BERKOVICH SKELETA AND BIRATIONAL GEOMETRY

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ABSTRACT. We give a survey of joint work with Mircea Mustaţă and Chenyang Xu on the connections between the geometry of Berkovich spaces over the field of Laurent series and the birational geometry of one-parameter degenerations of smooth projective varieties. The central objects in our theory are the *weight function* and the *essential skeleton* of the degeneration. We tried to keep the text self-contained, so that it can serve as an introduction to Berkovich geometry for birational geometers.

1. INTRODUCTION

Let R be a complete discrete valuation ring with residue field k and quotient field K. The main example to keep in mind is $R = \mathbb{C}[\![t]\!]$. The discrete valuation on K gives rise to a non-archimedean absolute value on K that one can use to develop a theory of analytic geometry over K. The theory that we will use is the one introduced by Berkovich in [Be90]. The principal purpose of these notes is to describe some interactions between Berkovich geometry over K and the birational geometry of degenerations of algebraic varieties over R. For a nice introduction to related results over trivially valued base fields, we refer to [Pa14].

In fact, we will use only a small part of the theory of Berkovich spaces: we are mainly interested in the underlying topological space of the analytification of an algebraic K-variety. The structure of this space can be described in terms of classical valuation theory; this will be explained in Section 2.

Let X be a connected, smooth and proper K-variety of dimension n. We denote by X^{an} the Berkovich analytification of K. An *sncd*-model of X is a regular scheme $\mathscr{X}$ of finite type over R, endowed with an isomorphism of K-schemes $\mathscr{X}_K \to X$, such that the special fiber $\mathscr{X}_k$ is a divisor with strict normal crossings. To any proper *sncd*-model $\mathscr{X}$ of X over R one can attach a subspace $\mathrm{Sk}(\mathscr{X})$ of X^{an}, called the Berkovich skeleton of X, which is canonically homeomorphic to the dual intersection complex of the strict normal crossings divisor $\mathscr{X}_k$. This skeleton can be viewed as the space of real valuations on the function field of X that extend the discrete valuation on K and that are monomial with respect to $\mathscr{X}_k$. The most important property of the skeleton $\mathrm{Sk}(\mathscr{X})$ is that it controls the homotopy type of X^{an}: it is a strong deformation retract of X^{an}. This provides an interesting link between the geometry of X^{an} and the birational geometry of models of X.

If X has dimension one and genus at least one, then X has a unique minimal *sncd*-model, and thus a canonical Berkovich skeleton. In higher dimensions, minimal *sncd*-models no longer exist. Nevertheless, one can ask whether it is still possible to construct a canonical skeleton in X^{an}. We will present two constructions, which we developed in collaboration with Mircea Mustaţă [MN13] and Chenyang Xu [NX13], respectively. The first construction is based on work of Kontsevich

and Soibelman on degenerations of Calabi-Yau varieties and Mirror Symmetry [KS06]; the second one relies on the Minimal Model Program, and in particular on the results in [dFKX12]. As we will see, both approaches yield the same result. We assume in the remainder of this introduction that the residue field k has characteristic zero.

The main idea behind the first approach is the following. Each proper $sncd$-model $\mathscr{X}$ of X gives rise to a Berkovich skeleton $\mathrm{Sk}(\mathscr{X})$ in X^{an}. We will use pluricanonical forms ω on X to single out certain essential faces of the simplicial complex $\mathrm{Sk}(\mathscr{X})$, which must be contained in the skeleton of *every* proper *sncd*-model of X. The union of these ω-essential faces is called the Kontsevich-Soibelman skeleton of (X, ω) and denoted by $\mathrm{Sk}(X, \omega)$. It only depends on X and ω, but not on the choice of $\mathscr{X}$. Taking the union of the skeleta $\mathrm{Sk}(X, \omega)$ over all non-zero pluricanonical forms ω on X, we obtain a subspace of X^{an} with piecewise affine structure that we call the *essential skeleton* of X and that we denote by $\mathrm{Sk}(X)$.

This construction has the merit of being quite natural and elementary, but it is not at all clear from the definition what $\mathrm{Sk}(X)$ looks like or whether $\mathrm{Sk}(X)$ is still a strong deformation retract of X^{an}. Therefore, we will also consider a second approach. As we have already mentioned, minimal *sncd*-models usually do not exist if the dimension of X is at least two, but we can enlarge our class of models in such a way that minimal models exist and such that we can still use the members of this class to describe the homotopy type of X^{an}. The Minimal Model Program suggests to consider so-called *dlt*-models of X, which should be viewed as proper *sncd*-models with mild singularities. We can define the skeleton $\mathrm{Sk}(\mathscr{X})$ of such a *dlt*-model by simply ignoring the singularities. The theory of minimal models guarantees that minimal *dlt*-models exist if the canonical sheaf of X is semi-ample, which means that some tensor power is generated by global sections. Minimal *dlt*-models are not unique, but the skeleton $\mathrm{Sk}(\mathscr{X})$ does not depend on the choice of a minimal *dlt*-model $\mathscr{X}$. By a careful analysis of the steps in the Minimal Model Program, it was proven in [dFKX12] that the skeleton $\mathrm{Sk}(\mathscr{X})$ can be obtained from the skeleton of any proper *sncd*-model of X by a sequence of elementary collapses. This implies that $\mathrm{Sk}(\mathscr{X})$ is still a strong deformation retract of X^{an}.

One of the main results of [NX13] is that these two constructions yield the same result: if the canonical sheaf of X is semi-ample, then the essential skeleton $\mathrm{Sk}(X)$ of X coincides with the skeleton of any minimal *dlt*-model of X. In particular, $\mathrm{Sk}(X)$ is a strong deformation retract of X^{an}. The semi-ampleness condition can be understood as follows: it guarantees that X has enough pluricanonical forms to detect all the important pieces of the skeleton of a proper *sncd*-model.

Notation. We denote by R a complete discrete valuation ring with maximal ideal $\mathfrak{m}$, residue field k and quotient field K. We denote by v_K the discrete valuation $K^* \twoheadrightarrow \mathbb{Z}$. We define an absolute value on K by setting $|x|_K = \exp(-v_K(x))$ for every element x of K^*. A variety over a field F is a separated F-scheme of finite type. If α and β are elements of $\mathbb{R}^m$ for some positive integer m, then we denote by $\alpha \cdot \beta$ their scalar product $\sum_{i=1}^{m} \alpha_i \beta_i$.

2. The Berkovich skeleton of an *sncd*-model

2.1. **Birational points.**

(2.1.1) Let X be a connected and smooth K-variety of dimension n. We denote by X^{an} the Berkovich analytification of X and by $i\colon X^{\mathrm{an}} \to X$ the analytification morphism. We will mainly be interested in the underlying topological space of X^{an}, which is easy to describe. As a set, X^{an} consists of the couples $(x, |\cdot|)$ where x is a scheme-theoretic point of X and $|\cdot|\colon \kappa(x) \to \mathbb{R}$ is an absolute value on the residue field $\kappa(x)$ of X at x that extends the absolute value $|\cdot|_K$ on K. The analytification map $i\colon X^{\mathrm{an}} \to X$ is simply the forgetful map that sends a couple $(x, |\cdot|)$ to x. The topology on X is the coarsest topology such that the following two properties are satisfied:

(1) the topology on X^{an} is finer than the Zariski topology, that is, the map $i\colon X^{\mathrm{an}} \to X$ is continuous;

(2) for every Zariski-open subset U of X and every regular function f on U, the map
$$|f|\colon i^{-1}(U) \to \mathbb{R}^+ \colon (x, |\cdot|) \mapsto |f(x)|$$
is continuous.

