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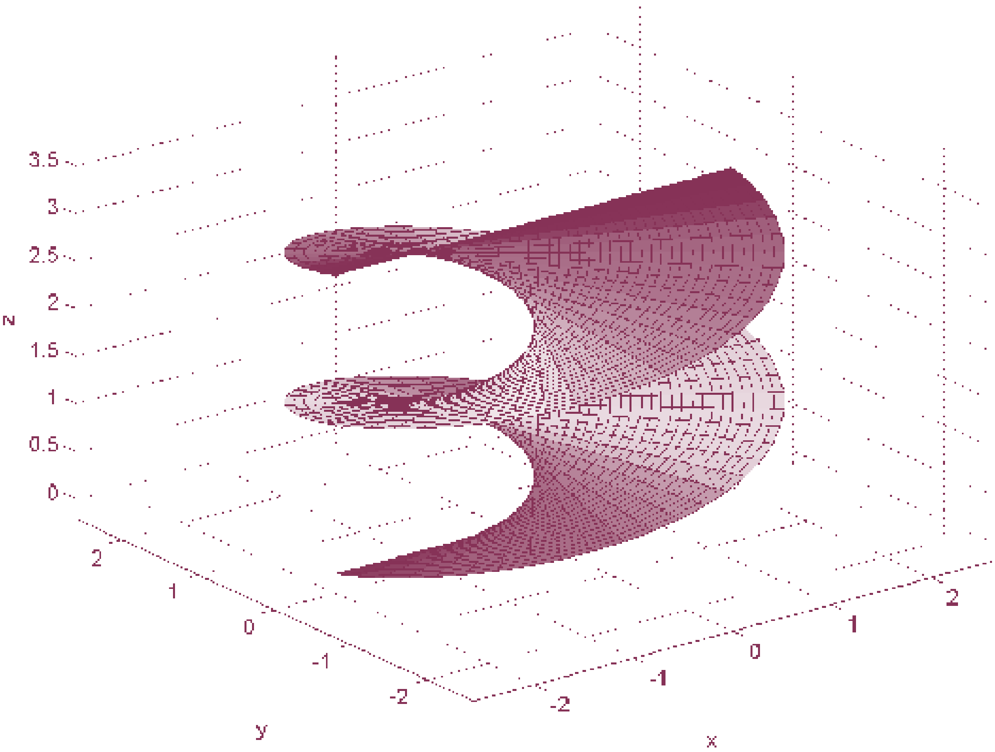

MATHEMATICA

4/2009

Anul LIV **2009**

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SUMAR - CONTENTS - SOMMAIRE

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

BERNSTEIN-TYPE OPERATORS ON TETRAHEDRONS

PETRU BLAGA, TEODORA CĂTINAŞ, AND GHEORGHE COMAN

Abstract. The aim of the paper is to construct Bernstein-type operators on tetrahedron with all straight edges and on tetrahedron with three curved edges defined by some given functions. We study the interpolation properties, the approximation accuracy (degree of exactness, precision set) and the remainder of the corresponding approximation formulas. The accuracy is also illustrated by numerical examples.

1. **Introduction**

In some previous papers were constructed and applied some interpolation operators on triangle with one curved edge respectively on tetrahedron with straight edges ([1, 6, 7, 8, 9, 12]), as well as Bernstein-type operators on triangle with all straight edges, respectively on triangle with one curved edge ([4, 5]). There were studied the interpolation properties and the accuracy of these operators respectively the remainders of the corresponding approximation formulas.

The order of an approximation operator P is given by the degree of exactness $(\mathrm{dex}(P))$ and by the precision set $(\mathrm{pres}(P))$. Remind that $\mathrm{dex}\,(P) = r$ if $Pf = f$ for all $f \in \mathcal{P}_r^n$ and there exists $g \in \mathcal{P}_{r+1}^n$ such that $Pg \neq g$, where $\mathcal{P}_r^n$ denotes the space of polynomials in n variables of global degree at most r. The precision set of an approximation operator is the set of all monomials for which the approximation is exact [2].

Received by the editors: 01.09.2009.

2000 *Mathematics Subject Classification.* 41A35,41A36,41A80.

Key words and phrases. Bernstein operator, product operator, Boolean sum operator, approximation accuracy, tetrahedron.

The goal of this paper is to study Bernstein-type operators on tetrahedrons with straight edges respectively with three curved edges given by functions.

2. **Bernstein-type operators on tetrahedrons with straight edges**

By affine invariance it is sufficient to consider only the standard tetrahedron $\mathcal{T}_h$ with vertices $V_0 = (0,0,0)$, $V_1 = (h,0,0)$, $V_2 = (0,h,0)$ and $V_3 = (0,0,h)$, with three edges τ_1, τ_2, τ_3 along the coordinate axes and with the edges Γ_1, Γ_2, Γ_3 (opposite to the vertex V_0). Also, one denotes by σ_{012}, σ_{013}, σ_{023} and σ_{123} the tetrahedron faces from the planes $V_0V_1V_2$, $V_0V_1V_3$, $V_0V_2V_3$ and $V_1V_2V_3$ respectively (see the left side of Figure 1).

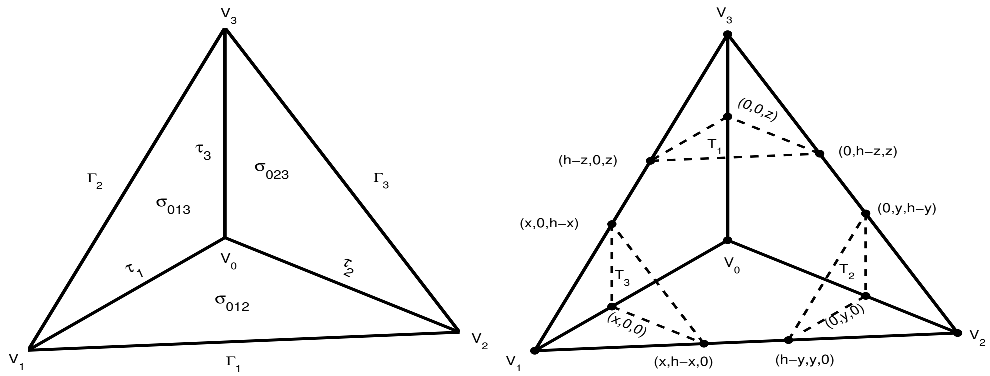

FIGURE 1. Tetrahedron with straight edges

Let Π_i, $i = 1,2,3$, be the parallel planes to the tetrahedron faces that intersect the tetrahedron edges in three points and T_i, $i = 1,2,3$, be the triangles in which the planes Π_i, $i = 1,2,3$, intersect the tetrahedron faces respectively (see the right side of Figure 1).

2.1. **Univariate operators.** On each triangle one defines two Bernstein-type operators.

Remark 1. *We shall study, in detail, only the Bernstein-type operators on the triangle T_1. For the triangles T_2 and T_3 there are obtained analogous results.*

Let us consider the triangle T_1 (see Figure 2).

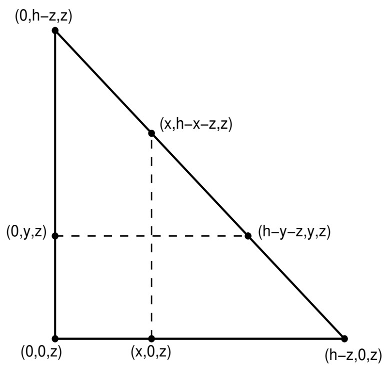

FIGURE 2. Triangle T_1

For the uniform partitions

$$\Delta_m^x = \left\{ \left(i\frac{h-y-z}{m}, y, z \right) \;\middle|\; i = \overline{0,m} \right\}$$

and

$$\Delta_n^y = \left\{ \left(x, j\frac{h-x-z}{n}, z \right) \;\middle|\; j = \overline{0,n} \right\},$$

of the intervals $[(0,y,z), (h-y-z,y,z)]$ $[(x,0,z), (x,h-x-z,z)]$ respectively, one considers the Bernstein-type operators B_m^{xy} and B_n^{yx} defined by

$$\left(B_m^{xy}F\right)(x,y,z) = \sum_{i=0}^{m} p_{m,i}(x,y,z)\, F\left(i\frac{h-y-z}{m}, y, z \right)$$

with

$$p_{m,i}(x,y,z) = \binom{m}{i} \left(\frac{x}{h-y-z} \right)^i \left(1 - \frac{x}{h-y-z} \right)^{m-i}$$

and

$$\left(B_n^{yx}F\right)(x,y,z) = \sum_{j=0}^{n} q_{n,j}(x,y,z)\, F\left(x, j\frac{h-x-z}{n}, z \right)$$

with

$$q_{n,j}(x,y,z) = \binom{n}{j}\left(\frac{y}{h-x-z}\right)^j\left(1-\frac{y}{h-x-z}\right)^{n-j},$$

where F is a real-valued function defined on $\mathcal{T}_h$.

Theorem 2.1. *If $F\colon \mathcal{T}_h \to \mathbb{R}$ then:*

(i) $B_m^{xy}F = F$ *on* $\sigma_{023}\cup\sigma_{123}$,

$B_n^{yx}F = F$ *on* $\sigma_{013}\cup\sigma_{123}$;

(ii) $\operatorname{dex}(B_m^{xy}) = \operatorname{dex}(B_n^{yx}) = 1$;

(iii) $\operatorname{pres}(B_m^{xy}) = \left\{x^i y^j z^k \;\middle|\; i=0,1; j,k\in\mathbb{N}\right\}$,

$\operatorname{pres}(B_n^{yx}) = \left\{x^i y^j z^k \;\middle|\; j=0,1; i,k\in\mathbb{N}\right\}$;

(iv) $(B_m^{xy}e_{2jk})(x,y,z) = \left[x^2 + \dfrac{x(h-x-y-z)}{m}\right]y^j z^k$,

$(B_n^{yx}e_{i2k})(x,y,z) = \left[y^2 + \dfrac{y(h-x-y-z)}{n}\right]x^i z^k, \quad i,j,k\in\mathbb{N}.$

Proof. The relations

$$\begin{aligned} p_{m,i}(0,y,z) &= \begin{cases} 1, & \text{for } i=0,\\ 0, & \text{for } i>0; \end{cases}\\ p_{m,i}(h-y-z,y,z) &= \begin{cases} 1, & \text{for } i=m,\\ 0, & \text{for } i<m; \end{cases} \end{aligned} \tag{1}$$

respectively

$$\begin{aligned} q_{n,j}(x,0,z) &= \begin{cases} 1, & \text{for } j=0,\\ 0, & \text{for } j>0; \end{cases}\\ q_{n,j}(x,h-x-z,z) &= \begin{cases} 1, & \text{for } j=n,\\ 0, & \text{for } j<n; \end{cases} \end{aligned} \tag{2}$$

imply that

$$(B_m^{xy}F)(0,y,z) = F(0,y,z),$$

$$(B_m^{xy}F)(h-y-z,y,z) = F(h-y-z,y,z),$$

and

$$\begin{aligned}
&\left(B_n^{yx}F\right)(x,0,y) = F\,(x,0,z)\,,\\
&\left(B_n^{yx}F\right)(x,h-x-z,z) = F\,(x,h-x-z,z)\,,
\end{aligned}$$

i.e., the interpolation properties (i). Regarding the approximation accuracy, we have

$$\begin{aligned}
\left(B_m^{xy}e_{000}\right)(x,y,z) &= \sum_{i=0}^{m} p_{m,i}\,(x,y,z) = 1,\\
\left(B_m^{xy}e_{100}\right)(x,y,z) &= \sum_{i=0}^{m} \binom{m}{i}\left(\frac{x}{h-y-z}\right)^{i}\left(1-\frac{x}{h-y-z}\right)^{m-i}\frac{i\,(h-y-z)}{m}\\
&= x\sum_{i=0}^{m-1}\binom{m-1}{i}\left(\frac{x}{h-y-z}\right)^{i}\left(1-\frac{x}{h-y-z}\right)^{m-1-i} = x,\\
\left(B_m^{xy}e_{200}\right)(x,y,z) &= \sum_{i=0}^{m}\binom{m}{i}\left(\frac{x}{h-y-z}\right)^{i}\left(1-\frac{x}{h-y-z}\right)^{m-i} i^2\left(\frac{h-y-z}{m}\right)^2\\
= \left(\frac{h-y-z}{m}\right)^2 &\sum_{i=2}^{m}\binom{m}{i}\left(\frac{x}{h-y-z}\right)^{i}\left(1-\frac{x}{h-y-z}\right)^{m-i} + \frac{x\,(h-y-z)}{m}\\
&= \frac{m-1}{m}x^2 + \frac{x\,(h-y-z)}{m} = x^2 + \frac{x\,(h-x-y-z)}{m},\\
\left(B_m^{xy}e_{ijk}\right)(x,y,z) &= y^j z^k\left(B_m^{xy}e_{i00}\right)(x,y,z)\,,\quad i=0,1,2,\ j,k\in\mathbb{N},
\end{aligned}$$

respectively

$$\begin{aligned}
&\left(B_n^{yx}e_{000}\right)(x,y,z) = \sum_{j=0}^{n} q_{n,j}\,(x,y,z) = 1,\\
&\left(B_n^{yx}e_{010}\right)(x,y,z) = y,\\
&\left(B_n^{yx}e_{i2k}\right)(x,y,z) = y^2 + \frac{y\,(h-x-y-z)}{n},\\
&\left(B_n^{yx}e_{ijk}\right)(x,y,z) = x^i z^k\left(B_n^{yx}e_{0j0}\right)(x,y,z)\,,\quad j=0,1,2,\ i,k\in\mathbb{N},
\end{aligned}$$

that are proved in the same way, which imply (ii)-(iv). □

Let

$$F = B_m^{xy}F + R_m^{xy}F$$

be the approximation formula generated by the operator B_m^{xy}.

Theorem 2.2. *If $F(\cdot,y,z)\in C[0,h-y-z]$ then*

$$\left|(R_m^{xy}F)(x,y,z)\right|\leqslant\left(1+\frac{h}{2\delta\sqrt{m}}\right)\omega\left(F(\cdot,y,z);\delta\right),\quad y+z\leqslant h,$$

where $\omega(F(\cdot,y,z);\delta)$ is the modulus of continuity of the function F with regard to the variable x.

Moreover, if $\delta=1/\sqrt{m}$ then

$$\left|(R_m^{xy}F)(x,y,z)\right|\leqslant\left(1+\frac{h}{2}\right)\omega\left(F(\cdot,y,z);\frac{1}{\sqrt{m}}\right).\tag{3}$$

Proof. We have

$$\begin{aligned}\left|(R_m^{xy}F)(x,y,z)\right|&\leqslant\sum_{i=0}^{m}p_{m,i}(x,y,z)\left|F(x,y,z)-F\left(i\frac{h-y-z}{m},y,z\right)\right|\\&\leqslant\sum_{i=0}^{m}p_{m,i}(x,y,z)\left(\frac{1}{\delta}\left|x-i\frac{h-y-z}{m}\right|+1\right)\omega\left(F(\cdot,y,z);\delta\right)\\&\leqslant\left[1+\frac{1}{\delta}\left(\sum_{i=0}^{m}p_{m,i}(x,y,z)\left(x-i\frac{h-y-z}{m}\right)^2\right)^{1/2}\right]\omega\left(F(\cdot,y,z);\delta\right)\\&\leqslant\left[1+\frac{1}{\delta}\sqrt{\frac{x(h-x-y-z)}{m}}\right]\omega\left(F(\cdot,y,z);\delta\right).\end{aligned}$$

Since,

$$\max_{T_1}\Big[x(h-x-y-z)\Big]\leqslant\frac{h^2}{4},\quad z\in[0,h],\tag{4}$$

we have

$$\left|(R_m^{xy}F)(x,y,z)\right|\leqslant\left(1+\frac{h}{2\delta\sqrt{m}}\right)\omega\left(F(\cdot,y,z);\delta\right)$$

respectively (for $\delta=1/\sqrt{m}$)

$$\left|(R_m^{xy}F)(x,y,z)\right|\leqslant\left(1+\frac{h}{2}\right)\omega\left(F(\cdot,y,z);\frac{1}{\sqrt{m}}\right).\qquad\square$$

We also have

$$\left|(R_n^{yx}F)(x,y,z)\right|\leqslant\left(1+\frac{h}{2}\right)\omega\left(F(x,\cdot,z);\frac{1}{\sqrt{n}}\right).\tag{5}$$

Theorem 2.3. *If $F(\cdot,y,z)\in C^2[0,h]$ then*

$$(R_m^{xy}F)(x,y,z)=-\frac{x(h-x-y-z)}{2m}F^{(2,0,0)}(\xi,y,z),$$

for $0 \leqslant \xi \leqslant h - y - z;\ y, z \in [0, h]$, *and*

$$\left|\left(R_m^{xy} F\right)(x, y, z)\right| \leqslant \frac{h^2}{8m} M_{200} F, \tag{6}$$

where

$$M_{ijk} F = \max_{\mathcal{T}_h} \left| F^{(i,j,k)}(x, y, z) \right|.$$

Proof. Since $\operatorname{dex}\left(B_m^{xy}\right) = 1$, by Peano's kernel theorem, follows that

$$\left(R_m^{xy} F\right)(x, y, z) = \int_0^{h-y-z} K_{200}(x, y, z; s)\, F^{(2,0,0)}(s, y, z)\, ds,$$

where the kernel

$$K_{200}(x, y, z; s) = (x - s)_+ - \sum_{i=0}^{m} p_{m,i}(x, y, z) \left(i \frac{h - y - z}{m} - s \right)_+$$

does not change the sign $\left(K_{200}(x, y, z; s) \leqslant 0,\ \ s \in [0, h - y - z]\right)$. By mean value theorem, one obtains

$$\begin{aligned} \left(R_m^{xy} F\right)(x, y, z) &= F^{(2,0,0)}(\xi, y, z) \int_0^{h-y-z} K_{200}(x, y, z; s)\, ds \\ &= -\frac{x\,(h-x-y-z)}{2m} F^{(2,0,0)}(\xi, y, z),\ \ 0 \leqslant \xi \leqslant h - y - z. \end{aligned}$$

Now, the inequality of (4) implies (6). □

Remark 2. *On the same way it is proved the evaluations of the remainder in the formula*

$$F = B_n^{yx} F + R_n^{yx} F$$

i.e., for $F(x, \cdot, z) \in C[0, h - x - z]$

$$\left|\left(R_n^{yx} F\right)(x, y, z)\right| \leqslant \left(1 + \frac{h}{2}\right) \omega \left(F(x, \cdot, z); \frac{1}{\sqrt{n}} \right) \tag{7}$$

respectively, for $F(x, \cdot, z) \in C^2[0, h]$

$$\left|\left(R_n^{yx} F\right)(x, y, z)\right| \leqslant \frac{h^2}{8n} M_{020} F \tag{8}$$

on $\mathcal{T}_h$.

2.2. **Product operators.** Let $P^1_{mn} = B^{xy}_m B^{yx}_n$ and $Q^1_{nm} = B^{yx}_n B^{xy}_m$ be the products of the operators B^{xy}_m and B^{yx}_n.

We have

$$\left(P^1_{mn}F\right)(x,y,z) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y,z)\, q_{n,j}\left(i\frac{h-y-z}{m}, y, z\right) \times$$
$$\times F\left(i\frac{h-y-z}{m}, j\frac{(m-i)(h-z)+iy}{mn}, z\right),$$

respectively

$$\left(Q^1_{nm}F\right)(x,y,z) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}\left(x, j\frac{h-x-z}{n}, z\right) q_{n,j}(x,y,z) \times$$
$$\times F\left(i\frac{(n-j)(h-z)+jx}{mn}, j\frac{h-x-z}{n}, z\right).$$

Theorem 2.4. *If F is a real-valued function defined on $\mathcal{T}_h$ then*

$$P^1_{mn}F = F \tag{9}$$

and

$$Q^1_{nm}F = F \tag{10}$$

on $\tau_3 \cup \sigma_{123}$.

Proof. Taking into account (1) and (2), one obtains

$$\left(P^1_{mn}F\right)(0,0,z) = F(0,0,z),$$
$$\left(P^1_{mn}F\right)(h-y-z,y,z) = F(h-y-z,y,z),$$

respectively

$$\left(Q^1_{nm}F\right)(0,0,z) = F(0,0,z),$$
$$\left(Q^1_{nm}F\right)(h-y-z,y,z) = F(h-y-z,y,z),$$

for all $y,z \in [0,h]$. □

For the approximation error of the operators P^1_{mn} and Q^1_{nm}, we have the following theorem.

Theorem 2.5. *If* $F(\cdot,\cdot,z) \in C([0,h]\times[0,h])$ *then*

$$\left|\left(F - P^1_{mn}F\right)(x,y,z)\right| \leqslant (1+h)\,\omega\left(F(\cdot,\cdot,z);\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right) \tag{11}$$

and

$$\left|\left(F - Q^1_{nm}F\right)(x,y,z)\right| \leqslant (1+h)\,\omega\left(F(\cdot,\cdot,z);\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right) \tag{12}$$

on $\mathcal{T}_h$.

Proof. We have

$$\begin{aligned}
\left|\left(F - P^1_{mn}F\right)(x,y,z)\right| \leqslant & \left[\frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y,z)\, q_{n,j}\left(i\frac{h-y-z}{m},y,z\right)\times\right.\\
&\qquad\times\left|x - i\frac{h-y-z}{m}\right|\\
&+\frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y,z)\, q_{n,j}\left(i\frac{h-y-z}{m},y,z\right)\times\\
&\qquad\times\left|y - j\frac{(m-i)(h-z)+iy}{mn}\right|\\
&\left.+\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y,z)\, q_{n,j}\left(i\frac{h-y-z}{m},y,z\right)\right]\omega\left(F(\cdot,\cdot,z);\delta_1,\delta_2\right)\\
\leqslant & \left(\frac{1}{\delta_1}\sqrt{\frac{x(h-x-y-z)}{m}}+\frac{1}{\delta_2}\sqrt{\frac{y(h-x-y-z)}{n}}+1\right)\omega\left(F(\cdot,\cdot,z);\delta_1,\delta_2\right).
\end{aligned}$$

As,

$$\begin{aligned}
x(h-x-y-z) &\leqslant \frac{(h-y-z)^2}{4} \quad \text{on} \quad [0,h-y-z],\\
y(h-x-y-z) &\leqslant \frac{(h-x-z)^2}{4} \quad \text{on} \quad [0,h-x-z],
\end{aligned}$$

one obtains

$$\begin{aligned}
\left|\left(F - P^1_{mn}F\right)(x,y,z)\right| &\leqslant \left(\frac{1}{\delta_1}\frac{h-y-z}{2\sqrt{m}}+\frac{1}{\delta_2}\frac{h-x-z}{2\sqrt{n}}+1\right)\omega\left(F(\cdot,\cdot,z);\delta_1,\delta_2\right)\\
&\leqslant \left(\frac{1}{\delta_1}\frac{h}{2\sqrt{m}}+\frac{1}{\delta_2}\frac{h}{2\sqrt{n}}+1\right)\omega\left(F(\cdot,\cdot,z);\delta_1,\delta_2\right).
\end{aligned}$$

Now, for $\delta_1 = 1/\sqrt{m}$ and $\delta_2 = 1/\sqrt{n}$, one obtains (11). The inequality (12) is proved in the same way. $\square$

2.3. **Boolean sum operators.** Let

$$S^1_{mn} := B^{xy}_m \oplus B^{yx}_n = B^{xy}_m + B^{yx}_n - B^{xy}_m B^{yx}_n \tag{13}$$

and

$$T^1_{nm} := B^{yx}_n \oplus B^{xy}_m = B^{yx}_n + B^{xy}_m - B^{yx}_n B^{xy}_m \tag{14}$$

be the Boolean sums of the operators B^{xy}_m and B^{yx}_n.

Theorem 2.6. *If F is a real-valued function defined on $\mathcal{T}_h$ then*

$$S^1_{mn}F = F \quad \textit{and} \quad T^1_{nm}F = F$$

on $\sigma_{013} \cup \sigma_{023} \cup \sigma_{123}$.

Proof. We have:

$$\left(B^{xy}_m F\right)(0,y,z) = F(0,y,z), \quad \left(P^1_{mn}F\right)(0,y,z) = \left(B^{yx}_n F\right)(0,y,z)$$

which imply that

$$S^1_{mn}F = F \quad \text{on} \quad \sigma_{023};$$

$$\left(B^{yx}_n F\right)(x,0,z) = F(x,0,z), \quad \left(P^1_{mn}F\right)(x,0,z) = \left(B^{xy}_m F\right)(x,0,z)$$

which imply that

$$S^1_{mn}F = F \quad \text{on} \quad \sigma_{013};$$

and

$$B^{xy}_m F = F, \quad B^{yx}_n F = F, \quad P^1_{mn}F = F, \quad \text{on} \quad \sigma_{123}$$

which imply that

$$S^1_{mn}F = F \quad \text{on} \quad \sigma_{123}.$$

Analogously, it is proved that $T^1_{nm}F = F$ on $\sigma_{013} \cup \sigma_{023} \cup \sigma_{123}$. $\square$

Theorem 2.7. *If $F \in C(\mathcal{T}_h)$ then*

$$\begin{aligned}\left|\left(F - S^1_{mn}F\right)(x,y,z)\right| \leqslant \left(1+\frac{h}{2}\right)\omega\left(F(\cdot,y,z);\frac{1}{\sqrt{m}}\right) + \\ + \left(1+\frac{h}{2}\right)\omega\left(F(x,\cdot,z);\frac{1}{\sqrt{n}}\right) + (1+h)\,\omega\left(F(\cdot,\cdot,z);\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\end{aligned}$$

on $\mathcal{T}_h$.

Proof. From the identity

$$F - S_{mn}^1 F = (F - B_m^{xy} F) + (F - B_n^{yx} F) - \big(F - P_{mn}^1 F\big)$$

one obtains

$$\begin{aligned}\left|\big(F - S_{mn}^1 F\big)(x,y,z)\right| \leqslant{}& \left|(R_m^{xy} F)(x,y,z)\right| + \left|(R_n^{yx} F)(x,y,z)\right| \\ &+ \left|\big(F - P_{mn}^1 F\big)(x,y,z)\right|\end{aligned}$$

and from (3), (5), (11), the proof follows. □

Remark 3. *The same inequality is obtained for the error* $\big(F - T_{nm}^1 F\big)(x,y,z)$ *using instead of* (11) *the inequality* (12).

3. Bernstein-type operators on tetrahedrons with three curved edges

One considers, also, the standard tetrahedron $\mathcal{T}_h$ with vertices $V_0 = (0,0,0)$, $V_1 = (h,0,0)$, $V_2 = (0,h,0)$ and $V_3 = (0,0,h)$, with three straight edges τ_1, τ_2, τ_3 along the coordinate axes and with three curved edges γ_1, γ_2, γ_3 (opposite to the vertex V_0), defined, respectively, by the one-to-one functions f_i and g_i, where g_i is the inverse of the function f_i, $i = 1,2,3$. Also, one denotes by s_{012}, s_{013}, s_{023} and the tetrahedron faces from the planes $V_0V_1V_2$, $V_0V_1V_3$, $V_0V_2V_3$ and $V_1V_2V_3$ respectively, by s_{123} the curved faced (opposite to the vertex V_0) (see the left side of Figure 3) and by t_i, $i = 1,2,3$, the triangles with one curved edge in which the planes Π_i, $i = 1,2,3$, intersect the faces of the tetrahedron $\mathcal{T}_h$, respectively (see left side of Figure 3).

Next, one considers the particular case when the face s_{123} is on the sphere $x^2 + y^2 + z^2 = h^2$, i.e., $f_i(u) = \sqrt{h^2 - u^2}$ and $g_i(v) = \sqrt{h^2 - v^2}$, $i = 1,2,3$ (see right side of Figure 3)

3.1. Univariate operators.

One each triangle t_i, $i = 1,2,3$, one defines two Bernstein-type operators.

We discuss here only on the triangle t_1 (Figure 4).

We have

$$(B_m^{xy} F)(x,y,z) = \sum_{i=0}^{m} p_{m,i}(x,y,z)\, F\left(i\frac{\sqrt{h^2 - y^2 - z^2}}{m}, y, z\right)$$

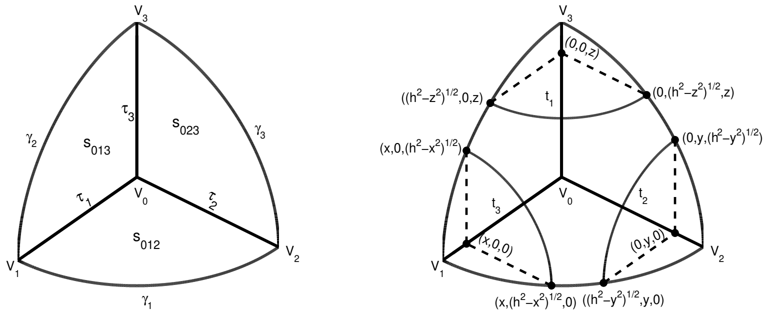

FIGURE 3. Tetrahedron with three curved edges

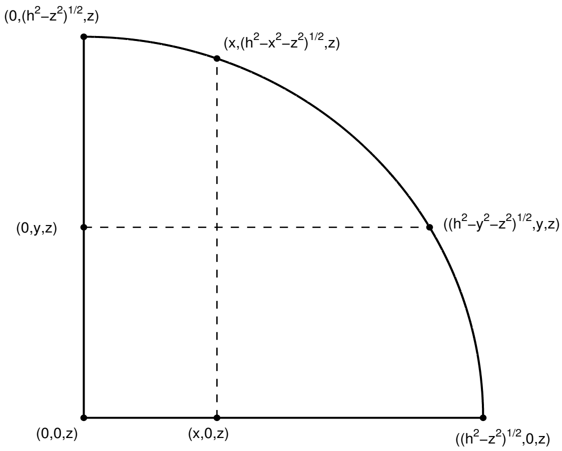

FIGURE 4. Triangle t_1

and

$$\left(B_n^{yx}F\right)(x,y,z)=\sum_{j=0}^{n} q_{n,j}\,(x,y,z)\,F\left(x,j\frac{\sqrt{h^2-x^2-z^2}}{n},z\right)$$

with

$$p_{m,i}\,(x,y,z)=\binom{m}{i}\left(\frac{x}{\sqrt{h^2-y^2-z^2}}\right)^{i}\left(1-\frac{x}{\sqrt{h^2-y^2-z^2}}\right)^{m-i},$$

respectively

$$q_{n,j}(x,y,z)=\binom{n}{j}\left(\frac{y}{\sqrt{h^2-x^2-z^2}}\right)^j\left(1-\frac{x}{\sqrt{h^2-x^2-z^2}}\right)^{n-j},$$

where F is a real-valued function defined on $\mathcal{T}_h$.

Following the way used in the Section 2 one can prove the corresponding theorems:

Theorem 3.1. *If $F\colon \mathcal{T}_h\to\mathbb{R}$ then:*

(i') $B_m^{xy}F=F$ *on* $s_{023}\cup s_{123}$,
$B_n^{yx}F=F$ *on* $s_{013}\cup s_{123}$;

(ii') $\operatorname{dex}(B_m^{xy})=\operatorname{dex}(B_n^{yx})=1$;

(iii') $\operatorname{pres}(B_m^{xy})=\left\{x^iy^jz^k \mid i=0,1;j,k\in\mathbb{N}\right\}$,
$\operatorname{pres}(B_n^{yx})=\left\{x^iy^jz^k \mid j=0,1;i,k\in\mathbb{N}\right\}$;

(iv') $(B_m^{xy}e_{2jk})(x,y,z)=\left[x^2+\dfrac{x\left(\sqrt{h^2-y^2-z^2}-x\right)}{m}\right]x^iz^k,$
$(B_n^{yx}e_{i2k})(x,y,z)=\left[y^2+\dfrac{y\left(\sqrt{h^2-x^2-z^2}-y\right)}{n}\right]x^iz^k,\ i,j,k\in\mathbb{N}.$

Let

$$F=B_m^{xy}F+R_m^{xy}F$$

be the approximation formula generated by the operator B_m^{xy}.

Theorem 3.2. *If $F(\cdot,y,z)\in C\left[0,\sqrt{h^2-y^2-z^2}\right]$ then*

$$\left|(R_m^{xy}F)(x,y,z)\right|\leqslant\left(1+\frac{h}{2\delta\sqrt{m}}\right)\omega\left(F(\cdot,y,z);\delta\right),\quad y+z\leqslant h,$$

respectively

$$\left|(R_m^{xy}F)(x,y,z)\right|\leqslant\left(1+\frac{h}{2}\right)\omega\left(F(\cdot,y,z);\frac{1}{\sqrt{m}}\right).$$

Theorem 3.3. *If $F(\cdot,y,z)\in C^2[0,h]$ then*

$$\left|(R_m^{xy}F)(x,y,z)\right|=-\frac{x\left(\sqrt{h^2-y^2-z^2}-x\right)}{2m}F^{(2,0,0)}(\xi,y,z),$$

for $0\leqslant\xi\leqslant\sqrt{h^2-y^2-z^2}$, $y,z\in[0,h]$, and

$$\left|(R_m^{xy}F)(x,y,z)\right|\leqslant\frac{h^2}{8m}M_{200}F.$$

Remark 4. *Analogous results take place for the remainder in the approximation formula*

$$F = B_n^{yx} F + R_n^{yx} F.$$

3.2. **Product operators.** Let $P_{mn} = B_m^{xy} B_n^{yx}$ and $Q_{nm} = B_n^{yx} B_m^{xy}$ be the products of the operators B_m^{xy} and B_n^{yx}, i.e.,

$$\begin{aligned}(P_{mn}F)(x,y,z) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y,z)\, q_{n,j}\left(i\frac{\sqrt{h^2-y^2-z^2}}{m}, y, z\right) \times \\ \times F\left(i\frac{\sqrt{h^2-y^2-z^2}}{m}, j\frac{\sqrt{(m^2-i^2)(h^2-z^2)+i^2y^2}}{mn}, z\right),\end{aligned}$$

respectively

$$\begin{aligned}(Q_{nm}F)(x,y,z) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}\left(x, j\frac{\sqrt{h^2-x^2-z^2}}{n}, z\right) q_{n,j}(x,y,z) \times \\ \times F\left(i\frac{\sqrt{(n^2-j^2)(h^2-z^2)+j^2x^2}}{mn}, j\frac{\sqrt{h^2-x^2-z^2}}{n}, z\right).\end{aligned}$$

Theorem 3.4. *If* $F\colon \mathcal{T}_h \to \mathbb{R}$ *then*

$$P_{mn}F = F \quad \textit{and} \quad Q_{nm}F = F \quad \textit{on} \quad \tau_3 \cup s_{123}.$$

Theorem 3.5. *If* $F(\cdot,\cdot,z) \in C([0,h]\times[0,h])$, *then*

$$\left|(F - P_{mn}F)(x,y,z)\right| \leqslant (1+h)\,\omega\left(F(\cdot,\cdot,z); \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right)$$

and

$$\left|(F - Q_{nm}F)(x,y,z)\right| \leqslant (1+h)\,\omega\left(F(\cdot,\cdot,z); \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right).$$

3.3. **Boolean sum operators.** If $S_{mn} = B_m^{xy} \oplus B_n^{yx}$ and $T_{nm} = B_n^{yx} \oplus B_m^{xy}$ are the Boolean sums of the operators B_m^{xy} and B_n^{yx}, then we have:

Theorem 3.6. *If* $F\colon \mathcal{T}_h \to \mathbb{R}$ *then*

$$S_{mn}F = F \quad \textit{and} \quad T_{nm}F = F \quad \textit{on} \quad s_{013} \cup s_{023} \cup s_{123}.$$

Theorem 3.7. *If $F \in C(\mathcal{T}_h)$ then*

$$\begin{aligned}\left|(F - S_{mn}F)(x,y,z)\right| \leqslant & \left(1+\frac{h}{2}\right)\omega\left(F(\cdot,y,z);\frac{1}{\sqrt{m}}\right)\\ &+\left(1+\frac{h}{2}\right)\omega\left(F(x,\cdot,z);\frac{1}{\sqrt{n}}\right)\\ &+\left(1+h\right)\omega\left(F(\cdot,\cdot,z);\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right),\end{aligned}$$

and a similar inequality holds for the error $F - T_{nm}F$.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

SUBORDINATION CHAINS AND QUASICONFORMAL EXTENSIONS OF HOLOMORPHIC MAPPINGS IN $\mathbb{C}^n$

PAULA CURT

Abstract. Let B be the unit ball in $\mathbb{C}^n$ with respect to the Euclidean norm. In this paper, by using the method of subordination chains, we obtain a sufficient condition for a normalized quasiregular mapping f to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

1. **Introduction and preliminaries**

J.A. Pfaltzgraff [12] proved that if $0 \le q < 1$ and $f \in \mathcal{H}(B)$ is a quasiregular mapping, which satisfies the condition

$$(1 - \|z\|^2)\|[Df(z)]^{-1}D^2f(z)(z, \cdot)\| \le q, \quad z \in B,$$

then f is biholomorphic on B and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

The problem of quasiconformal extensions for quasiregular holomorphic mappings on the unit ball in $\mathbb{C}^n$ has been studied by H. Hamada and G. Kohr [11], P. Curt [5], P. Curt and G. Kohr [7], [8], [9].

