Bernstein-type operators on a triangle with one curved side

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Abstract. We construct Bernstein-type operators on a triangle with one curved side. We study univariate operators, their product and Boolean sum, as well as their interpolation properties, the order of accuracy (degree of exactness, precision set) and the remainder of the corresponding approximation formulas. We also give some illustrative examples.

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Keywords. Bernstein operator, product and boolean sum operators, modulus of continuity, degree of exactness, error evaluation.

1. Introduction

Starting with the paper [2] of R.E. Barnhill, G. Birkhoff and W.J. Gordon, there have been constructed interpolation operators of Lagrange, Hermite and Birkhoff type, that interpolate the values of a given function or the values of the function and of certain of its derivatives on the boundary of a triangle with straight sides (see, e.g., [3], [10], [11], [14], [15], [21]). In order to match all the boundary information on a curved domain (as in Dirichlet, Neumann or Robin boundary conditions for differential equation problems) there were considered interpolation operators on a triangle with curved sides (see, e.g., [16], [17]).

Since the Bernstein-type operators interpolate a given function at the endpoints of the interval, these operators can also be used as interpolation operators both on triangles with straight sides (see, e.g., [8], [25], [26]) and with curved sides. The aim of this paper is to construct Bernstein-type operators, and also their product and boolean sum, (see, e.g., [18], [19], [20]), for a triangle with one curved side and to study such operators especially from the theoretical point of view. We study here only the local problem and not consider the global problem of assembling the curved elements in a triangulation of a domain with curved boundaries, as there are, for example, in [6],

[7], [13], [12]. Using modulus of continuity, respectively Peano's theorem we also study the remainders of the corresponding approximation formulas.

As in the case of a triangle with straight sides, by affine invariance, it is sufficient to consider the standard triangle $\widetilde{T}_h$ with vertices $V_1 = (0, h)$, $V_2 = (h, 0)$ and $V_3 = (0, 0)$, with two straight sides Γ_1, Γ_2, along the coordinate axes, and with the third edge Γ_3 (opposite to the vertex V_3) defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e., $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = h$, for $h > 0$. Also, we have $f(x) \leq h$ and $g(y) \leq h$, for $x, y \in [0, h]$. The functions f and g are defined as in [3]. In the sequel we denote by $e_{ij}(x, y) = x^i y^j$, for $i, j \in \mathbb{N}$.

2. Univariate operators

Let F be a real-valued function defined on $\widetilde{T}_h$ and $(0, y)$, $(g(y), y)$, respectively, $(x,\ 0)$, $(x, f(x))$ be the points in which the parallel lines to the coordinate axes, passing through the point $(x, y) \in \widetilde{T}_h$, intersect the sides Γ_i, $i = 1, 2, 3$. (See Figure 1.)

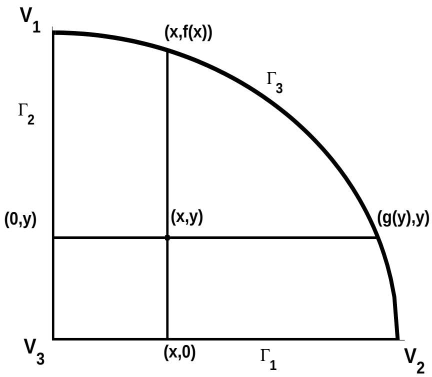


Figure 1. Triangle $\widetilde{T}_h$.

One considers the Bernstein-type operators B_m^x and B_n^y defined by

$$(B_m^x F)(x, y) = \sum_{i=0}^{m} p_{m,i}(x, y)\, F\left(\frac{i}{m} g(y), y\right),$$

with

$$p_{m,i}(x, y) = \binom{m}{i} \left(\frac{x}{g(y)}\right)^i \left(1 - \frac{x}{g(y)}\right)^{m-i}, \quad 0 \leqslant x + y \leqslant g(y),$$

respectively,

$$(B_n^y F)(x, y) = \sum_{j=0}^{n} q_{n,j}(x, y)\, F\left(x, \tfrac{j}{n} f(x)\right)$$

