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The Bernstein Form of a Polynomial

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1. Introduction

Let P denote a polynomial with real coefficients. In this paper we devise an algorithm for determining upper and lower bounds for the set $\{P(x): 0 \leq x \leq 1\}$.

2. Some Preliminary Observations

For a polynomial P with real coefficients, it is possible, given $\epsilon > 0$, to compute, by means of arithmetical operations whose number can be readily determined, upper and lower bounds for $\{P(x): 0 \leq x \leq 1\}$ which differ from the corresponding sharp bounds by less than ϵ. This is an immediate consequence of the simple observation that, *if* $P(x) \equiv a_0 + a_1 x + \ldots + a_n x^n (n \geq 1)$ *is a polynomial with real coefficients, then, for each positive integer* m,

$$\min \{P(k/m): k=1, 2, \ldots, m\} - \frac{1}{m} \sum_{k=1}^{n} k|a_k|$$

$$\leq P(x) \leq \max \{P(k/m): k=1, 2, \ldots, m\}$$

$$+ \frac{1}{m} \sum_{k=1}^{n} k|a_k| (0 \leq x \leq 1),$$

which follows from the law of the mean and the inequality $|P'(x)| \leq \sum_{k=1}^{n} k|a_k| (0 \leq x \leq 1)$.

This procedure has the disadvantage that an extremely large number of arithmetical operations may be required to obtain moderately good bounds. In this paper we consider an algorithm which sacrifices accuracy in order to gain tractability.

3. The Bernstein Form of a Polynomial

THEOREM. *Let* $P(x) \equiv a_0 + a_1 x + \ldots + a_n x^n$ *be a polynomial of exact degree* n (≥ 0) *with real coefficients. Then*

$$\min \{b_k : k=0, 1, \ldots, n\} \leq P(x)$$

$$\leq \max \{b_k : k=0, 1, \ldots, n\} \quad (0 \leq x \leq 1) \qquad (1)$$

where

$$b_k = \sum_{r=0}^{k} a_r \binom{k}{r} \Big/ \binom{n}{r} \qquad (k=0, 1, \ldots, n). \qquad (2)$$

The upper (lower) bound is sharp if and only if it is equal to b_0 *or to* b_n.

Preliminary remark. It is obvious from (2) that the bounds given by (1) are always at least as good as the crude bounds $\pm \sum_{k=0}^{n} |a_k|$.

PROOF: The following representation (to be established below, but essentially due to S. N. Bernstein[1]) holds:

$$P(x) \equiv \sum_{k=0}^{n} b_k \binom{n}{k} x^k (1-x)^{n-k} \qquad (3)$$

where b_k $(k=0, 1, \ldots, n)$ is given by (2).[2] Since $\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \equiv 1$ and $\binom{n}{k} x^k (1-x)^{n-k} \geq 0$ $(0 \leq x$

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[1] Bernstein, S. N., On the best approximation of continuous functions by polynomials of a given degree (Russian), Communications of the Kharʼkov Mathematical Society, Series 2, **13**, 49–194 (1912).
[2] One easily sees that $(x^k(1-x)^{n-k})_{k=0}^{n}$ forms a basis for the space of all polynomials $\sum_{k=0}^{n} c_k x^k$, c_k real. Consequently, a representation (3) with real b_k's uniquely determined must hold.

$\leqslant 1;\ k=0, 1, \ldots, n$), (3) implies (1). We refer to $P(x)$ written in the form $\sum_{k=0}^{n} b_k \binom{n}{k} x^k(1-x)^{n-k}$ as the "Bernstein form of $P(x)$." The last sum is the Bernstein polynomial of order n of every real function f defined on $[0, 1]$ for which $f(k/n)=b_k$ $(k=0, 1, \ldots, n)$.[3]

If $\max\{b_k : k=0, 1, \ldots, n\}$ is equal to b_0 or to b_n, then the upper bound given by (1) is sharp since $P(0)=b_0$ and $P(1)=b_n$. Conversely, if the upper bound given by (1) is sharp, then it is equal to b_0 or to b_n since, if $0<x<1$, either $b_0=b_1=\ldots=b_n$ or $\sum_{k=0}^{n} b_k \binom{n}{k} x^k(1-x)^{n-k} < \max\{b_k : k=0, 1, \ldots, n\}$. The sharpness of the lower bound can be treated in an analogous fashion.

In order to establish (3), we observe that

$$\sum_{r=0}^{n} a_r x^r \equiv \sum_{r=0}^{n} a_r x^r\{x+(1-x)\}^{n-r}$$

$$\equiv \sum_{r=0}^{n}\sum_{s=0}^{n-r} a_r \binom{n-r}{s} x^{n-s}(1-x)^s$$

$$\equiv \sum_{s=0}^{n}\sum_{r=0}^{n-s} a_r \binom{n-r}{s} x^{n-s}(1-x)^s$$

$$\equiv \sum_{k=0}^{n}\left\{\sum_{r=0}^{k} a_r \binom{n-r}{n-k}\Big/\binom{n}{k}\right\}\binom{n}{k} x^k(1-x)^{n-k}$$

$$\equiv \sum_{k=0}^{n}\left\{\sum_{r=0}^{k} a_r \binom{k}{r}\Big/\binom{n}{r}\right\}\binom{n}{k} x^k(1-x)^{n-k}.$$

This completes the proof of the theorem.