In this paper we shall generalize the results due to J.A. Pfaltzgraff [12], P. Curt [5].

Let $\mathbb{C}^n$ denote the space of n-complex variables $z = (z_1, \dots, z_n)$ with the usual inner product $\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w}_i$ and Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$. Let B denote the open unit ball in $\mathbb{C}^n$ and let U be the unit disc in $\mathbb{C}$.

Received by the editors: 11.10.2008.

2000 *Mathematics Subject Classification.* Primary 32H, Secondary 30C45.

Key words and phrases. Loewner chains, quasiregular mapping, quasiconformal mapping.

Let $\mathcal{H}(\Omega)$ be the set of holomorphic mappings from a domain Ω in $\mathbb{C}^n$ into $\mathbb{C}^n$. If $f \in \mathcal{H}(B)$, let $J_f(z) = \det Df(z)$ be the complex jacobian determinant of f at z. Also let $\mathcal{L}(\mathbb{C}^n)$ be the space of continuous linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the standard operator norm

$$\|A\| = \sup\{\|Az\| : \|z\| = 1\},$$

and let I be the identity in $\mathcal{L}(\mathbb{C}^n)$. A mapping $f \in \mathcal{H}(B)$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$.

We say that a mapping $f \in \mathcal{H}(B)$ is K-quasiregular, $K \geq 1$, if

$$\|Df(z)\|^n \leq K|\det Df(z)|, \quad z \in B.$$

A mapping $f \in \mathcal{H}(B)$ is called quasiregular if f is K-quasiregular for some $K \geq 1$. It is well known that quasiregular holomorphic mappings are locally biholomorphic.

Definition 1.1. Let G and G' be domains in $\mathbb{R}^m$. A homeomorphism $f : G \to G'$ is said to be K-quasiconformal if it is differentiable a.e., ACL (absolutely continuous on lines) and

$$\|Df(z)\|^m \leq K|\det Df(x)| \text{ a.e. } x \in G,$$

where $Df(x)$ denotes the real Jacobian matrix of f and K is a constant.

Note that a K-quasiregular biholomorphic mapping is K^2-quasiconformal.

If $f, g \in \mathcal{H}(B)$, we say that f is subordinate to g (and write $f \prec g$) if there exists a Schwarz mapping v (i.e. $v \in \mathcal{H}(B)$ and $\|v(z)\| \leq \|z\|$, $z \in B$) such that $f(z) = g(v(z))$, $z \in B$.

Definition 1.2. A mapping $L : B \times [0, \infty) \to \mathbb{C}^n$ is called a subordination chain if the following conditions hold:

(i) $L(0, t) = 0$ and $L(\cdot, t) \in \mathcal{H}(B)$ for $t \geq 0$;

(ii) $L(\cdot, s) \prec L(\cdot, t)$ for $0 \leq s \leq t < \infty$.

If $L(z, t)$ is a subordination chain such that $L(\cdot, t)$ is biholomorphic on B for $t \in [0, \infty)$, then we say that $L(z, t)$ is a univalent subordination chain (or a Loewner chain).

If $L(z,t)$ is a univalent subordination chain such that $DL(0,t) = e^t I$, we say that $L(z,t)$ is a normalized Loewner chain.

An important role in our discussion is played by the n-dimensional version of the class of holomorphic functions on the unit disc with positive real part

$$\mathcal{N} = \{h \in \mathcal{H}(B):\ h(0) = 0,\ \mathrm{Re}\ \langle h(z), z\rangle > 0,\ z \in B \setminus \{0\}\}$$

$$\mathcal{M} = \{h \in \mathcal{N},\ Dh(0) = I\}.$$

The authors ([10, Theorem 1.10]) and [6, Theorem 2.3]) proved that normalized univalent subordination chains satisfy the generalized Loewner differential equation. Using an elementary change of variable, it is not difficult to reformulate the mentioned result in the case of normalized subordination chains $L(z,t) = a(t)z + \ldots$, where $a : [0,\infty) \to \mathbb{C}$, $a \in C^1([0,\infty))$, $a(0) = 1$, and $a(t) \to \infty$ as $t \to \infty$.

Theorem 1.1. *Let $L(z,t) : B \times [0,\infty) \to \mathbb{C}^n$ be a Loewner chain such that $L(z,t) = a(t)z + \ldots$, where $a \in C^1([0,\infty))$, $a(0) = 1$, and $\lim_{n\to\infty} |a(t)| = \infty$. Then there exists a mapping $h = h(z,t) : B \times [0,\infty) \to \mathbb{C}^n$ such that $h(\cdot,t) \in \mathcal{N}$ for $t \geq 0$, $h(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in B$, and*

$$\frac{\partial L}{\partial z}(z,t) = DL(z,t)h(z,t),\ \ a.e.\ t \geq 0,\ z \in B.$$

Recently P. Curt and G. Kohr [9] proved the following result.

Theorem 1.2. *Let $L(z,t) : B \times [0,\infty) \to \mathbb{C}^n$, $L(z,t) = a(t)z + \ldots$, be a Loewner chain such that $a(\cdot) \in C^1[0,\infty)$, $a(0) = 1$ and $\lim_{t\to\infty} |a(t)| = \infty$. Assume that the following conditions hold:*

(i) There exists $K > 0$ such that $L(\cdot,t)$ is K-quasiregular for each $t \geq 0$.

(ii) There exist some constants $M > 0$ and $\alpha \in [0,1)$ such that

$$\|DL(z,t)\| \leq \frac{M|a(t)|}{(1-\|z\|)^\alpha},\quad z \in B,\ t \in [0,\infty).$$

(iii) There exists a sequence $\{t_m\}_{m\in\mathbb{N}}$, $t_m > 0$, $\lim_{m\to\infty} t_m = \infty$, and a mapping $F \in \mathcal{H}(B)$ such that

$$\lim_{m\to\infty} \frac{L(z,t_m)}{a(t_m)} = F(z)$$

locally uniformly on B.

Further, assume that the mapping $h(z,t)$ defined by Theorem 2 satisfies the following conditions:

(iv) There exists a constant $C > 0$ such that

$$C\|z\|^2 \leq R\langle h(z,t), z\rangle, \quad z \in B,\ t \in [0,\infty).$$

(v) There exists a constant $C_1 > 0$ such that

$$\|h(z,t)\| \leq C_1, \quad z \in B,\ t \in [0,\infty).$$

Then $f = L(\cdot, 0)$ extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

In this paper we obtain a sufficient condition for a normalized quasiregular holomorphic mapping on B, which can be embedded as the first element of a nonnormalized univalent subordination chain, to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

2. **Main results**

Theorem 2.1. *Let $f, g \in \mathcal{H}(B)$ be such that $f(0) = g(0) = 0$, $Df(0) = Dg(0) = I$ and g is quasiregular in B. Also let $a \geq 2$. If there is $q \in [0,1)$ such that $1 - \frac{2}{\alpha} \leq q < \frac{2}{\alpha}$,*

$$\frac{2}{\alpha}\left\|[Dg(z)]^{-1}Df(z) - \frac{\alpha}{2}I\right\| \leq q < 1 \tag{2.1}$$

and

$$\frac{2}{\alpha}\Big\|\|z\|^{\alpha}\{[Dg(z)]^{-1}Df(z) - I\} \tag{2.2}$$

$$+(1-\|z\|^{\alpha})[Dg(z)]^{-1}D^2g(z)(z,\cdot) + \left(1 - \frac{\alpha}{2}\right)I\Big\| \leq q < 1,\ z \in B,$$

then f extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. We shall show that the conditions (2.1) and (2.2) enable us to embed f as the initial element $f(z) = L(z,0)$ of a suitable subordination chain.

We define

$$L(z,t) = f(e^{-t}z) + (e^{\alpha t} - 1)e^{-t}Df(ze^{-t})(z),\ t \in [0,\infty),\ z \in B. \tag{2.3}$$

In [4] the authors proved that the mapping L defined by (2.3) is a subordination chain. In the same paper the authors showed that the subordination chain defined by (2.3) satisfies the generalized Loewner equation where the mapping h is defined by:

$$h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z),\ q \in B,\ t \in [0,\infty) \tag{2.4}$$

and the mapping $E : B \times [0,\infty) \to \mathcal{L}(\mathbb{C}^n)$ is defined by

$$E(z,t) = -\frac{2}{\alpha}e^{-\alpha t}\{[Dg(ze^{-t})]^{-1}Df(ze^{-t}) - I\} \tag{2.5}$$

$$-\frac{2}{\alpha}(1-e^{-\alpha t})[Dg(ze^{-t})]^{-1}D^2g(ze^{-t})(ze^{-t},\cdot) - I\left(\frac{2}{\alpha}-1\right).$$

Further, we shall show that $\|E(z,t)\| \le q$ for all $(z,t) \in B \times [0,\infty)$.

We have

$$\|E(z,0)\| = \frac{2}{\alpha}\left\|[Dg(z)]^{-1}Df(z) - \frac{\alpha}{2}I\right\| \le q < 1, \quad z \in B,$$

by the condition (2.1). Next, fix $t \in (0,\infty)$. In view of the maximum principle for holomorphic mappings into complex Banach spaces, we obtain that

$$\|E(z,t)\| \le \max_{\|w\|=1}\|E(w,t)\|$$

$$= \frac{2}{\alpha}\max_{\|w\|=1}\Big\|\|we^{-t}\|^{\alpha}[Dg(we^{-t})]^{-1}[Df(we^{-t}) - I_n]$$

$$+(1-\|we^{-t}\|^{\alpha})[Dg(we^{-t})]^{-1}D^2g(we^{-t})(we^{-t},\cdot) + I\left(1-\frac{\alpha}{2}\right)\Big\|,\ z \in B.$$

Hence, we deduce from the condition (2.2) that

$$\|E(z,t)\| \le q < 1, \quad z \in B^n.$$

Therefore

$$\|E(z,t)\| \le q < 1, \quad z \in B,\ t \in [0,\infty)$$

and hence $I - E(z,t)$ is an invertible linear operator.

Further calculations show that

$$\frac{\partial L(z,t)}{\partial t} = \frac{\alpha}{2}e^{(\alpha-1)t}Dg(ze^{-t})[I + E(z,t)](z) \tag{2.6}$$

$$= DL(z,t)[I - E(z,t)]^{-1}[I + E(z,t)](z)$$

$$= DL(z,t)h(z,t), \quad t \in [0,\infty),\ z \in B.$$

On the other hand, taking into account the conditions (i) and (ii) in the hypothesis, we deduce that

$$\begin{aligned}
&(1-\|z\|^{\alpha})\|[Dg(z)]^{-1}D^2g(z)(z,\cdot)\| \qquad (2.7)\\
&\le q\cdot\frac{\alpha}{2}+\|z\|^{\alpha}\cdot\frac{\alpha}{2}\cdot q+(1-\|z\|^{\alpha})\left(\frac{\alpha}{2}-1\right)\\
&=\|z\|^{\alpha}\left(q\frac{\alpha}{2}-\frac{\alpha}{2}+1\right)+q\frac{\alpha}{2}+\frac{\alpha}{2}-1\\
&\le \max_{x\in[0,1]}\left\{x\left(q\frac{\alpha}{2}-\frac{\alpha}{2}+1\right)+q\frac{\alpha}{2}+\frac{\alpha}{2}-1\right\}\\
&=\max\left\{q\frac{\alpha}{2}+\frac{\alpha}{2}-1, q\alpha\right\}=q\alpha=2\beta, \quad z\in B,
\end{aligned}$$

where $\beta = \dfrac{q\alpha}{2} < 1$. Since $\alpha \ge 2$, we deduce from the above relation that

$$(1-\|z\|^2)\|[Dg(z)]^{-1}D^2g(z)(z,\cdot)\| \le 2\beta, \quad \beta \le \|z\| < 1.$$

From (2.6), by using a similar argument with that used in the proof of Theorem 2.1 [9] we obtain that there exists $M > 0$ such that

$$|\det Dg(z)| \le \frac{M}{(1-\|z\|)^{n\beta}}, \quad z \in B, \tag{2.8}$$

and hence

$$\|Dg(z)\| \le \frac{L}{(1-\|z\|)^{\beta}}. \tag{2.9}$$

It remains to prove that the mappings $L(\cdot,t)$, $t \ge 0$ are quasiregular. For the subordination chain defined by (2.3) we have

$$DL(z,t) = e^{(\alpha-1)t}\frac{\alpha}{2}Dg(ze^{-t})[I-E(z,t)], \quad z \in B,\ t \ge 0$$

where $L = \sqrt[n]{ML}$.

Since g is a quasiregular holomorphic mapping and the following inequality holds

$$1-q \le \|I-E(z,t)\| \le 1+q, \quad z \in B,\ t \ge 0$$

we easily obtain

$$\|DL(z,t)\| \le \frac{\alpha}{2}e^{(\alpha-1)t}(1+q)\frac{L}{(1-\|z\|)^{\beta}} \tag{2.10}$$

$$= \frac{L^* a(t)}{(1-\|z\|)^\beta}, \quad z \in B,\ t \in [0,\infty).$$

On the other hand, we have

$$\|DL(z,t)\|^n \le \left(\frac{\alpha}{2}\right)^n e^{n(\alpha-1)t}\|Dg(ze^{-t})\|^n(1+q)^n \tag{2.11}$$

$$\le \left(\frac{\alpha}{2}\right)^n e^{n(\alpha-1)t}K|\det Dg(ze^{-t})|(1+q)^n$$

$$\le \left(\frac{1+q}{1-q}\right)^n K|\det DL(z,t)|, \quad z \in B,\ t \ge 0.$$

Since the conditions of Theorem 1.2 are satisfied we obtain that the function $f(z) = L(z,\cdot)$ admits a quasiconformal extension defined on $\mathbb{R}^{2n}$.

Observe that:

a) if $f = g$ and $\alpha = 2$ we obtain Theorem 3.1 of [12],

b) if $\alpha = 2$ we obtain Theorem 2.1 of [5].

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

ON APPLICATIONS OF THE REPRODUCING KERNEL METHOD FOR CONSTRUCTION OF CUBATURE FORMULAS

EMIL A. DANCIU

Abstract. In this paper we use the method of Reproducing Kernel and *Gegenbauer* polynomials for constructing cubature formulas on the unit ball B^d, and on the standard simplex. Also we study the relation between interpolation polynomials based on the zeros of quasi-orthogonal *Chebyshev* polynomials and the nodes of near minimal degree cubature formulas.

1. **Introduction**

1) The Reproducing Kernel of a Hilbert space of functions

One calls reproducing Kernel of the Hilbert space H of functions defined on D, real valued ($D \subset \mathbb{R}^d$), a function $K = K(x, y) : D \times D \to \mathbb{R}$, which verifies the following conditions

i) $K(\cdot, y) \in H$, for any fixed $y \in D$,

ii) $< f, K(\cdot, y) > = f(y), \ \forall f \in H$.

It is known that in the Hilbert space H are stated the following results.

Theorem 1.1. If the Hilbert space H has a Reproducing Kernel, then this kernel is unique and symmetric with respect to its arguments.

Theorem 1.2. If L is a linear and bounded functional defined on the Hilbert space H, which has a Reproducing Kernel, then the representation function corresponding to L is $g(x) = L_y[K(x, y)]$.

Received by the editors: 05.01.2009.

2000 *Mathematics Subject Classification.* 41A25, 41A36, 65D32.

Key words and phrases. Reproducing Kernel, cubature formulas, Gegenbauer polynomials on simplex, on ball.

We consider now, $H = \mathbb{P}_n^d$ the space of all polynomials of degree at most n, and $D \subset \mathbb{R}^d$.

It is known that $dim\mathbb{P}_n^d(D) = \binom{n+d}{d}$, if and only if $int(D) \neq \emptyset$.

Let $f \in \mathbb{P}_n^d$ be a polynomial of degree exact n, and we denote

$$\mu = \mu(d,n) = \binom{n+d}{d} = \frac{(n+d)!}{n!d!}.$$

It was shown that the number of terms in the representation of the polynomial f is equal to $\mu(d,n)$ and this number represents the number of the monomials in the expression of $f = f(x)$.

Let $W = W(x) : D \to \mathbb{R}_0^+$, $(D \subset \mathbb{R}^d)$, be a weight function.

Theorem 1.3. For a given region (domain) D, $D \subset \mathbb{R}^d$ and a given weight function $W = W(x) : D \to \mathbb{R}_0^+$, exists and are unique $r(d,n) = \mu(d,n-1) = \frac{(n-1+d)!}{(n-1)!d!}$ orthogonal polynomials of degree n, which are linearly independent.

Let now, $\{e_i(x)\}_{i=0}^{\infty}$, be the monomials which are ordered increasing, and for the same degree for certain terms, we use the lexicographic order.

So, the set $\{e_i(x)\}$, $i = \overline{1,\mu(d,n)}$ represents all the monomials of degree at most n.

By applying the Gram-Schmidt orthonormalization process, we can obtain an *orthonormalized* set with respect to the scalar product

$$(f,g) = I(f \cdot g) = \int_D f(x)g(x)W(x)dx. \tag{1.1}$$

2) The *Gegenbauer* (ultraspherical) orthogonal polynomials

We present now, some of the properties of *Gegenbauer* polynomials, which play an important role in the applications of the cubature formulas theory by using the Reproducing Kernel method.

The *Gegenbauer* polynomials are usually defined by the following generating function:

$$(1-2tz+z^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(t)z^n, \tag{1.2}$$

where $|z| < 1$, $|t| \le 1$, $\lambda > 0$.

The coefficients $C_n^{(\lambda)}(t)$ are algebraic polynomials of degree n which are called the *Gegenbauer* polynomials associated with λ. One can prove that the family of polynomials $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$ is a complete orthogonal system for the weighted space $L_2(I,W)$, $I=[-1,1]$, $W(t)=W_\lambda(t):=(1-t^2)^{\lambda-\frac{1}{2}}$, and we have

$$\int_{[-1,1]} C_n^{(\lambda)}(t)C_m^{(\lambda)}(t)W(t)dt = \begin{cases} 0, & m\neq n \\ \gamma_{n,\lambda}=\frac{\pi^{1/2}(2\lambda)_n\Gamma(\lambda+1/2)}{(n+\lambda)n!\Gamma(\lambda)}, & m=n \end{cases}$$

where we use $(a)_\lambda$, the *Pockhammer* symbol,

$$(a)_0:=0,\quad (a)_n:=a(a+1)\dots(a+n-1)=\Gamma(a+n)/\Gamma(a).$$

Also we have,

$$C_n^{(\lambda)}(-t)=(-1)^nC_n^{(\lambda)}(t),\quad C_n^{(\lambda)}(1)=\frac{(2\lambda)n}{n!}\ \text{ and } C_0^{(\lambda)}=1. \tag{1.3}$$

The *Gegenbauer* polynomials can also be defined by the well known *Rodrigues's* formula (see [7] *Szegö*)

$$C_n^{(\lambda)}(t)=(-1)^n\alpha_{n,\lambda}(1-t^2)^{-\lambda+\frac{1}{2}}\frac{d^n}{dt^n}[(1-t^2)^{n+\lambda-\frac{1}{2}}]$$

where,

$$\alpha_{n,\lambda}=\frac{(2\lambda)_n}{n!2^n(\lambda+\frac{1}{2})_n}.$$

It is known that there exists the following identity which relates *Gegenbauer* polynomials with different weights

$$\frac{d^k}{dt^k}C_n^{(\lambda)}(t)=2^k(\lambda)_kC_{n-k}^{(\lambda+k)},\ k=1,2,\dots n. \tag{1.4}$$

For $\lambda=1/2$, we can obtain the *Legendre* polynomial

$$P_n(t)=\frac{(-1)^n}{2^nn!}\frac{d^n}{dt^n}[(1-t^2)^n]=C_n^{(1/2)}(t)$$

and for $\lambda=1$ we obtain the *Chebyshev* polynomial of second kind U_n,

$$U_n=\frac{sin[(n+1)arccost]}{\sqrt{1-t^2}}=C_n^{(1)}(t).$$

Also, we can obtain the *Chebyshev* polynomial of the first kind

$$T_n(t):=cos(narccost)=C_n^{(0)},$$

by considering $C_n^{(0)}$ associated with the weight function $W_0(t) = (1-t^2)^{-1/2}$.

We can also consider the *Gegenbauer* polynomials $C_n^{(\lambda)}$, for $\lambda < 0$, $\lambda \in \mathbb{Z}^-$ namely,

$$C_n^{(\lambda)}(t) := \alpha(1-t^2)^{-\lambda+\frac{1}{2}} \frac{d^n}{dt^n}\left[(1-t^2)^{n+\lambda-\frac{1}{2}}\right],\ \lambda < 0$$

where α is an constant independent of t and we can write the identity

$$\frac{d^k}{dt^k} C_n^{(\lambda)}(t) = c C_{n-k}^{(\lambda+k)}(t), \quad k = 1, 2 \ldots, n,$$

where c is independent of t.

3) The relation between Cubature Formulas and the Reproducing Kernels

The Reproducing Kernel method was first used by $I.PMysovskikh$ ([3]) and later studied by $M\ddot{o}ller$ ([2]).

Let a given weight function $W = W(x)$ be defined on a subset $D \subset \mathbb{R}^d$. Then, a cubature formula is a linear combination of function values on some points, that approximates $\int_D f(x)W(x)dx$.

Let $I^d[f] = \int_D f(x)W(x)dx$, $f \in C(D)$, $D \subset \mathbb{R}^d$ for which the moments $I^d[x^\alpha]$, $\alpha \in \mathbb{N}^d$ exists and W=W(x) is nonnegative.

We say that the cubature formula has the degree of exactness m, if it yields the exact value of the integrals for any function $f \in \mathbb{P}_m^d$, which is a polynomials of degree at most m.

We denote the space of polynomials of degree at most n by $\mathbb{P}_n^d$.

Let

$$\{P_k^n : 1 \le k \le r(d,n)\},\ 0 \le n < \infty,$$

(where $r(d,n) = \mu(d, n-1) = \binom{d+n-1}{d}$), denote a sequence of orthonormal polynomials of degree n with respect to the inner product (1.1), which are linearly independent, where the superscript n means that $P_k^n \in \mathbb{P}_n^d$ and let denote by $\mathbf{P}_n = (P_1^n, \ldots, P_{r(d,n)}^n)$, the vector of all these polynomials.

The $n-th$ Reproducing Kernel $K_n(x,y)$ of the Hilbert space $H = \mathbb{P}_n^d$ is defined by:

$$K_n(x,y) = \sum_{k=0}^{n} \mathbf{P}_k^T(x)\mathbf{P}_k(y) = \sum_{k=0}^{n} \sum_{j=1}^{r(d,k)} P_j^k(x)P_j^k(y),\ \forall\ x,y \in R^d. \qquad (1.5)$$

The method of Reproducing Kernel requires to choose d points: $a^{(1)},\dots,$ $a^{(d)} \in \mathbb{R}^d$, such that the hypersurfaces $H_1,\dots H_d$, where H_i is the surface defined by $H_i = \{x \in \mathbb{R}^d : K_n(x,a^{(i)}) = 0\}$, intersect at n^d points. The points $a^{(1)},\dots,a^{(d)}$ are chosen as follows.

For $a^{(1)}$ we choose any point that is not a common zero of the polynomial set $\mathbf{P}_n$. If the points $a^{(1)},\dots,a^{(r-1)}$ have been chosen, then we choose $a^{(r)} \in \bigcap_{k=1}^{r-1} H_k$, and $a^{(r)}$ may be any point of this set, which is not a common zero of $\mathbf{P}_n$.

We assume that the infinity is not a common point of $H_1,\dots,H_d$.

We present now the following results.

a) The Method of Reproducing Kernel

If $H_1,\dots,H_d$ defined by $a^{(1)},\dots,a^{(d)}$, intersect at n^d distinct points: $\{x^{(i)},\ i = \overline{1,n^d}\}$, then there is a cubature formula of degree $2n$,

$$Q_n(f) = \sum_{i=1}^{d} \lambda_i f(a^{(i)}) + \sum_{j=1}^{n^d} \mu_j f(x^{(j)}),\ \forall f \in \mathbb{P}_{2n}^d, \qquad (1.6)$$

where $\lambda_i = 1/K_n(a^{(i)},a^{(i)})$.

If the weight function $W = W(x)$ is centrally symmetric, that is, $W = W(x)$ and its support set D satisfy $\forall x \in D \Rightarrow -x \in D,\ W(-x) = W(x)$, then there is a modified method of Reproducing Kernel due to *Möller* ([2]).

Let $\widetilde{K_n}$ denote:

$$\widetilde{K_n}(x,y) = \sum_{k=0}^{n}{}' \sum_{j=0}^{r(d,k)}{}' P_j^k(x)P_j^k(y),\ \forall x,y \in R^d, \qquad (1.7)$$

where $\sum'$ means that the summation is taken over those j so that the corresponding P_j^k has the same parity as n. We choose the points $a^{(i)}$ as before except that we replace H_i by the *hypersurface* $\widetilde{H_i}$ defined by $\widetilde{H_i} = \{x \in \mathbb{R}^d : \widetilde{K_n}(x,a^{(i)}) = 0\}$ and we suppose that the infinity is not a common point of $\widetilde{H_1}\ldots,\widetilde{H_d}$. Then we have, if $W = W(x)$ is centrally symmetric on $D \subset \mathbb{R}^d$.

b) The Modified method of Reproducing Kernel

If $\widetilde{H_1},\ldots,\widetilde{H_d}$ defined by $a^{(1)},\ldots,a^{(d)}$ intersect at n^d distinct points: $\{x^{(i)}, i=\overline{1,n^d}\}$, then there is a cubature formula of degree $2n+1$,

$$Q_n(f)=\sum_{i=1}^{d}\lambda_i[f(a^{(i)})+f(-a^{(i)})]/2+\sum_{j=1}^{n^d}\mu_j f(x^{(j)}),\ \forall f\in\mathbb{P}^d_{2n+1}, \tag{1.8}$$

where $\lambda_i = 1/\widetilde{K_n}(a^{(i)},a^{(i)})$.

If $d=2$, then the method requires to choose two points $a^{(1)}$ and $a^{(2)}$ so that the polynomial surface $\widetilde{K_n}(x,a^{(1)})$ and $\widetilde{K_n}(x,a^{(2)})$ have n^2 common zeros.

In the paper [12] Y. Xu was presented a compact formula of the Reproducing Kernel for the *Jacobi* type weight functions on the unit ball and on the standard simplex.

The method of Reproducing Kernel yields cubature formulas of degree $2n+1$ or $2n$ with n^d+dn or n^d+dn-1 nodes, which is greater than the theoretic lower bound for the number of nodes.

2. Cubature formulas on the unit ball using the reproducing kernel method

Let $x,y\in\mathbb{R}^d$ and we use the following notations:

$<x,y>=x_1y_1+\cdots+x_dy_d$, the usual Euclidian inner product,

$|x|^2=\|x\|^2 =<x,x>$, the Euclidian norm.

We consider cubature formulas on the unit ball $B^d=\{x\in\mathbb{R}^d:\ \ \|x\|\le 1\}$, with respect to the normalized weight function

$$W_\mu(x)=w_\mu(1-\|x\|^2)^{\mu-\frac{1}{2}},\ \mu\ge 0,\ x\in B^d, \tag{2.1}$$

where w_μ is a constant chosen so that the integral $\int_{B^d}W_\mu(x)dx=1$, and we have

$$w_\mu=\frac{2}{\omega_{d-1}}\frac{\Gamma(\mu+\frac{d+1}{2})}{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{d}{2})}=\frac{\Gamma(\mu+\frac{d+1}{2})}{\pi^{d/2}\Gamma(\mu+\frac{1}{2})},$$

where $\omega_{d-1}=2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in $\mathbb{R}^d$.

Let $K_n(.,.)$ be the $n-th$ Reproducing Kernel with respect to weight function W_μ. In [12] is presented the following compact formula for the representation of this kernel.

$$K_n(W_\mu;x,y) = c_\mu \int_{-1}^{1} \Big[C_n^{(\mu+\frac{d+1}{2})}(< x,y > + \sqrt{1- \parallel x \parallel^2}\sqrt{1- \parallel y \parallel^2}\ t) + \tag{2.2}$$

$$+ C_{n-1}^{(\mu+\frac{d+1}{2})}(< x,y > + \sqrt{1- \parallel x \parallel^2}\sqrt{1- \parallel y \parallel^2}\ t)\Big](1-t^2)^{\mu-1}dt,$$

where $c_\mu = 1/\int_{-1}^{1}(1-t^2)^{\mu-1}dt$ and $C_n^{(\lambda)}$ is the *Gegenbauer* polynomial of degree n defined by the generating function (1.2), which have the property

$$C_n^{(\lambda)}(-t) = (-1)^n C_n^{(\lambda)}(t).$$

If we take in consideration the expressions: $K_n(W_\mu;x,y) \pm K_n(W_\mu;x,-y)$ for n being even and odd, respectively then it follows from the formula (1.5) and (1.7) that the modified Reproducing Kernel function $\widetilde{K_n}(W_\mu;...)$ is given by the formula

$$\widetilde{K_n}(W_\mu;x,y) = c_\mu \int_{-1}^{1} C_n^{(\mu+\frac{d+1}{2})}(< x,y > + \sqrt{1- \parallel x \parallel^2}\sqrt{1- \parallel y \parallel^2}\ t)(1-t^2)^{\mu-1}dt. \tag{2.3}$$

For $\mu \to 0$, in (2.2) and (2.3), one can use the limit

$$\lim_{\mu \to 0} c_\mu \int_{-1}^{1} f(t)(1-t^2)^{\mu-1}dt = \frac{f(1)+f(-1)}{2}. \tag{2.4}$$

In the case $\mu = \frac{1}{2}$, we have: $W_{1/2}(x) = d/\omega_{d-1}$.

If $\mu = 0$ we have: $W_0(x) = w_0(1- \parallel x \parallel)^{-1/2}$ and we obtain:

$$\widetilde{K_n}(W_0;x,y) = \frac{1}{2}\Big[C_n^{(3/2)}(< x,y > + \sqrt{1- \parallel x \parallel^2}\sqrt{1- \parallel y \parallel^2}) + \tag{2.5}$$

$$+ C_n^{(3/2)}(< x,y > - \sqrt{1- \parallel x \parallel^2}\sqrt{1- \parallel y \parallel^2})\Big].$$

If we consider $\parallel a \parallel = 1$, we have

$$\widetilde{K_n}(W_\mu;x,a) = C_n^{(\mu+(d+1)/2)}(< x,a >). \tag{2.6}$$

In this case, if $\parallel a \parallel = 1$ then a is not a common zero of the polynomial set $\mathbf{P}_n$, because that $\mathbf{P}_n$ has no common zeros if n is even, and it has only origin as common zero if n is odd.

2.1 The construction of a family of cubature formulas on B^d by using the *Gegenbauer* polynomials

One can use the properties of the *Gegenbauer* polynomial $C_n^{(\lambda)}(t),\ \lambda=\mu+(d+1)/2$, that all its zeros are inside $(-1,1)$, and we denote these zeros by:

$$-1<t_{1,n}<t_{2,n}<\cdots<t_{n,n}<1,\ \text{where}\ \lambda=\mu+(d+1)/2.$$

It is known that these zeros are symmetric with respect to the origin, that is, they satisfy the relation $t_{i,n}=-t_{n-(i-1),n}$. So, in [12] was given the following strategy to choose the points $a^{(1)},\dots,a^{(d)}$ as follows.

Let n be fixed and let $t_{*,n}$ be a fixed zero of $C_n^{(\mu+(d+1)/2)}(t)$, and let define:

$$a^{(1)}=(1,0,\dots,0),\quad a^{(k)}=(b_1,\ \dots\ ,b_{k-1},\sqrt{1-b_1^2-\cdots-b_{k-1}^2},0,\dots,0),$$

$0\le k\le d$, where the components $b_1,\dots,b_{d-1}$ are determined inductively by the conditions: $<a^{(k)},a^{(k+1)}>=t_{*,n}$, which is equivalent with

$$b_1^2+\cdots+b_{k-1}^2+\sqrt{1-b_1^2-\cdots-b_{k-1}^2}\ b_k=t_{*,n},\ k=\overline{1,d-1},$$

from which are obtained:

$$b_1=t_{*,n},\ b_2=(t_{*,n}-b_1^2)/\sqrt{1-b_1^2},\ \dots,\ \text{and we have}\ b_k\le\sqrt{1-b_1^2-\cdots-b_{k-1}^2},$$

because $t_{*,n}<1$, hence $a^{(k+1)}$ is well defined. It follows that

$$\bigcap_{i=1}^{k}H_k=\{x\in R^d:<x,a^{(1)}>=t_{i_1,n},\dots,<x,a^{(k)}>=t_{i_k,n},\ 1\le i_1,\dots,i_k\le n\}$$

for $k=\overline{2,d}$. If we assume that $a^{(k)}\in H_1\bigcap\cdots\bigcap H_{k-1}$, and we require that $a^{(k+1)}\in\bigcap_{i=1}^{k}H_i$, one observe that $a^{(2)}=(t_{*,n},\sqrt{1-t_{*,n}^2},0,\dots,0)\in H_1$.

Inductively, if we assume that $a^{(k)}\in\bigcap_{i=1}^{k-1}H_i$, that is

$$<a^{(i)},a^{(k)}>=t_{*,n},\ 1\le i\le k-1.$$

Since $a^{(k+1)}$ satisfies $<a^{(k)},a^{(k+1)}>=t_{*,n}$ it follows that:

$$<a^{(i)},a^{(k+1)}>=t_{*,n},\quad i=\overline{1,k},$$

that is $a^{(k+1)}\in H_1\bigcap\cdots\bigcap H_k$.

One can observe that $H_1 \bigcap \cdots \bigcap H_d$ contains n^d distinct points, which are given by the relations:

$$< x, a^{(1)} > = \ t_{i_1,n}, \ \ldots, \ < x, a^{(d)} > = \ t_{i_d,n}, \ 1 \le i_1, \ldots, i_d \le n. \tag{2.7}$$

Theorem 2.1. Let $a^{(1)}, \ldots, a^{(d)}$ be defined as above and let H_k be the surface defined by $H_k = \{x \in \mathbb{R}^d : \widetilde{K_n}(W_\mu; x, a^{(k)}) = 0\}$. Then the modified method of the Reproducing Kernel yields, a cubature formula of degree $2n+1$, based on $a^{(1)}, \ldots, a^{(d)}$ and the n^d distinct points determined by (2.7) and have the form

$$Q_n(f) = \sum_{i=1}^{d} \lambda_i [f(a^{(i)}) + f(-a^{(i)})]/2 + \sum_{j=1}^{n^d} \mu_j f(x^{(j)}), \ \forall f \in \mathbb{P}^d_{2n+1}, \tag{2.8}$$

where $\lambda_i = 1/\widetilde{K_n}(a^{(i)}, a^{(i)})$.

We obtain by using (2.6) that

$$\lambda_i = 1/\widetilde{K_n}(W_\mu; a^{(i)}, a^{(i)}) = 1/C_n^{(\mu+(d+1)/2)}(1) = 1/\binom{n+2\mu+d}{n}. \tag{2.9}$$

For fixed d and n, the others weights μ_j in (2.8) can be determined by solving a linear system of equations.