Note that the definition of $|f|$ makes sense because $f(x)$ is an element of the residue field $\kappa(x)$. We will often denote a point of X^{an} simply by x, leaving the absolute value $|\cdot|$ implicit in the notation. It is convenient in many situations to switch between the multiplicative and additive viewpoint: we will denote by v_x the real valuation
$$v_x\colon \kappa(x)^* \to \mathbb{R} \colon a \mapsto -\ln|a|.$$
As usual, we extend it to zero by setting $v_x(0) = +\infty$. The *residue field* of X^{an} at a point x is defined as the completion of the residue field $\kappa(i(x))$ of X at $i(x)$ with respect to the absolute value $|\cdot|$. It is a complete valued extension of the field K, which we denote by $\mathscr{H}(x)$. The valuation ring of $\mathscr{H}(x)$ will be denoted by $\mathscr{H}(x)^o$. If f is a rational function on X that is defined at $i(x)$, then we can think of $f(i(x)) \in \kappa(i(x))$ as an element of $\mathscr{H}(x)$, and we denote this element by $f(x)$.

(2.1.2) The topological space X^{an} is Hausdorff, and it is compact if and only if X is proper over K. If X is a curve, then there exists a simple classification of the points on X^{an} and one can draw a fairly explicit picture of X^{an}; see for instance Section 5.1 in [Ba08]. If the dimension of X is at least two, it is much more difficult to give a precise description of the whole space X^{an}. We will see in the following sections, however, that one can produce many interesting points in this space using the birational geometry of X, and that these points suffice to control the homotopy type of X^{an}.

(2.1.3) The set X^{bir} of birational points of X^{an} is defined as the inverse image under $i\colon X^{\mathrm{an}} \to X$ of the generic point of X. In the additive notation, this is simply the set of real valuations on the function field $K(X)$ of X that extend the discrete valuation v_K on K. We endow X^{bir} with the topology induced by the Berkovich topology on X^{an}. We will see in (2.4.13) that the inclusion $X^{\mathrm{bir}} \to X^{\mathrm{an}}$ is a homotopy equivalence if k has characteristic zero. By its very definition, X^{bir} is a

birational invariant of X, so we can hope to recover interesting birational invariants of X from this topological space.

2.2. Models.

(2.2.1) We will define certain subclasses of birational points using the geometry of R-models of X. An R-model of X is a flat separated R-scheme of finite type $\mathscr{X}$ endowed with an isomorphism of K-schemes $\mathscr{X}_K \to X$. Note that we do not impose any properness condition on X or $\mathscr{X}$. We say that $\mathscr{X}$ is an *sncd*-model of X if $\mathscr{X}$ is regular and its special fiber $\mathscr{X}_k$ is a divisor with strict normal crossings.

(2.2.2) Let $\mathscr{X}$ be an R-model of X and let x be a point of X^{an}. We say that x has a *center* on $\mathscr{X}$ if the canonical morphism $\mathrm{Spec}\,\mathscr{H}(x) \to X$ extends to a morphism $\mathrm{Spec}\,\mathscr{H}(x)^o \to \mathscr{X}$. Such an extension is unique if it exists, by the valuative criterion of separatedness. If it exists, the center of x on $\mathscr{X}$ is defined as the image of the closed point of $\mathrm{Spec}\,\mathscr{H}(x)^o$ in $\mathscr{X}$ and denoted by $\mathrm{sp}_{\mathscr{X}}(x)$. Note that the point $\mathrm{sp}_{\mathscr{X}}(x)$ always lies on the special fiber $\mathscr{X}_k$ of $\mathscr{X}$ because $\mathfrak{m}$ is contained in the maximal ideal of $\mathscr{H}(x)^o$. We denote by $\widehat{\mathscr{X}_\eta}$ the set of points on X^{an} that have a center on $\mathscr{X}$. If $\mathscr{X}$ is proper over R, then $\widehat{\mathscr{X}_\eta} = X^{\mathrm{an}}$ by the valuative criterion of properness. The map

$$\mathrm{sp}_{\mathscr{X}} : \widehat{\mathscr{X}_\eta} \to \mathscr{X}_k$$

is called the reduction map or specialization map. It has the peculiar property of being *anti-continuous*, which means that the inverse image of an open set is closed.

Example 2.2.3. If $X = \mathbb{A}_K^1 = \mathrm{Spec}\,K[T]$ and $\mathscr{X} = \mathbb{A}_R^1$, then

$$\widehat{\mathscr{X}_\eta} = \{x \in X^{\mathrm{an}} \,|\, |T(x)| \le 1\}$$

and $\mathrm{sp}_{\mathscr{X}}(x)$ is the reduction of $T(x) \in \mathscr{H}(x)^o$ modulo the maximal ideal of $\mathscr{H}(x)^o$ (viewed as a point of $\mathscr{X}_k = \mathrm{Spec}\,k[T]$).

(2.2.4) Assuming a bit more technology [Be96, §1], we can give an equivalent description of $\widehat{\mathscr{X}_\eta}$ and $\mathrm{sp}_{\mathscr{X}}$. If we denote by $\widehat{\mathscr{X}}$ the formal $\mathfrak{m}$-adic completion of $\mathscr{X}$, then the generic fiber of $\widehat{\mathscr{X}}$ is a compact analytic domain in X^{an} whose underlying set is precisely $\widehat{\mathscr{X}_\eta}$. The reduction map $\mathrm{sp}_{\mathscr{X}}$ is the map underlying the specialization morphism of locally ringed spaces

$$\mathrm{sp}_{\mathscr{X}} : \widehat{\mathscr{X}_\eta} \to \widehat{\mathscr{X}}.$$

2.3. Divisorial and monomial points.

(2.3.1) Let $\mathscr{X}$ be a normal R-model of X. If E is an irreducible component of the special fiber $\mathscr{X}_k$ with generic point ξ, then the fiber $\mathrm{sp}_{\mathscr{X}}^{-1}(\xi)$ consists of a unique point, which we call the divisorial point of X^{an} associated with $(\mathscr{X}, E)$. It is the birational point x on X^{an} that corresponds to the discrete valuation v_x on $K(X)$ with valuation ring $\mathcal{O}_{\mathscr{X}, \xi}$, normalized in such a way that v_x extends the discrete valuation v_K on K. Thus if f is a non-zero rational function on X, then

$$v_x(f) = \frac{1}{N} \mathrm{ord}_E f$$

where N denotes the multiplicity of E in the Cartier divisor $\mathscr{X}_k$ on $\mathscr{X}$ and $\mathrm{ord}_E f$ is the order of f along E. A point of X^{an} is called divisorial if it is the divisorial point associated with some couple $(\mathscr{X}, E)$ as above. We will denote the set of divisorial points by $X^{\mathrm{div}} \subset X^{\mathrm{bir}}$. It is not difficult to show that the set X^{div} is dense in X^{an}; see for instance [MN13, 2.4.12].

(2.3.2) The set of divisorial points is totally disconnected. We will define a more general class of points that should be viewed as some kind of interpolations between divisorial points: the monomial points. Let $\mathscr{X}$ be an *sncd*-model of X and let $E_1, \ldots, E_r$ be distinct irreducible components of the special fiber $\mathscr{X}_k$ with respective multiplicities $N_1, \ldots, N_r$ in $\mathscr{X}_k$, and assume that the intersection $\cap_{i=1}^r E_i$ is non-empty. Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a tuple of positive real numbers such that $\sum_{i=1}^r \alpha_i N_i = 1$ and let ξ be a generic point of $\cap_{i=1}^r E_i$ (by the definition of an *sncd*-model, this intersection is regular and of pure dimension $n + 1 - r$, but it is not necessarily connected).

Proposition 2.3.3. *There exists a unique minimal real valuation*

$$v \colon \mathcal{O}_{\mathscr{X}, \xi} \setminus \{0\} \to \mathbb{R}^+$$

such that $v(T_i) = \alpha_i$ for every i in $\{1, \ldots, r\}$ and every local equation $T_i = 0$ for E_i in $\mathscr{X}$ at ξ.