Next, we seek a more efficient method than the direct use of formula (2) for computing $b_k (k=0, 1, \ldots, n)$. To this end, we note (footnote 3, p. 12) that, if $P(x)\equiv \sum_{r=0}^{n} b_r \binom{n}{r} x^r(1-x)^{n-r} (n\geqslant 1)$, then, as a simple calculation shows,

$$P'(x)\equiv n\sum_{r=0}^{n-1}\{\Delta b_r\}\binom{n-1}{r} x^r(1-x)^{n-1-r}.$$

Repeating this process, we obtain

$$P^{(k)}(x)\equiv n(n-1)\ldots(n-k+1)\sum_{r=0}^{n-k}\{\Delta^k b_r\}$$
$$\binom{n-k}{r} x^r(1-x)^{n-k-r} (k=1, 2, \ldots, n). \quad (4)$$

Since $a_k=P^{(k)}(0)/k!$ $(k=1, 2, \ldots, n)$ if $P(x)\equiv a_0+a_1x+\ldots+a_nx^n$, (4) yields

$$a_k=\binom{n}{k}\Delta^k b_0 \qquad (k=1, 2, \ldots, n). \quad (5)$$

Equation (5) suggests a synthetic method for determining the Bernstein form of a polynomial $P(x)\equiv a_0+a_1x+\ldots+a_nx^n$ (a_r real, $n\geqslant 1$, $a_n\neq 0$) without using (2) explicitly. One simply writes the numbers $\Delta^k b_0 = a_k\Big/\binom{n}{k}$ $(k=0, 1, \ldots, n)$ in their usual positions in a difference table (see table 1) and then completes the table to obtain $b_0, b_1, \ldots, b_n$. In view of (4), the completed difference table also yields bounds for $\{P^{(k)}(x): 0\leqslant x\leqslant 1\}$ $(k=1, 2, \ldots, n)$.

For example, if $P(x)\equiv 1+10x-20x^2+30x^3+15x^4-x^5$, we start with the incomplete difference table (table 2) and complete it to obtain table 3, which yields $b_0=1$, $b_1=3$, $b_2=3$, $b_3=4$, $b_4=12$, and $b_5=35$. Table 3 also yields the bounds 0 and 115 for P', -40 and 300 for P'', 180 and 480 for P''', and 240 and 360 for $P^{(4)}$ on [0, 1].

TABLE 1

b_0	b_1	b_2	$\cdots$	b_{n-2}	b_{n-1}	b_n
Δb_0	Δb_1		$\cdots$	Δb_{n-2}	Δb_{n-1}	
$\Delta^2 b_0$			$\cdots$	$\Delta^2 b_{n-2}$		
.				.		
.				.		
.				.		
$\Delta^n b_0$						

TABLE 2

1	*	*	*	*	*
2	*	*	*	*	
−2	*	*	*		
3	*	*			
3	*				
−1					

TABLE 3

1	3	3	4	12	35
2	0	1	8	23	
−2	1	7	15		
3	6	8			
3	2				
−1					

4. Conclusion

We conclude with some miscellaneous remarks.

Formula (2) follows from (5) and the Newton interpolation formula.

One can obviously consider the "Bernstein form" $\sum_{k=0}^{n} b_k\binom{n}{k}x^k(1-x)^{n-k}$ of a polynomial P with complex

[3] Lorentz, G. G., Bernstein Polynomials, University of Toronto Press, Toronto, 1953.

coefficients. It is clear that P maps [0, 1] into the convex hull of the set $\{b_0, b_1, \ldots, b_n\}$.

In view of (5), it is easy to state simple sufficient conditions for a polynomial $P(x) \equiv a_0 + a_1 x + \ldots + a_n x^n \equiv \sum_{r=0}^{n} b_r \binom{n}{r} x^r (1-x)^{n-r}$ with real coefficients to have certain properties. For example, if $n \geq 1$, $a_0 > 0$, and $ka_k + (n-k+1)a_{k-1} > 0 (k=1, 2, \ldots, n)$, then $P(x) > 0$ whenever $0 \leq x \leq 1$. Indeed, $ka_k + (n-k+1)a_{k-1} > 0$ is equivalent to $\Delta^k b_0 + \Delta^{k-1} b_0 = a_k \Big/ \binom{n}{k} + a_{k-1} \Big/ \binom{n}{k-1} > 0$, and, consequently, all the entries in table 1 are positive except possibly $\Delta^k b_0 (k=1, 2, \ldots, n)$.

Let n be a positive integer. An analog of (1) obviously holds if, for each $k (k=0, \ldots, n)$, one takes instead of $\binom{n}{k} x^k (1-x)^{n-k}$ a polynomial $q_k(x) \equiv \sum_{r=0}^{n} c_r^{(k)} x^r$ with real coefficients such that (a) $q_k(x) \geq 0$ if $0 \leq x \leq 1$, (b) $q_0, q_1, \ldots, q_n$ are linearly independent, and (c) $\sum_{k=0}^{n} q_k(x) \equiv 1$. It is easy to verify that there are many such sequences $(q_k(x))_{k=0}^{n}$ of polynomials. For example, for each α, $0 < \alpha \leq 1$, the polynomials $P_k(x) \equiv \binom{n}{k} (\alpha x)^k (1 - \alpha x)^{n-k}$ $(k=0, 1, \ldots, n)$ form such a sequence. Moreover, if $q_0, q_1, \ldots, q_n$ form such a sequence, then so do $Q_0, Q_1, \ldots, Q_n$ where

$$Q_k(x) \equiv \frac{1}{x} \int_0^x q_k(t) dt \qquad (k=0, 1, \ldots, n).$$

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