From the fact that in definition of $a^{(k)}$, if we use the condition

$$< a^{(k-1)}, a^{(k)} >= t_{*,n},$$

we remark that one can choose $t_{*,n}$ to be any zero of the polynomial $C_n^{(\mu+(d+1)/2)}(t)$ and we can get many different formulas from this method.

Remark 2.1. When n is an odd integer, then $C_n^{(\mu+(d+1)/2)}$ is an odd polynomial, and it follows that $t = 0$ is a zero of this polynomial.

If we take $t_{*,n} = 0$ in the definition of $a^{(k)}$ in the above construction, then we obtain: $a^{(1)} = e_1, \ldots, a^{(d)} = e_d$, where $\{e_i, \ i = \overline{1,d}\}$ is the standard basis of $\mathbb{R}^d$, that is, $e_1 = (1, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$.

But from (2.6) it follows that

$$\widetilde{K_n}(W_\mu; x, e_k) = C_n^{(\mu+(d+1)/2)}(x_k), \ k = \overline{1,d}$$

and we observe that the n^d intersection points of $H_1 \bigcap \cdots \bigcap H_d$, namely $\{x^{(i)},\ i = \overline{1, n^d}\}$, are the tensor product of the zeros obtained from (2.7).

Let n be an odd integer and let $t_{1,n}, \ldots, t_{n,n}$ be the zeros of $C_n^{(\mu+(d+1)/2)}(t)$. Then there is a cubature of degree $2n+1$ on B^d of the form:

$$\int_{B^d} f(x) W_\mu(x) dx = \frac{1}{\begin{pmatrix} n+2\mu+d \\ n \end{pmatrix}} \sum_{k=1}^{n} \Big[f(e_k) + f(-e_k) \Big] / 2 + \tag{2.10}$$

$$+ \sum_{k_1=1}^{n} \cdots \sum_{k_d=1}^{n} \mu_{k_1,\ldots,k_d} f(t_{k_1,n}, \ldots, t_{k_d,n}),\ \forall f \in \mathbb{P}_{2n+1}^d.$$

The weights μ_j in the formula (2.10) can be computed by solving a linear system equations for a given n and d.

In the case $d = 2$, we can consider the polynomials $l_{k,n}$ defined by:

$$l_{k,n} = \prod_{i=1, i\neq k}^{n} \frac{x - t_{i,n}}{t_{k,n} - t_{i,n}} = \frac{C_n^{(\mu+(d+1)/2)}(t)}{(2\mu+d+1) C_{n-1}^{\mu+(d+3)/2)}(t_{k,n})(x - t_{k,n})},$$

which are the fundamental interpolation polynomials based on the zeros of $C_n^{(\mu+(d+1)/2)}(t)$ which satisfies the interpolation conditions: $l_{k,n}(t_{j,n}) = \delta_{k,j}$, by using (1.4).

One observe that the polynomial $l_{k_1,n}(x_1) l_{k_2,n}(x_2)(1 - x_1^2 - x_2^2)$ is of degree $2(n-1) + 2 = 2n$, then it will be integrated exactly by the cubature formula (2.10), and from the interpolation property of $l_{k,n}$ we will obtain the values of the weights are

$$\mu_{k_1,k_2} = \int_{B^2} l_{k_1,n}(x_1) l_{k_2,n}(x_2)(1 - x_1^2 - x_2^2) W_\mu(x_1, x_2) dx_1 dx_2.$$

The formula (2.10) uses the tensor product of nodes of an one variable quadrature rule. The points $\{t_{1,n}, \ldots, t_{n,n}\}$ are nodes of a Gaussian quadrature formula of degree $2n-1$ on $[-1,1]$ for the measure: $W(x) = (1-x^2)^{\mu+d/2} dx$ on $[-1,1]$. Moreover, $\{-1, t_{1,n}, \ldots, t_{n,n}, 1\}$ form the nodes of a $Gauss - Lobatto$ type quadrature formula of degree $2n+1$,

$$\int_{-1}^{1} f(x)(1-x^2)^{\mu+d/2} dx = Af(-1) + \sum_{k=1}^{n} \lambda_k f(t_{k,n}) + Af(1),\ \forall f \in \mathbb{P}_{2n+1}^1. \tag{2.11}$$

The tensor product of $\{-1, t_{1,n}, ..., t_{n,n}, 1\}$, can be used as nodes in the following product formula of degree $2n+1$ for the product weight function:

$$W(x) = \prod_{k=1}^{d} (1-x_k)^{\mu+d/2} \text{ on } [-1,1]^d,$$

$$\int_{[-1,1]^d} f(x) \prod_{k=1}^{d} (1-x_k)^{\mu+d/2} dx = \sum_{k_1=0}^{n+1} \cdots \sum_{k_d=0}^{n+1} \lambda_{k_1} \dots \lambda_{k_d} f(t_{k_1,n}, \dots, t_{k_d,n}), \quad (2.12)$$

for $\forall f \in \mathbb{P}^d_{2n+1}$, where $t_{0,n} = -1, \quad t_{n+1,n} = 1$ and $\lambda_0 = \lambda_{n+1} = A$.

It was showed that some nodes of the cubature formulas constructed above can lie outside of the unit ball B^d. But we can choose different values $a^{(k)}$ in order to construct formulas with all nodes inside of B^d.

2.2 Samples of cubature formulas of lower degree with nodes inside B^d

We use the modified method of the Reproducing Kernel to construct cubature formulas of lower degree with nodes inside B^d.

a. Formulas of degree 5

We choose $a^{(1)} = (0, 0, \dots, 0)$ the origin of $\mathbb{R}^d$ and we define $a^{(k+1)}$, $1 \le k \le d-1$ by

$$a^{(k+1)} = (\sqrt{\frac{1}{2\mu+d+3}}, \dots, \sqrt{\frac{1}{2\mu+d+3}}, \sqrt{\frac{d+3-k}{2\mu+d+3}}, 0 \dots, 0) \quad (2.13)$$

which has $d-k$ zero components.

From the properties of the *Gegenbauer* polynomials [7], we have:

$$C_2^{(\lambda)}(t) = \lambda[2(\lambda+1)t^2 - 1], \text{ for } n = 2,$$

where $\lambda = \mu + (d+1)/2$, and follows that

$$\tilde{K}_2(W_\mu; x, y) = \lambda\Big[(2\mu+d+3) < x, y >^2 + (2\mu+d+3)(1-|x|^2)(1-|y|^2)/(2\mu+1) - 1\Big]. \quad (2.14)$$

If we take, $a^{(1)} = (0, \dots, 0)$, it follows from the formula of $\tilde{K}_2(W_\mu; x, y)$ that $H_1 = \{x : K_2(x, a^{(1)}) = 0\} = \{x : |x|^2 = (d+2)/(2\mu+d+3)\}$ and we require that the chosen point $a^{(k+1)}$ from (2.13), belongs to H_k and we obtain:

$$\tilde{K}_2(W_\mu; x, a^{k+1}) = (\mu + \frac{d+1}{2})\Big[x_1^2 + \dots + x_{k-1}^2 + (d+3-k)x_k^2 - \|x\|^2\Big],$$

from which we obtain $a^{(k+1)} \in \bigcap_{i=1}^{k} H_i$ and

$$\bigcap_{i=1}^{d} H_i = \{(\pm\sqrt{\frac{1}{2\mu+d+3}}, \ldots, \pm\sqrt{\frac{1}{2\mu+d+3}}, \pm\sqrt{\frac{3}{2\mu+d+3}})\} \tag{2.15}$$

Thus, $\bigcap_{i=1}^{d} H_i$ has the 2^d intersection points obtained from (2.15).

Now we can apply the modified Reproducing Kernel method, from which one results that the nodes of the cubature formula are $\{a^{(i)},\ i = \overline{1,d}\}$ from (2.13) and $(x^{(j)},\ j = 1, 2^d)$ from (2.15) and these nodes generates a cubature formula of degree 5 on B^d of the form (2.8). Using the formula of $\tilde{K}_2(W_\mu; x, y)$ one get the coefficients of the formula for this choice of the nodes $a^{(k+1)}$, if we consider $n = 2$

$$\lambda_1 = 1/\tilde{K}_2(W_\mu; 0, 0) = \frac{2(2\mu+1)}{(2\mu+d+1)(d+2)}, \tag{2.16}$$

$$\lambda_{k+1} = 1/\tilde{K}_2(W_\mu; a_{k+1}, a_{k+1}) = \frac{2(2\mu+d+3)}{(2\mu+d+1)(d+2-k)(d+3-k)},\ k = \overline{2,d}.$$

Then there exists the weights μ_ξ such that the following cubature formula is of degree 5 for W_μ on B^d [12].

$$\int_{B^d} f(x)W_\mu(x)dx = \frac{2(2\mu+1)}{(2\mu+d+1)(d+2)} f(0)$$

$$+\frac{2\mu+d+3}{2\mu+d+1} \sum_{k=1}^{d-1} \frac{f(a^{(k+1)}) + f(-a^{(k+1)})}{(d+2-k)(d+3-k)}$$

$$+\sum_{\xi\in\{-1,1\}^d} \mu_\xi f\left(\xi_1\sqrt{\frac{1}{2\mu+d+3}}, \ldots, \xi_{d-1}\sqrt{\frac{1}{2\mu+d+3}}, \xi_d\sqrt{\frac{3}{2\mu+d+3}}\right). \tag{2.17}$$

In this formula the weights $\mu_\xi,\ \xi = (\xi_1, \ldots, \xi_d) \in \{-1, 1\}^d$ can be determined by the condition that the formula must be exact for polynomials of degree 5.

In the case of $d = 2$, we have the explicit formula

$$\int_{B^2} f(x)W_\mu(x)dx = \frac{2(2\mu+1)}{4(2\mu+3)} f(0) + \frac{2\mu+5}{12(2\mu+3)} \big[f(2/\sqrt{2\mu+5}, 0) \tag{2.18}$$

$$+f(-2/\sqrt{2\mu+5}, 0)\big] + \frac{2\mu+5}{12(2\mu+3)} \sum f(\pm 1/\sqrt{2\mu+5}, \pm\sqrt{3}/\sqrt{2\mu+5}).$$

The formula on B^d uses $N = 2^d + 2d - 1$ nodes. According with *Möller's* lower bound [2], the cubature formula of degree 5 must have at least $N^* \geq d(d+1)+1$ nodes, then the formula (2.18) which have $N = 2^2 + 2 \cdot 2 - 1 = 7$ is minimal.

For $d = 3$, the cubature formula on B^3, which was constructed by using (2.10) in [12], have $N = 13$ nodes and is minimal; for $d = 5$, $N = 2^5 + 2 \cdot 5 - 1 = 41$ nodes which is more that the lower bound of $N^* = 5(5+1) + 1 = 31$.

Finally we obtain the formula (2.17). To determine the other coefficients, one can require that the formula be exact for the polynomials of degree at most 5.

For $d = 3$, we can choose $f(x)$ to be the test functions $x_1,\ x_1x_2,\ x_1^2,\ x_1x_2x_3$.

For the case of $d > 3$, it is useful the following formula for the nonzero moments of the weight function $W_\mu = W_\mu(x)$ ([12])

$$\int_{B^d} x_1^{2k_1} \dots x_d^{2k_d} W_\mu(x)dx = \frac{\Gamma(\mu + (d+1)/2)\ \Gamma(k_1 + 1/2) \dots\ \Gamma(k_d + 1/2)}{\pi^{d/2}\Gamma(\mu + (d+1)/2 + k_1 + \dots + k_d)}.$$

3. **Cubature formulas on the triangle using the reproducing kernel method**

We consider now, cubature formulas on the triangle using the compact formula in [12], for a family of weight functions on a d-dimensional simplex.We use the following notations: $x \in \mathbb{R}^d,\ |x|_1 = |x_1| + \dots + |x_d|,$ the l^1 norm of x, $|\alpha|_1 = \alpha_1 + \dots + \alpha_d$, the length of multiindex $\alpha \in \mathbb{N}^d$ and the standard simplex:

$$T^d = \{x \in \mathbb{R}^d :\ x_1 \geq 0, \dots, x_d \geq 0,\ 1 - |x|_1 \geq 0\}.$$

We remark that, for $d = 2$ we have T^2 which is the triangle with vertices $(0,0), (1,0)$ and $(0,1)$.

In [12] was found the compact formula for the Reproducing Kernel with respect to the weight function:

$$W_\alpha(x) = w_\alpha x_1^{\alpha_1 - 1/2} \dots x_d^{\alpha_d - 1/2}(1 - |x|_1)^{\alpha_{d+1} - 1/2},\ \alpha_i \geq 0, \tag{3.1}$$

where w_α is the normalization constant such that $\int_{T^d} W_\alpha(x)dx = 1$, namely,

$$w_\alpha = \frac{\Gamma(|\alpha|_1 + (d+1)/2)}{\Gamma(\alpha_1 + 1/2) \dots \Gamma(\alpha_{d+1} + 1/2)}.$$

Then the reproducing Kernel $K_n(W_\alpha)$ given in terms of *Gegenbauer* polynomials, has the expression [12]:

$$K_n(W_\alpha; x, y) = \int_{[-1,1]^{d+1}} C_{2n}^{(|\alpha|_1+(d+1)/2)}(\sqrt{x_1y_1}\ t_1 + \cdots + \sqrt{x_{d+1}y_{d+1}}\ t_{d+1}). \quad (3.2)$$

$$\cdot \prod_{i=1}^{d+1} c_{\alpha_i}(1-t_i^2)^{\alpha_i-1}dt,$$

where

$$x, y \in T^d,\ x_{d+1} = 1 - |x|_1,\ y_{d+1} = 1 - |y|_1,$$

and we use limit (2.4) in the case when have one $\alpha_i = 0$.

If we take $y = e_i = (0, \ldots 0, 1, 0, \ldots 0)$, the i-th element of the standard basis, with the i-th component =1, of $\mathbb{R}^d$, $1 \leq i \leq d$, then we have the following explicit formula:

$$K_n(W_\alpha; x, e_i) = A_{\alpha,i} P_n^{(|\alpha|_1+d/2-\alpha_i, \alpha_i-1/2)}(2x_i - 1)$$

where

$$A_{\alpha,i} = C_{2n}^{(|\alpha|_1+(d+1)/2)}(0)/P_n^{(|\alpha|_1+d/2-\alpha_i,\alpha_i-1/2)}(-1)$$

(see [12]).

This formula was derived in [14] from (3.2) using a product formula for *Jacobi* polynomials.

We observe that, e_i is not a common zero of $\mathbf{P}_n$. This follows from the expression of $\mathbf{P}_n^T(x)\mathbf{P}_n(y) = \sum_k P_k^n(x)P_k^n(y)$.

Let $d = 2$ and $\alpha_1 = \alpha_2 = \alpha_3 = 1/2$. Then the weight function W_α becomes a multiple of unit weight function, denoted by $W_{1/2}$, and we have: $W_{1/2}(x) = 2$.

In this case, the Reproducing Kernel takes the form:

$$K_n(W_{1/2}; x, y) = \frac{1}{\pi^3} \int_{[-1,1]^3} C_{2n}^{(3)}(\sqrt{x_1y_1}t_1 + \sqrt{x_2y_2}t_2 + \sqrt{x_3y_3}t_3) \prod_{i=1}^{3}(1-t_i^2)^{-1/2}dt$$

For $\alpha = 0,$ we have $W_0(x) = (x_1x_2x_3)^{-1/2}/2\pi$.

In [11] was shown that any cubature formula for W_0 with all nodes inside T^2 corresponds to a cubature formula on a sphere S^2. In this case, the Reproducing

Kernel can be represented in the following simple form:

$$K_n(W_0; x, y) = \frac{1}{4} \sum C_{2n}^{(3/2)}(\sqrt{x_1 y_1} \pm \sqrt{x_2 y_2} \pm \sqrt{x_3 y_3}),$$

where the sum is over all possible sign changes, and this formula follows from (3.2) by taking limits (2.4).

Samples of cubature formulas on the triangle

For $n = 2$, we have the following explicit formula for $K_n(W_{1/2}; x, y)$

$$K_2(W_{1/2}; x, y) = 6(1 - 10(x_1 y_1 + x_2 y_2 + x_3 y_3) + 60(x_1 x_2 y_1 y_2 + x_1 x_3 y_1 y_3 +$$

$$+ x_2 x_3 y_2 y_3) + 15(x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2)).$$

If we take $a^{(1)} = (1, 0)$, one obtain that $K_2(W_{1/2}, x, (1, 0))$ has two zeros,

$$z_1 = (5 - \sqrt{10})/15, \ z_2 = (5 + \sqrt{10})/15.$$

From this fact, it follows that $K_2(W_{1/2}, x, (1, 0))$ and $K_2(W_{1/2}, x, (z_1, 0))$ have 4 distinct common zeros:

$$\left((5 - \sqrt{10})/15, (70 - 7\sqrt{10} \pm \sqrt{10(233 - 62\sqrt{10})}/90\right)$$

$$\left((5 + \sqrt{10})/15, (30 - 3\sqrt{10} \pm \sqrt{3(110 - 20\sqrt{10})}/90\right).$$

4. The construction of cubature formulas by using the *Chebyshev* orthogonal polynomials and the reproducing kernel method

Let us consider, the *Chebyshev* polynomial of degree n,

$$T_n^*(x) = cos n\theta, \ x = cos\theta,$$

that is

$$T_n^*(x) = cos(n arcos x).$$

The zeros of T_n^* are $x_k = \frac{(2k-1)\pi}{2n}$, $k = \overline{1, n}$, and T_n^* are orthogonal with respect to the *Chebyshev* weight function $w_1(x) = (1 - x^2)^{-1/2}$ on $[-1, 1]$.

The zeros of T_n^* can be selected as the nodes of the Gaussian quadrature formula with respect to $w(x)$ and these zeros can be used to construct a compact interpolation formula. Let we denote the classical *Chebyshev* weight of the first kind

$$w_1(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}},\ x \in (-1,1)$$

Then the orthonormal polynomials with respect to w_1 are

$$T_0(x) = 1,\ T_k(x) = \sqrt{2}cosk\theta,\ k \geq 1,\ x = cos\theta \quad \text{and} \quad \int_{-1}^{1} w_1(x)dx = 1.$$

Next, we can consider the product *Chebyshev* weight function on $[-1,1]^2$ defined by

$$W^{(2)}(x,y) = w_1(x)w_1(y) = \frac{1}{\pi^2}\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}},\ (x,y) \in [-1,1]^2. \tag{4.1}$$

One can verify that the polynomials defined by

$$P_k^n(x,y) = T_{n-k}(x)T_k(y),\ k = \overline{0,n},\ n \in \mathbb{N}_0, \tag{4.2}$$

where P_k^n is of degree exactly n are orthogonal with respect to $W^{(2)}(x,y)$.

In [10] was established the following relations. If we denote $\mathbf{P}_n = (P_0^n, ..., P_n^n)^T, n \in \mathbb{N}_0$, the vector of the polynomials of degree exactly n in (4.2) and the matrices,

$$A_{n,1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \sqrt{2} & 0 \end{bmatrix}, \qquad A_{n,2} = \frac{1}{2}\begin{bmatrix} 0 & \sqrt{2} & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \dots & 1 \end{bmatrix},$$

it can be verified that product *Chebyshev* polynomials satisfy the three-term relation

$$x_i\mathbf{P}_n(x) = A_{n,i}\mathbf{P}_{n+1}(x) + A_{n-1,i}^T\mathbf{P}_{n-1}(x),\ i = 1,2, x = (x_1,x_2) \text{ or } x = (x,y) \tag{4.3}$$

For $x, y \in \mathbb{R}^2$, the Reproducing Kernel of the product *Chebyshev* polynomials is defined by

$$K_n(x,y) = \sum_{k=0}^{n-1}\sum_{j=0}^{k} P_j^k(x)P_j^k(y) = \sum_{k=0}^{n-1}\mathbf{P}_k^T(x)\mathbf{P}_k(y)$$

and $\mathbf{P}_n^T(x)\mathbf{P}_n(y) = K_n(x,y) - K_{n-1}(x,y)$.

If one consider $x = (cos\theta_1, cos\theta_2),\ y = (cos\varphi_1, cos\varphi_2)$, then we have the compact formula [10].

$$K_n(x,y) = D_n(\theta_1+\varphi_1, \theta_2+\varphi_2) + D_n(\theta_1+\varphi_1, \theta_2-\varphi_2) + D_n(\theta_1-\varphi_1, \theta_2+\varphi_2)$$

$$+D_n(\theta_1-\varphi_1, \theta_2-\varphi_2),$$

where the function D_n has the form

$$D_n(\theta_1,\theta_2) = \frac{1}{2}\frac{cos(n-\frac{1}{2})\theta_1 cos\frac{\theta_1}{2} - cos(n-\frac{1}{2})\theta_2 cos\frac{\theta_2}{2}}{cos\theta_1 - cos\theta_2}.$$

One can use these formulas in order to obtain a compact formula for the Lagrange interpolation, which will be used to construct a cubature formula of degree $2n-1$ with respect to $W^{(2)}(x,y)$ of the form

$$I_n(f) = \int_{[-1,1]^2} f(x,y)W^{(2)}(x,y)dxdy \simeq Q_n(f), \tag{4.4}$$

where $Q_n(f) = \sum_{k=0}^{N} \lambda_k f(x_k),\ \lambda_k > 0,\ x_k \in \mathbb{R}^2$, so that we have

$$I_n(P) = Q_n(P),\ \forall P \in \mathbb{P}^2_{2n-1}.$$

According to a general result of *Möller* for centrally symmetric weight functions, for example one can consider $W^{(2)}(x,y) = w_1(x)w_1(y)$, the number of nodes in the cubature formula satisfies

$$N \geq dim\mathbb{P}^2_{n-1} + [n/2] = \binom{n+1}{2} + [n/2].$$

Let consider z_k be the points $z_k = z_{k,n} = cos\frac{k\pi}{n},\ k = \overline{0,n}.$

In [10] was stated, based on the three-term recurrence relation (4.3), that a cubature formula exists when the following matrix equations in the variable V are solvable

$$A_{n-1,1}(VV^T - I)A^T_{n-1,2} = A_{n-1,2}(VV^T - I)A^T_{n-1,1} \tag{4.5}$$

$$\text{and} \quad V^T A^T_{n-1,1} A_{n-1,2} V = V^T A^T_{n-1,2} A_{n-1,1} V,$$

where V is a matrix of size $(n+1)\times\sigma,\ \sigma = [n/2]$ or $\sigma = [n/2]+1$.

If $n = 2m$ in [10] was showed that a solution of (4.5) is

$$T_{n-(k-1)}(x)T_{k-1}(y) - T_{k-1}(x)T_{n-(k-1)}(y), 1 \leq k \leq n/2 + 1, \tag{4.6}$$

which corresponds to (A), and if $n = 2m - 1$, a solution of (4.5) is

$$T_{n-(k-1)}(x)T_{k-1}(y) - T_{k-1}(x)T_{n-(k-1)}(y), \quad 1 \leq k \leq (n+1)/2, \tag{4.7}$$

corresponds to (B).

If a cubature formula exists, we can consider the Lagrange interpolation problem based on the nodes of the cubature formula which consists in construction of a unique polynomial which is the solution of the problem to determining $P = P(x)$ so that $P(x_k) = f(x_k),\ k = \overline{1, N}$.

In [8], was proved that one can consider the subspace

$$\mathcal{V}_n^2 = \mathbb{P}_{n-1}^2 \bigcup span\{V^+\mathbf{P}_n\},$$

where V^+ is the unique *Moore-Penrose* generalized inverse of V, and in our case we have V with full rank and we have $V^+ = (V^T V)^{-1} V^T$.

For $(x, y) \in \mathbb{R}^2$, was used the following expression of the Reproducing Kernel

$$K_n^*(x, y) = K_n(x, y) + [V^+\mathbf{P}_n(x)]^T V^+\mathbf{P}_n(y). \tag{4.8}$$

Using a modified *Christoffel-Darboux* formula, was showed in [10] that $K_n^*(x_k, x_j) = 0$ for $k \neq j$ and $K_n^*(x_k, x_k) \neq 0$.

Finally, it follows that

$$(L_n f)(x) = \sum_{k=1}^{N} \frac{K_n^*(x, x_k)}{K_n^*(x_k, x_k)} f(x_k) \tag{4.9}$$

and we have

$$\int_{[-1,1]^2} (L_n f)(x) W_0(x) dx = \sum_{k=1}^{N} \lambda_k f(x_k) = I_n(f).$$

From the condition on P_j^k and the definition of $K_n^*(\cdot, \cdot)$ it follows that the coefficients in the cubature formula are given by the expression $\lambda_k = 1/K_n^*(x_k, x_k)$

If $n = 2m$, the interpolation nodes are

$$x_{2i,2j+1} = (z_{2i}, z_{2j+1}), \quad i = \overline{0, m},\ j = \overline{0, m-1} \tag{4.10}$$

$$x_{2i+1,2j} = (z_{2i+1}, z_{2j}), \quad i = \overline{0, m-1},\ j = \overline{0, m}.$$

From (4.7) and the expression of $K_n^*(x, y)$ one can obtain

$$K_n^*(x, x_{k,l}) = \frac{1}{2}[K_n(x, x_{k,l}) + K_{n-1}(x, x_{k,l})] - \frac{1}{2}(-1)^k[T_n(x) - T_n(y)].$$

Finally, one can obtain

$$K_n^*(x_{0,2j+1}, x_{0,2j+1}) = n^2,\ K_n^*(x_{2i,2j+1}, x_{2i,2j+1}) = n^2/2,$$

$$K_n^*(x_{2i+1,0}, x_{2i+1,0}) = n^2,\ K_n^*(x_{2i+1,2j}, x_{2i+1,2j}) = n^2/2,\ i > 0, j > 0.$$

If $n = 2m - 1$, the interpolation nodes are

$$x_{2i,2j} = (z_{2i}, z_{2j}), \quad i, j = \overline{0, m-1}$$

$$x_{2i+1,2j+1} = (z_{2i+1}, z_{2j+1}), \quad i, j = \overline{0, m-1},$$

from which, was derived

$$K_n^*(x, x_{k,l}) = \frac{1}{2}[K_n(x, x_{k,l}) + K_{n-1}(x, x_{k,l})] - \frac{1}{2}(-1)^k[T_n(x) + T_n(y)],$$

from which was obtained

$$K_n^*(x_{2i,2j}, x_{2i,2j}) = \begin{cases} n^2/2, & \text{if } 0 < i, j \leq m-1 \\ n^2, & \text{if } i = 0 \text{ or } j = 0,\ i + j > 0 \\ 2n^2, & \text{if } i = j = 0, \end{cases}$$

$$K_n^*(x_{2i+1,2j+1}, x_{2i+1,2j+1}) = \begin{cases} n^2/2, & \text{if } 0 \leq i, j < m-1 \\ n^2, & \text{if } i = m-1 \text{ or } j = m-1,\ i + j < 2m-2 \\ 2n^2, & \text{if } i = j = m-1. \end{cases}$$

In [14] was proved the mean convergence of Lagrange interpolation formula corresponding to the weight function $W^{(2)}(x, y)$ and by integrating this formula one can arrive to the following cubature formulas

Based on the nodes (x_i, x_j), we obtain the cubature formulas:

A) For $n = 2m$,

$$(A)\ \frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} f(x,y)\frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}} = \frac{2}{n^2}\sum_{i=0}^{\frac{n}{2}}{}'' \sum_{j=0}^{\frac{n}{2}-1} f(z_{2i}, z_{2j+1}) +$$

$$+\frac{2}{n^2}\sum_{i=0}^{\frac{n}{2}-1}\sum_{j=0}^{\frac{n}{2}}{}'' f(z_{2i+1}, z_{2j}), \forall f \in \mathbb{P}_{2n-1}^2$$

B) For $n = 2m - 1$,

$$(B)\ \frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} f(x,y)\frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}} = \frac{2}{n^2}\sum_{i=0}^{\frac{n-1}{2}}\sum_{j=0}^{\frac{n-1}{2}} f(z_{2i}, z_{2j}) +$$

$$+\frac{2}{n^2}\sum_{i=0}^{\frac{n-1}{2}}\sum_{j=0}^{\frac{n-1}{2}} f(z_{n-2i}, z_{n-2j}), \forall f \in \mathbb{P}_{2n-1}^2,$$

where Σ' means that the first term in summation is halved.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

OPTIMIZATION PROBLEMS AND η-APPROXIMATED OPTIMIZATION PROBLEMS

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Abstract. In this paper, a so-called η-approximated optimization problem (Ref. [1] and [3]) associated to an optimization problem is considered. The equivalence between the saddle points of the lagrangian of the η-approxiated optimization problem and optimal solutions of the original optimization problem is established.

1. **Introduction**

We consider the optimization problem

$$\begin{aligned} &\min \ f(x) \\ &\text{s.t.} \ \ x \in X \\ &\qquad g_i(x) \leqq 0, \quad i \in \{1, ..., m\}, \end{aligned} \tag{P}$$

where X is a subset of $\mathbb{R}^n$ and $f, g_1, ..., g_m : X \to \mathbb{R}$ are functions.

Let

$$\mathfrak{F}(P) := \{x \in X : \ g_i(x) \leqq 0, \ i \in \{1, ..., m\}\}$$

denote the set of all feasible solutions of Problem (P).

For solving optimization problem (P), there are various manners to approach. One of these manners is that for Problem (P) one attaches another optimization problem, problem whose solutions give us the (information about) optimal solutions of the initial problem (P).

Received by the editors: 01.09.2009.

2000 *Mathematics Subject Classification.* 90C26, 90C30, 90C46.

Key words and phrases. saddle points, invex functions, pseudoinvex functions, $\eta-$approximation.

Assuming that X is open, and that f and g are differentiable on X, Mangasarian (Ref. [11]) attached to Problem (P) and the point $x^0 \in X$, the problem

$$\begin{aligned} \min\ & f\left(x^0\right) + \left\langle u, \nabla f\left(x^0\right)\right\rangle \\ \text{s.t. }\ & u \in \mathbb{R}^n \\ & g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right](u) \leqq 0. \end{aligned}$$

He took the dual of this linear optimization problem and then considered x^0 to be a variable in X. This last problem is precisely the classical dual of the nonlinear optimization problem, introduced in a different way by Wolfe (Ref. [13]) and investigated extensively (see, for example Ref. [10]). Connections between optimal solutions of the dual and the primal are known (see, for example Ref. [10]).

The above process is repeated but taking nonlinear instead of linear approximation of f and g around some fixed $x^0 \in X$ and taking the dual of the resulting optimization problem. One takes the dual of this nonlinear optimization problem and then one considers x^0 be a variable in X. One obtains the so called higher-order dual problem of Problem (P). In Ref. [11], there are given connections between the optimal solutions of higher-order dual and initial problem (P). D.I. Duca (Ref. [7]) used this idea for optimization problems in complex space.

Another idea came from Antczak (Ref. [3], [2], [1]), who attached to Problem (P) and the point $x^0 \in X$, the following problem

$$\begin{aligned} \min\ & f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta(x)\right\rangle \\ \text{s.t. }\ & x \in X \\ & g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right](\eta(x)) \leqq 0, \end{aligned} \qquad \left(P_\eta\left(x^0\right)\right)$$

where $\eta = \eta_{x^0} : X \to X$ is a function. He studied the connections between the saddle points of Problem $\left(P_\eta\left(x^0\right)\right)$ and optimal solutions of Problem (P).

We attach to Problem (P), the Lagrange function (or the lagrangian) $L : X \times \mathbb{R}^m_+ \to \mathbb{R}$ defined by

$$L(x, v) := f(x) + \langle v, g(x)\rangle, \text{ for all } (x, v) \in X \times \mathbb{R}^m_+,$$

where $g = (g_1, ..., g_m)$.

Definition 1. *We say that* $\left(x^0, v^0\right) \in X \times \mathbb{R}_+^m$ *is a saddle point of the lagrangian* L (*or of Problem* (P)) *if*

$$L\left(x^0, v\right) \leqq L\left(x^0, v^0\right) \leqq L\left(x, v^0\right), \text{ for all } (x, v) \in X \times \mathbb{R}_+^m.$$

The saddle points of the lagrangian L of Problem (P) have been studied by many authors (see for example Ref. [10], [4] and others). A fundamental result of optimization theory is that, in certain conditions, the point x^0 is an optimal solution of Problem (P) if and only if there exists a point $v^0 \in \mathbb{R}_+^m$ such that $\left(x^0, v^0\right)$ is a saddle point of its lagrangian.

More precisely, we have the following results, results which play an important role in optimization theory and economics.

Theorem 2. *If* $\left(x^0, v^0\right) \in X \times \mathbb{R}_+^m$ *is a saddle point of the lagrangian* L *of Problem* (P) *then* x^0 *is an optimal solution of Problem* (P).

Proof. See, for example, Ref. [10]. □

Theorem 3. *Let* x^0 *be an optimal solution of Problem* (P). *Assume that* $f, g_1, ..., g_m$ *are convex at* x^0 *and a suitable constraint qualification (CQ, Ref.* [10]*) is satisfied at* x^0. *Then there exists a point* $v^0 \in \mathbb{R}_+^m$ *such that* $\left(x^0, v^0\right)$ *is a saddle point of the lagrangian of Problem* (P).

Proof. See, for example, Ref. [10]. □

In the last few years, attempts have been made to weaken the convexity hypotheses and thus to explore the existence of optimality conditions applicability. Various classes of generalized convex functions have been suggested for the purpose of weakening the convexity limitation in this result. Among these, the concept of an invex function proposed by Hanson (Ref. [9]) has received more attention. The name of invex (invariant convex) function was given by Craven (Ref. [6])

Definition 4. *Let* X *be a subset of* $\mathbb{R}^n$, x^0 *be an interior point of* X, $f : X \to \mathbb{R}$ *be a differentiable function at* x^0 *and* $\eta = \eta_{x^0} : X \to \mathbb{R}^n$ *be a function. We say that* f *is invex at* x^0 *with respect to* η *if*

$$f(x) - f\left(x^0\right) \geqq \left\langle \nabla f\left(x^0\right), \eta(x)\right\rangle, \text{ for all } x \in X. \tag{1}$$

Hanson defined invex functions which allow the use of the Kuhn-Tucker conditions as sufficient conditions for optimality in constrained optimization problems. Later, Martin (Ref. [12]) proved that invexity hypotheses are not only sufficient but also necessary when using the Kuhn-Tucker optimality conditions for unconstrained optimization problems.

After the works of Hanson and Craven, other types of differentiable functions have appeared with the intent of generalizing invex function from different points of view.

Ben-Israel and Mond (Ref. [5]) defined the so-called pseudoinvex functions, generalizing pseudoconvex functions in the same way that invex functions generalize convex functions.

Definition 5. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta = \eta_{x^0} : X \to \mathbb{R}^n$, and $f : X \to \mathbb{R}$ be a differentiable function at x^0. We say that f is pseudoinvex at x^0 with respect to η if, for each $x \in X$ with the property that*

$$\left\langle \nabla f\left(x^0\right), \eta\left(x\right)\right\rangle \geqq 0,$$

we have

$$f\left(x\right) \geqq f\left(x^0\right).$$

Definition 6. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta = \eta_{x^0} : X \to \mathbb{R}^n$, and $f : X \to \mathbb{R}$ be a differentiable function at x^0. We say that f is quasiinvex at x^0 with respect to η if, for each $x \in X$ with the property that*

$$f\left(x\right) \leqq f\left(x^0\right),$$

we have

$$\left\langle \nabla f\left(x^0\right), \eta\left(x\right)\right\rangle \leqq 0.$$

Remark 7. *Note that, in general, there exists no unique function η such that the function f is invex, respectively pseudoinvex and quasiinvex at the point $x^0 \in X$.*

Indeed, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f\left(x\right) = \exp x, \text{ for all } x \in \mathbb{R},$$

is invex at $x^0 = 0$ *with respect to the function* $\eta : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\eta(x) = x - x_0 = x, \text{ for all } x \in \mathbb{R}.$$

Also, the function f *is invex at* $x^0 = 0$ *with respect to the function* $\eta : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\eta(x) = x + \frac{x^2}{2} + \frac{x^3}{6}, \text{ for all } x \in \mathbb{R}.$$

And also, the function f *is invex at* x^0 *with respect to the function* $\eta : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\eta(x) = x - 2, \text{ for all } x \in \mathbb{R}.$$

In this paper, in more general hypotheses that in Ref. [3], the equivalence between the saddle points of the lagrangian of the η-approximated optimization problem and optimal solutions of the original optimization problem is established.