(2.3.4) Proposition 2.3.3 can be proven by combining [MN13, 2.4.6 and 3.1.6]. We will not give a complete proof here, but only sketch how the valuation v can be constructed. For every i in $\{1, \ldots, r\}$ we choose a local equation $T_i = 0$ of E_i in $\mathscr{X}$ at ξ. Then the elements T_i form a regular system of local parameters in the local ring $\mathcal{O}_{\mathscr{X}, \xi}$. It is not difficult to show that every element f in $\mathcal{O}_{\mathscr{X}, \xi}$ can be written in the completed local ring $\widehat{\mathcal{O}}_{\mathscr{X}, \xi}$ as a power series

$$(2.3.5) \qquad\qquad f = \sum_{\beta \in \mathbb{N}^r} c_\beta T^\beta$$

where each coefficient c_β is either zero or a unit in $\widehat{\mathcal{O}}_{\mathscr{X}, \xi}$. Such an expansion is not unique, but one can show that the expression

$$(2.3.6) \qquad\qquad v(f) := \min\{\alpha \cdot \beta \mid \beta \in \mathbb{N}^r, \, c_\beta \neq 0\}$$

does not depend on any choices and that it defines a valuation v with the required properties. If R has equal characteristic, then the arguments can be simplified by using the fact that $\widehat{\mathcal{O}}_{\mathscr{X}, \xi}$ is isomorphic to the power series ring $\kappa(\xi)\llbracket T_1, \ldots, T_r \rrbracket$ by Cohen's structure theorem.

(2.3.7) The valuation v in Proposition 2.3.3 extends to a real valuation $v \colon K(X)^* \to \mathbb{R}$. It extends the discrete valuation v_K on K: if π is a uniformizer in R, then in the ring $\mathcal{O}_{\mathscr{X}, \xi}$ we can write

$$\pi = u \prod_{i=1}^r T_i^{N_i}$$

with u a unit, so that $v(\pi) = \sum_{i=1}^r \alpha_i N_i = 1$. Thus v defines a birational point x on X^{an}, which we call the monomial point associated with the data

$$(2.3.8) \qquad\qquad (\mathscr{X}, (E_1, \ldots, E_r), \alpha, \xi).$$

The point x belongs to $\widehat{\mathscr{X}_\eta}$, and $\mathrm{sp}_{\mathscr{X}}(x) = \xi$. We remark for later use that formula (2.3.6) can be generalized as follows: if

$$f = \sum_{\beta \in \mathbb{N}^r} c_\beta d_\beta T^\beta$$

where each coefficient c_β is either zero or a unit in $\widehat{\mathcal{O}}_{\mathscr{X},\xi}$ and each coefficient d_β belongs to K, then

(2.3.9) $v(f) = \min\{v_K(d_\beta) + \alpha \cdot \beta \,|\, \beta \in \mathbb{N}^r,\ c_\beta \neq 0\}$

since we can rewrite d_β as the product of

$$\pi^{v_K(d_\beta)} = (u \prod_{i=1}^r T_i^{N_i})^{v_K(d_\beta)}$$

with a unit in R to get an expansion for f of the form (2.3.5).

(2.3.10) A point on X^{an} is called monomial if it is the monomial point associated with a tuple of data as in (2.3.8); we will also say that such a point is monomial with respect to the model $\mathscr{X}$. If $r = 1$, then we get precisely the divisorial point associated with $(\mathscr{X}, E_1)$. Thus every divisorial point is monomial. Conversely, the monomial point associated with (2.3.8) is divisorial (possibly with respect to a different model $\mathscr{X}$) if and only if the parameters α_i all belong to $\mathbb{Q}$ (see [MN13, 2.4.1]). The set of monomial points on X^{an} will be denoted by X^{mon}. We have the following inclusions:

$$X^{\mathrm{div}} \subset X^{\mathrm{mon}} \subset X^{\mathrm{bir}} \subset X^{\mathrm{an}}.$$

2.4. The Berkovich skeleton.

(2.4.1) Let $\mathscr{X}$ be an *sncd*-model of X. We define the Berkovich skeleton of $\mathscr{X}$ as the set of all points of X^{an} that are monomial with respect to $\mathscr{X}$, and we denote it by $\mathrm{Sk}(\mathscr{X})$. By construction, the Berkovich skeleton $\mathrm{Sk}(\mathscr{X})$ is a subspace of $\widehat{\mathscr{X}_\eta} \cap X^{\mathrm{mon}}$. The importance of this object is that we can give an explicit description of the topology on $\mathrm{Sk}(\mathscr{X})$ and that this suffices to understand the homotopy type of $\widehat{\mathscr{X}_\eta}$, as we will now explain.

(2.4.2) We first need to recall the definition of the *dual complex* of the strict normal crossings divisor $\mathscr{X}_k$. We write $\mathscr{X}_k = \sum_{i \in I} N_i E_i$ and for every non-empty subset J of I, we set $E_J = \cap_{i \in J} E_i$. The dual complex of $\mathscr{X}_k$ is a simplicial complex[1] $|\Delta(\mathscr{X}_k)|$ whose simplices of dimension d correspond bijectively to the connected components of the regular k-varieties E_J where J runs through the set of subsets of I of cardinality $d + 1$. If J and J' are non-empty subsets of I, and C and C' are connected components of E_J and $E_{J'}$, respectively, then the simplex corresponding to C is a face of the simplex corresponding to C' if and only if C contains C'. Thus the vertices of $|\Delta(\mathscr{X}_k)|$ correspond to the irreducible components E_i of $\mathscr{X}_k$, and we will denote the vertices accordingly by ν_i, $i \in I$. If i and j are

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[1]To be precise, $|\Delta(\mathscr{X}_k)|$ is not a simplicial complex in the strict sense, because we allow for instance multiple edges between two vertices. This has no importance for the present exposition (and can always be remediated by blowing up $\mathscr{X}$ at connected components of the subvarieties E_J, which gives rise to a stellar subdivision of the corresponding face of $|\Delta(\mathscr{X}_k)|$). In any case, $|\Delta(\mathscr{X}_k)|$ is the topological realization of a finite simplicial set.

distinct elements of I then the number of edges between ν_i and ν_j is the number of connected components of $E_i \cap E_j$, and so on. In this way, the dual complex $|\Delta(\mathscr{X}_k)|$ encodes the combinatorial structure of the intersections of prime components in $\mathscr{X}_k$. The dimension of $|\Delta(\mathscr{X}_k)|$ is at most n, the dimension of X. If X has dimension one, then the dual complex $|\Delta(\mathscr{X}_k)|$ is more commonly known as the *dual graph* of the special fiber $\mathscr{X}_k$.

Example 2.4.3. Assume that X has dimension one and that $\mathscr{X}_k$ has four irreducible components E_1, E_2, E_3, E_4 such that E_1 intersects each of the other components in precisely one point and there are no other intersection points. Then $|\Delta(\mathscr{X}_k)|$ is a graph with four vertices v_1, v_2, v_3, v_4 with one edge between v_1 and v_i for $i = 2, 3, 4$ and no other edges.

If X has dimension two and $\mathscr{X}_k$ is isomorphic to the union of the coordinate planes in $\mathbb{A}_k^3$, then $|\Delta(\mathscr{X}_k)|$ is the standard 2-simplex.

(2.4.4) We will now construct a map

$$\Phi \colon |\Delta(\mathscr{X}_k)| \to \mathrm{Sk}(\mathscr{X}).$$

For each $i \in I$, our map Φ sends the vertex ν_i of $|\Delta(\mathscr{X}_k)|$ to the divisorial point associated with $(\mathscr{X}, E_i)$. In order to define Φ on the higher-dimensional faces of $|\Delta(\mathscr{X}_k)|$, we use monomial valuations to interpolate between these divisorial valuations, as follows. Let y be a point of $|\Delta(\mathscr{X}_k)|$. Then there exists a unique face τ of $|\Delta(\mathscr{X}_k)|$ such that y lies in the interior τ^o of τ. By the construction of $|\Delta(\mathscr{X}_k)|$, the face τ corresponds to a connected component C of an intersection E_J for some subset J of I. We denote by ξ the generic point of C. The vertices of τ correspond precisely to the irreducible components E_i with $i \in J$. We can represent the point y by a tuple of barycentric coordinates $\beta \in \mathbb{R}^J$ where each coordinate β_i is a positive real number and their sum is equal to one. Now we define $\Phi(y)$ as the monomial point of X^{an} associated with the data

$$(\mathscr{X}, (E_i)_{i \in J}, (\beta_i/N_i)_{i \in J}, \xi).$$

(2.4.5) It is easy to see that Φ is a bijection, since we can give the following description of the map Φ^{-1}. Let x be a point of $\mathrm{Sk}(\mathscr{X})$. Then $\mathrm{sp}_{\mathscr{X}}(x)$ is a generic point ξ of E_J, for some uniquely determined non-empty subset J of I. For each $i \in J$ we choose a local equation $T_i = 0$ for E_i in $\mathscr{X}$ at ξ, and we set $\alpha_i = v_x(T_i)$. Then $\Phi^{-1}(x)$ lies in the interior of the face τ of $|\Delta(\mathscr{X}_k)|$ corresponding to the connected component of ξ in E_J, and its tuple of barycentric coordinates is equal to $(\alpha_i N_i)_{i \in J}$. In fact, we can say more.