with

$$q_{n,j}(x, y) = \binom{n}{j} \left(\frac{y}{f(x)}\right)^j \left(1 - \frac{y}{f(x)}\right)^{n-j}, \quad 0 \leq x + y \leq f(x),$$

where

$$\Delta_m^x = \left\{ \frac{i}{m} g(y) \middle| \; i = \overline{0,m} \right\} \text{ and } \Delta_n^y = \left\{ \tfrac{j}{n} f(x) \middle| \; j = \overline{0,n} \right\}$$

are uniform partitions of the intervals $[0, g(y)]$ and $[0, f(x)]$.

Theorem 2.1. *If F is a real-valued function defined on $\widetilde{T}_h$ then:*

(i) $B_m^x F = F$ *on* $\Gamma_2 \cup \Gamma_3$,
(ii) $B_n^y F = F$ *on* $\Gamma_1 \cup \Gamma_3$,
and
(iii) $(B_m^x e_{ij})(x,y) = x^i y^j, \quad i = 0,1;\ j \in \mathbb{N}$,
$(B_m^x e_{2j})(x,y) = \left[x^2 + \frac{x(g(y)-x)}{m} \right] y^j, \quad j \in \mathbb{N}$,
(iv) $(B_n^y e_{ij})(x,y) = x^i y^j, \quad i \in \mathbb{N},\ j = 0,1$,
$(B_n^y e_{i2})(x,y) = x^i \left[y^2 + \frac{y(f(x)-y)}{n} \right], \quad i \in \mathbb{N}$.

Proof. The interpolation properties (i) and (ii) follow from the relations:

$$p_{m,i}(0,y) = \begin{cases} 1, & \text{for } i = 0, \\ 0, & \text{for } i > 0, \end{cases}$$

and

$$p_{m,i}(g(y),y) = \begin{cases} 0, & \text{for } i < m, \\ 1, & \text{for } i = m, \end{cases}$$

respectively by

$$q_{n,j}(x,0) = \begin{cases} 1, & \text{for } j = 0, \\ 0, & \text{for } j > 0, \end{cases}$$

and

$$q_{n,j}(x,f(x)) = \begin{cases} 0, & \text{for } j < n, \\ 1, & \text{for } j = n. \end{cases}$$

Regarding the properties (iii), we have

$$(B_m^x e_{ij})(x,y) = y^j (B_m^x e_{i0})(x,y), \qquad j \in \mathbb{N}$$

and

$$(B_m^x e_{00})(x,y) = \left(\frac{x}{g(y)} + 1 - \frac{x}{g(y)} \right)^m = 1,$$

$$\begin{aligned} B_m^x e_{10}(x,y) &= \sum_{i=0}^{m} \binom{m}{i} \left(\frac{x}{g(y)} \right)^i \left(1 - \frac{x}{g(y)} \right)^{m-i} \frac{i}{m} g(y) \\ &= x \sum_{i=0}^{m-1} \binom{m-1}{i} \left(\frac{x}{g(y)} \right)^i \left(1 - \frac{x}{g(y)} \right)^{m-1-i} \\ &= x \left(\frac{x}{g(y)} + 1 - \frac{x}{g(x)} \right)^{m-1} = x, \end{aligned}$$

$$
\begin{aligned}
B_m^x e_{20}(x,y) &= \sum_{i=0}^{m} \binom{m}{i} \left(\frac{x}{g(y)}\right)^i \left(1-\frac{x}{g(y)}\right)^{m-i} i^2 \left(\frac{g(y)}{m}\right)^2 \\
&= \left(\frac{g(y)}{m}\right)^2 \sum_{i=0}^{m} \binom{m}{i} i(i-1) \left(\frac{x}{g(y)}\right)^i \left(1-\frac{x}{g(y)}\right)^{m-i} + x\frac{g(y)}{m} \\
&= \frac{m-1}{m}x^2 + x\frac{g(y)}{m} = x^2 + \frac{x[g(y)-x]}{m}.
\end{aligned}
$$

Properties (iv) are proved in the same way. □

Now, let us consider the approximation formula

$$F = B_m^x F + R_m^x F.$$

Theorem 2.2. *If $F(\cdot,y) \in C[0,g(y)]$ then*

$$\left|\,(R_m^x F)(x,y)\,\right| \le \left(1 + \tfrac{h}{2\delta\sqrt{m}}\right) \omega(F(\cdot,y);\delta), \quad y \in [0,h],$$

where $\omega(F(\cdot,y);\delta)$ is the modulus of continuity of the function F with regard to the variable x.