2. $\eta-$approximated optimization problem

In what follows x^0 is an interior point of X, and f and g are differentiable at x^0.

For the function $\eta = \eta_{x^0} : X \to \mathbb{R}^n$, we attach to Problem (P) the optimization problem $\left(P_\eta\left(x^0\right)\right)$, called η-approximated at x^0 of Problem (P).

Remark 8. *If* $X = \mathbb{R}^n$ *and* $\eta(x) = x - x_0$, *for all* $x \in X$, *then Problem* $\left(P_\eta\left(x^0\right)\right)$ *is linear.*

Let

$$\mathfrak{F}\left(P_\eta\left(x^0\right)\right) := \{x \in X : \ g_i\left(x^0\right) + \left\langle \nabla g_i\left(x^0\right), \eta(x)\right\rangle \leqq 0, \ i \in \{1, ..., m\}\},$$

denote the set of all feasible solutions of Problem $\left(P_\eta\left(x^0\right)\right)$.

The lagrangian of Problem $\left(P_\eta\left(x^0\right)\right)$ will be denoted by L_η, i.e. $L_\eta : X \times \mathbb{R}^m_+ \to \mathbb{R}$ is defined by

$$L_\eta(x, v) := f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta(x)\right\rangle + \left\langle v, g\left(x^0\right)\right\rangle + \left\langle v, \left[\nabla g\left(x^0\right)\right](\eta(x))\right\rangle,$$

for all $(x, v) \in X \times \mathbb{R}^m_+$.

Example 9. *Let us consider the optimization problem*

$$\begin{aligned} &\min f(x) = \exp x \\ &s.t. \quad x \in X = \mathbb{R} \\ &\qquad g_1(x) = x^2 - x \leqq 0. \end{aligned} \tag{$\overline{P}$}$$

We have that $\mathfrak{F}\left(\overline{P}\right) = [0,1]$ *and* $x^0 = 0$ *is the unique optimal solution of Problem* $\left(\overline{P}\right)$.

The functions f *and* g_1 *are invex at* $x^0 = 0$ *with respect to the function* $\eta = \eta_{x^0} : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\eta(x) = x, \text{ for all } x \in \mathbb{R}.$$

Then the η*-approximated optimization problem is*

$$\begin{aligned} &\min (1 + x) \\ &s.t. \quad x \in X = \mathbb{R} \\ &\qquad - x \leqq 0, \end{aligned} \tag{$\overline{P}_\eta\left(x^0\right)$}$$

which has the optimal solution $x^0 = 0$.

On the other hand, the lagrangian $\overline{L}_\eta$ *of Problem* $\left(\overline{P}_\eta\left(x^0\right)\right)$ *is defined by*

$$\overline{L}_\eta(x, v) = 1 + x - vx, \text{ for all } (x, v) \in \mathbb{R} \times \mathbb{R}_+.$$

Obviously, $\left(x^0, v^0\right) = (0, 1)$ *is a saddle point of the lagrangian* $\overline{L}_\eta$.

In this section we show the equivalence between saddle points of the lagrangian L_η, of Problem $\left(P_\eta\left(x^0\right)\right)$, and optimal solutions of Problem $\left(P_\eta\left(x^0\right)\right)$.

By Theorem 2, the following saddle point theorem follows:

Theorem 10. *If* $\left(x^0, v^0\right) \in X \times \mathbb{R}_+^m$ *is a saddle point of the lagrangian* L_η *of Problem* $\left(P_\eta\left(x^0\right)\right)$, *then* x^0 *is an optimal solution of Problem* $\left(P_\eta\left(x^0\right)\right)$.

Remark 11. *We established Theorem 10, without any assumption about the functions involved in Problem* $\left(P_\eta\left(x^0\right)\right)$.

In order to prove that if $x^0 \in X$ is an optimal solution of Problem $\left(P_\eta\left(x^0\right)\right)$, then there exists a point $v^0 \in \mathbb{R}_+^m$ such that $\left(x^0, v^0\right)$ is a saddle point of the lagrangian of Problem $\left(P_\eta\left(x^0\right)\right)$, let us denote by $F, G_1, ..., G_m : X \to \mathbb{R}$ the functions defined by

$$F(x) := f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta(x)\right\rangle,$$

$$G_i(x) := g_i\left(x^0\right) + \left\langle \nabla g_i\left(x^0\right), \eta(x)\right\rangle, \ i \in \{1, ..., m\},$$

for all $x \in X$.

Obviously, Problem $\left(P_\eta\left(x^0\right)\right)$ can be written as

$$\begin{aligned} &\min F(x) \\ &\text{s.t.} \quad x \in X \\ &\qquad G_i(x) \leqq 0, \quad i \in \{1, ..., m\}. \end{aligned}$$

Now, we can state the converse theorem of Theorem 10.

Theorem 12. *Let $x^0 \in X$ be an optimal solution of Problem $\left(P_\eta\left(x^0\right)\right)$, $\mu = \mu_{x^0} : X \to \mathbb{R}^n$ be a function. Assume that $\eta : X \to \mathbb{R}^n$ is differentiable at x^0, the functions $F, G_1, ..., G_m : X \to \mathbb{R}$ are invex at x^0 with respect to μ and a suitable constraint qualification (CQ, Ref* [10]*) is satisfied at x^0. Then there exists a point $v^0 \in \mathbb{R}^m_+$ such that $\left(x^0, v^0\right)$ is a saddle point of Problem $\left(P_\eta\left(x^0\right)\right)$.*

Proof. Let $G = (G_1, ..., G_m)$. In view of Karush-Kuhn-Tucker theorem, there exists a point $v^0 \in \mathbb{R}^m_+$ such that

$$\nabla F\left(x^0\right) + \left[\nabla G\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right) = 0, \tag{2}$$

$$\left\langle v^0, G\left(x^0\right)\right\rangle = 0, \tag{3}$$

i.e.

$$\nabla f\left(x^0\right) + \left\langle v^0, \left[\nabla g\left(x^0\right)\right]\left(\nabla \eta\left(x^0\right)\right)\right\rangle = 0,$$

$$\left\langle v^0, g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right)\right\rangle = 0.$$

The functions F, $G_1, ..., G_m$ are invex at x^0 with respect to μ, then, for each $x \in X$, we have

$$F(x) - F\left(x^0\right) \geqq \left\langle \nabla F\left(x^0\right), \mu(x)\right\rangle, \tag{4}$$

$$G_i(x) - G_i\left(x^0\right) \geqq \left\langle \nabla G_i\left(x^0\right), \mu(x)\right\rangle, \ i \in \{1, ..., m\}. \tag{5}$$

Since $v^0 \in \mathbb{R}^m_+$, by (5), we obtain

$$\left\langle v^0, G(x)\right\rangle - \left\langle v^0, G\left(x^0\right)\right\rangle \geqq \left\langle v^0, \left[\nabla G\left(x^0\right)\right](\mu(x))\right\rangle, \text{ for all } x \in X. \tag{6}$$

Then, for each $x \in X$,

$$\begin{aligned} L_\eta\left(x, v^0\right) - L_\eta\left(x^0, v^0\right) = \\ = F(x) + \left\langle v^0, G(x)\right\rangle - F\left(x^0\right) - \left\langle v^0, G\left(x^0\right)\right\rangle \geqq \text{ (by (4), and (6))} \\ \geqq \left\langle \nabla F\left(x^0\right), \mu(x)\right\rangle + \left\langle v^0, \left[\nabla G\left(x^0\right)\right](\mu(x))\right\rangle = \\ = \left\langle \nabla F\left(x^0\right) + \left[\nabla G\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right), \mu(x)\right\rangle = \text{ (by (2))} \\ = 0. \end{aligned}$$

Consequently, the second inequality in the definition of saddle point is satisfied.

In order to prove the first inequality of the definition of saddle point, let $v \in \mathbb{R}^m_+$. Then

$$\begin{aligned} L_\eta\left(x^0, v^0\right) - L_\eta\left(x^0, v\right) = \\ = \left\langle v^0, G\left(x^0\right)\right\rangle - \left\langle v, G\left(x^0\right)\right\rangle = \text{ (by (3))} \\ = -\left\langle v, G\left(x^0\right)\right\rangle \geqq \\ \geqq 0, \end{aligned}$$

because $G\left(x^0\right) \leqq 0$ $\square$

3. Equivalence between saddle points of η-approximated problem and of the original problem

In this section we will prove the equivalence between the original optimization problem (P) and its associated η-approximated optimization problem $\left(P_\eta\left(x^0\right)\right)$. We establish the results where one assumes that the function $\eta = \eta_{x^0}$ satisfies only the condition $\eta\left(x^0\right) = 0$.

In Ref. [1] one proves the following statement:

Theorem 13. *Let x^0 be a feasible solution of Problem (P). We assume that f and g are invex at x^0 on $\mathfrak{F}(P)$ with respect to $\eta = \eta_{x^0} : X \to \mathbb{R}^n$ satisfying the condition $\eta(x^0) = 0$. If $(x^0, v^0) \in \mathfrak{F}(P) \times \mathbb{R}^m_+$ is a saddle point of the η-approximated optimization problem $(P_\eta(x^0))$, then x^0 is an optimal solution of the original optimization problem (P).*

This theorem is true in more general hypotheses.

If x^0 is a feasible solution of Problem (P), then

$$I(x^0) = \{i \in \{1, ..., m\} : \ g_i(x^0) = 0\}$$

denote the indices of the active restrictions at x^0.

The following statement is true

Theorem 14. *Let $x^0 \in X$, $\eta = \eta_{x^0} : X \to \mathbb{R}^n$ such that $\eta(x^0) = 0$, $f : X \to \mathbb{R}$ be pseudoinvex at x^0 with respect to η and $g_1, ..., g_m : X \to \mathbb{R}$ such that g_i, $i \in I(x^0)$ are quasiinvex at x^0 with respect to η.*

If $(x^0, v^0) \in X \times \mathbb{R}^m_+$ is a saddle point of the lagrangian L_η of Problem $(P_\eta(x^0))$, then x^0 is an optimal solution of the original problem (P).

Proof. The point $(x^0, v^0) \in X \times \mathbb{R}^m_+$ is a saddle point of the lagrangian L_η of Problem $(P_\eta(x^0))$; then

$$L_\eta(x^0, v) \leqq L_\eta(x^0, v^0), \text{ for all } v \in \mathbb{R}^m_+,$$

i.e.

$$(v - v^0) g(x^0) \leqq 0, \text{ for all } v \in \mathbb{R}^m_+, \tag{7}$$

because $\eta(x^0) = 0$.

Let $i \in \{1, ..., m\}$, and $e^i = (0, ..., 1, ..., 0) \in \mathbb{R}^m$ be the i-th unit vector of $\mathbb{R}^m$. Then, for $v = e^i + v^0 \in \mathbb{R}^m_+$, relation (7) becomes $g_i(x^0) \leqq 0$. Hence

$$g_i(x^0) \leqq 0, \text{ for all } i \in \{1, ..., m\}.$$

Consequently,

$$x^0 \in \mathfrak{F}(P).$$

If follows that

$$\left\langle v^0, g\left(x^0\right)\right\rangle \leqq 0, \tag{8}$$

because $v^0 \in \mathbb{R}^m_+$. But, from (7) we deduce

$$\left\langle v^0, g\left(x^0\right)\right\rangle \geqq 0, \tag{9}$$

because $v = 0 \in \mathbb{R}^m_+$.

Thus, by (8) and (9)

$$\left\langle v^0, g\left(x^0\right)\right\rangle = 0. \tag{10}$$

From (10) it follows that

$$v_i^0 = 0, \text{ for all } i \in \{1, ..., m\} \backslash I\left(x^0\right). \tag{11}$$

On the other hand, from

$$L_\eta\left(x^0, v^0\right) \leqq L_\eta\left(x, v^0\right), \text{ for all } x \in X,$$

we deduce that

$$\left\langle \nabla f\left(x^0\right), \eta\left(x\right)\right\rangle + \left\langle v^0, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x\right)\right)\right\rangle \geqq 0, \text{ for all } x \in X. \tag{12}$$

In order to prove that x^0 is an optimal solution of Problem (P), let $x \in \mathfrak{F}(P)$. Then

$$g_i\left(x\right) \leqq 0, \text{ for all } i \in \{1, ..., m\}.$$

Let $i \in I\left(x^0\right)$. Since

$$g_i\left(x\right) - g_i\left(x^0\right) = g_i\left(x\right) \leqq 0,$$

and g_i is quasiinvex at x^0 with respect to η, we have

$$\left\langle \nabla g_i\left(x^0\right), \eta\left(x\right)\right\rangle \leqq 0,$$

hence

$$v_i^0 \left\langle \nabla g_i\left(x^0\right), \eta\left(x\right)\right\rangle \leqq 0,$$

because $v_i^0 \geqq 0$. Then

$$\left\langle v^0, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x\right)\right)\right\rangle \leqq 0, \tag{13}$$

because $v_i^0 = 0$, for all $i \in \{1, ..., m\} \backslash I\left(x^0\right)$.

From (12) and (13) it follows that

$$\left\langle \nabla f\left(x^0\right), \eta(x)\right\rangle \geqq -\left\langle v^0, \left[\nabla g\left(x^0\right)\right](\eta(x))\right\rangle \geqq 0. \tag{14}$$

But, the function f is pseudoinvex at x^0 with respect to η, and then, by (14), we deduce that

$$f(x) \geqq f\left(x^0\right).$$

Consequently, x^0 is an optimal solution of the original problem (P). The theorem is proved. □

Remark 15. *If the functions $f, g_1, ..., g_m$ are invex at x^0 with respect to η, then the hypotheses that f is pseudoinvex at x^0 with respect to η and g_i, $i \in I\left(x^0\right)$ are quasiinvex at x^0 with respect to η are satisfied.*

Remark 16. *The assumption that the function η satisfies the condition $\eta\left(x^0\right) = 0$ is essential in order to have the equivalence between the saddle points of the lagrangian L_η of Problem $\left(P_\eta\left(x^0\right)\right)$, and the optimal solutions of the original problem (P). (see Example 3.4 from Ref.* [1]*)*

Now, we show that, if x^0 is an optimal solution of the original problem (P), then under certain conditions, there exists a point $v^0 \in \mathbb{R}_+^m$ such that $\left(x^0, v^0\right)$ is a saddle point of the η-approximated problem $\left(P_\eta\left(x^0\right)\right)$.

More exactly, the following statement is true:

Theorem 17. *Let $x^0 \in X$ be an optimal solution of the original problem (P) and assume that a suitable constraint qualification is satisfied at x^0 (CQ in Ref.* [10]*). If the function $\eta = \eta_{x^0} : X \to \mathbb{R}^n$ satisfies:*

(i) $\left\langle \nabla f\left(x^0\right), \eta\left(x^0\right)\right\rangle \leqq 0$;

(ii) $g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right) \leqq 0$ *(i.e.* $x^0 \in \mathfrak{F}\left(P_\eta\left(x^0\right)\right)$*)*,

then there exists a point $v^0 \in \mathbb{R}_+^m$ such that $\left(x^0, v^0\right)$ is a saddle point of the lagrangian L_η of the η-approximated problem $\left(P_\eta\left(x^0\right)\right)$.

Proof. Since x^0 is an optimal solution of Problem (P), and some suitable constraint qualification at x^0 is satisfied, by Karush-Kuhn-Tucker' Theorem, there exists a point $v^0 \in \mathbb{R}_+^m$ such that

$$\nabla f\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right) = 0, \tag{15}$$

$$\left\langle v^0, g\left(x^0\right)\right\rangle = 0. \tag{16}$$

Let $x \in X$. Then, from (15), we have

$$L_\eta\left(x, v^0\right) - L_\eta\left(x^0, v^0\right) = \left\langle \nabla f\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right), \eta\left(x^0\right)\right\rangle = 0.$$

Consequently, the second inequality from the saddle point definition is true.

In order to prove the first inequality from the saddle point definition, let $v \in \mathbb{R}_+^m$. Then

$$L_\eta\left(x^0, v^0\right) - L_\eta\left(x^0, v\right) =$$

$$= \left\langle v^0, g\left(x^0\right)\right\rangle - \left\langle v, g\left(x^0\right)\right\rangle + \left\langle v^0, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right)\right\rangle - \left\langle v, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right)\right\rangle =$$

$$= -\left\langle v, g\left(x^0\right)\right\rangle + \left\langle \eta\left(x^0\right), \left[\nabla g\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right)\right\rangle - \left\langle v, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right)\right\rangle =$$

$$= -\left\langle v, g\left(x^0\right)\right\rangle - \left\langle \nabla f\left(x^0\right), \eta\left(x^0\right)\right\rangle - \left\langle v, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right)\right\rangle =$$

$$= -\left\langle \nabla f\left(x^0\right), \eta\left(x^0\right)\right\rangle - \left\langle v, g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right)\right\rangle \geqq$$

$$\geqq -\left\langle \nabla f\left(x^0\right), \eta\left(x^0\right)\right\rangle \geqq$$

$$\geqq 0.$$

Consequently, $\left(x^0, v^0\right)$ is a saddle point of the lagrangian of Problem $\left(P_\eta\left(x^0\right)\right)$. □

Remark 18. *If $\eta\left(x^0\right) = 0$, then the hypotheses (i) and (ii) from Theorem 17 are satisfied.*

Remark 19. *If $f, g_1, ..., g_m$ are invex at x^0 with respect to η, then the hypotheses (i) and (ii) from Theorem 17 are satisfied.*

Remark 20. *The hypothesis that the original problem (P) satisfies a suitable constraint qualification at x^0 is essential. Indeed, for the problem*

$$\begin{aligned} &\min\ f(x) = x_2 \\ &s.t.\quad x \in X = \mathbb{R}^2 \\ &\qquad g_1(x) = x_1 + x_2^2 \leqq 0, \\ &\qquad g_2(x) = -x_1 + x_2^2 \leqq 0, \end{aligned} \tag{$\widehat{P}$}$$

we have the set of all feasible solutions $\mathfrak{F}\left(\widehat{P}\right) = \{(0,0)\}$, and hence $x^0 = (0,0)$ is the unique optimal solution. Let us remark that Problem $\left(\widehat{P}\right)$ is convex, and then the functions f, g_1, g_2 are invex at $x^0 = (0,0)$ with respect to $\eta : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\eta(x) = x,\ \textit{for all}\ x \in \mathbb{R}^2.$$

In this case, the η-approximated optimization problem is

$$\begin{aligned} &\min\ x_2 \\ &s.t.\quad (x_1, x_2) \in \mathbb{R}^2 \\ &\qquad -x_1 \leqq 0, \\ &\qquad\quad x_1 \leqq 0. \end{aligned} \tag{$\widehat{P}_\eta(x^0)$}$$

Thus, $\widehat{L}_\eta : \mathbb{R}^2 \times \mathbb{R}^2_+ \to \mathbb{R}$ is defined by

$$\widehat{L}_\eta(x, v) = x_2 - v_1 x_1 + v_2 x_1,\ \textit{for all}\ (x, v) = ((x_1, x_2), (v_1, v_2)) \in \mathbb{R}^2 \times \mathbb{R}^2_+,$$

and $\left(x^0, v^0\right)$, where $v^0 = \left(v_1^0, v_2^0\right) \geqq 0$, is not a saddle point of the lagrangian of Problem $\left(\widehat{P}_\eta\left(x^0\right)\right)$.

4. Conclusions

In this paper one shows that the invexity hypotheses from paper [3] can be weaker.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

THERMAL STRESSES IN A THIN POROUS PLATE

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Abstract. The thermal stresses that appear in a thin porous plate are analized and numerical results are obtained with FreeFem++.

1. **Introduction**

The porous plates have been recently studied, in particular, the silicon thin porous plates which are component parts of electronic engines (integrate circuits, transistors) and are often used in nanotechnology.

An existence and uniqueness result for the problem with initial data and boundary conditions was established by Bîrsan [1], using the logarithmic convexity method. Kumar and Rani [6] determined an analytical solution for the equilibrium equations for the generalized thermoelastic half-space with voids using the Laplace and Fourier transforms.

In what follows, based on the representation theory, we shall establish an existence and uniqueness theorem, using the theory of semigroups [8]. In order to obtain numerical results modeled with FreeFem++, it is necessary to give the variational formulation of the limit problem (1.1).

We are interested to study the thermal effect on a thin porous plate (deformation and thermal stresses), not taking into account the chemical and physical phenomena that appear under the action of the thermal field. In order to do that, we need a representation theorem of the solution of the limit problem (1.1).

Received by the editors: 27.04.2009.

2000 *Mathematics Subject Classification.* 35J20, 37K05, 74A10, 74A15, 74A35.

Key words and phrases. porous plates, thermal stresses.

This paper was presented at the International Conference on Nonlinear Operators, Differential Equations and Applications, July 4-8, 2007, Cluj-Napoca, Romania.

Consider a porous media having the form of a rectangular plate that fulfills a domain $B \subset \mathbb{R}^3$. The geometry of the plate is described with respect to an orthonormal positively oriented frame $Ox_1x_2x_3$, having the axis Ox_1 and Ox_2 in the median plane Σ of the plate.

We shall reduce the study of the system to the 2-dimensional case (in the median plane), using the micropolar theory of thermoelastic media introduced by Eringen [3].

Following Lord and Shulman [7], Green and Lindsay [4] and Ieşan [5], the field equations and constitutive relations in a generalized thermoelastic solid with voids, without body forces, heat sources and extrinsic equilibrated body force are:

$$\begin{cases} (\lambda+\mu)\dfrac{\partial}{\partial x_i}(\operatorname{div}\bar{u}) + \mu\Delta u_i + b\dfrac{\partial\phi}{\partial x_i} - \beta\dfrac{\partial\theta}{\partial x_i} + \rho_0 f_i^* = \rho_0\ddot{u}_i, \quad i=\overline{1,3} \\ \alpha\Delta\phi - b(\operatorname{div}\bar{u}) - \xi\phi + m\theta + \rho_0 l^* = \rho_0\chi\ddot{\phi} \\ T_0[\beta(\operatorname{div}\dot{\bar{u}}) + m\dot{\phi} + a\dot{\theta}] = k\Delta\theta + \rho_0 S^* \end{cases}, \tag{1.1}$$

on $B\times(0,t_0)$, where by $\bar{u}$ we denoted the displacement field, θ stands for the variation of the absolute temperature, Φ is the change in volume fraction field, ρ_0 is the density of the medium, λ, μ are the Lame's constants, k is the thermal conduction coefficient and a, b, m, α, β, ξ are the constitutive coefficients.

Denote by $f_i = \rho_0 f_i^*$ the density of the body forces. Assume that $\bar{f} \in C^0(\bar{B}\times(0,t_0))$ and $\bar{f} \in C^{2,1}(B\times(0,t_0))$. Then

$$\bar{f} = \operatorname{grad} Q + \operatorname{rot}\gamma,$$

where $Q,\gamma \in C^{2,1}(B\times(0,t_0))$ and $\operatorname{div}\gamma = 0$. Assume that $\beta \neq 0$. Put

$$\bar{u} = \operatorname{grad}\Phi + \operatorname{rot}\psi. \tag{1.2}$$

The first equation of the system (1.1) becomes

$$\mu\Delta\bar{u} + (\lambda+\mu)\operatorname{grad}(\operatorname{div}\bar{u}) + b\operatorname{grad}\phi - \beta\operatorname{grad}\theta - \rho_0\frac{\partial^2\bar{u}}{\partial t^2} = -\bar{f}$$

$$\Longleftrightarrow \mu\Delta(\operatorname{grad}\Phi + \operatorname{rot}\psi) + (\lambda+\mu)\operatorname{grad}(\operatorname{div}(\operatorname{grad}\Phi + \operatorname{rot}\psi)) + b\operatorname{grad}\phi -$$
$$-\beta\operatorname{grad}\theta - \rho_0\frac{\partial^2}{\partial t^2}(\operatorname{grad}\Phi + \operatorname{rot}\psi) = -(\operatorname{grad}Q + \operatorname{rot}\gamma)$$

$$\Longleftrightarrow \operatorname{grad}[(\lambda+2\mu)\Delta\Phi - \rho_0\ddot{\Phi} + b\phi - \beta\theta + Q] + \operatorname{rot}[\mu\Delta\phi - \rho_0\ddot{\psi} + \gamma] = 0.$$

The first equation of the system (1.1) is satisfied if we take

$$\Delta\Phi - \frac{\rho_0}{\lambda+2\mu}\frac{\partial^2\Phi}{\partial t^2} = \frac{1}{\lambda+2\mu}(\beta\theta - b\phi - Q)$$

$$\Longleftrightarrow (\Delta - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi = \frac{1}{\lambda+2\mu}(\beta\theta - b\phi - Q), \tag{1.3}$$

where $c_1 = \sqrt{\dfrac{\lambda+2\mu}{\rho_0}}$, and respectively,

$$\Delta\psi - \frac{\rho_0}{\mu}\frac{\partial^2\psi}{\partial t^2} = -\frac{1}{\mu}\gamma$$

$$\Longleftrightarrow (\Delta - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2})\psi = -\frac{1}{\mu}\gamma,$$

where $c_2 = \sqrt{\dfrac{\mu}{\rho_0}}$.

We obtain

$$\theta = \frac{1}{\beta}[(\lambda+2\mu)(\Delta - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi + b\psi + Q],$$

and respectively,

$$\psi = \frac{1}{b}[-(\lambda+2\mu)(\Delta - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi + b\theta - Q].$$

Replacing $\bar{u}$ and θ in the last equation of the system (1.1) we get

$$T_0[\beta\frac{\partial}{\partial t}\operatorname{div}(\operatorname{grad}\Phi + \operatorname{rot}\psi) + m\dot{\psi} + \frac{\alpha}{\beta}\frac{\partial}{\partial t}((\lambda+2\mu)(\delta - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi + b\phi + Q)] =$$

$$= \frac{k}{\beta}\Delta((\lambda+2\mu)(\delta - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi + b\phi + Q) + \rho_0 S^*.$$

Multiplying this relation by $\dfrac{\beta}{a(\lambda+2\mu)}$, it becomes

$$[(\frac{k}{a}\Delta - \frac{\partial}{\partial t})(\Delta - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}) - \frac{\beta^2 T_0}{a\rho_0 c_1^2}\frac{\partial}{\partial t}\Delta]\Phi = -(\frac{k}{a}\Delta - \frac{\partial}{\partial t})Q - \frac{b}{\lambda+2\mu}[\frac{k}{a}\Delta -$$

$$-(1 + \frac{mT_0\beta}{ab})\frac{\partial}{\partial t}]\phi - \frac{\beta\rho_0}{a(\lambda+2\mu)}S^*. \tag{1.4}$$

Replacing $\bar{u}$ and ϕ in the second equation of the system (1.1) we get

$$\frac{\alpha}{b}\Delta[-(\lambda+2\mu)(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi+\beta\theta-Q]-b\Delta\Phi-\frac{\xi}{b}[-(\lambda+2\mu)(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi+\beta\theta-Q]+$$

$$+m\Theta-\frac{\rho_0\chi}{b}\frac{\partial^2}{\partial t^2}[-(\lambda+2\mu)(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi+\beta\theta-Q]=-\rho_0 l^*.$$

Multiplying this relation by $\dfrac{b}{\rho_0\chi(\lambda+2\mu)}$, it becomes

$$[(\frac{\partial^2}{\partial t^2}-\frac{\alpha}{\rho_0\chi}\Delta)(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})-(\frac{b^2-\xi(\lambda+2\mu)}{\rho_0\chi}\Delta+\frac{\xi}{\chi}\frac{\partial^2}{\partial t^2})]\Phi=\frac{1}{\lambda+2\mu}(\beta\frac{\partial^2}{\partial t^2}-$$

$$-\frac{\alpha\beta}{\rho_0\chi}\Delta+(\xi\beta-bm))\theta+\frac{1}{\lambda+2\mu}[-\frac{\partial^2}{\partial t^2}+\frac{\alpha}{\rho_0\chi}\Delta-\frac{\xi}{\rho_0\chi}]Q-\frac{b}{\chi(\lambda+2\mu)}l^*. \quad (1.5)$$

Therefore, it holds a Deresiewicz [2] - Zorski [9] theorem:

Theorem 1.1. *Let* $\bar{u}=\operatorname{grad}\phi+\operatorname{rot}\psi$ *and* $\theta=\frac{1}{\beta}[(\lambda+2\mu)(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi+b\psi+Q]$, *where* $\Phi\in C^{4,4}(\bar{B}\times(0,t_0))$ *and* $\psi\in C^{3,2}(\bar{B}\times(0,t_0))$ *satisfy the relations (1.3), (1.4), (1.5). Then* $\bar{u},\theta$ *and* ψ *satisfy the system (1.1).*

2. Existence and uniqueness

According to the micropolar theory of thermoelasticity for elastic media with voids introduced by Eringen [3], we shall assume

$$\bar{u}^{(1)}=(x_3v_1,x_3v_2,w)$$

$$\bar{u}^{(2)}=\operatorname{grad}(x_3\Phi)$$

$$\phi=x_3\psi,\ \theta=x_3T$$

where the functions v_1,v_2,w,Φ,ψ,T depend on x_1,x_2,t $[(x_1,x_2)\in\Sigma,\ t\in\mathfrak{T}]$.

Using the representation theorem 1.1, the equilibrium equations can be reduced to the following systems:

$$\begin{cases}\dfrac{\partial^2 v_1}{\partial t^2}-\dfrac{\mu}{\rho_0}\Delta v_1=0\\ \dfrac{\partial^2 v_2}{\partial t^2}-\dfrac{\mu}{\rho_0}\Delta v_2=0\\ \dfrac{\partial^2 w}{\partial t^2}-\dfrac{\mu}{\rho_0}\Delta w=0\end{cases} \quad (2.1)$$

and

$$\begin{cases} \ddot{\Phi} = \frac{\lambda+2\mu}{\rho_0}\Delta\Phi + \frac{b}{\rho_0}\psi - \frac{\beta}{\rho_0}T \\ \ddot{\psi} = \frac{\alpha}{\rho_0\chi}\Delta\psi - \frac{b}{\rho_0}\Delta\Phi - \frac{\xi}{\rho_0}\psi + \frac{m}{\rho_0}T \\ \dot{T} = \frac{k}{\rho_0 c_l}\Delta T - \frac{\beta T_0}{\rho_0 c_l}\Delta\dot{\Phi} - \frac{mT_0}{\rho_0 c_l}\dot{\psi} \end{cases} . \tag{2.2}$$

The last one can be decomposed into

$$\begin{cases} \dot{\Phi} = \zeta \\ \dot{\zeta} = \frac{\lambda+2\mu}{\rho_0}\Delta\Phi + \frac{b}{\rho_0}\psi - \frac{\beta}{\rho_0}T \\ \dot{\psi} = \tau \\ \dot{\tau} = \frac{\alpha}{\rho_0\chi}\Delta\psi - \frac{b}{\rho_0}\Delta\Phi - \frac{\xi}{\rho_0}\psi + \frac{m}{\rho_0}T \\ \dot{T} = \frac{k}{\rho_0 c_l}\Delta T - \frac{\beta T_0}{\rho_0 c_l}\Delta\dot{\Phi} - \frac{mT_0}{\rho_0 c_l}\dot{\psi} \end{cases} \tag{2.3}$$

which is equivalent to

$$\begin{cases} \dot{\Phi} = \zeta \\ \dot{\zeta} - \frac{\lambda+2\mu}{\rho_0}\Delta\Phi = \frac{b}{\rho_0}\psi - \frac{\beta}{\rho_0}T \\ \dot{\psi} = \tau \\ \dot{\tau} - \frac{\alpha}{\rho_0\chi}\Delta\psi + \frac{b}{\rho_0}\Delta\Phi = -\frac{\xi}{\rho_0}\psi + \frac{m}{\rho_0}T \\ \dot{T} - \frac{k}{\rho_0 c_l}\Delta T + \frac{\beta T_0}{\rho_0 c_l}\Delta\dot{\Phi} = -\frac{mT_0}{\rho_0 c_l}\dot{\psi} \end{cases} . \tag{2.4}$$

Write the system (2.2) as an evolution system of order 1 associated to a strongly elliptic operator A on a Hilbert space.

Define $D(A) := (H^2(\Sigma) \times H^1(\Sigma)) \times (H^2(\Sigma) \times H^1(\Sigma)) \times H^2(\Sigma) =: V(\Sigma)$ and for $W = (\Phi, \zeta, \psi, \tau, T)^t \in D(A)$, let

$$AW := M\Delta W,$$

where $M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{\lambda+2\mu}{\rho_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{b}{\rho_0} & 0 & \frac{\alpha}{\rho_0\chi} & 0 & 0 \\ 0 & -\frac{\beta T_0}{\rho_0 c_l} & 0 & 0 & \frac{k}{\rho_0 c_l} \end{pmatrix}$.

Denote by $|||\cdot|||$ the norm $||\cdot||_{V(\Sigma)}$ in the product space $V(\Sigma)$.

The system (2.4) is an evolution system associated to the operator $-\Delta$ [8] and can be written in the operatorial form:

$$\frac{\partial W}{\partial t} - AW = \bar{F}(t, x_1, x_2, W), \tag{2.5}$$

for $\bar{F}(t, x_1, x_2, W) := NW$, where $N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{b}{\rho_0} & 0 & -\frac{\beta}{\rho_0} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{\xi}{\rho_0} & 0 & \frac{m}{\rho_0} \\ 0 & 0 & 0 & -\frac{mT_0}{\rho_0 c_l} & 0 \end{pmatrix}$.

Consider the initial data $W(0, x_1, x_2) = W_0(x_1, x_2)$ on Σ and the boundary condition $W(t, x_1, x_2) = 0$ for $(x_1, x_2) \in \partial\Sigma$. Following Pazy [8] (chapters 7, 8), we can state:

Proposition 2.1. *Let Σ be a domain in $\mathbb{R}^2$ with smooth boundary and $\bar{F} = (F_1, F_2, F_3, F_4, F_5, F_6)$ with every component continuous locally Lipschitz function of all its arguments. Assume that there is some continuous functions $\eta_i : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}_+$, $1 \leq i \leq 5$, such that*

$$|F_i(t, x_1, x_2, W)| \leq \eta_i(t, |||W|||), \quad 1 \leq i \leq 5$$

and

$$|F_i(t, x_1, x_2, W_1) - F_i(t, x_1, x_2, W_2)| \leq \eta_i(t, |||W_1||| + |||W_2|||), \quad 1 \leq i \leq 5.$$

For every $W_0 \in (H^2(\Sigma) \times H_0^1(\Sigma)) \times ... \times (H^2(\Sigma) \times H_0^1(\Sigma))$, the initial value problem

$$\begin{cases} \dfrac{\partial W}{\partial t} - AW = \bar{F}(t, x_1, x_2, W) \\ W(0, x_1, x_2) = W_0(x_1, x_2) \quad \text{on} \quad \Sigma \end{cases}$$

i): *has a unique solution $W = (\Phi, \zeta, \psi, \tau, T)^t \in (H^2(\Sigma) \times L^2(\Sigma)) \times (H^2(\Sigma) \times L^2(\Sigma)) \times H^2(\Sigma)$, if $F_i \in C_0^\infty(\Sigma)$ for every $1 \leq i \leq 5$;*

ii): *has a unique solution $W = (\Phi, \zeta, \psi, \tau, T)^t \in V(\Sigma)$, if $\bar{F} \in (H^1(\Sigma) \times L^2(\Sigma)) \times (H^1(\Sigma) \times L^2(\Sigma)) \times H^1(\Sigma)$.*

3. **Numerical results**

We shall give a numerical modeling and simulation for the thermal stress of an elastic, thin, porous plate made up of magnesium, using FreeFem++.

Assume that the heat transport is realized by conduction as long as there exists an internal thermic source whose temperature is constant.