Proposition 2.4.6. *The map*

$$\Phi \colon |\Delta(\mathscr{X}_k)| \to \mathrm{Sk}(\mathscr{X})$$

is a homeomorphism.

Proof. Since the source of Φ is compact and the target is Hausdorff, we only need to prove that Φ is continuous. This is not difficult: using the definition of the Berkovich topology in (2.1.1) and the explicit description of monomial valuations in (2.3.4), one immediately checks that Φ is continuous on the interior of each of the faces of $|\Delta(\mathscr{X}_k)|$. To get the continuity at the boundary faces, one uses the following easy observation. Suppose that some of the α_i are zero in the construction of the

valuation v in (2.3.4), say, $\alpha_1, \ldots, \alpha_s \neq 0$ and $\alpha_{s+1} = \ldots = \alpha_r = 0$ for some $s < r$. Then the formula we gave in (2.3.6) defines the monomial valuation associated with $\mathscr{X}$, the components $E_1, \ldots, E_s$, the parameters $\alpha_1, \ldots, \alpha_s$ and the unique generic point of $E_1 \cap \ldots \cap E_s$ whose closure contains ξ. For details, see [MN13, 2.4.9 and 3.1.4]. $\qquad\square$

(2.4.7) We can use the homeomorphism Φ to endow $\mathrm{Sk}(\mathscr{X})$ with a piecewise $\mathbb{Z}$-affine structure; see [MN13, §3.2]. This structure can be defined intrinsically on X^{an} and is independent of the choice of the model $\mathscr{X}$. The induced piecewise $\mathbb{Q}$-affine structure is simply the one inherited from the faces of $|\Delta(\mathscr{X})|$. We will not use the finer $\mathbb{Z}$-affine structure so we will not recall its definition here. If f is a non-zero rational function on X, then the function

$$\mathrm{Sk}(\mathscr{X}) \to \mathbb{R} \colon x \mapsto \ln|f(x)|$$

is continuous and piecewise affine.

(2.4.8) Proposition 2.4.6 gives an explicit description of the topological space $\mathrm{Sk}(\mathscr{X})$. We will now explain how one can use this description to determine the homotopy type of $\widehat{\mathscr{X}_\eta}$. First, we construct a retraction

$$\rho_{\mathscr{X}} \colon \widehat{\mathscr{X}_\eta} \to \mathrm{Sk}(\mathscr{X})$$

for the embedding of $\mathrm{Sk}(\mathscr{X})$ in $\widehat{\mathscr{X}_\eta}$. Let x be a point of $\widehat{\mathscr{X}_\eta}$. Let J be the set of indices $i \in I$ such that E_i contains the center $\mathrm{sp}_{\mathscr{X}}(x)$ of x on $\mathscr{X}$. We denote by C the connected component of x in E_J and by ξ the generic point of C. For each $i \in J$ we choose a local equation $T_i = 0$ for E_i in $\mathscr{X}$ at $\mathrm{sp}_{\mathscr{X}}(x)$, and we set $\alpha_i = v_x(T_i)$. Then $\rho_{\mathscr{X}}(x)$ is the monomial point in X^{an} associated with the data

$$(\mathscr{X}, (E_i)_{i \in J}, (\alpha_i)_{i \in J}, \xi).$$

In other words, it is the unique point of the skeleton such that the Zariski closure of its center contains the center of x and which gives the same valuation to each local defining equation of an irreducible component E_i of $\mathscr{X}_k$ passing through $\mathrm{sp}_{\mathscr{X}}(x)$. It is an easy exercise to verify that $\rho_{\mathscr{X}}$ is continuous. The most fundamental result about Berkovich skeleta is the following theorem.

Theorem 2.4.9 (Berkovich, Thuillier)**.** *There exists a continuous map*

$$H : \widehat{\mathscr{X}_\eta} \times [0,1] \to \widehat{\mathscr{X}_\eta}$$

such that $H(\cdot, 0)$ is the identity, $H(\cdot, 1)$ is the map $\rho_{\mathscr{X}} \colon \widehat{\mathscr{X}_\eta} \to \mathrm{Sk}(\mathscr{X})$, and $H(x, t) = x$ for every point x of $\mathrm{Sk}(\mathscr{X})$ and every t in $[0,1]$. Thus $\mathrm{Sk}(\mathscr{X})$ is a strong deformation retract of $\widehat{\mathscr{X}_\eta}$.

Corollary 2.4.10. *If X is proper and $\mathscr{X}$ is a proper sncd-model of X, then $\mathrm{Sk}(\mathscr{X})$ is a strong deformation retract of X^{an}. In particular, X^{an} has the same homotopy type as the simplicial complex $|\Delta(\mathscr{X}_k)|$.*

Proof. This follows from the fact that $\widehat{\mathscr{X}_\eta} = X^{\mathrm{an}}$ if $\mathscr{X}$ is proper, and from Proposition 2.4.6. $\qquad\square$

(2.4.11) Giving a proof of Theorem 2.4.9 goes beyond the scope of this survey, but we will work out an elementary example in Section 2.5. The origins of Theorem 2.4.9 are the results by Berkovich on skeleta of so-called poly-stable formal schemes [Be99]. Berkovich used these skeleta to prove that smooth non-archimedean analytic spaces are locally contractible. An *sncd*-model $\mathscr{X}$ (or rather, its formal $\mathfrak{m}$-adic completion) is not poly-stable unless the special fiber $\mathscr{X}_k$ is reduced[2]. Thus we cannot directly apply Berkovich's result here. If R has equal characteristic and $\mathscr{X}$ is defined over an algebraic curve, we explained in [NX13, 3.1.3] how one can deduce Theorem 2.4.9 from results by Thuillier on skeleta over trivially valued fields [Th07]. The general case can be proven by translating Thuillier's toroidal methods into the language of log-geometry; details will be given in a forthcoming publication.

(2.4.12) If X is proper over K, then the existence of a proper *sncd*-model $\mathscr{X}$ is known if k has characteristic zero (by Hironaka's resolution of singularities), and also if k has arbitrary characteristic and X is a curve (by Lipman's resolution of singularities for excellent schemes of dimension two). Most experts believe that it should exist in general, but at this moment, resolution of singularities in positive and mixed characteristic remains one of the big open problems in algebraic geometry. Corollary 2.4.10 implies in particular that the homotopy type of the dual complex $|\Delta(\mathscr{X}_k)|$ does not depend on the choice of the proper *sncd*-model $\mathscr{X}_k$. This is an analog of Thuillier's generalization of Stepanov's theorem in [Th07], saying that the homotopy type of the dual complex of a log resolution of a pair of algebraic varieties over a perfect field is independent of the choice of the log resolution.

(2.4.13) If $\overline{X}$ is a smooth compactification of X and $\mathscr{X}$ is a proper *sncd*-model of $\overline{X}$, then the explicit construction of the strong deformation retract H from Theorem 2.4.9 shows that it restricts to strong deformation retracts of X^{an} and X^{bir} onto $\mathrm{Sk}(\mathscr{X})$ (in fact, $H(x,t)$ lies in X^{bir} for every x in $\overline{X}^{\mathrm{an}}$ and every $t > 0$). Thus the inclusions $X^{\mathrm{bir}} \to X^{\mathrm{an}}$ and $X^{\mathrm{an}} \to \overline{X}^{\mathrm{an}}$ are homotopy equivalences. In particular, the homotopy type of the analytification of a smooth K-variety is a birational invariant if k has characteristic zero.