Moreover, if $\delta = 1/\sqrt{m}$ then

$$\left|\,(R_m^x F)\,(x,y)\right| \le \left(1 + \tfrac{h}{2}\right) \omega(F(\cdot,y);\tfrac{1}{\sqrt{m}}), \quad y \in [0,h]. \tag{2.1}$$

Proof. From the property $(B_m^x e_{00})(x,y) = 1$, it follows that

$$\left|\,(R_m^x F)\,(x,y)\right| \le \sum_{i=0}^{m} p_{m,i}(x,y) \left| F(x,y) - F(\frac{i}{m}g(y),y) \right|.$$

Using the inequality

$$\left| F(x,y) - F(\frac{i}{m}g(y),y) \right| \le \left(\tfrac{1}{\delta} \left| x - \frac{i}{m}g(y) \right| + 1 \right) \omega(F(\cdot,y);\delta)$$

one obtains

$$
\begin{aligned}
|(R_m^x F)(x,y)| &\le \sum_{i=0}^{m} p_{m,i}(x,y) \left(\tfrac{1}{\delta} \left| x - \frac{i}{m}g(y) \right| + 1 \right) \omega(F(\cdot,y);\delta) \\
&\le \left[1 + \frac{1}{\delta} \Big(\sum_{i=0}^{m} p_{m,i}(x,y) \big(x - \frac{i}{m}g(y)\big)^2 \Big)^{1/2} \right] \omega(F(\cdot,y);\delta) \\
&= \left[1 + \frac{1}{\delta} \sqrt{\tfrac{x(g(y)-x)}{m}} \right] \omega(F(\cdot,y);\delta).
\end{aligned}
$$

Since,

$$\max_{0 \le x \le g(y)} \big[x(g(y)-x)\big] = \tfrac{g^2(y)}{4} \text{ and } \max_{0 \le y \le h} g^2(y) = h^2,$$

it follows that

$$\max_{\widetilde{T}_h} \big[x(g(y)-x)\big] = \frac{h^2}{4},$$

hence

$$\left|(R_m^x F)(x,y)\right| \le \left(1 + \frac{h}{2\delta\sqrt{m}}\right) \omega(F(\cdot,y);\delta).$$

Now, for $\delta = 1/\sqrt{m}$, one obtains (2.1). $\square$

Theorem 2.3. *If $F(\cdot, y) \in C^2[0,h]$ then*

$$(R_m^x F)(x,y) = \frac{x[x-g(y)]}{2m} F^{(2,0)}(\xi, y), \quad \text{for } \xi \in [0, g(y)]$$

and

$$\left|(R_m^x F)(x,y)\right| \leq \frac{h^2}{8m} M_{20}F, \ \ (x,y) \in \widetilde{T}_h,$$

where

$$M_{ij}F = \max_{\widetilde{T}_h} \left| F^{(i,j)}(x,y) \right|.$$

Proof. Taking into account that the degree of exactness of the operator B_m^x is 1, i.e., $\mathrm{dex}(B_m^x) = 1$, by Peano's theorem, it follows

$$(R_m^x F)(x,y) = \int_0^{g(y)} K_{20}(x,y;s) F^{(2,0)}(s,y) ds,$$

where

$$K_{20}(x,y;s) = (x-s)_+ - \sum_{i=0}^{m} p_{m,i}(x,y) \big(\frac{i}{m} g(y) - s \big)_+ .$$