Consider the following initial conditions:

$S_0 = 998$ K - thermal source

$T_0 = 298$ K - initial temperature of plate

$\Phi(x_1, x_2, 0) = 0$

$\psi(x_1, x_2, 0) = 1,0011507$ - porosity

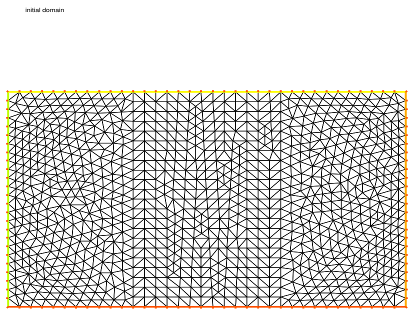
The physical constants of the material and the parameters of voids can be found in [6].

We shall model the case when the body forces are uniformly distributed orthogonal to the median plane of the plate.

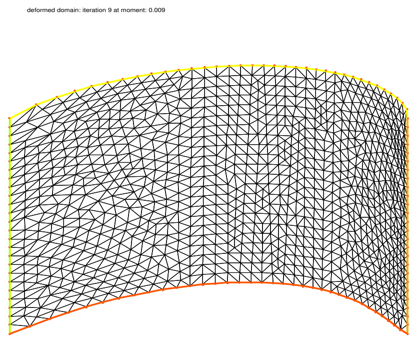
The dependence on temperature of the deformations, of the stresses and of the change in volume fraction field are further showed.

One can notice that after a number of iterations, the plate reached a thermal equilibrium state.

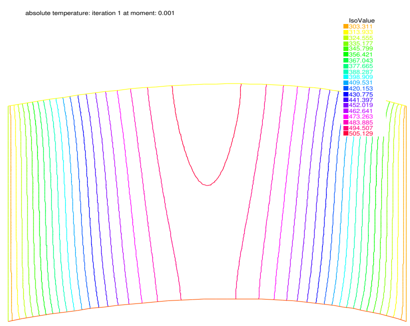
The numerical results are presented in the nearby graphics.



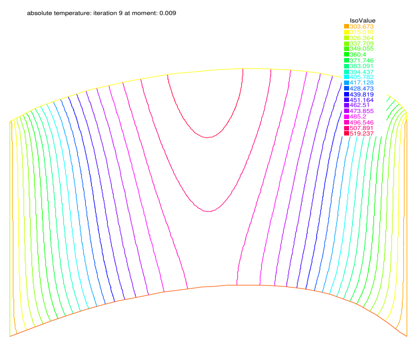
initial domain



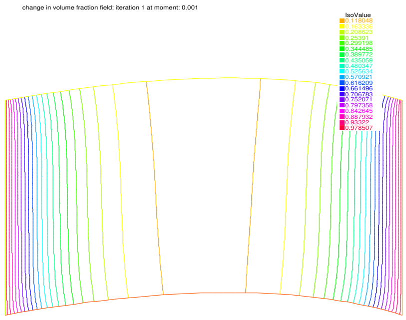
deformed domain: iteration 9



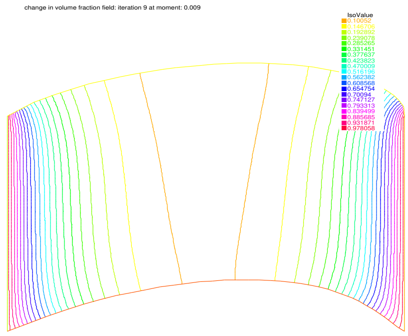
absolute temperature: iteration 1



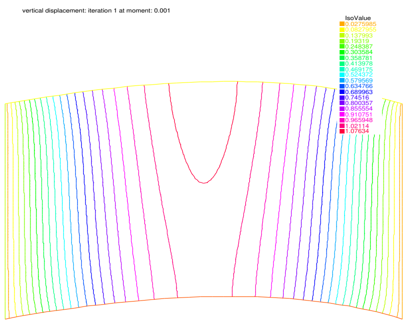
absolute temperature: iteration 9



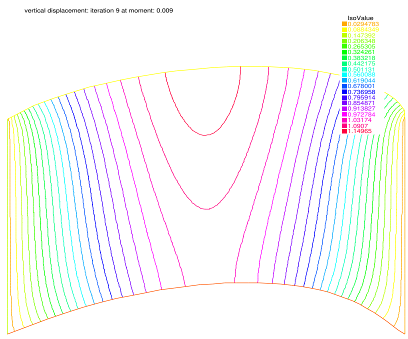
porosity: iteration 1



porosity: iteration 9



vertical displacement: iteration 1



vertical displacement: iteration 9

Final remarks

As the absolute temperature inside the plate is growing, the plate is deforming more and more until it reaches the thermal equilibrium state.

Acknowledgments

The first author was supported by the CNCSIS grant ET 3/2006.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

NEW ESTIMATE FOR THE NUMERICAL RADIUS OF A GIVEN MATRIX AND BOUNDS FOR THE ZEROS OF POLYNOMIALS

MOHAMMAD AL-HAWARI

Abstract. In this paper we find new estimate for the numerical radius of a given matrix, and we prove that, this estimate is better than any estimate for the numerical radius. We present also new bounds for the zero of polynomials by using new estimate for the numerical radius of a companion matrix of a given polynomial and matrix inequalities.

1. **Introduction**

Numerical radii estimate of companion matrices have been invoked by Linden (1999) [7] and Kittaneh [6]. Also, Kittaneh (2003) found that

$$w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}}).$$

Also, we know that

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|.$$

In this paper, we find that

$$w(A) \leq \|A^2\|^{\frac{1}{2}} \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{\frac{1}{2}}\right) \leq \|A\|,$$

whenever, A^2does not converge to the zero matrix.

Also, if $A^2 = [0]_{n\times n}$, then $w(A) = \frac{1}{2}\|A\|$, and from the new estimate and matrix inequalities we find new bounds for the zeros of polynomials.

In this work, let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices.

Definition 1.1 *If* $A \in M_n(\mathbb{C})$, *then*

Received by the editors: 23.03.2009.

2000 *Mathematics Subject Classification.* 47A12, 47A30, 15A48.

Key words and phrases. Companion matrix, numerical radius, spectral norm, unitary matrix and zeros of polynomials.

(i) The spectral norm (or the operator norm) is defined by

$$\|A\| = \max\{\|Ax\| : \|x\| = 1\} = \max\left\{\frac{\|Ax\|}{\|x\|} : \|x\| \neq 0\right\}.$$

(ii) *The numerical radius of A is defined by*

$$w(A) = \max\{|(Ax, x)| : x \in \mathbb{C}^n, \|x\| = 1\}.$$

(iii) *The spectral radius of A is defined by*

$$r(A) = \max\{|\lambda| : \lambda \textit{ is an eigenvalue of } A\}.$$

Now, we list some known results as a background and reminder for the reader.

Theorem 1.1

(i) If $A \in M_n(\mathbb{C})$, then $\frac{1}{2}|A\| \leq w(A) \leq \|A\|$ (see e.g.[2]).

(ii) If $A \in M_n(\mathbb{C})$, then $w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}})$ (see e.g.[6]).

(iii) If $A \in M_n(\mathbb{C})$, then there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that $A = U|A|$, where $|A| = (A^*A)^{\frac{1}{2}}$.

(iv) Let $A = [a_{ij}] \in M_n(\mathbb{C})$ be written in partitioned form as

$$A = \begin{bmatrix} \widetilde{A} & x \\ x^* & a_{nn} \end{bmatrix},$$

where $x \in \mathbb{C}^{n-1}$ and $\widetilde{A} \in M_{n-1}(\mathbb{C})$, then

$$det(A) = a_{nn}det(\widetilde{A}) - x^*(adj\widetilde{A})x,$$

where $adj\widetilde{A}$ is the classical adjoint of $\widetilde{A}$.(see e.g[4]).

(v) Let $A = [a_{ij}] \in M_n(\mathbb{C})$ be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{ij} is an $n_i \times n_j$ matrix for $i, j = 1, 2$, with $n_1 + n_2 = n$.

If

$$\widetilde{A} = \begin{bmatrix} \|A_{11}\| & \|A_{12}\| \\ \|A_{21}\| & \|A_{22}\| \end{bmatrix},$$

then

$$\|A\| \leq \|\widetilde{A}\|, r(A) \leq r(\widetilde{A}), \textit{and } w(A) \leq w(\widetilde{A}).$$

2. Main results

In the following theorem, we find a new estimate for the numerical radius of a given matrix.

Theorem 2.1 *Let* $A \in M_n(\mathbb{C})$*, then*

(i) $w(A) \leq \|A^2\|^{\frac{1}{2}}$,*if* A^2does not converge to the zero matrix.

(ii) $w(A) = \frac{1}{2}\|A\|$,*if* A^2*is the zero matrix*

Proof. (i) Let $A = u\,|A|$, for some unitary matrix $u \in M_n(\mathbb{C})$.

Now,

$$w(A) = \max\left\{|(Ax, x)| : x \in \mathbb{C}^n, \|x\| = 1\right\}.$$

So

$$\begin{aligned}
(Ax, x) &= (u\,|A|\,x, x) = (|A|\,x, u^*x) \\
&\leq (|A|\,x, x)^{\frac{1}{2}} \,(|A|\,u^*x, u^*x)^{\frac{1}{2}} \\
&\leq (|A|\,x, x)^{\frac{1}{2}} \,(u\,|A|\,u^*x, x)^{\frac{1}{2}} \\
&\leq (|A|\,x, x)^{\frac{1}{2}} \,(|A^*|\,x, x)^{\frac{1}{2}} \\
&\leq (x^*\,|A|\,xx^*\,|A^*|\,x)^{\frac{1}{2}} \\
&\leq (x^*\,|A|\,|A^*|\,x)^{\frac{1}{2}} \\
&\leq (|A|\,|A^*|\,x, x)^{\frac{1}{2}} \\
&\leq \||A|\,|A^*|\|^{\frac{1}{2}},
\end{aligned}$$

since

$$S_1^2\,(|A|\,|A^*|) = S_1^2\left(A^2\right),$$

where $S_1^2\left(A^2\right)$ denotes the largest singular value of A^2 and

$$S_1\left(A^2\right) = \|A^2\|.$$

So we get

$$\||A|\,|A^*|\|^2 = \|A^2\|^2 \text{ and } \||A|\,|A^*|\| = \|A^2\|,$$

hence

$$w(A) \leq \|A^2\|^{\frac{1}{2}}. \qquad \square$$

(ii) We know that

$$\frac{1}{2}\|A\| \leq w(A), \tag{1}$$

since $A^2 = [0]_{n\times n}$, so

$$w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}}) \leq \frac{1}{2}\|A\|. \tag{2}$$

From (1) and (2) we get the result. $\square$

In Theorem 2.1, the estimate of the numerical radius is a uniform estimate, because

$$w(A) \leq \|A^2\|^{\frac{1}{2}} \leq \|A\|,$$

also, since

$$\|A^2\|^{\frac{1}{2}} \leq \|A\|,$$

we have

$$w(A) \leq \|A^2\|^{\frac{1}{2}} \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{\frac{1}{2}}\right).$$

Corollary 2.1 *Since $r(A) \leq w(A)$, we get that,*

$$r(A) \leq \|A^2\|^{\frac{1}{2}}, \text{if } A^2 \textit{does not converge to the zero matrix.}$$

Also,

$$r(A) \leq \frac{1}{2}\|A\|, \text{if } A^2 \text{is the zero matrix.}$$

Corollary 2.2 *If $A \in M_n(\mathbb{C})$ is a normal matrix, then*

$$r(A) = w(A) = \|A^2\|^{\frac{1}{2}} = \|A\|.$$

3. New bounds for the zeros of polynomials

In this section, we find new bounds for the zeros of the monic polynomials

$$p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \ldots + a_z z + a_1 \tag{3}$$

with complex coefficients $a_1, a_2, a_3, ..., a_n$ where, $a_1 \neq 0$ and $n \geq 3$ by using numerical radius, and matrix inequalities of the companion matrix

$$C(p) = \begin{bmatrix} -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{4}$$

In [6] Kittaneh found $\|C^2(p)\|$,where

$$C^2(p) = \begin{bmatrix} a_n^2 - a_{n-1} & a_n a_{n-1} - a_{n-2} & a_n a_{n-2} - a_{n-3} & \cdots & a_n a_1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots 1 & 0 \end{bmatrix}. \tag{5}$$

And hence by using Theorem 2.1, we find new bounds for zeros of polynomials as in the following theorem.

Theorem 3.1 *If z is a zero of $p(z)$ as in (3), then*

$$|z| \leq \sqrt[4]{\frac{(\delta+1)\left[(\delta+1)^2 + 4\delta'\right]^{\frac{1}{2}}}{2}},$$

where

$$\delta = \frac{1}{2}\left[(\alpha+\beta) + \left((\alpha+\beta)^2 + 4|\gamma|^2\right)^{\frac{1}{2}}\right],$$

and

$$\alpha = \sum_{j=1}^{n} |a_j|^2, \beta = \sum_{j=1}^{n} |L_j|^2,$$

$$\gamma = -\sum_{j=1}^{n} L_j \overline{a_j}, \ \ L_j = a_n a_j - a_{j-1}$$

for $j = 1, 2, \ldots, n$ *with* $a_0 = 0$,*and*

$$\delta' = \frac{1}{2}\left[(\alpha' + \beta') + \sqrt{(\alpha' + \beta')^2 + 4|\gamma'|^2}\right],$$

where

$$\alpha' = \sum_{j=3}^{n} |a_j|^2, \beta' = \sum_{j=3}^{n} |L_j|^2 \text{ and } \gamma' = -\sum_{j=3}^{n} L_j a_j.$$

Proof. Let $C(p)$ be the companion matrix of $p(z)$.Since $C^2(p) \neq [0]_{n\times n}$,we have

$$wC(p) \leq \left\|C^2(p)\right\|^{\frac{1}{2}}.$$

Kittaneh found $\left\|C^2(p)\right\|$ in [6], and hence we get the result. ☐

Now since

$$\left\|C^2(p)\right\|^{\frac{1}{2}} \leq \frac{1}{2}\left(\|C(p)\| + \left\|C^2(p)\right\|^{\frac{1}{2}}\right).$$

Therefore, the bound in Theorem 3.1, is better than Kittaneh bound in [6].

Kittaneh found new bound for the zeros of a polynomial $p(z)$ by using matrix inequality as in the following theorem.

Theorem 3.2 *(see[5]) If z is a zero of $p(z)$ as in (3), then*

$$|z| \leq \frac{1}{2}\left[(1 + |a_n|) + \sqrt{(1 + |a_n|)^2 + 4\left(\sum_{j=1}^{n-1} |a_j|^2\right)^{\frac{1}{2}}}\right].$$

In the following theorem, we find new bounds for the zeros of $p(z)$ by using matrix inequalities.

Theorem 3.3. *If z is any zero of $p(z)$ as in (3), then*

$$|z| \leq \frac{1}{2}\left[\beta + \sqrt{\beta^2 + 4|a_1|}\right],$$

where

$$\beta = \left[\frac{\left(1 + \sum_{j=2}^{n} |a_j|^2\right) + \sqrt{\left(\sum_{j=2}^{n} |a_j|^2 - 1\right)^2 + 4|a_2|^2}}{2}\right]^{\frac{1}{2}}.$$

Proof. Let $C(p)$ be the companion matrix of $p(z)$ as in (4). Then

$$C(p) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

where

$$C_{11} = \begin{bmatrix} -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots 1 & 0 \end{bmatrix},$$

$$C_{12} = \begin{bmatrix} -a_1 & 0 & 0 & \cdots & 0 \end{bmatrix}^t,$$

$$C_{21} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$C_{22} = \begin{bmatrix} 0 \end{bmatrix}.$$

Known,

$$rC(p) \leq r\left(\begin{bmatrix} \beta & |a_1| \\ 1 & 0 \end{bmatrix}\right).$$

By using (iv) in Theorem 1.2, we get

$$\beta = \|C_{11}\| = \left[\frac{\left(1 + \sum_{j=2}^{n} |a_j|^2\right) + \sqrt{\left(\sum_{j=2}^{n} |a_j|^2 - 1\right)^2 + 4\,|a_2|^2}}{2}\right]^{\frac{1}{2}}. \qquad \square$$

That is, the desired result.

Acknowledgment. The author would like to thank the referee's comments.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

FUNCTIONALS ON NORMED SEQUENCE SPACES AND UNIFORM EXPONENTIAL INSTABILITY OF EVOLUTION OPERATORS

MIHAIL MEGAN AND LARISA BIRIŞ

Abstract. In this paper, we present necessary and sufficient conditions for uniform exponential instability of evolution operators in Banach spaces. Variants for uniform exponential instability of some well-known results due to Datko, Neerven and Zabczyk are given. As consequences, some results proved in [6] are obtained.

1. **Introduction**

One of the most remarkable result in stability theory of evolution operators in Banach spaces has been obtained by Datko in [3]. An important generalization of Datko's result was proved by Rolewicz in [12]. A new and interesting idea has been presented by Neerven in [9], where an unified treatment of the preceding results is given and the exponential stability of C_0-semigroups has been characterized in terms of functionals on Banach function spaces. Some generalizations of these results for the case of linear evolution operators have been presented in [1], [5] and [6].
In this paper, we shall present characterizations for exponential instability of linear evolution operators in the spirit of Neerven's approach. Thus we obtain the versions of the theorems due to Datko, Zabczyck and Neerven for the case of exponential instability. As consequences, we obtain some results presented in [6].

Received by the editors: 01.09.2009.
2000 *Mathematics Subject Classification.* 34D05, 34D20.
Key words and phrases. Exponential instability, evolution operator, functionals on function spaces.

2. **Linear evolution operators**

Let X be a Banach space. The norm on X and on the space $\mathcal{B}(X)$ of all bounded linear operators from X into itself will be denoted by $\|.\|$.
Let T be the set defined by

$$T = \{(t, t_0) \in \mathbb{R}_+^2 : t_0 \leq t\}.$$

Definition 2.1. An application $E : T \to \mathcal{B}(X)$ is called *evolution operator* on X, if it satisfies the following conditions :

(i) $E(t, t) = I$ (the identity operator on X);
(ii) $E(t, s)E(s, t_0) = E(t, t_0)$, for all $(t, s) \in T$ and $(s, t_0) \in T$;
(iii) there exist $M, \omega > 0$ such that

$$\|E(t, t_0)\| \leq Me^{\omega(t-t_0)}, \quad \text{for all} \quad (t, t_0) \in T.$$

Particular classes of evolution operators are given by:

Definition 2.2. An evolution operator E is said to be

(i) *strongly measurable*, if for every $(t_0, x) \in \mathbb{R}_+ \times X$ the mapping $E(\cdot, t_0)x$ is measurable;
(ii) *injective*, if for every $(t, t_0) \in \mathbb{R}_+^2$ the linear operator $E(t, t_0)$ is injective.
(iii) *uniformly exponentially instable*, if there are $N, \nu > 0$ such that

$$\|E(t, t_0)x\| \geq Ne^{\nu(t-t_0)}\|x\|, \quad \text{for all} \quad (t, t_0, x) \in T \times X.$$

A characterization of the exponential instability property is given by:

Proposition 2.1. *An evolution operator E is uniformly exponentially instable if and only if there exists a nondecreasing sequence $f : \mathbb{N} \to \mathbb{R}_+^* = (0, \infty)$ with $\lim_{n\to\infty} f(n) = \infty$ and*

$$\|E(n + t_0, t_0)x\| \geq f(n)\|x\|, \quad \textit{for all} \quad m, n \in \mathbb{N} \quad \textit{and} \quad x \in X.$$

Proof. *Necessity* is trivial.
Sufficiency. Let $(t, t_0) \in T$ and $n \in \mathbb{N}$ such that $n \leq t - t_0 < n + 1$. Then for every

$x \in X$ with $\|x\| = 1$, we have that

$$\begin{aligned}\|E(t,t_0)x\| &= \|E(t,t_0+n)E(t_0+n,t_0)x\| \geq f(t-t_0-n)\|E(t_0+n,t_0)x\| \\ &\geq f(0)f(1)\|E(t_0+(n-1),t_0)x\| \geq \cdots \geq f(0)f(1)^n\|x\| \\ &= f(0)e^{\nu n}\|x\| \geq Ne^{\nu(n+1)}\|x\| \geq Ne^{\nu(t-t_0)}\|x\|,\end{aligned}$$

where $\nu = lnf(1) > 0$ and $N = f(0)/f(1) > 0$, which shows that E is uniformly exponentially instable. $\square$

3. Main results

Let $\mathcal{S}(\mathbb{R})$ the set of all real sequences. By $\mathcal{S}^+(\mathbb{R})$ we denote the set of all $s \in \mathcal{S}(\mathbb{R})$ with $s(n) \geq 0$, for all $n \in \mathbb{N}$.

Let $\mathcal{F}$ be the set of all functions $F : \mathcal{S}^+(\mathbb{R}) \to [0,\infty]$ with the properties:

(f_1) if s_1, $s_2 \in \mathcal{S}^+(\mathbb{R})$ with $s_1 \leq s_2$ then $F(s_1) \leq F(s_2)$;

(f_2) there exists $\alpha > 0$ such that $F(c\chi_{\{n\}}) \geq \alpha c$, for all $c > 0$ and $n \in \mathbb{N}$;

(f_3) there exists $f \in \mathcal{S}^+(\mathbb{R}_+)$ with $\lim\limits_{n\to\infty} f(n) = \infty$ such that $F(c\chi_{\{0,\ldots,n\}}) \geq f(n)$, for all $c > 0$ and $n \in \mathbb{N}$.

Here χ_A denotes the characteristic function of the set A.

For every injective evolution operator E and every $x \in X$ with $\|x\| = 1$, we associate the following sequences:

$$e_x^{t_0}(n) = \frac{1}{\|E(n+t_0,t_0)x\|}, \quad e_x^{m,t_0}(n) = e_x^{t_0}(m+n), \quad v_x^{m,t_0}(n) = \frac{e_x^{m,t_0}(n)}{e_x^{t_0}(m)},$$

for all $m, n \in \mathbb{N}$.

Remark 3.1. If the evolution operator E is uniformly exponentially instable then there exists $F \in \mathcal{F}$ with the properties:

(i) $\sup\limits_{\substack{\|x\|=1 \\ t_0\geq 0}} F(e_x^{t_0}) < \infty$;

(ii) there exists $N > 0$ such that $F(e_x^{m,t_0}) \leq Ne_x^{t_0}(m)$, for all $m \in \mathbb{N}$, $t_0 \geq 0$ and $x \in X$ with $\|x\| = 1$;

(iii) $\sup\limits_{\substack{\|x\|=1 \\ (m,t_0)\in\mathbb{N}\times\mathbb{R}_+}} F(e_x^{m,t_0}) < \infty$;

(iv) $\sup\limits_{\substack{\|x\|=1 \\ (m,t_0)\in\mathbb{N}\times\mathbb{R}_+}} F(v_x^{m,t_0}) < \infty.$

Indeed, if we consider the function $F : \mathcal{S}^+(\mathbb{R}) \to [0, \infty]$ defined by

$$F(s) = \sum_{n=0}^{\infty} s(n)$$

then it is easy to verify that the uniformly exponentially instability property of E implies the conditions (i), (ii), (iii) and (iv).

The main result of this paper is :

Proposition 3.1. *An injective evolution operator E is uniformly exponentially instable if and only if there is $F \in \mathcal{F}$ such that*

$$\sup_{\substack{\|x\|=1 \\ t_0\geq 0}} F(e_x^{t_0}) < \infty.$$

Proof. *Necessity.* It results from Remark 3.1.

Sufficiency. We observe that

$$e_x^{t_0} = \sum_{k=0}^{\infty} e_x^{t_0}(k)\chi_{\{k\}} \geq \sum_{k=0}^{n} e_x^{t_0}(k)\chi_{\{k\}} \geq e_x^{t_0}(n)\chi_{\{n\}}.$$

Let $M = \sup\limits_{\substack{\|x\|=1 \\ t_0\geq 0}} F(e_x^{t_0}) < \infty$. Using the hypothesis, we obtain that

$$M \geq F(e_x^{t_0}) \geq F(e_x^{t_0}(n)\chi_{\{n\}}) \geq \alpha \cdot e_x^{t_0}(n),$$

and hence $e_x^{t_0}(n) \leq M/\alpha$, for all $t_0 \geq 0, n \in \mathbb{N}$ and $x \in X$ with $\|x\| = 1$.

The last inequality becomes

$$\|E(n + t_0, t_0)x\| \geq \frac{\alpha}{M},$$

for all $n \in \mathbb{N}, t_0 \geq 0$ and all $\|x\| = 1$.

This implies that

$$\|E(n + t_0, k + t_0)E(k + t_0, t_0)x\| \geq \frac{\alpha}{M}\|x\|, \quad \text{for all} \quad x \in X \quad \text{with} \quad \|x\| = 1,$$

and hence

$$\|E(n + t_0, t_0)x\| \geq \frac{\alpha}{M}\|E(k + t_0, t_0)x\|, \quad \text{for all} \quad k, n \in \mathbb{N}, \quad k \leq n, \quad t_0 \geq 0$$

and $x \in X$ with $\|x\| = 1$.

We have that

$$e_x^{t_0} = \sum_{k=0}^{\infty} e_x^{t_0}(k)\chi_{\{k\}} \geq \sum_{k=0}^{n} e_x^{t_0}(k)\chi_{\{k\}} \geq \frac{\alpha}{M} e_x^{t_0}(n)\chi_{\{0,\dots,n\}}.$$

It follows that

$$M \geq F(e_x^{t_0}) \geq \frac{\alpha}{M} e_x^{t_0}(n) f(n)$$

and hence

$$\|E(n+t_0,t_0)x\| \geq \frac{\alpha}{M^2} f(n), \quad \text{for all} \quad x \in X \quad \text{with} \quad \|x\| = 1.$$

By Proposition 2.1 it results that E is uniformly exponentially instable. $\square$

Corollary 3.1. *An injective evolution operator E is uniformly exponentially instable if and only if there exist $N > 0$ and $F \in \mathcal{F}$ such that*

$$F(e_x^{m,t_0}) \leq N e_x^{t_0}(m),$$

for all $m \in \mathbb{N}$, $t_0 \geq 0$ and $x \in X$ with $\|x\| = 1$.

Proof. *Necessity.* It results from Remark 3.1.

Sufficiency. We observe that

$$\sup_{\substack{\|x\|=1\\ t_0\geq 0}} F(e_x^{t_0}) = \sup_{\substack{\|x\|=1\\ t_0\geq 0}} F(e_x^{0,t_0}) \leq N e_x^{t_0}(0) = N < \infty$$

and by Proposition 3.1 it follows that E is uniformly exponentially instable. $\square$

Corollary 3.2. *An injective evolution operator E is uniformly exponentially instable if and only if there is $F \in \mathcal{F}$ such that*

$$\sup_{\substack{\|x\|=1\\ (m,t_0)\in\mathbb{N}\times\mathbb{R}_+}} F(e_x^{m,t_0}) < \infty.$$

Proof. *Necessity.* It results from Remark 3.1.

Sufficiency. It results from Proposition 3.1 taking into account that

$$\sup_{\substack{\|x\|=1\\ t_0\geq 0}} F(e_x^{t_0}) = \sup_{\substack{\|x\|=1\\ t_0\geq 0}} F(e_x^{0,t_0}) \leq \sup_{\substack{\|x\|=1\\ (m,t_0)\in\mathbb{N}\times\mathbb{R}_+}} F(e_x^{m,t_0}) < \infty.$$

$\square$

Similarly, we obtain:

Corollary 3.3. *An injective evolution operator E is uniformly exponentially instable if and only if there is $F \in \mathcal{F}$ such that*

$$\sup_{\substack{\|x\|=1 \\ (m,t_0)\in\mathbb{N}\times\mathbb{R}_+}} F(v_x^{m,t_0}) < \infty.$$

Remark 3.2. The preceding results are discrete versions of a Neerven's theorem ([9]) for the case of instability property.

We shall denote by Φ the set of all nondecreasing functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and $\varphi(t) > 0$, for every $t > 0$.

Corollary 3.4. *An injective evolution operator E is uniformly exponentially instable if and only if there exists $\varphi \in \Phi$ such that*

$$\sup_{\substack{\|x\|=1 \\ t_0\geq 0}} \sum_{n=0}^{\infty} \varphi(e_x^{t_0}(n)) < \infty.$$

Proof. *Necessity.* It is trivial for $\varphi(t) = t$.

Sufficiency. It results from Proposition 3.1 for

$$F(s) = \sum_{n=0}^{\infty} \varphi(s(n)).$$

□

Remark 3.3. The preceding corollary extends a Zabczyk's theorem ([13]) for the case of exponential instability.

For the particular case $\varphi(t) = t^p$, we obtain:

Corollary 3.5. *An injective evolution operator E is uniformly exponentially instable if and only if there exists $p \in [1, \infty)$ such that*

$$\sup_{\substack{\|x\|=1 \\ t_0\geq 0}} \sum_{n=0}^{\infty} [e_x^{t_0}(n)]^p < \infty.$$

Remark 3.4. Corollary 3.5 is a discrete version of Datko's theorem ([3]) for the case of exponential instability. It can be also considered as a variant for the exponential instability of a theorem proved by Przyluski and Rolewicz in [11] for the case of exponential stability.

Let $\mathcal{B}(\mathbb{N})$ be the set of all normed sequence spaces B ([8]) with the properties:

(i) $\chi_{\{0,\dots,n\}} \in B$, for all $n \in \mathbb{N}$;

(ii) $\lim\limits_{n\to\infty} |\chi_{\{0,\dots,n\}}|_B = \infty$;

(iii) there exists $\alpha > 0$ such that $|\chi_{\{n\}}|_B \geq \alpha$, for all $n \in \mathbb{N}$.

Corollary 3.6. *An injective evolution operator E is uniformly exponentially instable if and only if there exists a normed sequence space $B \in \mathcal{B}(\mathbb{N})$ such that for every $x \in X$ with $\|x\| = 1$, we have that $e_x^{t_0} \in B$ and*

$$\sup_{\substack{\|x\|=1 \\ t_0 \geq 0}} |e_x^{t_0}|_B < \infty.$$

Proof. *Necessity.* It is immediate for $B = l^1$.

Sufficiency. Let $F : \mathcal{S}^+(\mathbb{R}) \to [0, \infty]$ be the function defined by

$$F(s) = \sup_{n\in\mathbb{N}} |s \cdot \chi_{\{0,\dots,n\}}|_B.$$

It is easy to see that $F \in \mathcal{F}$ and

$$e_x^{t_0}\chi_{\{0,\dots,n\}} \leq e_x^{t_0}, \quad \text{for all} \quad n \in \mathbb{N}, \quad t_0 \geq 0 \quad \text{and} \quad x \in X \quad \text{with} \quad \|x\| = 1.$$

Then

$$\sup_{\substack{\|x\|=1 \\ t_0 \geq 0}} F(e_x^{t_0}) \leq \sup_{\substack{\|x\|=1 \\ t_0 \geq 0}} |e_x^{t_0}|_B < \infty.$$

By Proposition 3.1 it results that E is uniformly exponentially instable. $\square$

Remark 3.5. The Corollary 3.6 is a discrete variant for exponential instability of Theorem 3.1.5 from [8].

As a particular case, for the Banach sequence space

$$B = \{s \in \mathcal{S}^+(\mathbb{R}) : \beta s \in l^p\}$$

we obtain:

Corollary 3.7. *An injective evolution operator E is uniformly exponentially instable if and only if there are $p \in [1, \infty)$ and $\beta \in \mathcal{S}^+(\mathbb{R})$ with $\beta > 0$ and $\sum\limits_{n=0}^{\infty} \beta(n) = \infty$ such that*

$$\sup_{\substack{\|x\|=1 \\ t_0 \geq 0}} \sum_{n=0}^{\infty} \beta^p(n)[e_x^{t_0}(n)]^p < \infty.$$

Remark 3.6. The preceding corollary is an extension of Corollary 3.1.6. from [8].

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

ON A GENERAL CLASS OF BETA APPROXIMATING OPERATORS OF SECOND KIND

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Abstract. We shall define a general linear transform, from which we obtain as special case the beta second kind transform. We obtain several positive linear operators as a special case of this beta second kind transform. We apply the beta second kind transform to Baskakov's operator B_n and we obtain different generalization of it.

1. **Introduction**

In this paper we continue our earlier investigations [5], [6], [7], [8], [9], [10] concerning to use Euler's beta function for constructing linear positive operators.

Euler's beta function is defined for $p, q > 0$ by the following formula

$$B(p,q) = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} du. \tag{1.1}$$

The beta second kind transform of the function f is defined by the following formula

$$T_{p,q}f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u) du. \tag{1.2}$$

We shall define a more general linear transform from which we obtain as special case the beta second kind transform.

Let us denote by $M[0,\infty)$ the linear space of functions defined for $t \geq 0$, bounded and Lebesgue measurable in each interval $[c,d]$, where $0 < c < d < \infty$.

Received by the editors: 22.02.2008.

2000 *Mathematics Subject Classification.* 41A36.

Key words and phrases. Euler's beta function, the beta second kind transform, positive linear operators.

For $a, b \in \mathbb{R}$ we define the (a, b)-beta transform of a function f (see [5])

$$\mathcal{T}_{p,q}^{(a,b)} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f\left(\frac{u^a}{(1+u)^{a+b}}\right) du, \tag{1.3}$$

where $B(\cdot,\cdot)$ is the beta function (1.1) and $f \in M[0,\infty)$ such that $\mathcal{T}_{p,q}^{(a,b)}|f| < \infty$.

If we consider in (1.3) $a + b = 0$ we obtain the second kind transform of function $f \in M[0,\infty)$

$$T_{p,q}^{(a)} = \mathcal{T}_{p,q}^{(a,-a)} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u^a) du \tag{1.4}$$

such that $T_{p,q}^{(a)}|f| < \infty$. Clearly $T_{p,q}^{(a)}$ is a positive linear functional.

We shall consider here only the special case $a = 1$.

2. **The beta second kind transform. Case** $a = 1$

If we put in (1.4) $a = 1$ we obtain the beta second kind transform

$$T_{p,q} f = T_{p,q}^{(1)} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u) du \tag{2.1}$$

for $f \in M[0,\infty)$ such that $T_{p,q}|f| < \infty$ considered by D.D. Stancu [13] (see also [7]).

Remark. If $a = -1$ we obtain $T_{p,q}^{(-1)} f = T_{p,q}^{(1)} f = T_{p,q} f$ (see [7]).

Theorem 2.1. [13] *The moment of order k $(1 \le k < q)$ of the functional $T_{p,q}$ has the following value*

$$T_{p,q} e_k = \frac{p(p+1)\ldots(p+k-1)}{(q-1)\ldots(q-k)}, \quad 1 \le k < q. \tag{2.2}$$

Consequently we obtain

$$T_{p,q} e_1 = \frac{p}{q-1}, \quad T_{p,q} e_2 = \frac{p(p+1)}{(q-1)(q-2)}, \quad q > 2. \tag{2.3}$$

We impose that $T_{p,q} e_1 = e_1$, that is $\dfrac{p}{q-1} = x$, or $p = \dfrac{\beta}{\alpha} x$, $q = 1 + \dfrac{\beta}{\alpha}$, $x > 0$, $\alpha, \beta > 0$ and we obtain the following linear positive operators

$$(\mathcal{T}^{(\alpha,\beta)} f)(x) = \frac{1}{B\left(\dfrac{\beta}{\alpha} x, 1 + \dfrac{\beta}{\alpha}\right)} \int_0^\infty \frac{u^{\frac{\beta}{\alpha} - 1}}{(1+u)^{1+\frac{\beta}{\alpha}(x+1)}} f(u) du. \tag{2.4}$$

Corollary 2.2. *One has*

$$\mathcal{T}^{(\alpha,\beta)}((t-x)^2;x) = \frac{\alpha}{\beta-\alpha}x(1+x), \quad \beta > \alpha > 0. \tag{2.5}$$

Proof. It is obtained from (2.3) for $p = \frac{\beta}{\alpha}x$, $q = 1 + \frac{\beta}{\alpha}$, $p+q = 1 + \frac{\beta}{\alpha}(1+x)$.

$$(\mathcal{T}^{(\alpha,\beta)}e_2)(x) = \frac{\beta x(\beta x+\alpha)}{\beta(\beta-\alpha)} = x^2 + \left(\frac{\beta x^2+\alpha x}{\beta-\alpha} - x^2\right) = x^2 + \frac{\alpha(x+x^2)}{\beta-\alpha}$$

and

$$\mathcal{T}^{(\alpha,\beta)}((t-x)^2;x) = \frac{\alpha}{\beta-\alpha}x(1+x). \quad \beta > \alpha > 0.$$

Special cases

1. Let $\mathcal{T}_1^{(\alpha)}$ be the beta second kind operator defined by

$$(\mathcal{T}_1^{(\alpha)}f)(x) = \frac{1}{B\left(\frac{x}{\alpha}, 1+\frac{1}{\alpha}\right)}\int_0^\infty \frac{u^{\frac{x}{\alpha}-1}}{(1+u)^{\frac{1+x}{\alpha}+1}}f(u)du. \tag{2.6}$$

The operator (2.6) has been considered by Stancu [13] (see also [1], [2], [7], [11]) and it is obtained by (2.4) if we choose in (2.4) $\beta = 1$ and $\alpha \in (0,1)$.