2.5. The deformation retraction in a basic example.

(2.5.1) We will give an explicit construction of the map H from Theorem 2.4.9 for the following elementary example:

$$\mathscr{X} = \mathrm{Spec}\, R[T_1, T_2]/(T_1^{N_1} T_2^{N_2} - \pi),$$

with π a uniformizer in R and N_1, N_2 positive integers. Then $\mathscr{X}$ is an *sncd*-model for its generic fiber $X = \mathscr{X}_K$, and $\widehat{\mathscr{X}_\eta}$ is the set of points x in X^{an} such that $|T_1(x)| \leq 1$ and $|T_2(x)| \leq 1$. We denote by E_i the component of $\mathscr{X}_k$ defined by $T_i = 0$ for $i = 1, 2$ and by O the unique intersection point of E_1 and E_2. The dual complex $|\Delta(\mathscr{X}_k)|$ is the standard 1-simplex

$$\Delta_1 = \{(\lambda, 1 - \lambda) \in \mathbb{R}^2 \,|\, 0 \leq \lambda \leq 1\}$$

---

[2]However, the class of poly-stable formal schemes is much larger than the class of *sncd*-models with reduced special fiber.

and the morphism Φ constructed in (2.4.4) sends $(1,0)$ to the divisorial point associated with $(\mathscr{X}, E_1)$, $(0,1)$ to the divisorial point associated with $(\mathscr{X}, E_2)$, and $(\lambda, 1 - \lambda)$ to the monomial point associated with

$$(\mathscr{X}, (E_1, E_2), (\frac{\lambda}{N_1}, \frac{1 - \lambda}{N_2}), O)$$

for all $\lambda \in \,]0, 1[$.

(2.5.2) The construction of the map H is best understood in terms of torus actions. We set $c = \gcd(N_1, N_2)$ and $M_i = N_i/c$ for $i = 1, 2$, and we choose integers a_1 and a_2 such that $a_1 M_1 + a_2 M_2 = 1$. For every complete valued field extension $(L, |\cdot|_L)$ of K we set

$$\mathbb{G}_L = \{x \in (\operatorname{Spec} L[U_1, U_2]/(U_1^{M_1} U_2^{M_2} - 1))^{\mathrm{an}} \mid |U_1(x)| = 1\}$$

with the group structure given by componentwise multiplication. For every element t in the interval $[0, 1]$ we define a point $\gamma_L(t)$ in $\mathbb{G}_L$ as follows. We will make use of the isomorphism of L-algebras

$$L[V, V^{-1}] \to L[U_1, U_2]/(U_1^{M_1} U_2^{M_2} - 1) \colon V \mapsto U_1^{a_2} U_2^{-a_1}$$

whose inverse is given by $U_1 \mapsto V^{M_2}$ and $U_2 \mapsto V^{-M_1}$. We can write every polynomial f in $L[V]$ as a Taylor expansion

$$f = \sum_{i \geq 0} c_i (V - 1)^i$$

around the point 1, where the coefficients c_i lie in L. Then the point $\gamma_L(t)$ is fully determined by the property that

$$|f(\gamma_L(t))| = \max_{i \geq 0} |c_i|_L t^i.$$

In other words, the point $\gamma_L(t)$ is the sup-norm on the closed disc of radius t around the point 1 in a completed algebraic closure $\widehat{L^a}$ of L. Note that

$$|U_1(\gamma_L(t))| = |V^{M_2}(\gamma_L(t))| = 1$$

for every t so that $\gamma_L(t)$ is indeed a point of $\mathbb{G}_L$. The map

$$\gamma_L \colon [0, 1] \to \mathbb{G}_L : t \mapsto \gamma(t)$$

is a continuous path from $\gamma_L(0) = 1$ to $\gamma_L(1)$.

(2.5.3) The torus $\mathbb{G}_K$ acts on $\widehat{\mathscr{X}_\eta}$ by componentwise multiplication, and we can use this action together with the paths γ_L to produce paths in $\widehat{\mathscr{X}_\eta}$. For every point x of $\widehat{\mathscr{X}_\eta}$, the action of $\mathbb{G}_K$ gives rise to a continuous map

$$\mathbb{G}_{\mathscr{H}(x)} \to (\widehat{\mathscr{X}_\eta}) \times_K \mathscr{H}(x) : g \mapsto g \cdot x.$$

For every t in $[0, 1]$, we define $H(x, t)$ as the image of $\gamma_{\mathscr{H}(x)}(t) \cdot x$ under the projection map

$$(\widehat{\mathscr{X}_\eta}) \times_K \mathscr{H}(x) \to \widehat{\mathscr{X}_\eta}.$$

In this way, we obtain a map

$$H \colon \widehat{\mathscr{X}_\eta} \times [0, 1] \to \widehat{\mathscr{X}_\eta} : (x, t) \mapsto H(x, t).$$

The map H is continuous by continuity of the paths γ_L and of the torus action on $\widehat{\mathscr{X}_\eta}$.

(2.5.4) We can also give a more explicit and down-to-earth (but less conceptual) description of the map H. Let x be a point of $\widehat{\mathscr{X}_\eta}$ and let t be an element of $[0,1]$. For notational convenience, we set $x_1 = T_1(x)$ and $x_2 = T_2(x)$; these are elements of the residue field $\mathscr{H}(x)$ of X^{an} at x. Let f be an element of $K[T_1, T_2]$. Then we can write the Laurent polynomial $f(x_1 V^{M_2}, x_2 V^{-M_1})$ in $\mathscr{H}(x)[V, V^{-1}]$ as a rational function

$$f(x_1 V^{M_2}, x_2 V^{-M_1}) = \frac{1}{V^j} \sum_{i \geq 0} c_i (V - 1)^i,$$

where the coefficients c_i belong to the valued field $\mathscr{H}(x)$ and only finitely many of them are non-zero. The point $H(x,t)$ is fully characterized by the property

$$|f(H(x,t))| = \max_i |c_i| t^i.$$

We remark for later reference that

(2.5.5) $$|f(H(x,t))| \geq |c_0| = |f(x)|.$$

(2.5.6) Now we prove that H is a strong deformation retract onto $\mathrm{Sk}(\mathscr{X})$. Setting $t = 0$ we find $|f(H(x,0))| = |f(x)|$ so that $H(x,0) = x$. To compute $H(x,1)$ we use the fact that for every complete valued extension $(L, |\cdot|_L)$ of K, the closed disc with radius one around 1 in $\widehat{L^a}$ coincides with the closed disc with radius one around 0, so that

$$|g(\gamma_L(1))| = \max_i |c_i|_L$$

for every polynomial $g = \sum_{i \geq 0} c_i V^i$ in $L[V]$. In this way, we see that for every polynomial

$$f = \sum_{i,j \geq 0} c_{ij} T_1^i T_2^j$$

in $K[T_1, T_2]$, we have

$$|f(H(x,1))| = \max_{i,j} |c_{ij}|_K |x_1|^i |x_2|^j,$$

or, in additive notation:

$$v_{H(x,1)}(f) = \min_{i,j}\{v_K(c_{ij}) + i v_x(T_1) + j v_x(T_2)\}.$$

Thus we find by using formula (2.3.9) that $H(x,1)$ is the monomial point on X^{an} associated with

$$(\mathscr{X}, (E_1, E_2), (v_x(T_1), v_x(T_2)), O).$$

This is precisely the image $\rho_{\mathscr{X}}(x)$ of x under the retraction $\rho_{\mathscr{X}} \colon \widehat{\mathscr{X}_\eta} \to \mathrm{Sk}(\mathscr{X})$. Finally, we show that $H(x,t) = x$ for all t in $[0,1]$ when x is a point of the skeleton $\mathrm{Sk}(\mathscr{X})$, that is, a monomial point with respect to $\mathscr{X}$. Direct computation shows that $|T_1(H(x,t))| = |x_1|$ and $|T_2(H(x,t))| = |x_2|$. Combining the inequality (2.5.5) with the minimality property of monomial valuations in Proposition 2.3.3, we see at once that $H(x,t)$ must be equal to x.