For a given $\nu \in \{1, ..., m\}$ one denotes by $K_{20}^\nu(x,y;\cdot)$ the restriction of the kernel $K_{20}(x,y;\cdot)$ to the interval $\left[(\nu-1)\frac{g(y)}{m}, \nu\frac{g(y)}{m}\right]$, i.e.,

$$K_{20}^\nu(x,y;\nu) = (x-s)_+ - \sum_{i=\nu}^{m} p_{m,i}(x,y) \left(\frac{i}{m} g(y) - s \right),$$

whence,

$$K_{20}^\nu(x,y;s) = \begin{cases} x - s - \sum\limits_{i=\nu}^{m} p_{m,i}(x,y) \left(\frac{i}{m} g(y) - s \right), & s < x \\ -\sum\limits_{i=\nu}^{m} p_{m,i}(x,y) \left(\frac{i}{m} g(y) - s \right), & s \geq x. \end{cases}$$

It follows that $K_{20}^\nu(x,y;s) \leq 0$, for $s \geq x$. For $s < x$ we have

$$K_{20}^\nu(x,y;s) = x - s - \sum_{i=0}^{m} p_{m,i}(x,y) \left(\frac{i}{m} g(y) - s \right) + \sum_{i=0}^{\nu-1} p_{m,i}(x,y) \left(\frac{i}{m} g(y) - s \right).$$

As,

$$\sum_{i=0}^{m} p_{m,i}(x,y) \left(\frac{i}{m} g(y) - s \right) = x - s,$$

it follows that

$$K_{20}^\nu(x,y;s) = \sum_{i=0}^{\nu-1} p_{m,i}(x,y) \left(\frac{i}{m} g(y) - s \right) \leq 0.$$

So, $K_{20}^\nu(x,y;\cdot) \leq 0$ for any $\nu \in \{1, ..., m\}$, i.e., $K_{20}(x,y;s) \leq 0$, for $s \in [0, g(y)]$.

By mean value theorem, one obtains

$$(R_m^x F)(x,y) = F^{(2,0)}(\xi,y)\int_0^{g(y)} K_{20}(x,y;s)ds, \quad 0 \le \xi \le g(y).$$

Since,

$$\int_0^{g(y)} K_{20}(x,y;s)ds = \frac{x(x-g(y))}{2m}$$

and

$$\max_{0\le x\le g(y)} \frac{|x(x-g(y))|}{2m} = \frac{g^2(y)}{8m} \le \frac{h^2}{8m}, \quad y\in[0,h]$$

the conclusion follows. □

Remark 2.4. *Analogous results are obtained for the remainder of the formula*

$$F = B_n^y F + R_n^y F,$$

i.e.,

$$\left|(R_n^y F)(x,y)\right| \le \left(1+\tfrac{h}{2\delta\sqrt{n}}\right)\omega(F(x,\cdot);\delta), \qquad F(x,\cdot)\in C[0,f(x)]$$

and

$$(R_n^y F)(x,y) \le \left(1+\tfrac{h}{2}\right)\omega\left(F(x,\cdot);\tfrac{1}{\sqrt{n}}\right)$$

respectively,

$$(R_n^y F)(x,y) = \frac{y[y-f(x)]}{2n}F^{(0,2)}(x,\eta), \quad \eta\in[0,f(x)]$$

and

$$\left|(R_n^y F)(x,y)\right| \le \frac{h^2}{8n}M_{02}F, \qquad (x,y)\in\widetilde{T}_h.$$

3. Product operator

Let $P_{mn} = B_m^x B_n^y$, respectively, $Q_{nm} = B_n^y B_m^x$ be the products of the operators B_m^x and B_n^y.