Corollary 2.3. [7] *One has*

$$\mathcal{T}_1^{(\alpha)}((t-x)^2;x) = \frac{\alpha}{1-\alpha}x(1+x), \quad \alpha \in (0,1). \tag{2.7}$$

For $\alpha = \frac{1}{n}$ *we obtain*

$$\mathcal{T}_1^{(1/n)}((t-x)^2;x) = \frac{x(1+x)}{n-1}.$$

2. Another beta second kind operator it is obtained by (2.4) for $\beta = \frac{1}{1+x}$, $\beta > \alpha$, $x \in \left(0, \frac{1}{\alpha} - 1\right)$, $\alpha \in (0,1)$

$$(\mathcal{T}_2^{(\alpha)}f)(x) = \frac{1}{B\left(\frac{x}{\alpha(1+x)}, 1+\frac{1}{\alpha(1+x)}\right)}\int_0^\infty \frac{u^{\frac{x}{\alpha(1+x)}-1}}{(1+u)^{\frac{1}{\alpha}+1}}f(t)dt \tag{2.8}$$

where $f \in M[0,\infty)$ such that $\mathcal{T}_2^{(\alpha)}|f| < \infty$, considered by J. Adell [2] (see also [7]).

Corollary 2.4. [7] *One has*

$$\mathcal{T}_2^{(\alpha)}((t-x)^2;x)=\frac{\alpha x(1+x)^2}{1-\alpha(1+x)},\quad x<\frac{1}{\alpha}-1.$$

For $\alpha=1/n$, $n\in\mathbb{N}$ we obtain

$$\mathcal{T}_2^{(1/n)}((t-x)^2;x)=\frac{x(1+x)^2}{n-1-x},\quad x<n-1.$$

3. Let $\mathcal{T}_3^{(\alpha)}$ be the operator defined by

$$(\mathcal{T}_3^{(\alpha)}f)(x)=\frac{1}{B\left(\dfrac{1}{\alpha};1+\dfrac{1}{\alpha x}\right)}\int_0^\infty\frac{u^{\frac{1}{\alpha}-1}}{(1+u)^{\frac{1+x}{\alpha x}+1}}f(t)dt,\tag{2.9}$$

$x\in\left(0,\dfrac{1}{\alpha}\right)$, $\alpha\in(0,1)$.

The operator (2.9) is obtained by (2.4) if we choose in (2.4) $\beta=\dfrac{1}{x}$.

Corollary 2.5. *One has*

$$\mathcal{T}_3^{(\alpha)}((t-x)^2;x)=\frac{\alpha x^2(1+x)}{1-\alpha x},\quad x<\frac{1}{\alpha}.$$

For $\alpha=1/n$, $n\in\mathbb{N}$ we obtain

$$\mathcal{T}_3^{(1/n)}((t-x)^2;x)=\frac{x^2(1+x)}{n-x},\quad x<n.$$

4. For $\beta=\dfrac{x}{1+x}>\alpha$, $x\in\left(\dfrac{\alpha}{1-\alpha},\infty\right)$, $\alpha\in(0,1)$ we obtain by (2.4) the following operator

$$(\mathcal{T}_4^{(\alpha)}f)(x)=\frac{1}{B\left(\dfrac{x^2}{\alpha(1+x)},1+\dfrac{x}{\alpha(1+x)}\right)}\int_0^\infty\frac{u^{\frac{x^2}{\alpha(1+x)}-1}}{(1+u)^{\frac{x}{\alpha}-1}}f(u)du.\tag{2.10}$$

Corollary 2.6. *One has*

$$\mathcal{T}_4^{(\alpha)}((t-x)^2;x)=\frac{\alpha x(1+x)^2}{x-\alpha(1+x)},\quad x>\frac{\alpha}{1-\alpha}.$$

For $\alpha=1/n$, $n\in\mathbb{N}$ we obtain

$$\mathcal{T}_4^{(1/n)}((t-x)^2;x)=\frac{x(1+x)^2}{(n-1)x-1},\quad x>\frac{1}{n-1}.$$

5. Let $\mathcal{T}_5^{(\alpha)}$ be the operator

$$(\mathcal{T}_5^{(\alpha)}f)(x)=\frac{1}{B\left(\dfrac{1+x}{\alpha},\dfrac{1+x}{\alpha x}+1\right)}\int_0^\infty\frac{u^{\frac{1+x}{\alpha}-1}}{(1+u)^{\frac{(1+x)^2}{\alpha x}+1}}f(u)du \tag{2.11}$$

$\alpha\in(0,1)$, $\alpha>0$. The operator (2.11) is obtained by (2.4) if we put in (2.4) $\beta=\dfrac{1+x}{x}$.

Corollary 2.7. *One has*

$$\mathcal{T}_5^{(\alpha)}((t-x)^2;x)=\frac{\alpha x^2(1+x)}{1+(1-\alpha)x},\quad x>0.$$

For $\alpha=1/n$, $n\in\mathbb{N}$,

$$\mathcal{T}_5^{(1/n)}((t-x)^2;x)=\frac{1}{n(1+x)-x}.$$

6. For $\beta=x$, $x\in(\alpha,\infty)$, $\alpha\in(0,1)$ we obtain by (2.4) the following operator

$$(\mathcal{T}_6^{(\alpha)}f)(x)=\frac{1}{B\left(\dfrac{x^2}{\alpha},1+\dfrac{x}{\alpha}\right)}\int_0^\infty\frac{u^{\frac{x^2}{\alpha}-1}}{(1+u)^{\frac{x(1+x)}{\alpha}+1}}f(u)du \tag{2.12}$$

Corollary 2.8. *One has*

$$\mathcal{T}_6^{(\alpha)}((t-x)^2;x)=\frac{\alpha x(1+x)}{x-\alpha},\quad x>\alpha.$$

For $\alpha=1/n$, $n\in\mathbb{N}$ *we obtain*

$$\mathcal{T}_6^{(1/n)}((t-x)^2;x)=\frac{x(1+x)}{nx-1},\quad x>\frac{1}{n}.$$

7. Let $\mathcal{T}_7^{(\alpha)}$ be the beta operator defined by

$$(\mathcal{T}_7^{(\alpha)}f)(x)=\frac{1}{B\left(\dfrac{x(1+x)}{\alpha},1+\dfrac{1+x}{\alpha}\right)}\int_0^\infty\frac{u^{\frac{x(1+x)}{\alpha}-1}}{(1+u)^{\frac{(1+x)^2}{\alpha}+1}}f(u)du \tag{2.13}$$

$\alpha\in(0,1)$, $x>0$. The operator (2.13) is obtained by (2.4) if we put in (2.4) $\beta=1+x$.

Corollary 2.9. *One has*

$$\mathcal{T}_7^{(\alpha)}((t-x)^2;x)=\frac{\alpha x(1+x)}{1-\alpha+x},\quad x>0.$$

For $\alpha = 1/n$ we obtain

$$\mathcal{T}_7^{(1/n)}((t-x)^2;x) = \frac{x(1+x)}{nx+n-1}.$$

8. Another beta second kind operator is obtained for $\beta = \dfrac{1}{x(1+x)} > \alpha$, $x \in \left(0, \dfrac{\sqrt{1+4/\alpha}-1}{2}\right)$, $\alpha \in (0,1)$

$$(\mathcal{T}_8^{(\alpha)} f)(x) = \frac{1}{B\left(\dfrac{1}{\alpha(1+x)}, \dfrac{1}{\alpha x(1+x)} + 1\right)} \int_0^\infty \frac{u^{\frac{1}{\alpha(1+x)}-1}}{(1+u)^{\frac{1}{\alpha x}+1}} f(u)du. \tag{2.14}$$

Corollary 2.10. *One has*

$$\mathcal{T}_8^{(\alpha)}((t-x)^2;x) = \frac{\alpha x^2(1+x)^2}{1-\alpha x(1+x)}, \quad x < \frac{\sqrt{1+4/\alpha}-1}{2}.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$ we obtain

$$\mathcal{T}_8^{(1/n)}((t-x)^2;x) = \frac{x^2(1+x)^2}{n-x(1+x)}, \quad x(1+x) < n.$$

9. For $\beta = x(1+x) > \alpha$, $x \in \left(\dfrac{\sqrt{1+4\alpha}-1}{2}, \infty\right)$, $\alpha \in (0,1)$ we obtain by (2.4) the following operator

$$(\mathcal{T}_9^{(\alpha)} f)(x) = \frac{1}{B\left(\dfrac{x^2(1+x)}{\alpha}, \dfrac{x(1+x)}{\alpha} + 1\right)} \int_0^\infty \frac{u^{\frac{x^2(1+x)}{\alpha}-1}}{(1+u)^{\frac{x(1+x)^2}{\alpha}+1}} f(u)du. \tag{2.15}$$

Corollary 2.11. $\mathcal{T}_9^{(\alpha)}((t-x)^2;x) = \dfrac{\alpha x(1+x)}{x(1+x)-\alpha}$, $x(1+x) > \alpha$.

For $\alpha = 1/n$, $n \in \mathbb{N}$ we obtain

$$\mathcal{T}_9^{(1/n)}((t-x)^2;x) = \frac{x(1+x)}{nx(1+x)-1}, \quad nx(1+x) > 1.$$

3. The functional $B_n^{(p,q)} f = \mathcal{T}_{p,q}(B_n f)$

Now let us apply the transform (2.1) to the Baskakov operator B_n, defined by [3]

$$(B_n f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right). \tag{3.1}$$

Theorem 3.1. [7] *The $\mathcal{T}_{p,q}$ transform of $B_n f$ can be expressed by the following formula*

$$B_n^{(p,q)} f = \mathcal{T}_{p,q}(B_n f) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{(p)_k (q)_n}{(p+q)_{n+k}} f\left(\frac{k}{n}\right) \tag{3.2}$$

where $(a)_m = a(a+1)\dots(a+m-1)$.

Theorem 3.2. [7] *One has*

$$B_n^{(p,q)} e_1 = \frac{p}{q-1}, \quad B_n^{(p,q)} e_2 = \frac{p(p+1)}{(q-1)(q-2)} + \frac{1}{n} \cdot \frac{p(p+q-1)}{(q-1)(q-2)}, \quad q > 2. \tag{3.3}$$

We impose that $B_n^{(p,q)} e_1 = e_1$, that is $\frac{p}{q-1} = x$, or $p = \frac{\beta}{\alpha} x$, $q = 1 + \frac{\beta}{\alpha}$, $x > 0$; $\alpha, \beta > 0$, $\alpha < \beta$ and we obtain from Theorem 3.1 and Theorem 3.2 the following results.

Corollary 3.3. *One has*

$$(B_n^{(\alpha,\beta)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} b_{n,k}^{(\alpha,\beta)}(x) f\left(\frac{k}{n}\right) \tag{3.4}$$

where

$$b_{n,k}^{(\alpha,\beta)}(x) = \frac{\beta x(\beta x+\alpha)\dots(\beta x+(k-1)\alpha)(\beta+\alpha)(\beta+2\alpha)\dots(\beta+n\alpha)}{(\beta(1+x)+\alpha)(\beta(1+x)+2\alpha)\dots(\beta(1+x)+(n+k)\alpha)}.$$

Corollary 3.4. *One has*

$$(B_n^{(\alpha,\beta)} e_1)(x) = x, \quad (B_n^{(\alpha,\beta)} e_2)(x) = x^2 + \frac{\alpha n+\beta}{\beta-\alpha} \cdot \frac{x(1+x)}{n}$$

$$B_n^{(\alpha,\beta)}((t-x)^2; x) = \frac{\alpha n+\beta}{\beta-\alpha} \cdot \frac{x(1+x)}{n}, \quad \beta > \alpha. \tag{3.5}$$

Special cases

1. If we put in (3.4) $\beta = 1$, $\alpha \in (0,1)$, we obtain the operator considered by D.D. Stancu [13] (see also [1], [7])

$$(C_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} c_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.6}$$

$$c_{n,k}^{(\alpha)} = \frac{x(x+\alpha)\dots(x+(k-1)\alpha)(1+\alpha)\dots(1+n\alpha)}{(1+x+\alpha)(1+x+2\alpha)\dots(1+x+(n+k)\alpha)}$$

Corollary 3.5. *One has*

$$C_n^{(\alpha)}((t-x)^2; x) = \frac{1+\alpha n}{1-\alpha} \cdot \frac{x(1+x)}{n}. \tag{3.7}$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$C_n^{(1/n)}((t-x)^2; x) = \frac{2x(1+x)}{n-1}.$$

2. Another operator it is obtained by (3.4) for $\beta = \dfrac{1}{1+x}$, $\alpha \in (0,1)$, $x \in \left(0, \dfrac{1}{\alpha} - 1\right)$

$$(D_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} d_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.8}$$

$$d_{n,k}^{(\alpha)}(x) = \frac{x(x+\alpha(1+x))\dots(x+(k-1)\alpha(1+x))(1+\alpha(1+x)\dots(1+n\alpha(1+x))}{(1+\alpha)(1+2\alpha)\dots(1+(n+k)\alpha)(1+x)^{n+k}}$$

Corollary 3.6. *One has*

$$D_n^{(\alpha)}((t-x)^2; x) = \frac{1+n\alpha(1+x)}{1-\alpha(1+x)} \cdot \frac{x(1+x)}{n}, \quad x \in \left(0, \frac{1}{\alpha} - 1\right).$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$D_n^{(1/n)}((t-x)^2; x) = \frac{x(1+x)(2+x)}{n-1-x}, \quad x \in (0, n-1).$$

3. Let $E_n^{(\alpha)}$ be the operator defined by

$$(E_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} e_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.9}$$

$$e_{n,k}^{(\alpha)}(x) = \frac{(1+\alpha)(1+2\alpha)\dots(1+(k-1)\alpha)(1+\alpha x)\dots(1+n\alpha x)}{(1+x+\alpha x)\dots(1+x+(n+k)\alpha x)} \cdot x^k$$

$\alpha \in (0,1)$, $x \in (0, 1/\alpha)$. This operator is obtained by (3.4) for $\beta = 1/x$.

Corollary 3.7. *One has*

$$E_n^{(\alpha)}((t-x)^2; x) = \frac{\alpha n x + 1}{1 - \alpha x} \cdot \frac{x(1+x)}{n}, \quad x < \frac{1}{\alpha}.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$E_n^{(1/n)}((t-x)^2; x) = \frac{x(1+x)^2}{n-x}, \quad x < n.$$

4. For $\beta = \dfrac{x}{1+x}$, $\alpha \in (0,1)$, $x > \dfrac{\alpha}{1-\alpha}$ we obtain by (3.4) the following operator

$$(F_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} f_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.10}$$

$$f_{n,k}^{(\alpha)}(x) = \frac{x^2(x^2+\alpha(1+x))\dots(x^2+(k-1)\alpha(1+x))(x+\alpha(1+x))\dots(x+n\alpha(1+x))}{(x+\alpha)(x+2\alpha)\dots(x+(n+k)\alpha)(1+x)^{n+k}}$$

Corollary 3.8. *One has*

$$F_n^{(\alpha)}((t-x)^2; x) = \frac{\alpha n(1+x) + x}{x - \alpha(1+x)}, \quad x > \frac{\alpha}{1-\alpha}.$$

For $\alpha = \dfrac{1}{n}$, $n \in \mathbb{N}$, we obtain

$$F_n^{(1/n)}((t-x)^2; x) = \frac{x(1+x)(1+2x)}{(n-1)x - 1}, \quad x > \frac{1}{n-1}.$$

5. Let $G_n^{(\alpha)}$ be the operator

$$(G_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} g_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.11}$$

$$g_{n,k}^{(\alpha)}(x) = \frac{(1+x)(1+x+\alpha)\dots(1+x+(k-1)\alpha)(1+x+\alpha x)\dots(1+x+n\alpha x)}{((1+x)^2+\alpha x)((1+x)^2+2\alpha x)\dots((1+x)^2+(n+k)\alpha x)} \cdot x^k.$$

The operator (3.11) is obtained by (3.4) if we put in (3.4) $\beta = \dfrac{1+x}{x}$, $\alpha \in (0,1)$, $x > 0$.

Corollary 3.9. *One has*

$$G_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha n x + 1 + x}{1 + x - \alpha x} \cdot \frac{x(1+x)}{n}.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$G_n^{(1/n)}((t-x)^2;x) = \frac{x(1+x)(1+2x)}{n+(n-1)x}.$$

6. For $\beta = x$, $\alpha \in (0,1)$, $x \in (\alpha, \infty)$ we obtain by (3.4) the following operator

$$(H_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} h_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.12}$$

$$h_{n,k}^{(\alpha)}(x) = \frac{x^2(x^2+\alpha)\dots(x^2+(k-1)\alpha)(x+\alpha)(x+2\alpha)\dots(x+n\alpha)}{(x(1+x)+\alpha)\dots(x(1+x)+(n+k)\alpha)}.$$

Corollary 3.10. *One has*

$$H_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha n + x}{x - \alpha} \cdot \frac{x(1+x)}{n}, \quad x > \alpha.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$H_n^{(\alpha)}((t-x)^2;x) = \frac{x(1+x)^2}{nx-1}, \quad x > \frac{1}{n}.$$

7. Let $K_n^{(\alpha)}$ be the operator

$$(K_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} k_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.13}$$

$$k_{n,k}^{(\alpha)}(x) = \frac{x(1+x)(x(1+x)+\alpha)\dots(x(1+x)+(k-1)\alpha)(1+x+\alpha)\dots(1+x+n\alpha)}{((1+x)^2+\alpha)((1+x)^2+2\alpha)\dots((1+x)^2+(n+k)\alpha)}$$

The operator $K_n^{(\alpha)}$ is obtained by (3.4) if we put in (3.4) $\beta = 1+x$, $\alpha \in (0,1)$, $x \in (0,\infty)$.

Corollary 3.11. *One has*

$$K_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha n + x + 1}{1 + x - \alpha} \cdot \frac{x(1+x)}{n}.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$K_n^{(1/n)}((t-x)^2;x) = \frac{x(1+x)(2+x)}{n(1+x)-1}.$$

8. For $\beta = \dfrac{1}{x(1+x)}$, $\alpha \in (0,1)$, $\alpha x(1+x) < 1$ we obtain by (3.4) the following operator

$$(L_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} l_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.14}$$

$$l_{n,k}^{(\alpha)}(x) = \sum_{k=0}^{\infty} \frac{(1+\alpha(1+x))\dots(1+(k-1)\alpha(1+x))(1+\alpha x(1+x))\dots(1+n\alpha x(1+x))x^k}{(1+\alpha x)(1+2\alpha x)\dots(1+(n+k)\alpha x)(1+x)^{n+k}}$$

Corollary 3.12. *One has*

$$L_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha n x(1+x)+1}{1-\alpha x(1+x)} \cdot \frac{x(1+x)}{n}, \quad \alpha x(1+x) < 1.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$L_n^{(1/n)}((t-x)^2;x) = \frac{x(1+x)(1+x(1+x))}{n-x(1+x)}, \quad x(1+x) < n.$$

9. Another operator it is obtained for $\beta = x(1+x)$, $\alpha \in (0,1)$, $x(1+x) > \alpha$.

$$(M_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} m_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.15}$$

$$m_{n,k}^{(\alpha)}(x) = \frac{x^2(1+x)(x^2(1+x)+\alpha)\dots(x^2(1+x)+(k-1)\alpha)(x(1+x)+\alpha)\dots(x(1+x)+n\alpha)}{(x(1+x)^2+\alpha)\dots(x(1+x)^2+(n+k)\alpha)}.$$

Corollary 3.13. *One has*

$$M_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha n + x(1+x)}{x(1+x)-\alpha} \cdot \frac{x(1+x)}{n}, \quad x(1+x) > \alpha.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$M_n^{(1/n)}((t-x)^2;x) = \frac{x(1+x)(1+x(1+x))}{nx(1+x)-1}, \quad nx(1+x) > 1.$$

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

ON THE CONVERGENCE RATES OF PICARD, MANN AND ISHIKAWA ITERATIONS OF GENERALIZED CONTRACTIVE OPERATORS

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Abstract. The convergence rates of Picard, Mann and Ishikawa iterations have been compared by several authors for a class of quasi-contractive maps defined on an arbitrary closed convex subset of a Banach space (e.g. [1], [3] and [10]). In this paper, a comparison of the convergence rates of those iterations are studied for a more general class of operators called the generalized contractive operators.

1. Introduction

Let X be a real Banach space, and C a nonempty convex subset of X. Let T be a self map of C, and let $p_o, x_o, y_o, z_o \in C$. The Picard iteration is defined by

$$p_{n+1} = Tp_n. \tag{1}$$

The Mann iteration (see [7]) is defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n. \tag{2}$$

The Ishikawa iteration (see [6]) is defined by

$$y_{n+1} = (1-\alpha_n)y_n + \alpha_n T z_n \tag{3}$$

$$z_n = (1-\beta_n)y_n + \beta_n T y_n \tag{4}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$.

Received by the editors: 01.10.2008.
2000 *Mathematics Subject Classification.* 47H10.
Key words and phrases. Fixed point, quasi-contraction, Picard iteration, Mann iteration, Ishikawa iteration.

Definition 1. [18]. Let (X, d) be a metric space. $T : X \to X$ will be called a *Zamfirescu operator* if there exist the real numbers a, b, c satisfying $0 < a < 1,\ 0 < b,\ c < 1/2$ such that for each pair $x, y \in C$, at least one of the following is true:

(i) $d(Tx, Ty) \leq ad(x, y)$;

(ii) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$;

(iii) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Definition 2. [4]. Let T be a mapping of a metric space (X, d) into itself. A mapping T is called a $quasi - contraction$ if for some $0 \leq k < 1$ and all $x, y \in X$,

$$d(Tx, Ty) \leq k.max\{d(x, y); d(x, Tx); d(y, Ty); d(x, Ty); d(y, Tx)\}. \tag{5}$$

Clearly a Zamfirescu operator is a quasi-contraction map. Quasi-contraction map is one of the most general contractive maps. For results on quasi-contraction maps see [4-5],[14-15] and [17].

Definition 3. [8]. Let T be a mapping of a metric space (X, d) into itself. A mapping T will be called a *generalized contractive operator* if for some $0 \leq k < 1$ and all $x, y \in X$,

$$d(Tx, Ty) \leq k.max\{d(x, y); d(x, Tx); d(y, Ty); d(x, Ty) + d(y, Tx)\}. \tag{6}$$

A *generalized contractive operator* is more general than a *quasi-contraction* as can be seen from the following example.

Example. [8]. Let $X = R$ with the usual metric. Define $T : X \to X$ by $Tx = x$. Clearly T is a generalized contractive operator. In fact, $d(x, Ty)+d(y, Tx) = 2d(x, y)$, $d(Tx, Ty) = d(x, y)$. Let $k = \frac{3}{4}$. Then $d(Tx, Ty) \leq k\{d(x, Ty) + d(y, Tx)\}$. However T is not a quasicontraction.

The Ishikawa iteration and the Mann iteration converge to a fixed point of T when T is a *Zamfirescu operator* defined on a closed convex set of a Banach space (see [2], [9]). The Picard iteration converges faster than the Mann iteration [3] while the Mann iteration converges faster than the Ishikawa iteration [1] when dealing with the same class of Zamfirescu operators defined on a closed convex subset of a Banach space. In [5] it was shown that the Ishikawa iteration converges to the fixed point of T when T is a *quasi-contraction* map defined on a closed convex set of a Banach space. Also the Picard iteration converges faster to the fixed point of T than the Mann iteration [10] while the Mann iteration converges faster than the Ishikawa iteration [11] when T is a *quasi-contraction.* That answers the question posed in [3]. In this paper, we investigate the convergence rate of the Picard, Mann and Ishikawa iteration when dealing with a more general class of operators called the *generalized contractive operators* (6). It was proved that the Picard iteration converges to the fixed point of T faster than the Mann iteration and the Mann iteration converges faster than the Ishikawa iteration when T is a *generalized contractive operator.* It should be noted that the Picard iteration converges to the fixed point of T when T is a *generalized contractive operator* [13] while both the Ishikawa and consequently the Mann iterations of this class of maps also converges to the fixed point of T [12].

The definitions and the methodology of Berinde [3], also used in [1] and [10], will be adopted .

Definition 4. [3]. Let $\{a_n\}_{n=0}^{n=\infty}$ and $\{b_n\}_{n=0}^{n=\infty}$ be two sequences of real numbers that converge to a and b respectively, and assume there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}.$$

If $l = 0$, then we say that $\{a_n\}_{n=0}^{n=\infty}$ *converges faster* to a than $\{b_n\}_{n=0}^{n=\infty}$ to b.

Definition 5. [3]. Let $\{u_n\}_{n=0}^{n=\infty}$ and $\{v_n\}_{n=0}^{n=\infty}$ be two fixed point iteration procedures that converge to the same fixed point p on a normed space X such that the error estimates

$$\|u_n - p\| \le a_n,\ n = 0, 1, 2, ... \tag{7}$$

and

$$\|v_n - p\| \le b_n,\ n = 0, 1, 2, ... \tag{8}$$

are available, where $\{a_n\}_{n=0}^{n=\infty}$ and $\{b_n\}_{n=0}^{n=\infty}$ are two sequences of positive numbers (converging to zero). If $\{a_n\}_{n=0}^{n=\infty}$ converges faster than $\{b_n\}_{n=0}^{n=\infty}$, then we say that $\{u_n\}_{n=0}^{n=\infty}$ converges faster to p than $\{v_n\}_{n=0}^{n=\infty}$.

2. The main results

Theorem 1. *Let K be a nonempty closed convex subset of a Banach space X and let $T : K \to K$ be a generalized contractive operator (6). Then*

1) T has a unique fixed point p in X;

2) The Picard iteration $\{p_n\}$ defined by $Tp_n = p_{n+1}$ converges to p for any $p_o \in K$;

3) The Mann iteration $\{x_n\}$, defined by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n$, n=1,2,... such that $\sum \alpha_n = \infty$, converges strongly to p for any $x_o \in K$;

4) The Picard iteration converges to p faster than Mann iteration.

Proof. For the proofs of 1) and 2) see ([13]). The Ishikawa iteration of T converges to p [12]. By setting $\beta_n = 0$ for all n, it is clear that the Mann iteration converges too.

We now proof (4). Since T is a generalized contractive operator (6), then,

$$\begin{aligned} \|Ty - Tx\| \quad &\leq \quad k\ max\{\|y - x\|, \|x - Tx\|, \|y - Ty\|, \\ &\qquad \|x - Ty\| + \|y - Tx\|\}. \end{aligned}$$

If $\|Ty - Tx\| \leq k\|y - Ty\|$, then

$$\|Ty - Tx\| \leq k\{\|y - x\| + \|x - Tx\| + \|Tx - Ty\|\}$$

and so,

$$\|Ty - Tx\| \leq \frac{k}{1-k}\{\|y - x\| + \|x - Tx\|\}. \tag{9}$$

If $\|Ty - Tx\| \leq k\{\|x - Ty\| + \|y - Tx\|\}$, then,

$$\|Ty - Tx\| \leq k\{\|x - Tx\| + \|Tx - Ty\| + \|y - x\| + \|x - Tx\|\}$$

which, after computing, gives

$$\|Tx - Ty\| \leq \frac{k}{1-k}\{\|y - x\| + 2\|x - Tx\|\}. \tag{10}$$

Denote $\delta = \max\{k, \frac{k}{1-k}\} = \frac{k}{1-k}$. Then in view of (9) and (10), inequality (6) gives

$$\|Ty - Tx\| \leq \delta\{\|y - x\| + 2\|x - Tx\|\}. \tag{11}$$

Suppose p is a fixed point of T, then, if $x = p$ and $y = p_n$, from (11) we obtain

$$\|Tp_n - p\| \leq \delta \|p_n - p\|. \tag{12}$$

If we assume Picard approximation technique in (11) by assuming that $Tp_n = p_{n+1}$ for all n, we obtain

$$\|p_{n+1} - p\| \leq \delta \|p_n - p\|$$

which inductively gives

$$\|p_{n+1} - p\| \leq \delta^n \|p_1 - p\|, \ n \geq 0. \tag{13}$$

Following the same procedure in proving (10), it can be shown that

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|y - Tx\| \tag{14}$$

for all $x, y \in K$ where $\delta = \frac{k}{1-k}$.

Let $\{x_n\}_{n=0}^{n=\infty}$ be the Mann iteration as defined in the Theorem and $x_o \in K$ arbitrary. Then

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)x_n + \alpha_n Tx_n - (1 - \alpha_n + \alpha_n)p) \\ &= \|(1-\alpha_n)(x_n - p) + \alpha_n(Tx_n - p)\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|Tx_n - p). \qquad (*) \end{aligned}$$

If $x = p$ and $y = x_n$ in (14) we obtain

$$\|Tx_n - p\| \leq \delta \|x_n - p\| + 2\delta \|x_n - p\| = 3\delta \|x_n - p\|$$

and therefore by (*) we obtain

$$\|x_{n+1} - p\| \leq [1 - \alpha_n + 3\delta\alpha_n]\|x_n - p\| \leq [1 + 3\delta\alpha_n + \delta]\|x_n - p\|, \ n = 0, 1, 2, ..$$

which implies that

$$\|x_{n+1} - p\| \leq \prod_{k=1}^{n} [[1 + 3\delta\alpha_k + \delta]\|x_1 - p\|, \ n = 0, 1, 2, \ldots . \tag{15}$$

In order to compare $\{p_n\}$ and $\{x_n\}$ we must compare δ^n and $\prod_{k=1}^{n}[1 + 3\delta\alpha_k + \delta]$.

We first note that $\delta < 1 + 3\delta\alpha_k + \delta$ for each k. Therefore $\frac{\delta}{1+3\delta\alpha_k+\delta} < 1$ for each k. Hence

$$lim_{n\to\infty}\frac{\delta^n}{\prod_{k=1}^{n}[1+3\delta\alpha_k+\delta]} \to 0.$$

This shows that the Picard iteration converges faster than the Mann iteration.

Corollary 1. [10]. *Let K be a nonempty closed convex subset of a Banach space X and let $T : K \to K$ be a quasi-contraction map (5). Then*

1) T has a unique fixed point p in X;

2) The Picard iteration $\{p_n\}$ defined by $Tp_n = p_{n+1}$ converges to p for any $p_o \in K$;

3) The Mann iteration $\{x_n\}$, defined by $x_{n+1} = (1-\alpha_n)x_n+\alpha_n Tx_n$, n=1,2,... such that $\sum \alpha_n = \infty$, converges strongly to p for any $x_o \in K$;

4) The Picard iteration converges to p faster than Mann iteration.

Corollary 2. [3, Theorem 4]. *Let X be a Banach space, K a closed convex subset of X, and $T : K \to K$ a Zamfirescu operator. Then*

1) T has a unique fixed point p in X;

2) The Picard iteration $\{p_n\}$ defined by $Tp_n = p_{n+1}$ converges to p for any $p_o \in K$;

3) The Mann iteration $\{x_n\}$, defined by $x_{n+1} = (1-\alpha_n)x_n+\alpha_n Tx_n$, n=1,2,... such that $\sum \alpha_n = \infty$, converges strongly to p for any $x_o \in K$;

4) Picard iteration converges faster than Mann iteration.

Observe that Corollary 1 is more general than Corollary 2 which is the main result in [3].

Theorem 2. *Let K be a nonempty closed convex subset of a Banach space X and let $T : K \to K$ be a generalized contraction map (6). Let $\{x_n\}$ and $\{y_n\}$ be the Mann and Ishikawa iterations respectively defined by (2) and (3)-(4) for $x_o,\ y_o \in K$ with $\{\alpha_n\}$ and $\{\beta_n\}$ real sequences such that $0 \le \alpha_n,\ \beta_n \le 1$ and $\sum \alpha_n = \infty$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to the unique fixed point of T, and moreover, the Mann iteration converges to the fixed point of T faster than the Ishikawa iteration.*

Proof. The Ishikawa iteration (3)-(4) converges strongly to the unique fixed point of T (e.g. see [12]). Consequently, if $\beta_n = 0$ for all n, the Mann iteration converges strongly to the unique fixed point of T. Since the fixed point of T is unique [13], then both iterations must converge to the same fixed point which we denote by p.

It is not difficult to see that the quasi-contraction map satisfies the following inequalities

$$\|Tx - Ty\| \leq \delta\{\|x - y\| + 2\|x - Tx\|\} \tag{16}$$

$$\|Tx - Ty\| \leq \delta\{\|x - y\| + 2\|y - Tx\|\} \tag{17}$$

for all $x, y \in K$ where $\delta = \max\{k, \frac{k}{1-k}\} = \frac{k}{1-k}$.

Let $\{x_n\}$ be the Mann iteration associated with T, then, in view of (2), we have

$$\|x_{n+1} - p\| \leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - p\|. \tag{18}$$

Suppose $x = p$ and $y = x_n$, (16) becomes

$$\|Tx_n - p\| \leq \delta\|x_n - p\|. \tag{19}$$

In view of (18) and (19), we have

$$\|x_{n+1} - p\| \leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\delta\|x_n - p\| = [1 - \alpha_n(1 - \delta)]\|x_n - p\|. \tag{20}$$

Hence

$$\|x_{n+1} - p\| \leq \prod_{k=1}^{n}[1 - \alpha_k(1 - \delta)].\|x_1 - p\|,\ n = 0,\ 1,\ 2, ... \tag{21}$$

It is clear that

$$1 - \alpha_k(1 - \delta) > 0\ \forall\ k = 0,\ 1,\ 2, ... \tag{22}$$

Similarly, let $\{y_n\}$ be the Ishikawa iteration defined in (3)-(4), then, we have

$$\|y_{n+1} - p\| \leq (1 - \alpha_n)\|y_n - p\| + \alpha_n\|Tz_n - p\|. \tag{23}$$

If $x = p$ and $y = z_n$ in (17), we have

$$\|Tz_n - p\| \leq \delta\|z_n - p\| + 2\delta\|z_n - p\| = 3\delta\|z_n - p\|. \tag{24}$$

If $x = p$ and $y = y_n$ in (17), we have

$$\|Ty_n - p\| \leq \delta\|y_n - p\| + 2\delta\|y_n - p\| = 3\delta\|y_n - p\|. \tag{25}$$

We know by (4) that

$$\|z_n - p\| \leq (1 - \beta_n)\|y_n - p\| + \beta_n\|Ty_n - p\|. \tag{26}$$

In view of (23)-(26), we have

$$\begin{aligned}
\|y_{n+1} - p\| &\leq (1 - \alpha_n)\|y_n - p\| + 3\delta\alpha_n\|z_n - p\| \\
&\leq (1 - \alpha_n)\|y_n - p\| + 3\delta\alpha_n[(1 - \beta_n)\|y_n - p\| \\
&+ \beta_n\|Ty_n - p\|] \\
&= (1 - \alpha_n)\|y_n - p\| + 3\delta\alpha_n(1 - \beta_n)\|y_n - p\| \\
&+ 3\delta\alpha_n\beta_n\|Ty_n - p\| \\
&= [(1 - \alpha_n) + 3\delta\alpha_n(1 - \beta_n) + 9\alpha_n\beta_n\delta^2]\|y_n - p\| \\
&= [1 - \alpha_n(1 - 3\delta + 3\beta_n\delta - 9\beta_n\delta^2)].\|y_n - p\| \\
&= [1 - \alpha_n(1 - 3\delta)(1 + 3\beta_n\delta)].\|y_n - p\|. \qquad (**)
\end{aligned}$$

Since $(1 - 3\delta)(1 + 3\beta_n\delta) < 1 - 9\delta^2 \leq 1$, it is clear that

$$1 - \alpha_n(1 - 2\delta)(1 + 2\beta_n\delta) > 0 \ \forall\ n = 0, 1, 2, ... \tag{27}$$

We consider the following two cases.