3. Weight functions and the Kontsevich-Soibelman skeleton

3.1. The work of Kontsevich and Soibelman.

(3.1.1) In [KS06], Kontsevich and Soibelman proposed a new interpretation of mirror symmetry based on non-archimedean geometry over the field of complex Laurent series $\mathbb{C}((t))$. Their fundamental idea was to encode a part of the geometry of a one-parameter degeneration of complex Calabi-Yau varieties into a topological manifold endowed with a $\mathbb{Z}$-affine structure with singularities, and to interpret mirror symmetry as a certain combinatorial duality between such manifolds. They worked out in detail the case of degenerations of $K3$-surfaces. Similar ideas were developed by Gross and Siebert in their theory of *toric degenerations*. Gross and Siebert replaced the use of non-archimedean geometry by methods from tropical and logarithmic geometry and extended the results for $K3$ surfaces to higher-dimensional degenerations [Gr12].

(3.1.2) An essential ingredient of the construction of Kontsevich and Soibelman is the following. We denote by Δ a small disc around the origin of the complex plane and we set $\Delta^* = \Delta \setminus \{0\}$. We denote by t a local coordinate on Δ centered at 0. Let X be a smooth projective family of varieties over Δ^* and let ω be a relative differential form of maximal degree on the family $X \to \Delta^*$. Kontsevich and Soibelman associated to these data a skeleton $\mathrm{Sk}(X, \omega)$, which is a topological subspace of the Berkovich analytfication of the $\mathbb{C}((t))$-variety obtained from X by base change. If X is a family of Calabi-Yau varieties, then we set $\mathrm{Sk}(X, \omega) = \mathrm{Sk}(X)$ where ω is any relative volume form on X. This definition does not depend on the choice of ω.

(3.1.3) Kontsevich and Soibelman proved that $\mathrm{Sk}(X, \omega)$ can be explicitly computed on any strict normal crossings model $\mathscr{X}$ for X over Δ: it is a union of faces of the Berkovich skeleton $\mathrm{Sk}(\mathscr{X})$ of the model $\mathscr{X}$ on which ω is minimal in a suitable sense. Their proof relied on the Weak Factorization Theorem. It is interesting to note that, even though the Berkovich skeleton $\mathrm{Sk}(\mathscr{X})$ from Section 2.4 heavily depends on the chosen model $\mathscr{X}$, the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X, \omega)$ only depends on X and ω. It singles out certain faces of $\mathrm{Sk}(\mathscr{X})$ that must appear in the skeleton of *every* strict normal crossings model.

(3.1.4) In [MN13], Mircea Mustaţă and the author extended this construction to varieties over complete discretely valued fields K of arbitrary characteristic, and to pluricanonical forms ω. Our approach does not use the Weak Factorization Theorem but only relies on basic computations on valuations and canonical sheaves. Moreover, we proved that the skeleton of a Calabi-Yau variety over $\mathbb{C}((t))$ is always connected. An interesting gadget that appears in our work is the *weight function*

$$\mathrm{wt}_\omega : X^{\mathrm{an}} \to \mathbb{R} \cup \{+\infty\}$$

associated to a smooth and proper K-variety X and a pluricanonical form ω on X. This weight function is piecewise affine on the Berkovich skeleton of any strict normal crossings model of X and strictly increasing as one moves away from the Berkovich skeleton. The Kontsevich-Soibelman skeleton is precisely the set of points where wt_ω reaches its minimal value; see Section 3.3.

3.2. Log discrepancies in birational geometry.

(3.2.1) Our approach is inspired by interesting analogies with some fundamental invariants in birational geometry. Let X be a smooth complex variety and $\mathcal{I}$ a coherent ideal sheaf on X. Let v be a divisorial valuation on X, that is, a positive real multiple of the discrete valuation ord_E on the function field $\mathbb{C}(X)$ associated to a prime divisor E on a normal birational modification Y of X. We denote by N the multiplicity of the scheme $Z(\mathcal{I}\mathcal{O}_Y)$ along E and by ν the multiplicity of E in the relative canonical divisor $K_{Y/X}$. We set

$$\mathrm{wt}_{\mathcal{I}}(v) = \frac{\nu+1}{N}$$

and we call this positive rational number the weight of $\mathcal{I}$ at v. Then the infimum of the values $\mathrm{wt}_{\mathcal{I}}(v)$ at all divisorial valuations v on X is called the log-canonical threshold of the pair $(X, \mathcal{I})$ and denoted by $\mathrm{lct}(X, \mathcal{I})$. This is a measure for the singularities of the zero locus $Z(\mathcal{I})$ of $\mathcal{I}$ on X, and one of the most important invariants in birational geometry. We refer to [Ko95] for more background.

(3.2.2) It is a fundamental fact that the log-canonical threshold of $(X, \mathcal{I})$ can be computed on a single log-resolution of $(X, \mathcal{I})$, i.e., a proper birational morphism $h: Y \to X$ such that Y is smooth, h is an isomorphism over the complement of $Z(\mathcal{I})$, and $Z(\mathcal{I}\mathcal{O}_Y)$ is a strict normal crossings divisor on Y. Namely, we have

$$\mathrm{lct}(X, \mathcal{I}) = \min\{\mathrm{wt}_{\mathcal{I}}(v)\}$$

where v runs over the divisorial valuations associated to the prime components of $Z(\mathcal{I}\mathcal{O}_Y)$. If this minimum is reached on a prime component E of $Z(\mathcal{I}\mathcal{O}_Y)$, then we say that E computes the log-canonical threshold of $(X, \mathcal{I})$. If we denote by $\mathcal{E}$ the union of such prime components E, then the Connectedness Theorem of Shokurov and Kollár [Ko95, 17.4] states that for every point x of $Z(\mathcal{I})$ and every sufficiently small open neighbourhood U of x in $Z(\mathcal{I})$, the topological space $h^{-1}(U) \cap \mathcal{E}$ is connected. This was the main source of inspiration for our theorem on the connectedness of the skeleton of a Calabi-Yau variety over $\mathbb{C}((t))$ (Theorem 3.5.6).

(3.2.3) In [BFJ08] and [JM11], a function closely related to the weight function $\mathrm{wt}_{\mathcal{I}}$ was extended from the set of divisorial valuations on X to the non-archimedean link of $Z(\mathcal{I})$ in X, that is, the analytic space over the field $\mathbb{C}$ with the trivial absolute value that we obtain by removing the generic fiber of the $\mathbb{C}$-variety $Z(\mathcal{I})$ from the generic fiber of the formal completion of X along $Z(\mathcal{I})$. We have made a similar construction to define weight functions on analytic spaces over discretely valued fields; this construction will be explained in Section 3.4.

3.3. Definition of the Kontsevich-Soibelman skeleton.

(3.3.1) Let X be a connected, smooth and proper K-variety of dimension n, and let ω be a non-zero m-pluricanonical form on X, that is, a non-zero element of $\omega_{X/K}^{\otimes m}(X)$. Let $\mathscr{X}$ be a regular R-model of X, E an irreducible component of $\mathscr{X}_k$ and x the divisorial point on X^{an} associated with $(\mathscr{X}, E)$. The relative canonical sheaf $\omega_{\mathscr{X}/R}$ is a line bundle on $\mathscr{X}$ that extends the canonical line bundle $\omega_{X/K}$ on X. The differential form ω on X defines a rational section of $\omega_{\mathscr{X}/R}^{\otimes m}$ and thus a

divisor $\operatorname{div}_{\mathscr{X}}(\omega)$ on $\mathscr{X}$. We denote by N the multiplicity of E in $\mathscr{X}_k$ and by ν the multiplicity of E in $\operatorname{div}_{\mathscr{X}}(\omega)$. We define the weight of ω at x by the formula

$$\operatorname{wt}_\omega(x) = (\nu + m)/N.$$

This definition only depends on x and not on the choice of $\mathscr{X}$ and E. We define the weight of X with respect to ω by

$$\operatorname{wt}_\omega(X) = \inf\{\operatorname{wt}_\omega(x) \mid x \in X^{\operatorname{div}}\} \in \mathbb{R} \cup \{-\infty\}.$$

(3.3.2) A divisorial point x on X^{an} is called ω-*essential* if the weight function wt_ω reaches its minimal value at x, that is,

$$\operatorname{wt}_\omega(X) = \operatorname{wt}_\omega(x).$$

The skeleton $\operatorname{Sk}(X, \omega)$ of the pair (X, ω) is defined as the closure of the set of ω-essential divisorial points in the space of birational points X^{bir}. It is obvious from the definition that $\operatorname{Sk}(X, \omega)$ is a birational invariant of the pair (X, ω), since the spaces X^{bir} and X^{div} and the weight function wt_ω are birational invariants.