We have

$$(P_{mn}F)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)\,q_{n,j}\left(\tfrac{i}{m}g(y),y\right)F\left(\tfrac{i}{m}g(y),\tfrac{j}{n}f\left(\tfrac{i}{m}g(y)\right)\right)$$

and

$$(Q_{nm}F)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}\left(x,\tfrac{j}{n}f(x)\right)q_{n,j}(x,y)\,F\left(\tfrac{i}{m}g\left(\tfrac{j}{n}f(x)\right),\tfrac{j}{n}f(x)\right).$$

Remark 3.1. *The nodes of the operator P_{mn}, respectively, Q_{nm} are given in Figure 2.*

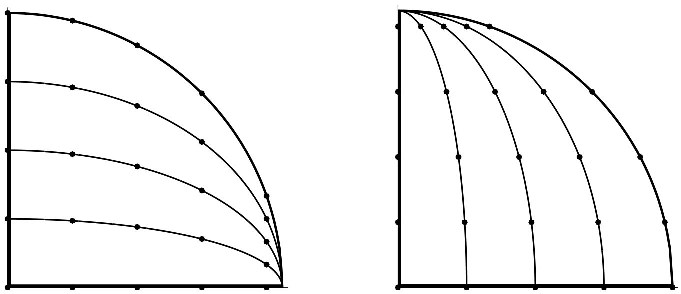


Figure 2. The nodes for P_{mn} and Q_{nm}, for $m = n = 4$.

Theorem 3.2. *If F is a real-valued function defined on $\widetilde{T}_h$ then:*

(i) $(P_{mn}F)(V_3) = F(V_3)$,
$P_{mn}F = F$, *on* Γ_3
and

(ii) $(Q_{nm}F)(V_3) = F(V_3)$,
$Q_{nm}F = F$, *on* Γ_3.

Proof. The proof follows from the properties

$$\begin{aligned}
&(P_{mn}F)(x,0) = (B_m^x F)(x,0),\\
&(P_{mn}F)(0,y) = (B_n^y F)(0,y),\\
&(P_{mn}F)(x,f(x)) = F(x,f(x)), \qquad x,y \in [0,h]
\end{aligned}$$

and

$$\begin{aligned}
&(Q_{nm}F)(x,0) = (B_m^x F)(x,0),\\
&(Q_{nm}F)(0,y) = (B_n^y F)(0,y),\\
&(Q_{nm}F)(g(y),y) = F(g(y),y), \qquad x,y \in [0,h],
\end{aligned}$$

which can be verified by a straightforward computation.

For example, the property $(P_{mn}F)(x,0) = (B_m^y F)(x,0)$ implies $(P_{mn}F)(0,0) = F(0,0)$. □

Remark 3.3. *The product operators P_{mn} and Q_{nm} interpolate the function F at the vertex $(0,0)$ and on the side $y = f(x)$ (or $x = g(y)$).*

Let us consider now the approximation formula

$$F = P_{mn}F + R_{mn}^P F,$$

where R_{mn}^P is the corresponding remainder operator.

Theorem 3.4. *If $F \in C(\widetilde{T}_h)$ then*

$$\left| \left(R_{mn}^P F\right)(x,y)\right| \le (1+h)\omega\left(F; \tfrac{1}{\sqrt{m}}, \tfrac{1}{\sqrt{n}}\right), \quad (x,y) \in \widetilde{T}_h$$

Proof. We have

$$\begin{aligned}\left|(R^P_{mn}F)(x,y)\right| \le \Bigg[\frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|x-\tfrac{i}{m}g(y)\right| \\ +\frac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|y-\tfrac{j}{n}f\left(\tfrac{i}{m}g(y)\right)\right| \\ +\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\Bigg]\omega(F;\delta_1,\delta_2).\end{aligned}$$

Since,

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|x-\tfrac{i}{m}g(y)\right| \le \sqrt{\frac{x(g(y)-x)}{m}},$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|y-\tfrac{j}{n}f\left(\tfrac{i}{m}g(y)\right)\right| \le \sqrt{\frac{y(f(x)-y)}{n}},$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right) = 1,$$

it follows that

$$\left|(R^P_{mn}F)(x,y)\right| \le \left(1+\frac{1}{\delta_1}\sqrt{\frac{x(g(y)-x)}{m}}+\frac{1}{\delta_2}\sqrt{\frac{y(f(x)-y)}{n}}\right)\omega(F;\delta_1,\delta_2).$$

