t)| \\
&= t \left| \int_{2k}^{2k+1} \frac{1}{(x-y)^2+t^2}\, dy - \int_{2k-1}^{2k} \frac{1}{(x-y)^2+t^2}\, dy \right| \\
&\le t \int_0^1 \frac{3+2|2k-x|}{((2k+y-x)^2+t^2)((2k-1+y-x)^2+t^2)}\, dy \\
&\lesssim \frac{t(1+|2k-x|)}{(|2k-x|^2+t^2)^2}, \\
&\lesssim \frac{1}{(|2k-x|+t)^2}
\end{aligned}$$

and thus, we have

$$|F(x,t)| \lesssim \sum_{k=1}^\infty \frac{k^{-\alpha}}{(|2k-x|+t)^2} \tag{5.13}$$

From this, for $x < 0$, the estimate

$$|F(x,t)| \lesssim \frac{1}{(1+|x|+t)^2} \le \frac{1}{t(1+|x|)}$$

is clear. Also, for $0 \le x \le 1$, it is immediate from (5.13) that

$$|F(x,t)| \lesssim \frac{1}{t^2} \lesssim \frac{1}{t(1+x)}$$

For $x > 1$, we have $3(x-1+t) \ge 1+x+t$ and therefore

$$\sum_{|2k-x|\ge(x-1)/2} \frac{k^{-\alpha}}{(|2k-x|+t)^2} \lesssim \frac{1}{(x-1+t)^2} \lesssim \frac{1}{(1+x+t)^2} \le \frac{1}{t(1+x)},$$

and

$$\begin{aligned}\sum_{|2k-x|<(x-1)/2} \frac{k^{-\alpha}}{(|2k-x|+t)^2} &\lesssim \frac{1}{(1+x)^\alpha} \sum_{k=1}^{\infty} \frac{1}{(|2k-x|+t)^2} \\ &\lesssim \frac{1}{(1+x)} \int_{-\infty}^{\infty} \frac{1}{(|y-x|+t)^2}\, dy \\ &= \frac{2}{t(1+x)}.\end{aligned}$$

Combining all these estimates we see from (5.13) that (5.12) holds as desired. The proof is complete. □

One can modify the function considered in the proof of Proposition 5.7 to obtain an example for the case $p = 2$.

PROPOSITION 5.8. *There exists a function $f \in L^1 \cap L^2$ such that*

$$\int_{-\infty}^{\infty} f(x)\, dx = 0, \quad \int_{-\infty}^{\infty} |f(x)|^2 |x| (\log^+ |x|)^2\, dx = \infty \tag{5.14}$$

and yet $P[f] \in b^2$.

Proof. We modify the construction in the proof of Proposition 5.7. For example, modify the function (5.11) and consider

$$f = \sum_{k=2}^{\infty} k^{-1} (\log k)^{-\alpha} (\mathcal{X}_{2k} - \mathcal{X}_{2k-1})$$

where $\mathcal{X}_k$ has the same meaning as before. By taking $1 < \alpha \le \frac{3}{2}$, we have $f \in L^1 \cap L^2$ and (5.14) is satisfied. Let $F = P[f]$. Then, for $t > 1$ and $x \in \mathbf{R}$, a straightforward modification of estimates in the proof of Proposition 5.7 yields

$$|F(x,t)| \lesssim \sum_{k=2}^{\infty} k^{-1} (\log k)^{-\alpha} \frac{1}{(|2k-x|+t)^2} \tag{5.15}$$

and thus

$$|F(x,t)| \lesssim \frac{1}{t(1+|x|)} + \frac{1}{t(1+|x|)(1+\log(1+|x|))^\alpha} \lesssim \frac{1}{t(1+|x|)}.$$

This implies $F \in b^2$ as before. The proof is complete. □

We also have an example of the same type for $p = 1$. Construction and estimates of such an example are to be a bit more complicated because of the additional first moment vanishing property.

Proposition 5.9. *There exists a function $f \in L^1 \cap L^1(|x|dx)$ such that*

$$\begin{aligned} &\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} xf(x)\,dx = 0, \\ &\int_{-\infty}^{\infty} |f(x)||x| \log^+ |x|\,dx = \infty \end{aligned} \tag{5.16}$$

and yet $P[f] \in b^1$.