Case (1). Let $\delta \in (0, 1/3]$. Hence

$$1 - \alpha_n(1 - 3\delta)(1 + 3\beta_n\delta) \leq 1 \ \forall\ n = 0, 1, 2, ... \tag{28}$$

(**) then becomes

$$\|y_{n+1} - p\| \leq \|y_n - p\| \ \forall\ n \tag{29}$$

and hence

$$\|y_{n+1} - p\| \leq \|y_1 - p\| \ \forall\ n. \tag{30}$$

If we compare the coefficients of (21) and (30), and using Definition 5 so that

$$a_n = \prod_{k=1}^{n}[1 - \alpha_k(1 - \delta)] \ and\ b_n = 1, \tag{31}$$

we have $lim_{n\to\infty}(a_n/b_n) = 0$

Case (ii). Let $\delta > 1/3$. In this case we have

$$1 - \alpha_n(1 - 3\delta)(1 + 3\beta_n\delta) \leq 1 - \alpha_n(1 - 9\delta^2) \tag{32}$$

and so (**) becomes

$$\|y_{n+1} - p\| \leq [1 - \alpha_n(1 - 9\delta^2)]\|y_n - p\| \;\forall\; n. \tag{33}$$

Hence

$$\|y_{n+1} - p\| \leq \prod_{k=1}^{n}[1 - \alpha_k(1 - 9\delta^2)]\|y_1 - p\|. \tag{34}$$

Comparing (21) and (34) and using Definition 5, we have

$$a_n = \prod_{k=1}^{n}[1 - \alpha_k(1 - \delta)] \; and \; b_n = \prod_{k=1}^{n}[1 - \alpha_k(1 - 9\delta^2)]. \tag{35}$$

Clearly, $a_n \geq 0$ and $b_n \geq 0 \;\forall\; n$ and $\frac{a_n}{b_n} = \prod_{k=1}^{n} \frac{1-\alpha_k(1-\delta)}{1-\alpha_k(1-9\delta^2)}$. Also

$$\frac{min[1 - \alpha_k(1 - \delta), k = 1, 2..n]}{max[1 - \alpha_k(1 - 9\delta), k = 1.2..n]} < 1.$$

Since $\prod_{k=1}^{n} \frac{1-\alpha_k(1-\delta)}{1-\alpha_k(1-9\delta^2)} < (\frac{min[1-\alpha_k(1-\delta),\; k=1,2....n]}{max[1-\alpha_k(1-9\delta),\; k=1,2....n]})^n$ then $lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Therefore in both cases $\{a_n\}$ converges faster than $\{b_n\}$ and hence the Mann iteration converges faster than the Ishikawa iteration to the fixed point p of T.

In view of Theorems 1 and 2, we have the following results.

Corollary 3. *Let K be a nonempty closed convex subset of a Banach space X and let $T : K \to K$ be a generalized contractive map (6). Then*

1) T has a unique fixed point p in X;

2) The Picard iteration $\{p_n\}$ defined by $Tp_n = p_{n+1}$ converges to p for any $p_o \in K$;

3) The Picard iteration converges faster to the fixed point of T than Mann iteration (2); and the Mann iteration converges faster than the Ishikawa iteration (3)-(4).

Corollary 4. *Let K be a nonempty closed convex subset of a Banach space X and let $T : K \to K$ be a quasi-contraction (5). Then*

1) T has a unique fixed point p in X;

2) The Picard iteration $\{p_n\}$ defined by $Tp_n = p_{n+1}$ converges to p for any $p_o \in K$;

3) The Picard iteration converges faster to the fixed point of T than Mann iteration (2); and the Mann iteration converges faster than the Ishikawa iteration (3)-(4).

Corollary 5. ([1],[3]). *Let K be a nonempty closed convex subset of a Banach space X and let $T : K \to K$ be a Zamfirescu operator. Then*

1) T has a unique fixed point p in X;

2) The Picard iteration $\{p_n\}$ defined by $Tp_n = p_{n+1}$ converges to p for any $p_o \in K$;

3) The Picard iteration converges faster to the fixed point of T than Mann iteration; and the Mann iteration converges faster than the Ishikawa iteration (3)-(4).

Remarks. **1.** The technique of our proofs is due to [3] and has been used by several authors, e.g. see [16].

2. Ishikawa iteration has two parameters, $\{\alpha_n\}$ and $\{\beta_n\}$; the Mann iteration has only one parameters $\{\alpha_n\}$ while the Picard iteration has none. It appears that the more the parameters for an iteration process, the slower the rate of convergence. At least this is true in the case of Picard, Mann and the Ishikawa iterations when applied to generalized contraction maps. It is therefore an open problem whether this conjecture is true for other known iteration procedures and for a more general class of operators.

3. A *generalized contraction* map (see [14-15]) is a map satisfying the inequality

$$\|Tx - Ty\| \leq Q(M(x, y)), \tag{36}$$

where Q is a real-valued function satisfying

(a) $0 < Q(s) < s$ for each $s > 0$ and $Q(0) = 0$,

(b) Q is non-decreasing on $(0, \infty)$,

(c) $g(s) = s/(s - Q(s)$ is non-increasing on $(0, \infty)$,

$$M(x, y) = max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}. \tag{37}$$

The Mann and the Ishikawa iterations are equivalent when dealing with generalized contraction maps [15] i.e. if the Mann iteration converges to the fixed point of T, then the Ishikawa iteration converges to the fixed point of T and if the Ishikawa iteration converges, then the Mann iteration converges to the fixed point of T. It is still an open problem as to which of the iterations converges faster when T is a generalized contraction map. Suppose (37) is replaced with $M(x, y) = max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\| + \|y - Tx\|\}$, will the Mann and the Ishikawa iterations still be equivalent? Will the Mann iteration still converge faster than the Ishikawa iteration to the unique fixed point of T?

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

ERROR BOUND FOR THE SOLUTION OF A POLYLOCAL PROBLEM WITH A COMBINED METHOD

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Abstract. Consider the problem:

$$
\begin{aligned}
-y''(t)+q(t)y(t)&=r(t), \qquad t\in[a,b]\\
y(c)&=\alpha\\
y(d)&=\beta, \qquad c,d\in(a,b).
\end{aligned}
$$

The aim of this paper is to give an error bound for the solution of this problem using a collocation with B-spline method combined with a Runge-Kutta method. A numerical example is also given.

1. **Introduction**

Consider the problem:

$$-y''(t)+q(t)y(t)=r(t), \qquad t\in[a,b] \tag{1}$$

$$y(d)=\alpha \tag{2}$$

$$y(e)=\beta, \qquad d,e\in(a,b), d<e. \tag{3}$$

where $q,r\in C[a,b]$, $\alpha,\beta\in\mathbb{R}$. *This is not a two-point boundary value problem* (BVP), since $d,e\in(a,b)$.

Received by the editors: 16.02.2009.

2000 *Mathematics Subject Classification.* 65L10.

Key words and phrases. Collocation method, Runge-Kutta method, B-spline.

If the solution of the two-point boundary value problem

$$
\begin{aligned}
-y''(t) + q(t)y(t) &= r(t), \qquad t \in (d, e) \\
y(d) &= \alpha \\
y(e) &= \beta,
\end{aligned}
\tag{4}
$$

exists and it is unique, then the requirement $y \in C^2[a, b]$ assures the existence and the uniqueness of (1)+(2)+(3).

We have two initial value problems on $[a, d]$ and $[e, b]$, respectively, and the existence and the uniqueness for (4) assure existence and uniqueness of these problems. It is possible to solve this problem by dividing it into the three above-mentioned problems and to solve each of these problem separately.

This decomposition strategy allows us to solve the problem using a new combined method (collocation + Runge-Kutta) and to give an error estimation.

2. **Principles of the method**

We decompose our problem into a two-point BVP:

$$-y''(t) + q(t)y(t) = r(t), \qquad t \in (d, e) \tag{5}$$

$$y(d) = \alpha \tag{6}$$

$$y(e) = \beta, \tag{7}$$

and two initial value problems (IVP)

$$-y''(t) + q(t)y(t) = r(t), \qquad t \in [a, d] \tag{8}$$

$$y(d) = \alpha \tag{9}$$

$$y'(d) = \alpha' \tag{10}$$

and

$$-y''(t) + q(t)y(t) = r(t), \qquad t \in [e, b] \tag{11}$$

$$y(e) = \beta \tag{12}$$

$$y'(e) = \beta'. \tag{13}$$

The values of the differential y' at d and e required for the solution of problems (8)+(9)+(10) and (11)+(12)+(13) are approximated during the solution of the problem (5)+(6)+(7).

For the first problem we use a collocation method with nonuniform B-splines of order $k+2$, $k \in \mathbb{N}^*$ [1, 10, 3]. For properties of B-spline and basic algorithms see [5].

Consider the mesh (see [2, 3])

$$\Delta : d = x_1 < x_2 < \cdots < x_N < x_{N+1} = e, \tag{14}$$

and the step sizes

$$h_i := x_{i+1} - x_i, \qquad i = 1, \ldots, N.$$

The multiplicity of e and d is $k+2$ and the inner points have the multiplicity k. Within each subinterval we consider k points

$$\xi_{i,j} := x_i + h_i \rho_j, \qquad j = 1, \ldots, k, \qquad i = 1, \ldots, N,$$

where

$$0 \leq \rho_1 < \rho_2 < \cdots < \rho_k \leq 1,$$

are the roots of the kth Legendre's orthogonal polynomial on $[0, 1]$ [7, 8]. We add the points d and e to the set of collocation points.

We shall choose the basis such that the following conditions hold:

(C1) the solution verifies the differential equation (1) at $\xi_{i,j}$;

(C2) the solution verifies the conditions (2), (3).

We need a basis having $n = Nk + 2$ cubic B-spline functions.

One renumbers the collocation points (ξ_k), such that the first point is d and the last is e.

The form of solution is

$$y_\Delta(t) = \sum_{i=1}^{n} c_i B_i(t), \tag{15}$$

where $B_i(t)$ is the $k+2$ order B-spline with knots $x_i, \ldots, x_{i+k}$.

The conditions (C1) +(C2) yield a linear system $Ac = \gamma$, with n equations and n unknowns (the coefficients c_i, $i = 1, \ldots, n$).

Its matrix is

$$A = [a_{ij}]_{i,j=1,\ldots,n},$$

where

$$a_{ij} = \begin{cases} -B_j''(\xi_i) + q(\xi_i)B_j(\xi_i), & \text{for } i = 2, ..., n-1 \\ B_j(d), & \text{for } i = 1 \\ B_j(e), & \text{for } i = n. \end{cases} \tag{16}$$

The system matrix is banded with at most $k+2$ nonzero elements on each line ($k+2$ nonzero splines at each inner collocation point and only one four at d and e), since a $k+2$ order B-splines is nonzero only on $k+2$ consecutive subintervals. The right-hand side of the system is

$$\gamma = [\alpha, r(\xi_2), \ldots, r(\xi_{n-1}), \beta]^T.$$

The paper [9] gives a Maple implementation based on a different B-spline basis.

For the solution of problems (8)+(9)+(10) and (11)+(12)+(13) we consider a Runge-Kutta method with sufficiently high order. For the left IVP we consider negative steps. The values α' and β' are obtained by differentiating the B-spline solution of the BVP at points d and e, respectively.

3. **Main result**

Our estimation is inspired from [3, Chapter 5]. If the mesh is sufficiently fine, the condition number of matrix A given by (16) is not too high and the order of

Runge-Kutta method is sufficiently high we can obtain an acceptable upper bound of error.

Theorem 1. *Suppose there exists a $p \geq k \geq 2$ such that*

(a) *The linear problem (5) with boundary conditions (6)+(7) is well-posed, that is, the equivalent problem*

$$\begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ q(x) & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} + \begin{bmatrix} 0 \\ -r(x) \end{bmatrix}$$

has a condition number $\kappa_k = cond(A)$ of moderate size, $q,\ r \in C^p[a,b]$;

(b) *The linear problem (5) with boundary conditions (6)+(7) has a unique solution;*

(c) *The collocation points $\rho_1, \ldots, \rho_k$ satisfy the orthogonality condition*

$$\int_0^1 \Phi(t) \prod_{\ell=1}^{k} (t-\rho_\ell) dt = 0, \qquad \Phi \in \mathbb{P}_{p-k},\ (p \leq 2k),$$

where $\mathbb{P}_{p-k}$ is a set of polynomials at most degree $p-k$.

Then, for $h = \max_{i=1,\ldots,N} h_i$ sufficiently small, our method (collocation+two Runge-Kutta) is stable with constant $\kappa_k N$ and leads to a unique solution $y_\Delta(x)$. Furthermore, at mesh points x_i it holds

$$\left| y^{(j)}(x_i) - y_\Delta^{(j)}(x_i) \right| = O(h^p), \qquad j = 0,1;\ i = 1,...,N+1, \tag{17}$$

while, on the other hands, for $i = 1, \ldots, N$, $x \in [x_i, x_{i+1}]$

$$\left| y^{(j)}(x) - y_\Delta^{(j)}(x) \right| = O(h_i^{k+2-j}) + O(h^p), \qquad j = 0, \ldots, k+1. \tag{18}$$

Remark 2. *The condition* (c) *means that (ρ_ℓ) are the roots of kth Legendre polynomial.*

Proof. Using a result from [3, Theorem 5.140, page 253] we obtain the estimations (17)+(18) for the BVP (5)+(6)+(7). The error obtained by approximating α' and β' with $y_\Delta^{'}(d)$ and $y_\Delta^{'}(e)$ is $O(h^p)$, then, we use [7, Theorem 5.4.1, page 293]. If we choose an embedded pair of Runge-Kutta method of order at least $(p, p+1)$, the conditions in the hypothesis of theorem are fulfilled and the final error is $O(h^p)$. So, if the mesh is sufficiently fine, the embedded pair of Runge-Kutta methods does not increase the order of error. □

Remark 3. *The condition number may grow rapidly when h is small. The paper* [2, page 129] *gives the following estimation*

$$\kappa_\Delta \approx K \sum_{i=1}^{N} h_i^{-2} \max_{j=1,\dots,N+1} \int_{x_i}^{x_{i+1}} |G(x_j,t)|\,dt,$$

where K is a generic constant and G is the Green's function for the BVP problem.

4. Numerical examples

Our implementation is based on ideas from [5, 4]. We implement the method in MATLAB[1], using the Spline Toolbox™ 3 [6]. If $d = a$ and $e = b$, our problem becomes a classical BVP. If $d = a$ or $e = b$, our problem is decomposed into a BVP and one IVP. As a numerical example, we chose a problem with oscillatory solution:

$$\begin{aligned} -y''(x) - 243y(x) &= x, \qquad x \in [0,1] \qquad (19)\\ y\left(\frac{1}{4}\right) &= \frac{1}{243}\frac{\sin\left(\frac{9\sqrt{3}}{4}\right)}{\sin 9\sqrt{3}} - \frac{1}{972}\\ y\left(\frac{3}{4}\right) &= \frac{1}{243}\frac{\sin\left(\frac{27\sqrt{3}}{4}\right)}{\sin 9\sqrt{3}} - \frac{1}{324}. \end{aligned}$$

If we chose $k = 3$, the order of spline will be 5, and $p = 4$. For the initial value problems we choose the solver `ode45` (order 4). The exact solution is

$$y(x) = \frac{1}{243}\frac{\sin 9\sqrt{3}x}{\sin 9\sqrt{3}} - \frac{1}{243}x.$$

We plot the exact solution and approximate solution and the error in a semi-logarithmic scale for $n = 2$ and $k = 3$ in Figures 1 and 2, respectively.

[1]MATLAB is a trademark of the MathWorks, Inc.

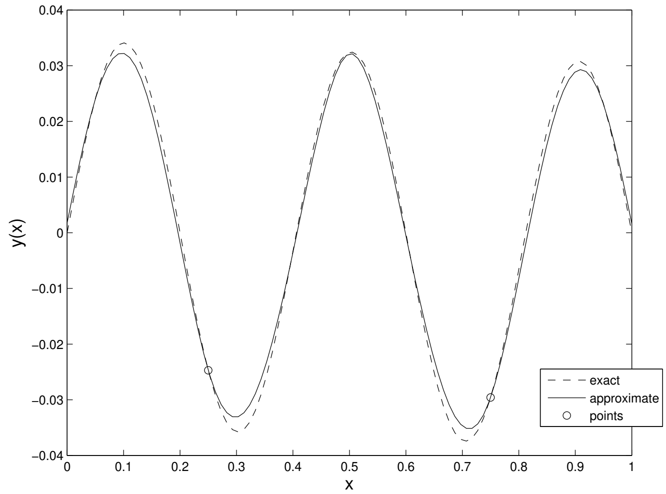

FIGURE 1. Exact and approximate solution of (19)

The collocation matrix for the BVP is

$$\begin{pmatrix}
1.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-301.782 & 187.397 & -91.380 & -35.996 & -1.239 & 0 & 0 & 0 \\
-63.187 & -60.750 & 4.875 & -92.343 & -31.593 & 0 & 0 & 0 \\
-2.477 & -34.757 & -91.380 & 36.506 & -150.891 & 0 & 0 & 0 \\
0 & 0 & 0 & -150.891 & 36.506 & -91.380 & -34.757 & -2.4779 \\
0 & 0 & 0 & -31.593 & -92.343 & 4.875 & -60.750 & -63.1875 \\
0 & 0 & 0 & -1.239 & -35.996 & -91.380 & 187.397 & -301.782 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.000
\end{pmatrix}$$

and its condition number is κ_Δ =2.5422e+003.

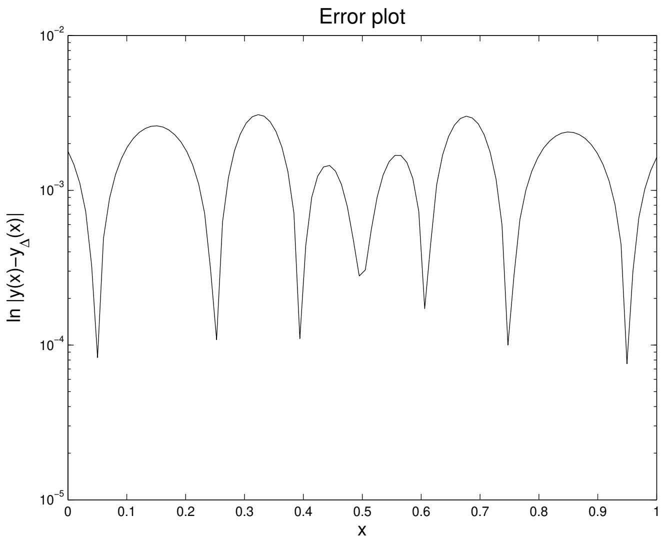

FIGURE 2. Error plot for Example (19)

5. Conclusions

The error estimation does not depend on the number of collocation points. Nevertheless, the Runge-Kutta method requires an order greater or equal to the order of error for the derivatives at d and e. We can conclude collocation combined with Runge-Kutta is an effective method for polylocal problem.

Aknowledgement. The author is indebted to Professor Ph.D. Damian Trif and Associate Professor Ph.D Radu Trîmbiţaş for their support and helpful hints and comments during the elaboration of this paper.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

ULAM STABILITY OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper we present four types of Ulam stability for ordinary differential equations: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. Some examples and counterexamples are given.

1. Introduction

The basic statements of data dependence in the theory of ordinary differential equations are the following (see for example [2], [5], [6], [8], [17], [20], [23], [24]): monotony w.r.t. data, continuity w.r.t. data, differentiability w.r.t. parameters, Liapunov stability, asymptotic behavior, structural stability, analiticity of solutions, regularity of solutions, G-convergences. On the other hand, in the theory of functional equations, there are some special kind of data dependence (see [9], [10], [4], [7], [3], [18], [19]). There are some results of this type for some differential equations ([8], [11], [12], [14]-[16]) and some integral equations ([13], [21] and [22]).

With these results in mind we shall present, in this paper, four types of Ulam stability for ordinary differential equations: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. Some examples and some counterexamples are given.

Received by the editors: 20.02.2009.

2000 *Mathematics Subject Classification.* 34G20, 34A40, 45N05, 47H10.

Key words and phrases. Differential equation, Ulam-Hyers stability, Ulam-Hyers-Rassias stability, differential inequation, integral inequation, Gronwall lemma, Cauchy problem, boundary value problem.

2. **General definitions and remarks**

Let $(\mathbb{B}, |\cdot|)$ be a (real or complex) Banach space, $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$, $a < b \leq +\infty$, ε a positive real number, $f : [a,b) \times \mathbb{B} \to \mathbb{B}$ be a continuous operator and $\varphi : [a,b) \to \mathbb{R}_+$ be a continuous function. We consider the following differential equation

$$x'(t) = f(t, x(t)),\ \forall\ t \in [a,b) \tag{2.1}$$

and the following differential inequations

$$|y'(t) - f(t, y(t))| \leq \varepsilon,\ \forall\ t \in [a,b) \tag{2.2}$$

$$|y'(t) - f(t, y(t))| \leq \varphi(t),\ \forall\ t \in [a,b) \tag{2.3}$$

and

$$|y'(t) - f(t, y(t))| \leq \varepsilon\varphi(t), \quad t \in [a,b). \tag{2.4}$$

Definition 2.1. The equation (2.1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C^1([a,b), \mathbb{B})$ of (2.2) there exists a solution $x \in C^1([a,b), \mathbb{B})$ of (2.1) with

$$|y(t) - x(t)| \leq c_f \varepsilon,\ \forall\ t \in [a,b).$$

Definition 2.2. The equation (2.1) is generalized Ulam-Hyers stable if there exists $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_f(0) = 0$, such that for each solution $y \in C^1([a,b), \mathbb{B})$ of the inequation (2.2) there exists a solution $x \in C^1([a,b), \mathbb{B})$ of the equation (2.1) with

$$|y(t) - x(t)| \leq \theta_f(\varepsilon),\ \forall\ t \in [a,b).$$

Definition 2.3. The equation (2.1) is Ulam-Hyers-Rassias stable with respect to φ if there exists $c_{f,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C^1([a,b), \mathbb{B})$ of (2.4) there exists a solution $x \in C^1([a,b), \mathbb{B})$ of (2.1) with

$$|y(t) - x(t)| \leq c_{f,\varphi}\varepsilon\varphi(t),\ \forall\ t \in [a,b).$$

Definition 2.4. The equation (2.1) is generalized Ulam-Hyers-Rassias stable with respect to φ if there exists $c_{f,\varphi} > 0$ such that for each solution $y \in C^1([a,b), \mathbb{B})$ of

(2.3) there exists a solution $x \in C^1([a,b),\mathbb{B})$ of (2.1) with

$$|y(t) - x(t)| \leq c_{f,\varphi}\varphi(t),\ \forall\ t \in [a,b).$$

Remark 2.1. A function $y \in C^1([a,b),\mathbb{B})$ is a solution of (2.2) if and only if there exists a function $g \in C([a,b),\mathbb{B})$ (which depend on y) such that

(i) $|g(t)| \leq \varepsilon,\ \forall\ t \in [a,b)$

(ii) $y'(t) = f(t,y(t)) + g(t),\ \forall\ t \in [a,b)$.

We have similar remarks for the inequations (2.3) and (2.4).

So, the Ulam stabilities of the differential equations are some special types of data dependence of the solutions of differential equations.

Remark 2.2. If $y \in C^1([a,b),\mathbb{B})$ is a solution of the inequation (2.2), then y is a solution of the following integral inequation

$$\left| y(t) - y(a) - \int_a^t f(s,y(s))ds \right| \leq (t-a)\varepsilon,\ \forall\ t \in [a,b).$$

Indeed, by Remark 2.1 we have that

$$y'(t) = f(t,y(t)) + g(t), \quad t \in [a,b).$$

This implies that

$$y(t) = y(a) + \int_a^t f(s,y(s))ds + \int_a^t g(s)ds, \quad t \in [a,b).$$

From this it follows that

$$\begin{aligned} \left| y(t) - y(a) - \int_a^t f(s,y(s))ds \right| &\leq \left| \int_a^t g(s)ds \right| \\ &\leq \int_a^t |g(s)|ds \leq \varepsilon(t-a). \end{aligned}$$

We have similar remarks for the solutions of the inequations (2.3) and (2.4).

Remark 2.3. A solution of the inequation (2.2) is called an ε-solution of the equation (2.1) (see for example [2], p. 94-95; [8], p. 14-18; [24], p. 233).

Remark 2.4. The case $b < +\infty$ and the case $b = +\infty$ are two distinct cases as the following example shows.

Example 2.1. We consider in the case $\mathbb{B} := \mathbb{R}$ the equation

$$x'(t) = 0, \quad t \in [a, b) \tag{2.5}$$

and the inequation

$$|y'(t)| \leq \varepsilon, \quad t \in [a, b). \tag{2.6}$$

Let $y \in C^1[a, b)$ be a solution of (2.6). Then there exists $g \in C[a, b]$ such that:

(i) $|g(t)| \leq \varepsilon,\ t \in [a, b)$

(ii) $y'(t) = g(t),\ t \in [a, b)$.

We have, for all $c \in \mathbb{R}$,

$$\begin{aligned} |y(t) - c| &\leq |y(0) - c| + \int_a^t |g(s)|ds \\ &\leq |y(0) - c| + \varepsilon(t - a),\ t \in [a, b). \end{aligned}$$

If we take $c := y(0)$, then

$$|y(t) - y(0)| \leq \varepsilon(t - a), \quad t \in [a, b).$$

If $b < +\infty$, then

$$|y(t) - y(0)| \leq (b - a)\varepsilon.$$

So, the equation (2.5) is Ulam-Hyers stable.

Let $b = +\infty$. The function $y(t) = \varepsilon t$ is a solution of the inequation (2.6) and

$$|y(t) - c| = |\varepsilon t - c| \to +\infty \text{ as } t \to +\infty.$$

So, the equation (2.5) is not Ulam-Hyers stable on the interval $[a, +\infty)$.

Let us consider the inequation

$$|y'(t)| \leq \varphi(t), \quad t \in [a, +\infty). \tag{2.7}$$

Let y be a solution of (2.7) and $x(t) = y(0)$, $t \in [a, +\infty)$ a solution of (2.5). We have that

$$|y(t) - x(t)| = |y(t) - y(0)| \leq \int_a^t \varphi(s)ds, \quad t \in [a, +\infty).$$

If there exists $c_\varphi \in \mathbb{R}_+$ such that

$$\int_a^t \varphi(s)ds \le c_\varphi \varphi(t), \quad t \in [a, +\infty)$$

then the equation (2.5) is generalized Ulam-Hyers-Rassias stable on $[a, +\infty)$ with respect to φ.

Remark 2.5. For the Ulam-Hyers-Rassias stability of the differential equation

$$y' - \lambda y = 0$$

in a Banach space see [16]. For other results see [1], [11], [12], [14] and [15].

3. Generalized Ulam-Hyers-Rassias stability

Let us consider the equation (2.1) and the inequation (2.3) in the case $b = \infty$. We suppose that:

(i) $f \in C([a, +\infty) \times \mathbb{B}, \mathbb{B})$ and $\varphi \in C([a, +\infty), \mathbb{R}_+)$ be an increasing function;

(ii) there exists $l_f \in L^1[a, +\infty)$ such that

$$|f(t,u) - f(t,v)| \le l_f(t)|u - v|, \ \forall\, u, v \in \mathbb{B}, \ \forall\, t \in [a, +\infty);$$

(iii) there exists $\lambda_\varphi > 0$ such that

$$\int_a^t \varphi(s)ds \le \lambda_\varphi \varphi(t), \ \forall\, t \in [0, a + \infty).$$

We have

Theorem 3.1. *In the conditions (i), (ii), (iii) the equation (2.1)* ($b = +\infty$) *is generalized Ulam-Hyers-Rassias stable.*

Proof. Let $y \in C^1([a, +\infty), \mathbb{B})$ be a solution of the inequation (2.3) ($b = +\infty$). Denote by x the unique solution of the Cauchy problem

$$x'(t) = f(t, x(t)), \ t \in [a, +\infty)$$
$$x(a) = y(a).$$

We have that

$$x(t) = y(a) + \int_a^t f(s, x(s))ds, \quad t \in [a, +\infty)$$

and

$$\left|y(t) - y(a) - \int_a^t f(s, y(s))ds\right| \leq \int_a^t \varphi(s)ds \leq \lambda_\varphi \varphi(t), \quad t \in [a, +\infty).$$

From these relation it follows

$$\begin{aligned}|y(t) - x(t)| &\leq \left|y(t) - y(a) - \int_a^t f(s, y(s))ds\right| \\ &+ \int_a^t |f(s, y(s)) - f(s, x(s))|ds \\ &\leq \lambda_\varphi \varphi(t) + \int_a^t l_f(s)|y(s) - x(s)|ds.\end{aligned}$$

By a Gronwall lemma (see [22], [23], [5]) we have that

$$\begin{aligned}|y(t) - x(t)| &\leq \lambda_\varphi \varphi(t) e^{\int_a^t l_f(s)ds} \\ &\leq [\lambda_\varphi l^{\int_a^{+\infty} l_f(s)ds}]\varphi(t) = c_{f,\varphi}\varphi(t),\ t \in [a, +\infty),\end{aligned}$$

i.e. the equation (2.1) ($b = +\infty$) is generalized Ulam-Hyers-Rassias stable.

Remark 3.1. For the case $\mathbb{B} := \mathbb{C}$ see [13], [15].

Remark 3.2. If we take $\mathbb{B}$ a Banach space of sequences in $\mathbb{K} = \mathbb{R} \vee \mathbb{C}$ $(C(\mathbb{K}), C_0(\mathbb{K}), l^p(\mathbb{K}), \dots)$ then we have some results for an infinite system of differential equations.

Remark 3.3. For the Ulam stability of some integral equations see [13] and [21].

Remark 3.4. If we have a differential equation of n-order in a Banach space $\mathbb{B}$ then we reduce it to a differential equation of first order in the Banach space $\mathbb{B}^n$. If the order n is even we can use the Green function technique as the following example shows.

For simplicity we shall consider the following second order differential equation

$$-x''(t) = f(t, x(t)), \quad t \in [a, b] \tag{3.1}$$

where $a < b < +\infty$ and $f \in C([a, b] \times \mathbb{R})$.

Let us denote by G the Green function of the following boundary value problem (see [6], [17], [20], [23])

$$-y'' = h(t)$$

$$y(a) = 0,\ y(b) = 0$$

The function $G : [a,b] \times [a,b] \to \mathbb{R}$ is defined by

$$G(t,s) := \begin{cases} \dfrac{(s-a)(b-t)}{b-a} & \text{if} \quad s \leq t, \\ \dfrac{(t-a)(b-s)}{b-a} & \text{if} \quad s \geq t. \end{cases}$$

We have

Theorem 3.2. *We suppose that:*

(i) $f \in C([a,b] \times \mathbb{R})$;

(ii) there exists $L_f > 0$ *such that*

$$|f(t,u) - f(t,v)| \leq L_f|u - v|,\ \forall\ t \in [a,b],\ \forall\ u,v \in \mathbb{R};$$

(iii) $L_f \dfrac{(b-a)^2}{4} < 1.$

Then the equation (3.1) is Ulam-Hyers stable.

Proof. Let $y \in C^2[a,b]$ be a solution of the inequation

$$|-y'' - f(t,y(t))| \leq \varepsilon,\ \forall\ t \in [a,b].$$

First of all we remark that y is a solution of the following inequation

$$\left| y(t) - \frac{t-a}{b-a}y(b) - \frac{b-t}{b-a}y(a) - \int_a^b G(t,s)f(s,y(s))ds \right|$$

$$\leq \varepsilon \left[\frac{t^2}{2} - \frac{a+b}{2}t + \frac{ab}{2} \right], \quad t \in [a,b].$$

Now we take x the solution of the following boundary value problem ([8], p. 186; [20], p. 99)

$$-x''(t) = f(t,x(t)), \quad t \in [a,b],$$

$$x(a) = y(a), \quad x(b) = y(b).$$

It is clear that

$$x(t) = \frac{t-a}{b-a}y(b) + \frac{b-t}{b-a}y(a) + \int_a^b G(t,s)f(s,x(s))ds, \quad t \in [a,b]$$

and we estimate $|y(t) - x(t)|$ in a similar way as in the proof of Theorem 3.1.

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COMPLETE SUBMANIFOLDS IN A HYPERBOLIC SPACE

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Abstract. In this paper, we study n-dimensional ($n \geq 3$) complete submanifolds M^n in a hyperbolic space $H^{n+p}(-1)$ with the scalar curvature $n(n-1)R$ and the mean curvature H being linearly related. Suppose that the normalized mean curvature vector field is parallel and the mean curvature is positive and obtains its maximum on M^n. We prove that if the squared norm $\|h\|^2$ of the second fundamental form of M^n satisfies $\|h\|^2 \leq nH^2 + (B_H)^2, (p \leq 2)$, and $\|h\|^2 \leq nH^2 + (\widetilde{B}_H)^2, (p \geq 3)$, then M^n is totally umbilical, or M^n is isometric to $S^{n-1}(r) \times H^1(-1/(r^2+1))$ for some $r > 0$, where B_H and $\widetilde{B}_H$ are denoted by (1.1) and (1.2), respectively.

1. Introduction

Let $M_p^{n+p}(c)$ be a $(n+p)$-dimensional space form of constant curvature c, M^n be an n-dimensional submanifold in $M^{n+p}(c)$ with parallel mean curvature vector. If $c = 0$, Cheng and Nonaka [3] obtained some intrinsic rigidity theorems of complete submanifolds with parallel mean vector in Euclidean space R^{n+p}. If $c > 0$, Xu [16] obtained the intrinsic rigidity theorems of these kind of submanifolds in a sphere $S^{n+p}(c)(c = 1)$. If $c < 0$, Yu [18] and Hu [10] proved some intrinsic rigidity theorems of complete hypersurfaces with constant mean curvature in a hyperbolic space $H^{n+1}(c)$

Let M^n be an n-dimensional complete submanifold with constant normalized scalar curvature in $M^{n+p}(c)$. If $c = 0$, for hypersurfaces ($p = 1$), Cheng and

Received by the editors: 13.05.2008.

2000 *Mathematics Subject Classification.* 53C40, 53C42.

Key words and phrases. Submanifold, scalar curvature, mean curvature, hyperbolic space.

Project supported by NSF of Shaanxi Province (SJ08A31) and NSF of Shaanxi Educational Committee (2008JK484).

Yau [6] obtained an intrinsic rigidity theorem of these kind of hypersurfaces in Euclidean space R^{n+1}, and for submanifolds ($p > 1$), Cheng [4] studied the problem and obtained a rigidity and classification theorem. If $c > 0$, Li [10] proved a rigidity and classification theorem of compact hypersurfaces with constant normalized scalar curvature in a sphere $S^{n+1}(c)(c = 1)$. As a generalization, Cheng [4] obtained a rigidity and classification theorem of higher codimension compact submanifolds in $S^{n+p}(c)(c = 1)$. If $c < 0$, the authors [15] studied the submanifolds with constant normalized scalar curvature in hyperbolic space $H^{n+p}(c)(c = -1)$ and obtained some rigidity and classification theorems.