3.4. Definition and properties of the weight function.

(3.4.1) Without suitable assumptions on the existence of resolutions of singularities, we cannot say much more about the skeleton $\operatorname{Sk}(X, \omega)$; for instance, we cannot prove that $\operatorname{Sk}(X, \omega)$ is non-empty. Therefore, we will assume from now on that k has characteristic zero or X is a curve. With the current state of affairs, these are the cases where resolution of singularities is known in the form that we need. In particular, it is known that every proper R-model of X can be dominated by a proper *sncd*-model of X.

(3.4.2) In Section 2.4 we have attached to each *sncd*-model $\mathscr{X}$ of X its Berkovich skeleton $\operatorname{Sk}(\mathscr{X})$. It was defined as the set of all birational points on X that are monomial with respect to the strict normal crossings divisor $\mathscr{X}_k$ on $\mathscr{X}$. We have shown in Proposition 2.4.6 that the skeleton $\operatorname{Sk}(\mathscr{X})$ is canonically homeomorphic to the dual complex $|\Delta(\mathscr{X}_k)|$ of the strict normal crossings divisor $\mathscr{X}_k$.

Theorem 3.4.3 (Proposition 4.4.5 in [MN13])**.** *There exists a unique smallest function*

$$\operatorname{wt}_\omega : X^{\operatorname{an}} \to \mathbb{R} \cup \{+\infty\}$$

with the following properties.

(1) *The function wt_ω is lower semi-continuous.*
(2) *Let $\mathscr{X}$ be an sncd-model for X and let x be a point of the Berkovich skeleton $\operatorname{Sk}(\mathscr{X})$. Let f be a rational function on $\mathscr{X}$ such that, locally at $\operatorname{sp}_{\mathscr{X}}(x)$, we have*

$$\operatorname{div}(f) = \operatorname{div}_{\mathscr{X}}(\omega) + m(\mathscr{X}_k)_{\operatorname{red}}.$$

Then

$$\operatorname{wt}_\omega(x) = -\ln|f(x)|.$$

In particular, wt_ω is continuous and piecewise affine on $\operatorname{Sk}(\mathscr{X})$, and we get the same value as in (3.3.1) on divisorial points. Moreover, for all x in $\widehat{\mathscr{X}_\eta}$, we have

$$\operatorname{wt}_\omega(x) \geq \operatorname{wt}_\omega(\rho_{\mathscr{X}}(x))$$

with equality if and only if $x \in \operatorname{Sk}(\mathscr{X})$.

(3) *The restriction of* wt_ω *to* X^{bir} *is a birational invariant of* (X, ω).

Proof. We only give a rough sketch of the arguments and refer to [MN13] for details. The formula in (2) can be used to extend the weight function wt_ω to the set X^{mon} of monomial points on X. Of course, each monomial point will belong to the Berkovich skeleta of several *sncd*-models, and one must show that the formula does not depend on the choice of an *sncd*-model. Next, one proves that the inequality in (2) holds for monomial points. When x is any point of X^{an}, one sets

$$\mathrm{wt}_\omega(x) = \sup_{\mathscr{X}} \{\mathrm{wt}_\omega(\rho_{\mathscr{X}})\}$$

where $\mathscr{X}$ runs through the set of proper *sncd*-models of X. Then one can prove that the resulting function wt_ω on X^{an} satisfies all the properties in the statement. $\square$

3.5. **Computation of the Kontsevich-Soibelman skeleton.**

(3.5.1) We can use the properties of the weight function in Theorem 3.4.3 to compute the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X, \omega)$ on a fixed proper *sncd*-model $\mathscr{X}$ of X. The divisorial points are dense in each face of $\mathrm{Sk}(\mathscr{X})$ (they are precisely the points with barycentric coordinates in $\mathbb{Q}$). Point (2) of the theorem immediately implies that $\mathrm{Sk}(X, \omega)$ is the subspace of the compact space $\mathrm{Sk}(\mathscr{X})$ consisting of the points where the continuous function $\mathrm{wt}_\omega|_{\mathrm{Sk}(\mathscr{X})}$ reaches its minimal value, because it says that the weight function is strictly increasing if we move away from the skeleton (recall that $\widehat{\mathscr{X}_\eta} = X^{\mathrm{an}}$ if $\mathscr{X}$ is proper over R). In particular, $\mathrm{Sk}(X, \omega)$ is a non-empty compact topological space. We can make this description much more explicit, as follows.

(3.5.2) We write $\mathscr{X}_k = \sum_{i \in I} N_i E_i$. For each $i \in I$, we denote by ν_i the multiplicity of E_i in the divisor $\mathrm{div}_{\mathscr{X}}(\omega)$. Recall that each face of $\mathrm{Sk}(\mathscr{X})$ corresponds to a connected component C of an intersection $E_J = \cap_{j \in J} E_j$ where J is a non-empty subset of I. We say that the face is ω-essential if

$$\frac{\nu_j + m}{N_j} = \min\left\{ \frac{\nu_i + m}{N_i} \,\Big|\, i \in I \right\}$$

for every j in J and C is not contained in the Zariski-closure in $\mathscr{X}$ of the pluricanonical divisor $\mathrm{div}_X(\omega)$ (the divisor of zeroes of ω on the K-variety X).

Theorem 3.5.3 (Theorem 4.5.5 in [MN13]). *The weight of X with respect to ω is given by*

$$\mathrm{wt}_\omega(X) = \min\left\{ \frac{\nu_i + m}{N_i} \,\Big|\, i \in I \right\}$$

and the skeleton $\mathrm{Sk}(X, \omega)$ *is the union of the ω-essential faces of* $\mathrm{Sk}(\mathscr{X})$. *In particular, this union only depends on X and ω, and not on the choice of the model $\mathscr{X}$.*

Proof. This follows easily from the properties of the weight function described in Theorem 3.4.3(2). We have already explained in (3.5.1) that $\mathrm{Sk}(X, \omega)$ is the locus of points in $\mathrm{Sk}(\mathscr{X})$ where wt_ω reaches its minimal value. Recall that for every $i \in I$, the value of wt_ω at the vertex of $\mathrm{Sk}(\mathscr{X})$ corresponding to E_i is given by $(\nu_i + m)/N_i$. The explicit formula for the weight function on $\mathrm{Sk}(\mathscr{X})$ implies that it is piecewise affine and concave on every face of $\mathrm{Sk}(\mathscr{X})$, and it is affine on a face

if and only if the corresponding subvariety C of $\mathscr{X}_k$ is not contained in the closure of $\mathrm{div}_X(\omega)$; see [MN13, 4.5.5] for details. $\square$

Example 3.5.4. Suppose that $R = \mathbb{C}[\![t]\!]$ and that X is a $K3$-surface over K. Assume that X has an *sncd*-model $\mathscr{X}$ such that $\mathscr{X}_k$ is reduced and $\omega_{\mathscr{X}/R}$ is trivial. Such models play an important role in the classification of semi-stable degenerations of $K3$-surfaces by Kulikov [Ku77] and Persson-Pinkham [PP81]. They have the special property that $\mathrm{Sk}(X,\omega) = \mathrm{Sk}(\mathscr{X})$ for every volume form ω on X, since all multiplicities N_i are equal to one, and all ν_i are equal by the triviality of $\omega_{\mathscr{X}/R}$.