But

$$x\big(g(y)-x\big) \le \frac{h^2}{4} \text{ and } y\big(f(x)-y\big) \le \frac{h^2}{4},$$

whence,

$$\left|(R^P_{mn}F)(x,y)\right| \le \left(1+\frac{1}{\delta_1}\frac{h}{2\sqrt{m}}+\frac{1}{\delta_2}\frac{h}{2\sqrt{n}}\right)\omega(F;\delta_1,\delta_2)$$

and

$$\left|(R^P_{mn}F)(x,y)\right| \le (1+h)\,\omega\left(F;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right).$$

□

4. Boolean sum operators

Finally, we consider the Boolean sums of the operators B^x_m and B^y_n, i.e.,

$$S_{mn} := B^x_m \oplus B^y_n = B^x_m + B^y_n - B^x_m B^y_n,$$

respectively,

$$T_{nm} := B^y_n \oplus B^x_m = B^y_n + B^x_m - B^y_n B^x_m.$$

Theorem 4.1. *If F is a real-valued function defined on $\widetilde{T}_h$ then*

$$S_{mn}F\left|_{\partial\widetilde{T}} = F\right|_{\partial\,\widetilde{T}}$$

and

$$T_{nm}F\left|_{\partial\widetilde{T}} = F\right|_{\partial\widetilde{T}}.$$

Proof. As,

$$\begin{aligned}&\left(P_{mn}F\right)(x,0) = \left(B_m^xF\right)(x,0),\\ &\left(P_{mn}F\right)(0,y) = \left(B_n^yF\right)(0,y),\\ &\left(B_m^xF\right)(x,h-x) = \left(B_n^yF\right)(x,h-x) = (P_{mn}F)(x,h-x) = F(x,h-x)\end{aligned}$$

the conclusion follows. □

For the remainder of the Boolean sum approximation formula,

$$F = S_{mn}F + R_{mn}^SF,$$

we have the following result.

Theorem 4.2. *If $F \in C(\widetilde{T}_h)$ then*

$$\begin{aligned}\left|(R_{mn}^SF)(x,y)\right| \leq & (1+\tfrac{h}{2})\omega(F(\cdot,y);\tfrac{1}{\sqrt{m}}) + (1+\tfrac{h}{2})\omega(F(x,\cdot);\tfrac{1}{\sqrt{n}})\\ & + (1+h)\omega(F;\tfrac{1}{\sqrt{m}},\tfrac{1}{\sqrt{n}}),\ (x,y)\in\widetilde{T}_h.\end{aligned} \tag{4.1}$$

Proof. The identity

$$F - S_{mn}F = F - B_m^xF + F - B_n^yF - (F - P_{mn}F)$$

implies that

$$\left|(R_{mn}^SF)(x,y)\right| \leq \left|\left(R_m^xF\right)(x,y)\right| + \left|\left(R_n^yF(x,y)\right|\right. + \left|(R_{mn}^PF)(x,y)\right|$$

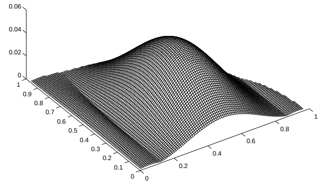
and the conclusion follows. □

5. Numerical examples

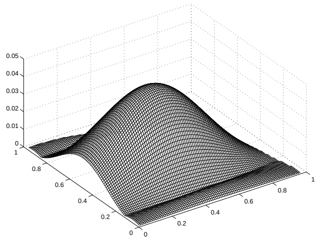
Example 5.1. *We consider the following test functions, generally used in the literature, (see, e.g., [22]):*

$$\begin{aligned}F_1(x,y) &= \tfrac{1}{3}\exp[-\tfrac{81}{16}((x-0.5)^2+(y-0.5)^2)],\\ F_2(x,y) &= \frac{1.25+\cos 5.4y}{6+6(3x-1)^2}.\end{aligned} \tag{5.1}$$

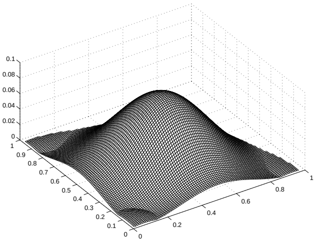
In Figure 3 and Figure 4 we plot the graphs of the maximum errors for approximating by $B_m^xF_i,\ B_n^yF_i, P_{mn}F_i,\ S_{mn}F_i,\ i=\overline{1,2}$, on $\tilde{T}_h$, considering $h=1, m=5,\ n=6$ and $f:[0,1]\to[0,1],\ f(x)=\sqrt{1-x^2}$.