Proof. Let

$$f = \sum_{k=2}^{\infty} (k \log k)^{-2} (\mathcal{X}_{2k} - \mathcal{X}_{2k-1})$$

where $\mathcal{X}_k$ denotes the characteristic function of the set $[k, k+1] \cup [-k-1, -k]$. Then it is not hard to see $f \in L^1 \cap L^1(|x|dx)$ and (5.16). Let $F = P[f]$. To show $F \in b^1$, we use the same argument as in the proof of Proposition 5.7. Thus, it is sufficient to show

$$|F(x,t)| \lesssim \frac{1}{(1+|x|+t)^3} + \frac{1}{t^2(1+|x|(\log^+|x|)^2)} \tag{5.17}$$

for $t > 1$ and $x \in \mathbf{R}$. To estimate (5.17), put

$$\begin{aligned} H(x,t,y) &= \frac{t}{(x-y)^2+t^2} + \frac{t}{(x+y)^2+t^2} \\ &= \frac{2t(x^2+t^2+y^2)}{((x-y)^2+t^2)((x+y)^2+t^2)}. \end{aligned}$$

Then, we see

$$\begin{aligned} P[\mathcal{X}_{2k} - \mathcal{X}_{2k-1}](x,t) &= \int_{2k}^{2k+1} H(x,t,y)\,dy - \int_{2k-1}^{2k} H(x,t,y)\,dy \\ &= \int_0^1 [H(x,t,y+2k) - H(x,t,y+2k-1)]\,dy \end{aligned}$$

and therefore

$$|F(x,t)| \le \sum_{k=2}^{\infty}(k\log k)^{-2}\int_0^1 |H(x,t,y+2k)-H(x,t,y+2k-1)|\,dy.$$

Note that $F(x,t)$ is even with respect to x and hence we only need to consider $x>0$. So assume $t>1$ and $x>0$. Then, one can obtain an estimate (uniform in y, $0<y<1$)

$$\begin{aligned}|H(x,t,y+2k)-H(x,t,y+2k-1)| &\lesssim \frac{kt}{(|2k-x|^2+t^2)^2}\\ &\le \frac{k}{(|2k-x|+t)^3}.\end{aligned}$$

It follows that

$$|F(x,t)| \lesssim \sum_{k=2}^{\infty} k^{-1}(\log k)^{-2}\frac{1}{(|2k-x|+t)^3}.$$

Now, repeating almost the same argument as in the proof of Proposition 5.7, we have

$$|F(x,t)| \lesssim \frac{1}{(1+x+t)^3}+\frac{1}{t^2(1+x)(1+\log(1+x))^2}.$$

So, we have (5.17) and the proof is complete. □

We finally give an example showing that our weighted integrability condition for $p>2$ is not necessary for the b^p-containment, either.

PROPOSITION 5.10. *For $p>2$, there exists a function $f\in L^p$ such that*

$$\int_{-\infty}^{\infty}|f(x)|^p|x|\,dx=\infty \tag{5.18}$$

and yet $P[f]\in b^p$.

Proof. Consider a function f defined in (5.11). This time we take $\frac{1}{p}<\alpha\le\frac{2}{p}$. It is easily verified that f satisfies (5.18) and $f\in L^p$. Put $F=P[f]$. We will see

$$|F(x,t)| \lesssim t^{-1+1/p}(1+|x|)^{-\alpha} \qquad (t>1,\ x\in\mathbf{R}), \tag{5.19}$$

which implies $F \in b^p$ as before. Assume $t > 1$ and let $x \in \mathbf{R}$. The estimate (5.13) is still available and thus we have by Hölder's inequality

$$|F(x,t)| \lesssim \left(\sum_{k=1}^{\infty} \frac{k^{-p\alpha}}{(|2k-x|+t)^{p\alpha}} \right)^{1/p} \left(\sum_{k=1}^{\infty} \frac{1}{(|2k-x|+t)^{q(2-\alpha)}} \right)^{1/q}$$

where q is the conjugate exponent of p. Note

$$\sum_{k=1}^{\infty} \frac{1}{(|2k-x|+t)^{q(2-\alpha)}} \lesssim \int_{-\infty}^{\infty} \frac{1}{(|y-x|+t)^{q(2-\alpha)}}\, dy \lesssim t^{1-q(2-\alpha)}.$$

Also, we have as in the proof of Proposition 5.7

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{k^{-p\alpha}}{(|2k-x|+t)^{p\alpha}} &\lesssim \frac{1}{(1+|x|+t)^{p\alpha}} + \frac{1}{t^{p\alpha-1}(1+|x|)^{p\alpha}} \\ &\lesssim \frac{1}{(1+|x|)^{p\alpha}}. \end{aligned}$$

It follows from these estimates

$$|F(x,t)| \lesssim (1+|x|)^{-\alpha} t^{\alpha-1-1/p},$$

which in turn implies (5.19), since $0 < \alpha - \frac{1}{p} \le \frac{1}{p}$. The proof is complete. □

Remark. We have seen that the converses of moment vanishing properties fail to hold. One can naturally expect the same failure in the sense of principal values. For an example in case $n = 1$, consider the function f defined in (5.2) with $\alpha = 1$. For any $1 < p \le 2$, one can easily check $f \in L^p$, $f \notin L^1$ and

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\, dx = 0$$

but $P[f] \notin b^p$. What we do not know is the other way round. That is, suppose $P[f] \in b^p$ for some $1 \le p \le 1 + \frac{1}{n}$. Does f necessarily have the relevant moment vanishing property in the sense of principal values?

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