(3.5.5) In [MN13] we proved the following variant of the Shokurov-Kollár Connectedness Theorem. Like the proofs of Shokurov and Kollár, our proof is based on vanishing theorems: we proved generalizations of Kawamata-Viehweg Vanishing and Kollár's Torsion-free Theorem for varieties over power series rings in characteristic zero by means of Greenberg approximation.

Theorem 3.5.6 (Theorem 5.3.3 in [MN13])**.** *Assume that the residue field k of K has characteristic zero. If X is a geometrically connected, smooth and proper K-variety of geometric genus one, and ω is a non-zero canonical form on X, then $Sk(X,\omega)$ is connected.*

One can say much more using advanced tools from the Minimal Model Program, as we will explain in the following section.

4. The essential skeleton and the Minimal Model Program

4.1. The essential skeleton.

(4.1.1) Throughout this section, we assume that the residue field k of K has characteristic zero. Let X be a connected, smooth and projective K-variety (the projectivity condition is needed to apply results from the Minimal Model Program). Let $\mathscr{X}$ be a proper *sncd*-model of X. We have seen in Theorem 3.5.3 that for every non-zero pluricanonical form ω on X, the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X,\omega)$ singles out certain faces of $\mathrm{Sk}(\mathscr{X})$ that do not depend on the model $\mathscr{X}$. In [MN13, §4.6] we defined the *essential skeleton* $\mathrm{Sk}(X)$ of X as the union of the Kontsevich-Soibelman skeleta $\mathrm{Sk}(X,\omega)$ for all non-zero pluricanonical forms ω on X.

(4.1.2) If $\omega_{X/K}$ is trivial and ω is a volume form on X, then it is not hard to see that $\mathrm{Sk}(X) = \mathrm{Sk}(X,\omega)$: multiplying ω with an element λ in K^* simply shifts the weight function by $v_K(\lambda)$, and for every $m > 0$, the space of m-pluricanonical forms on X is generated by the m-th tensor power of ω. It is not true for general X, however, that $\mathrm{Sk}(X) = \mathrm{Sk}(X,\omega)$ for some fixed pluricanonical form ω on X.

(4.1.3) Without suitable conditions on X, we cannot hope that $\mathrm{Sk}(X)$ is a strong deformation retract of X^{an}. For instance, if X is rational (e.g., a projective space $\mathbb{P}^n_K$) then all pluricanonical forms on X are zero and the essential skeleton is empty. However, Chenyang Xu and the author proved in [NX13] that $\mathrm{Sk}(X)$ is a strong deformation retract of X^{an} if X has "enough" pluricanonical forms. Our proof is based on the Minimal Model Program, and in particular on the results in [dFKX12]. We will now briefly explain the main ideas.

4.2. *dlt* **models.**

(4.2.1) If X is a curve of genus ≥ 1, then X has a unique minimal *sncd*-model $\mathscr{X}$, and thus a canonical Berkovich skeleton $\mathrm{Sk}(\mathscr{X})$. If X has dimension at least two, however, minimal *sncd*-models no longer exist in general. In order to get a good notion of minimal model, we have to enlarge the class of *sncd*-models to so-called (good) *dlt*-models. The abbreviation *dlt* stands for *divisorially log terminal*. The precise definition of a *dlt*-model $\mathscr{X}$ is quite technical; we do not give it here but refer to [NX13, §2.1] instead. The basic idea is that we allow certain mild singularities on $\mathscr{X}$, in accordance with the general philosophy of the Minimal Model Program. The set of points of $\mathscr{X}$ where $\mathscr{X}$ is regular and $\mathscr{X}_k$ is a strict normal crossings divisor is an open subscheme of $\mathscr{X}$ that we denote by $\mathscr{X}^{\mathrm{sncd}}$. The definition of a *dlt*-model guarantees that $\mathscr{X}^{\mathrm{sncd}}$ is still sufficiently large to capture all the important information about the skeleton; we set

$$\mathrm{Sk}(\mathscr{X}) := \mathrm{Sk}(\mathscr{X}^{\mathrm{sncd}}) \subset X^{\mathrm{bir}}.$$

(4.2.2) A *dlt*-model $\mathscr{X}$ of X is called minimal if the line bundle $\omega_{\mathscr{X}/R}((\mathscr{X}_k)_{\mathrm{red}})$ is semi-ample, which means that some power of this line bundle is generated by global sections. Fundamental theorems in birational geometry imply that X has a minimal *dlt*-model if and only if the canonical line bundle $\omega_{X/K}$ is semi-ample (we refer to [NX13, 2.2.6] for detailed references). To be precise, we should assume that X is defined over an algebraic curve because the necessary tools from the Minimal Model Program have only been developed under that assumption, but we will ignore this issue here; if $\omega_{X/K}$ is trivial one can get rid of the algebraicity condition by using tools from logarithmic geometry [NX13, §4.2].

(4.2.3) Minimal *dlt*-models are not unique, but they are closely related (birationally crepant) and the skeleton $\mathrm{Sk}(\mathscr{X})$ does not depend on the choice of the minimal *dlt*-model $\mathscr{X}$. One of the main results in [NX13] is the following.

Theorem 4.2.4 (Theorem 3.3.4 in [NX13]). *If $\omega_{X/K}$ is semi-ample and $\mathscr{X}$ is any minimal dlt-model of X, then the skeleton $\mathrm{Sk}(\mathscr{X})$ is equal to the essential skeleton $\mathrm{Sk}(X)$ of X.*

(4.2.5) If $\mathscr{X}'$ is any proper *sncd*-model of X, then the Minimal Model Program tells us how to modify $\mathscr{X}'$ into a minimal *dlt*-model $\mathscr{X}$ by a series of divisorial contractions and flips. The effect of the steps in the Minimal Model Program on the Berkovich skeleton $\mathrm{Sk}(\mathscr{X}')$ was carefully studied in [dFKX12], and these authors proved that $\mathrm{Sk}(\mathscr{X})$ can be obtained from $\mathrm{Sk}(\mathscr{X}')$ by means of a sequence of *elementary collapses*, combinatorial operations on simplicial complexes which are, in particular, strong deformation retracts. Since we already know that $\mathrm{Sk}(\mathscr{X}')$ is a strong deformation retract of X^{an} by Theorem 2.4.9, we obtain the following result.

Theorem 4.2.6 (Corollary 3.3.6 in [NX13]). *If $\omega_{X/K}$ is semi-ample then the essential skeleton $\mathrm{Sk}(X)$ is a strong deformation retract of X^{an}.*

(4.2.7) Under certain conditions on X, one can use further results from the Minimal Model Program to obtain information about the topological properties of $\mathrm{Sk}(X)$. For instance, if $\omega_{X/K}$ is trivial, the residue field k is algebraically closed and the skeleton $\mathrm{Sk}(X)$ has maximal dimension (that is, the same dimension as X) then results by Kollár and Kovács imply that $\mathrm{Sk}(X)$ is a closed pseudo-manifold (see [NX13, 4.1.7 and 4.2.4]).

(4.2.8) It would be quite interesting to have a proof of Theorem 4.2.6 that does not make use of the Minimal Model Program and the arguments in [dFKX12], but instead uses the properties of the weight function from Section 3.4 and the geometric structure of the Berkovich space X^{an}. A possible approach is the following. Assume that $\omega_{X/K}$ is trivial and let ω be a volume form on X. Then the essential skeleton $\mathrm{Sk}(X)$ of X is the locus where the weight function wt_ω on X^{an} reaches its minimal value, and we have seen in Theorem 3.4.3 that it is strictly increasing if one moves away from the Berkovich skeleton of any *sncd*-model of X. It is tempting to speculate that one can attach a gradient vector field on X^{an} to wt_ω that induces a flow that contracts X^{an} onto $\mathrm{Sk}(X)$. We refer to [NX14] for partial results in this direction and applications to the study of motivic zeta functions.

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