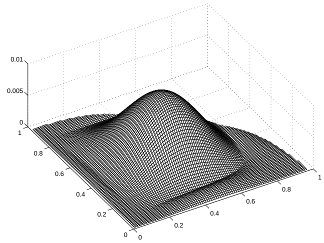
The approximation error for $B_m^x F_1$.



The approximation error for $B_n^y F_1$.

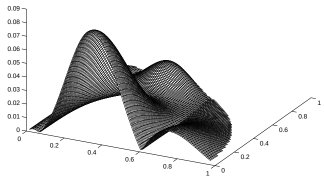


The approximation error for $P_{mn} F_1$.

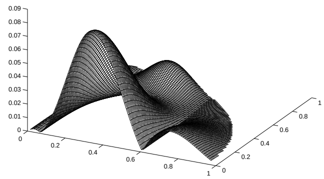


The approximation error for $S_{mn} F_1$.

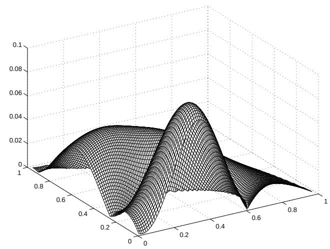
Figure 3. The maximum approximation errors for F_1.



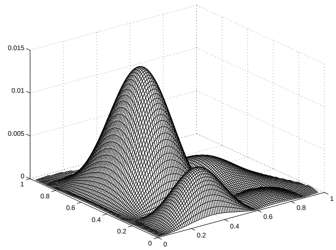
The approximation error for $B_m^x F_2$.



The approximation error for $B_n^y F_2$.



The approximation error for $P_{mn} F_2$.



The approximation error $S_{mn} F_2$.

Figure 4. The maximum approximation errors for F_2.

Table 1 contains the maximum approximation errors for the functions given in (5.1) for the Bernstein type operators and for some operators obtained in [16], namely Lagrange-type operators

$$(L_1F)(x,y)=\frac{g(y)-x}{g(y)}F(0,y)+\frac{x}{g(y)}F(g(y),y),$$

$$(P_{13}F)(x,y)=\frac{g(y)-x}{g(y)}F(0,y)+\frac{x}{g(y)[y+g(y)]}[g(y)F(y+g(y),0)+yF(0,y+g(y))],$$

$$\begin{aligned}(S_{12}F)(x,y)=&\frac{g(y)-x}{g(y)}F(0,y)+\frac{f(x)-y}{f(x)}F(x,0)+\frac{y}{f(x)}F(x,f(x))\\&-\frac{g(y)-x}{g(y)}\left[\frac{h-y}{h}F(0,0)+\frac{y}{h}F(0,h)\right],\end{aligned}$$

Hermite-type operator

$$(H_1F)(x,y)=\frac{[x-g(y)]^2}{g^2(y)}F(0,y)+\frac{x[2g(y)-x]}{g^2(y)}F(g(y),y)+\frac{x[x-g(y)]}{g(y)}F^{(1,0)}(g(y),y),$$

and Birkhoff-type operator

$$(B_1F)(x,y)=F(0,y)+xF^{(1,0)}(g(y),y).$$

Max error	F_1	F_2
B_m^xF	0.0525	0.0821
B_n^yF	0.0452	0.0692
$P_{mn}F$	0.0858	0.0943
$Q_{nm}F$	0.0857	0.0944
$S_{mn}F$	0.0095	0.0144
$T_{nm}F$	0.0095	0.0112
L_1F	0.2097	0.2259
$P_{13}F$	0.2943	0.2261
$S_{12}F$	0.1718	0.1809
H_1F	0.0758	0.2210
B_1F	0.6235	0.5302

Table 1. The approximation errors.

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