It is well-know that the investigation on hypersurfaces with the scalar curvature $n(n-1)R$ and the mean curvature H being linearly related is also important and interesting. Fox example, Cheng [5] and Li [11] obtained some characteristic theorems of such space-like hypersurfaces in a de Sitter space and such compact hypersurfaces in a unit sphere in terms of sectional curvature, respectively. It is natural and very important to study n-dimensional submanifolds with the scalar curvature $n(n-1)R$ and the mean curvature H being linearly related and with higher codimension in a space form $M^{n+p}(c)$. But there are few results about it. In this paper, we shall investigate n-dimensional complete submanifolds in a hyperbolic space $H^{n+p}(-1)$ with the scalar curvature and the mean curvature being linearly related. We shall prove the following:

Main Theorem. *Let M^n be a n-dimensional $(n \geq 3)$ complete submanifold with $n(n-1)R = k'H$, $(H^2 \geq 1)$ in a hyperbolic space $H^{n+p}(-1)$, where k' is a positive constant. Suppose that the normalized mean curvature vector field is parallel and the mean curvature H is positive and obtains its maximum on M^n. If the norm square $\|h\|^2$ of the second fundamental form of M^n satisfies*

$$\|h\|^2 \leq nH^2 + (B_H^+)^2, (p \leq 2),$$

and

$$\|h\|^2 \leq nH^2 + (\widetilde{B}_H^+)^2, (p \geq 3),$$

then M^n is totally umbilical, or M^n is isometric to $S^{n-1}(r) \times H^1(-1/(r^2+1))$ for some $r > 0$, where B_H^+ and $\widetilde{B}_H^+$ are denoted by

$$B_H^+ = -\frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}H + \sqrt{\frac{n^3H^2}{4(n-1)} - n}, \tag{1.1}$$

$$\widetilde{B}_H^+ = -\frac{1}{3}(n-2)\sqrt{\frac{n}{n-1}}H + \frac{1}{3}\sqrt{\frac{n}{n-1}(n^2+2n-2)H^2 - 6n}. \tag{1.2}$$

2. **Preliminaries**

Let M^n be a n-dimensional complete submanifold in a hyperbolic space $H^{n+p}(-1)$, we choose a local field of orthonormal frames $e_1, \cdots, e_{n+p}$ in $H^{n+p}(-1)$ such that at each point of $M^n, e_1, \cdots, e_n$ span the tangent space of M^n. Let $\omega_1, \cdots, \omega_{n+p}$ be the dual frame field, then the structure equations of $H^{n+p}(-1)$ are given by

$$d\omega_A = -\sum_{B=1}^{n+p} \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_{C=1}^{n+p} \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{C,D=1}^{n+p} K_{ABCD}\omega_C \wedge \omega_D, \tag{2.2}$$

$$K_{ABCD} = -(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.3}$$

Restricting these form to M^n, we have

$$\omega_\alpha = 0, \quad \alpha = n+1, \cdots, n+p. \tag{2.4}$$

$$\omega_{\alpha_i} = \sum_{j=1}^{n} h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha, \tag{2.5}$$

$$d\omega_i = -\sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.6}$$

$$d\omega_{ij} = -\sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l=1}^{n} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.7}$$

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{\alpha=n+1}^{n+p} (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha). \tag{2.8}$$

The normal curvature tensor $R_{\alpha\beta ij}$ and Ricci curvature are

$$R_{\alpha\beta ij} = \sum_{l=1}^{n} (h_{il}^{\alpha} h_{lj}^{\beta} - h_{jl}^{\alpha} h_{li}^{\beta}), \tag{2.9}$$

$$R_{jk} = -(n-1)\delta_{jk} + \sum_{\alpha=n+1}^{n+p} (\sum_{i=1}^{n} h_{ii}^{\alpha} h_{jk}^{\alpha} - \sum_{i=1}^{n} h_{ik}^{\alpha} h_{ji}^{\alpha}), \tag{2.10}$$

$$n(n-1)(R+1) = n^2H^2 - \|h\|^2, \tag{2.11}$$

where R is the normalized scalar curvature, H is the mean curvature of M^n, $\|h\|^2$ is the squared norm of the second fundamental form of M^n. Define the first and second covariant derivatives of h_{ij}^{α} by

$$\sum_{k=1}^{n} h_{ijk}^{\alpha} \omega_k = dh_{ij}^{\alpha} - \sum_{k=1}^{n} h_{ik}^{\alpha} \omega_{kj} - \sum_{k=1}^{n} h_{jk}^{\alpha} \omega_{ki} - \sum_{\beta=n+1}^{n+p} h_{ij}^{\beta} \omega_{\beta\alpha}, \tag{2.12}$$

$$\sum_{l=1}^{n} h_{ijkl}^{\alpha} \omega_l = dh_{ijk}^{\alpha} - \sum_{l=1}^{n} h_{ljk}^{\alpha} \omega_{li} - \sum_{l=1}^{n} h_{ilk}^{\alpha} \omega_{lj} - \sum_{l=1}^{n} h_{ijl}^{\alpha} \omega_{lk} - \sum_{\beta=n+1}^{n+p} h_{ijk}^{\beta} \omega_{\beta\alpha}. \tag{2.13}$$

The Codazzi equation and Ricci identities are

$$h_{ijk}^{\alpha} = h_{ikj}^{\alpha} = h_{jik}^{\alpha}, \tag{2.14}$$

$$h_{ijkl}^{\alpha} - h_{ijlk}^{\alpha} = \sum_{m=1}^{n} h_{mj}^{\alpha} R_{mikl} + \sum_{m=1}^{n} h_{im}^{\alpha} R_{mjkl} + \sum_{\beta=n+1}^{n+p} h_{ij}^{\beta} R_{\beta\alpha kl}. \tag{2.15}$$

The Laplacian of h_{ij}^{α} is defined by $\Delta h_{ij}^{\alpha} = \sum_{k=1}^{n} h_{ijkk}^{\alpha}$. From (2.14) and (2.15), we get

$$\Delta h_{ij}^{\alpha} = \sum_{k=1}^{n} h_{kkij}^{\alpha} + \sum_{k,m=1}^{n} h_{km}^{\alpha} R_{mijk} + \sum_{k,m=1}^{n} h_{mi}^{\alpha} R_{mkjk} + \sum_{k=1}^{n} \sum_{\beta=n+1}^{n+p} h_{ki}^{\beta} R_{\beta\alpha jk}. \tag{2.16}$$

Denote by ξ the mean curvature vector field. When $\xi \neq 0$, since we suppose $H > 0$, $e_{n+1} = \frac{\xi}{H}$ is the normal vector field on M^n. We define S_1 and S_2 by

$$S_1 = \sum_{i,j=1}^{n} (h_{ij}^{n+1} - H\delta_{ij})^2, \quad S_2 = \sum_{\alpha=n+2}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2. \tag{2.17}$$

Obviously, we have

$$\|h\|^2 = nH^2 + S_1 + S_2. \tag{2.18}$$

By the definition of the mean curvature vector ξ, we have

$$nH = \sum_{i=1}^{n} h_{ii}^{n+1}, \quad \sum_{i=1}^{n} h_{ii}^{\alpha} = 0, n+2 \le \alpha \le n+p. \tag{2.19}$$

From (2.11), (2.17) and (2.18), we get

$$\Delta(n^2H^2) = \Delta\|h\|^2 + n(n-1)\Delta R = \Delta(\mathrm{tr}H_{n+1}^2) + \Delta S_2 + n(n-1)\Delta R. \tag{2.20}$$

Hence, from (2.8), (2.9) and (2.16), by a direct and simple calculation we conclude

$$\begin{aligned}\frac{1}{2}\Delta(\mathrm{tr}H_{n+1}^2) = & \sum_{i,j,k=1}^{n} (h_{ijk}^{n+1})^2 + \sum_{i,j=1}^{n} h_{ij}^{n+1}\Delta h_{ij}^{n+1} \qquad (2.21)\\
= & \sum_{i,j,k=1}^{n} (h_{ijk}^{n+1})^2 + \sum_{i,j=1}^{n} h_{ij}^{n+1}(nH)_{ij} - n\sum_{i,j=1}^{n}(h_{ij}^{n+1})^2 - (\sum_{i,j=1}^{n}(h_{ij}^{n+1})^2)^2\\
& + nH\sum_{i,j,k=1}^{n} h_{ij}^{n+1}h_{jk}^{n+1}h_{ki}^{n+1} + n^2H^2 - \sum_{\beta=n+2}^{n+p}\{\sum_{i,j=1}^{n}(h_{ij}^{n+1} - H\delta_{ij})h_{ij}^{\beta}\}^2\\
& + \sum_{\beta=n+2}^{n+p}\{\sum_{i,j,k=1}^{n}[h_{ij}^{n+1}h_{kj}^{n+1} - (h_{ij}^{n+1})^2](h_{ik}^{\beta})^2\},\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\Delta S_2 = & \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 + \sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \qquad (2.22)\\
= & \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 - n\sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2 + nH\sum_{\alpha=n+2}^{n+p}\mathrm{tr}(H_{n+1}H_{\alpha}^2)\\
& - \sum_{\alpha=n+2}^{n+p}[\mathrm{tr}(H_{n+1}H_{\alpha})]^2 - \sum_{\alpha,\beta=n+2}^{n+p} N(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha})\\
& - \sum_{\alpha,\beta=n+2}^{n+p}[\mathrm{tr}(H_{\alpha}H_{\beta})]^2 + \sum_{\alpha=n+2}^{n+p}\mathrm{tr}(H_{n+1}H_{\alpha})^2 - \sum_{\alpha=n+2}^{n+p}\mathrm{tr}(H_{n+1}^2H_{\alpha}^2).\end{aligned}$$

We need the following lemmas:

Lemma 2.1 ([12], [1]). *Let μ_i, $i = 1, \cdots, n$ be real numbers, with $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2 \ge 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \le \sum_i \mu_i^3 \le \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{2.23}$$

and equality holds if and only if either $(n-1)$ of the numbers μ_i are equal to $\beta/\sqrt{n(n-1)}$ or $(n-1)$ of the numbers μ_i are equal to $-\beta/\sqrt{n(n-1)}$.

Lemma 2.2 ([14]). *Let A, B be symmetric $n\times n$ matrices satisfying $AB = BA$, and $\mathrm{tr}A = \mathrm{tr}B = 0$. Then*

$$|\mathrm{tr}A^2B| \le \frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}A^2)(\mathrm{tr}B^2)^{\frac{1}{2}}. \tag{2.24}$$

Lemma 2.3 ([4]). *Let $a_1, \cdots, a_n, b_{ij} (i, j = 1, 2, \cdots, n)$ be real numbers satisfying $\sum_{i=1}^n a_i = 0$, $\sum_{i=1}^n b_{ii} = 0$, $\sum_{i,j=1}^n b_{ij}^2 = b$ and $b_{ij} = b_{ji} (i, j = 1, 2, \cdots, n)$. Then*

$$-(\sum_{i=1}^n b_{ii}a_i)^2 + \sum_{i,j=1}^n b_{ij}^2 a_i a_j - \sum_{i,j=1}^n b_{ij}^2 a_i^2 \ge -\sum_{i=1}^n a_i^2 b. \tag{2.25}$$

Lemma 2.4 ([9]). *Let $A_1, A_2, \cdots, A_p$ be $(n\times n)$ symmetric matrices $(p\ge 2)$. Denote $S_{\alpha\beta} = trA_\alpha A'_\beta, S_\alpha = S_{\alpha\alpha} = N(A_\alpha), S = S_1 + \cdots + S_p$. Then*

$$\sum_{\alpha,\beta=1}^n N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha,\beta=1}^p S_{\alpha\beta}^2 \le \frac{3}{2}S^2, \tag{2.26}$$

and the equality holds if and only if one of the following conditions hold: (1) $A_1 = A_2 = \cdots = A_p = 0$; (2) Only two of $A_1, \cdots, A_p$ are different from zero. Assuming $A_1 \ne 0, A_2 \ne 0, A_3 = \cdots = A_p = 0$. Then $S_{11} = S_{22}$, and there exists $(n\times n)$ orthogonal matrix T such that

$$TA_1T' = \sqrt{\frac{S_{11}}{2}}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \end{pmatrix}, \quad TA_2T' = \sqrt{\frac{S_{22}}{2}}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

In order to represent our theorems, we need some notations, for details see Lawson [8] and Ryan [13]. First we give a description of the real hyperbolic space $H^{n+1}(c)$ of constant curvature $c(<0)$.

For any two vectors x and y in R^{n+2}, we set

$$g(x, y) = x_1y_1 + \cdots + x_{n+1}y_{n+1} - x_{n+2}y_{n+2},$$

(R^{n+2}, g) is the so-called Minkowski space-time. Denote $\rho = \sqrt{-1/c}$. We define

$$H^{n+1}(c) = \{x \in R^{n+2} \mid g(x, x) = -\rho^2, x_{n+2} > 0\}.$$

Then $H^{n+1}(c)$ is a simply-connected hypersurface of R^{n+2}. Hence, we obtain a model of a real hyperbolic space.

We define

$$\begin{aligned} M_1 =& \{x \in H^{n+1}(c) \mid x_1 = 0\}, \\ M_2 =& \{x \in H^{n+1}(c) \mid x_1 = r > 0\}, \\ M_3 =& \{x \in H^{n+1}(c) \mid x_{n+2} = x_{n+1} + \rho\}, \\ M_4 =& \{x \in H^{n+1}(c) \mid x_1^2 + \cdots + x_{n+1}^2 = r^2 > 0\}, \\ M_5 =& \{x \in H^{n+1}(c) \mid x_1^2 + \cdots + x_{k+1}^2 = r^2 > 0, \\ & x_{k+2}^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 = -\rho^2 - r^2\}. \end{aligned}$$

$M_1, \cdots, M_5$ are often called the standard examples of complete hypersurfaces in $H^{n+1}(c)$ with at most two distinct constant principal curvatures. It is obvious that $M_1, \cdots, M_4$ are totally umbilical. In the sense of Chen [2], they are called the hyperspheres of $H^{n+1}(c)$. M_3 is called the horosphere and M_4 the geodesic distance sphere of $H^{n+1}(c)$. Ryan [13] obtained the following:

Lemma 2.5 ([13]). *Let M^n be a complete hypersurface in $H^{n+1}(c)$. Suppose that, under a suitable choice of a local orthonormal tangent frame field of TM^n, the shape operator over TM^n is expressed as a matrix A. If M^n has at most two distinct constant principal curvatures, then it is congruent to one of the following:*

(1) M_1. *In this case, $A = 0$, and M_1 is totally geodesic. Hence M_1 is isometric to $H^n(c)$;*

(2) M_2. *In this case, $A = \frac{1/\rho^2}{\sqrt{1/\rho^2+1/r^2}} I_n$, where I_n denotes the identity matrix of degree n, and M_2 is isometric to $H^n(-1/(r^2 + \rho^2))$;*

(3) M_3. *In this case, $A = \frac{1}{\rho} I_n$, and M_3 is isometric to a Euclidean space R^n;*

(4) M_4. *In this case, $A = \sqrt{1/r^2 + 1/\rho^2} I_n, M_4$ is isometric to a round sphere $S^n(r)$ of radius r;*

(5) M_5. *In this case,* $A = \lambda I_k \oplus \mu I_{n-k}$, *where* $\lambda = \sqrt{1/\rho^2 + 1/r^2}$, *and* $\mu = \frac{1/\rho^2}{\sqrt{1/r^2+1/\rho^2}}$, M_5 *is isometric to* $S^k(r) \times H^{n-k}(-1/(r^2+\rho^2))$.

3. **Proof of main theorem**

For a C^2-function f defined on M^n, we defined its gradient and Hessian (f_{ij}) by

$$df = \sum_{i=1}^{n} f_i \omega_i, \quad \sum_{j=1}^{n} f_{ij}\omega_j = df_i + \sum_{j=1}^{n} f_j \omega_{ji}. \tag{3.1}$$

Let $T = \sum T_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor on M^n defined by

$$T_{ij} = nH\delta_{ij} - h_{ij}^{n+1}. \tag{3.2}$$

Follow Cheng-Yau [6], we introduce operator $\square$ associated to T acting on f by

$$\square f = \sum_{i,j=1}^{n} T_{ij} f_{ij} = \sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij}^{n+1}) f_{ij}. \tag{3.3}$$

By a simple calculation and from (2.20), we obtained

$$\begin{aligned}\square(nH) = &\sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij}^{n+1})(nH)_{ij} \\ =&\frac{1}{2}\Delta(n^2H^2) - n^2\|\nabla H\|^2 - \sum_{i,j=1}^{n} h_{ij}^{n+1}(nH)_{ij} \\ =&\frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta(\mathrm{tr}H_{n+1}^2) + \frac{1}{2}\Delta S_2 - n^2\|\nabla H\|^2 - \sum_{i,j=1}^{n} h_{ij}^{n+1}(nH)_{ij}.\end{aligned} \tag{3.4}$$

By making use of the similar method in [5], we prove the following:

Proposition 3.1. *Let* M^n *be an* n*-dimensional submanifold in a hyperbolic space* $H^{n+p}(-1)$ *with* $n(n-1)R = k'H(k' = const. > 0)$. *If the mean curvature* $H > 0$, *then the operator*

$$L = \square - (k'/2n)\Delta$$

is elliptic.

Proof. For a fixed α, we choose a orthonormal frame field $\{e_j\}$ at each point in M^n so that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$. From (2.19), we have, for any i,

$$\begin{aligned}
(nH-\lambda_i^{n+1} - k'/2n) = &\sum_j \lambda_j^{n+1} - \lambda_i^{n+1} \\
&- (1/2)[-\sum_{j,\alpha}(\lambda_j^\alpha)^2 + n^2H^2 - n(n-1)]/(nH) \\
\geq &\sum_j \lambda_j^{n+1} - \lambda_i^{n+1} \\
&- (1/2)[-\sum_j(\lambda_j^{n+1})^2 + (\sum_j \lambda_j^{n+1})^2 - n(n-1)]/(nH) \\
=&[(\sum_j \lambda_j^{n+1})^2 - \lambda_i^{n+1}(\sum_j \lambda_j^{n+1}) \\
&- (1/2)\sum_{l\neq j} \lambda_l^{n+1}\lambda_j^{n+1} + (1/2)n(n-1)](nH)^{-1} \\
=&[\sum_j(\lambda_j^{n+1})^2 + (1/2)\sum_{l\neq j}\lambda_l^{n+1}\lambda_j^{n+1} \\
&- \lambda_i^{n+1}(\sum_j \lambda_j^{n+1}) + (1/2)n(n-1)](nH)^{-1} \\
=&[\sum_{i\neq j}(\lambda_j^{n+1})^2 + (1/2)\sum_{\substack{l\neq j\\ l,j\neq i}}\lambda_l^{n+1}\lambda_j^{n+1} + (1/2)n(n-1)](nH)^{-1} \\
=&(1/2)[\sum_{j\neq i}(\lambda_j^{n+1})^2 + (\sum_{j\neq i}\lambda_j^{n+1})^2 + n(n-1)](nH)^{-1} > 0.
\end{aligned}$$

Thus, L is an elliptic operator. This completes the proof of Proposition 3.1.

Proposition 3.2. *Let M^n be a n-dimensional submanifold in a hyperbolic space $H^{n+p}(-1)$ with $n(n-1)R = k'H, (k' = const. > 0)$. If the mean curvature $H > 0$, then*

$$\|\nabla h\|^2 \geq n^2\|\nabla H\|^2.$$

Proof. Since $H > 0$, we have $\|h\|^2 \neq 0$. In fact, if $\|h\|^2 = \sum_{i,\alpha}(\lambda_i^\alpha)^2 = 0$ at a point of M^n, then $\lambda_i^\alpha = 0$ for all i and α at this point. This implies that $H = 0$ at this point. This is impossible.

From (2.11) and $n(n-1)R = k'H$, we have

$$k'\nabla_i H = 2n^2 H\nabla_i H - 2\sum_{j,k,\alpha} h^\alpha_{kj} h^\alpha_{kji},$$

$$(\frac{1}{2}k' - n^2 H)\nabla_i H = -\sum_{j,k,\alpha} h^\alpha_{kj} h^\alpha_{kji},$$

$$(\frac{1}{2}k' - n^2 H)^2\|\nabla H\|^2 = \sum_i(\sum_{j,k,\alpha} h^\alpha_{kj} h^\alpha_{kji})^2 \leq \sum_{i,j,\alpha}(h^\alpha_{ij})^2 \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 \quad = \|h\|^2\|\nabla h\|^2.$$

Therefore, we have

$$\begin{aligned}\|\nabla h\|^2 - n^2\|\nabla H\|^2 \geq &[(\frac{k'}{2} - n^2 H)^2 - n^2\|h\|^2]\|\nabla H\|^2 \frac{1}{\|h\|^2} \\ = &[\frac{(k')^2}{4} + n^3(n-1)]\|\nabla H\|^2 \frac{1}{\|h\|^2} \geq 0.\end{aligned}$$

This completes the proof of Proposition 3.2.

Proof of Main Theorem. By making use of the similar method in [4], we choose a local orthonornmal frame field $\{e_1, \cdots, e_n\}$ such that $h^{n+1}_{ij} = \lambda_i\delta_{ij}$. Let $\mu_i = \lambda_i - H$. Then $\sum\limits_{n=1}^{n}\mu_i = 0, \sum\limits_{i=1}^{n}\mu_i^2 = \sum\limits_{i=1}^{n}\lambda_i^2 - nH^2 = \mathrm{tr}H^2_{n+1} - nH^2 = S_1$. By Lemma 2.1, we get

$$\begin{aligned}nH\sum_{i,j,k=1}^{n} h^{n+1}_{ii}h^{n+1}_{jk}h^{n+1}_{ki} = &nH\sum_{i=1}^{n}\lambda_i^3 = 3nH^2S_1 + n^2H^4 + nH\sum_{i=1}^{n}\mu_i^3 \\ \geq &3nH^2S_1 + n^2H^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S_1)^{\frac{3}{2}}.\end{aligned} \tag{3.5}$$

From Lemma 2.3, we obtain

$$\begin{aligned}&-\sum_{\beta=n+2}^{n+p}\{\sum_{i,j=1}^{n}(h^{n+1}_{ij} - H\delta_{ij})h^\beta_{ij}\}^2 + \sum_{\beta=n+2}^{n+p}\{\sum_{i,j,k=1}^{n}[h^{n+1}_{ij}h^{n+1}_{kj} - (h^{n+1}_{ij})^2](h^\beta_{ik})^2\} \qquad (3.6)\\ &= -\sum_{\beta=n+2}^{n+p}\{\sum_{i=1}^{n}(\lambda_i - H)h^\beta_{ii}\}^2 + \sum_{\beta=n+2}^{n+p}\{\sum_{i,k=1}^{n}(\lambda_i\lambda_k - \lambda_i^2)(h^\beta_{ik})^2\} \\ &= \sum_{\beta=n+2}^{n+p}\{-(\sum_{i=1}^{n}\mu_i h^\beta_{ii})^2 + \sum_{i,k=1}^{n}(\mu_i\mu_k - \mu_i^2)(h^\beta_{ik})^2\} \\ &\geq \sum_{\beta=n+2}^{n+p}\{-\sum_{i=1}^{n}\mu_i^2\sum_{i,j=1}^{n}(h^\beta_{ij})^2\} = -S_1S_2.\end{aligned}$$

Hence from (2.21), (3.5), (3.6) we have

$$\begin{aligned}\frac{1}{2}\Delta(trH_{n+1}^2) \geq &\sum_{i,j,k=1}^{n}(h_{ijk}^{n+1})^2 + \sum_{i,j=1}^{n} h_{ij}^{n+1}(nH)_{ij} - n\sum_{i=1}^{n}\lambda_i^2 - (\sum_{i=1}^{n}\lambda_i^2)^2 \\ &+ n^2H^2 + 3nH^2S_1 + n^2H^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S_1)^{\frac{3}{2}} - S_1S_2 \\ = &\sum_{i,j,k=1}^{n}(h_{ijk}^{n+1})^2 + \sum_{i,j=1}^{n} h_{ij}^{n+1}(nH)_{ij} \\ &+ S_1\{-n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1} - S_1 - S_2\}.\end{aligned} \tag{3.7}$$

Let M^n be complete connect submanifold in $H^{n+p}(-1)$ with positive mean curvature. Suppose that the normalized mean curvature vector $\frac{\xi}{H}$ is parallel in $T^{\perp}M^n$. If we choose $e_{n+1} = \frac{\xi}{H}$, then $\omega_{\alpha n+1} = 0$, for all α. Consequently $R_{\alpha n+1jk} = 0$. From (2.9) we have

$$\sum_{i=1}^{n} h_{ij}^{\alpha}h_{ik}^{n+1} = \sum_{i=1}^{n} h_{ik}^{\alpha}h_{ij}^{n+1}. \tag{3.8}$$

Hence, we obtain

$$H_\alpha H_{n+1} = H_{n+1}H_\alpha. \tag{3.9}$$

We set $B = H_{n+1} - HI$, (I is the unit matrix) then $\mathrm{tr}B = 0$, since $\mathrm{tr}H_\alpha = 0 (\alpha > n+1)$. By (3.9) we get for $\alpha > n+1, H_\alpha B = BH_\alpha$. By virtue of Lemma 2.2, we see that

$$|\mathrm{tr}(H_\alpha^2 B)| \leq \frac{n-2}{\sqrt{n(n-1)}}\mathrm{tr}H_\alpha^2\sqrt{\mathrm{tr}B^2}, \quad \alpha > n+1. \tag{3.10}$$

Since

$$\mathrm{tr}(H_\alpha^2 B) = \mathrm{tr}(H_\alpha^2 H_{n+1}) - H\,\mathrm{tr}H_\alpha^2, \quad \alpha > n+1, \tag{3.11}$$

$$\mathrm{tr}B^2 = \mathrm{tr}H_{n+1}^2 - nH^2 = S_1. \tag{3.12}$$

By (3.10), (3.11) and (3.12), we have

$$\mathrm{tr}(H_\alpha^2 H_{n+1}) \leq (H + \frac{n-2}{\sqrt{n(n-1)}}\sqrt{S_1})\mathrm{tr}H_\alpha^2, \quad (\alpha > n+1). \tag{3.13}$$

From Lemma 2.4 and definition of S_2

$$-\sum_{\alpha,\beta=n+2}^{n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha,\beta=n+2}^{n+p}[\mathrm{tr}(H_\alpha H_\beta)]^2 \geq -\frac{3}{2}S_2^2. \tag{3.14}$$

When $p=2$, we have

$$-\sum_{\alpha,\beta=n+2}^{n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha,\beta=n+2}^{n+p} [\mathrm{tr}(H_\alpha H_\beta)]^2 = -S_2^2. \tag{3.15}$$

For a fixed $\alpha, n+2 \le \alpha \le n+p$, we choose a local orthonormal frame field $\{e_1, \cdots, e_n\}$ such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$. Thus, we have $\sum_{i=1}^n \lambda_i^\alpha = 0$ and $\mathrm{tr}H_\alpha^2 = \sum_{i=1}^n (\lambda_i^\alpha)^2$. Let $B = H_{n+1} - HI = (b_{ij})$. We have $b_{ij} = b_{ji}(i,j = 1,2,\cdots,n)$, $\sum_{i=1}^n b_{ii} = 0$ and $\sum_{i,j=1}^n b_{ij}^2 = S_1$. Since $\lambda_i^\alpha, b_{ij}(i,j=1,2,\cdots,n)$ satisfy Lemma 2.3, from Lemma 2.3, we get

$$\begin{aligned}
&-\sum_{\alpha=n+2}^{n+p}[\mathrm{tr}(H_{n+1}H_\alpha)]^2 + \sum_{\alpha=n+2}^{n+p}\mathrm{tr}(H_{n+1}H_\alpha)^2 - \sum_{\alpha=n+2}^{n+p}\mathrm{tr}(H_{n+1}^2 H_\alpha^2) \\
&= \sum_{\alpha=n+2}^{n+p}\{-[\mathrm{tr}((H_{n+1}-HI)H_\alpha)]^2 + \mathrm{tr}[(H_{n+1}-HI)H_\alpha]^2 - \mathrm{tr}[(H_{n+1}-HI)^2 H_\alpha^2]\} \\
&= \sum_{\alpha=n+2}^{n+p}\{-[\mathrm{tr}(BH_\alpha)]^2 + \mathrm{tr}(BH_\alpha)^2 - \mathrm{tr}(B^2H_\alpha^2)\} \\
&= \sum_{\alpha=n+2}^{n+p}\{-(\sum_{i=1}^n b_{ii}\lambda_i^\alpha)^2 + \sum_{i=1}^n b_{ij}^2(\lambda_i^\alpha)^2(\lambda_j^\alpha)^2 - \sum_{i=1}^n b_{ij}^2(\lambda_i^\alpha)^2\} \\
&\ge \sum_{\alpha=n+2}^{n+p}[-\sum_{i=1}^n(\lambda_i^\alpha)^2\sum_{i,j=1}^n b_{ij}^2] = -S_1\sum_{\alpha=n+2}^{n+p}\mathrm{tr}H_\alpha^2 = -S_1S_2.
\end{aligned} \tag{3.16}$$

Therefore, by (2.22), (3.13), (3.14) and (3.16), when $p \ge 3$, we get

$$\frac{1}{2}\Delta S_2 \ge \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^n (h_{ijk}^\alpha)^2 + S_2\{-n+nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1} - S_1 - \frac{3}{2}S_2\}. \tag{3.17}$$

When $p=2$, from (2.22), (3.13), (3.15), (3.16), we have

$$\frac{1}{2}\Delta S_2 \ge \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^n (h_{ijk}^\alpha)^2 + S_2\{-n+nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1} - S_1 - S_2\}. \tag{3.18}$$

Case 1. If $p = 1$, we have $S_2 = 0, S_1 = \|h\|^2 - nH^2$. Therefore, by (3.4), (3.7) and Proposition 3.2, we have

$$\begin{aligned}\Box(nH) =&\frac{1}{2}n(n-1)\Delta R + \|\nabla h\|^2 - n^2\|\nabla H\|^2 \\ &+ S_1\{-n + nH - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1} - S_1\} \\ \geq&\frac{1}{2}n(n-1)\Delta R + \|g\|^2\{-n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \|g\|^2\},\end{aligned} \tag{3.19}$$

where $\|g\|^2$ is a non-negative C^2-function on M^n defined by $\|g\|^2 = \|h\|^2 - nH^2$.

Therefore, from (3.19), we have

$$\begin{aligned}nLH =&n[\Box H - (k'/2n)\Delta H] \\ =&\Box(nH) - (1/2)n(n-1)\Delta R \\ \geq&\|g\|^2\{-n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \|g\|^2\} \\ =&\|g\|^2 P_H(\|g\|),\end{aligned} \tag{3.20}$$

where

$$P_H(\|g\|) = -n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \|g\|^2. \tag{3.21}$$

Since $H^2 \geq 1$, we know that $P_H(\|g\|)$ has two real roots B_H^+ and B_H^- given by

$$B_H^{\pm} = -\frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}H \pm \sqrt{\frac{n^3H^2}{4(n-1)} - n}. \tag{3.22}$$

Therefore, we know that

$$P_H(\|g\|) = (\|g\| - B_H^-)(-\|g\| + B_H^+).$$

Clearly, we know that $\|g\| - B_H^- > 0$. From the assumption of Main Theorem, we infer that $P_H(\|g\|) \geq 0$ on M^n. This implies that the right-hand side of (3.20) is non-negative. From Proposition 3.1, we know that L is elliptic. Since H obtains its maximum on M^n, from (3.20), we have $H = const.$ on M^n. From (3.20) again, we get $\|g\|^2 P_H(\|g\|) = 0$. Therefore, we have $\|g\|^2 = 0$ and M^n is totally umbilical, or $P_H(\|g\|) = 0$. In the latter case, we infer that the equalities hold in (3.20), (3.19) and (2.23) of Lemma 2.1. Therefore, we know that $(n-1)$ of the numbers $\lambda_i - H$ are equal to $\|g\|/\sqrt{n(n-1)}$. This implies that M^n has $(n-1)$ principal curvatures equal and constant. As H is constant, the other principal curvature is constant as well. Therefore we know that M^n is isoparametric. From the result of Lemma 2.5, M^n is isometric to $S^{n-1}(r) \times H^1(-1/(r^2+1))$ for some $r > 0$.

Case 2. If $p = 2$, from (2.18), we have

$$S_1 \leq \|h\|^2 - nH^2. \tag{3.23}$$

From (3.4), (3.7), (3.18), (3.23), Proposition 3.2 and (2.18) we have

$$\begin{aligned}\Box(nH) \geq &\frac{1}{2}n(n-1)\Delta R + (S_1+S_2)\{-n+nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1} - (S_1+S_2)\} \\ &\hspace{30em}(3.24) \\ \geq &\frac{1}{2}n(n-1)\Delta R + \|g\|^2\{-n+nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \|g\|^2\},\end{aligned}$$

where $\|g\|^2 = \|h\|^2 - nH^2$.

Therefore, from (3.22), we have

$$\begin{aligned} nLH = &\Box(nH) - (1/2)n(n-1)\Delta R \qquad (3.25)\\ \geq &\|g\|^2\{-n+nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \|g\|^2\} \\ = &\|g\|^2 P_H(\|g\|),\end{aligned}$$

where $P_H(\|g\|)$ is denoted by (3.21). $P_H(\|g\|)$ has two real roots B_H^+ and B_H^- denoted by (3.22). Therefore, we know that

$$P_H(\|g\|) = (\|g\| - B_H^-)(-\|g\| + B_H^+).$$

Since $\|g\| - B_H^- > 0$, from the assumption of Main Theorem, we infer that $P_H(\|g\|) \geq 0$ on M^n. This implies that the right-hand side of (3.25) is non-negative. By making use of the same method in Case 1, we can obtain $\|g\|^2 P_H(\|g\|) = 0$. Therefore, we have $\|g\|^2 = 0$ and M^n is totally umbilical, or $P_H(\|g\|) = 0$. If $P_H(\|g\|) = 0$, we infer that the equalities hold in (3.25), (3.24), (3.23) and (2.23) of Lemma 2.1. If the equality holds in (3.23), we have $S_1 = \|h\|^2 - nH^2$. This implies $S_2 = 0$. Since e_{n+1} is parallel on the normal bundle $T^{\perp}(M^n)$ of M^n, using the method of Yau [17], we know that M^n lies in a totally geodesic submanifold $H^{n+1}(-1)$ of $H^{n+p}(-1)$. If the equality holds in Lemma 2.1, by making use of the same assertion as in the proof of Case 1, we infer that M^n has two distinct principal curvatures and is isoparametric. Therefore, from Lemma 2.5, we know that M^n is isometric to $S^{n-1}(r) \times H^1(-1/(r^2+1))$ for some $r > 0$.

Case 3. If $p \geq 3$, from (3.4),(3.7),(3.17),(3.23) and Proposition 3.2, we have

$$\begin{aligned}\Box(nH) \geq& \frac{1}{2}n(n-1)\Delta R + (S_1+S_2)\{-n+nH^2 \\ &- \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1} - (S_1+S_2)\} - \frac{1}{2}S_2^2 \\ \geq& \frac{1}{2}n(n-1)\Delta R + (S_1+S_2)\{-n+nH^2 \\ &- \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1} - (S_1+S_2)\} - \frac{1}{2}(S_1+S_2)^2 \\ \geq& \frac{1}{2}n(n-1)\Delta R + (S_1+S_2)\{-n+nH^2 \\ &- \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{\|h\|^2-nH^2} - \frac{3}{2}(S_1+S_2)\} \\ =& \frac{1}{2}n(n-1)\Delta R + \|g\|^2\{-n+nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \frac{3}{2}\|g\|^2\},\end{aligned} \tag{3.26}$$

where $\|g\|^2 = \|h\|^2 - nH^2$.

Therefore, we have

$$\begin{aligned}nLH =& \Box(nH) - (1/2)n(n-1)\Delta R \\ \geq& \|g\|^2\{-n+nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \frac{3}{2}\|g\|^2\} \\ =& \frac{3}{2}\|g\|^2\{\frac{2}{3}(nH^2-n) - \frac{2}{3}\frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \|g\|^2\} \\ =& \frac{3}{2}\|g\|^2 Q_H(\|g\|),\end{aligned} \tag{3.27}$$

where

$$Q_H(\|g\|) = \frac{2}{3}(nH^2-n) - \frac{2}{3}\frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \|g\|^2.$$

Since $H^2 \geq 1$, we know that $Q_H(\|g\|)$ has two real roots $\widetilde{B}_H^+$ and $\widetilde{B}_H^-$ given by

$$\widetilde{B}_H^{\pm} = -\frac{1}{3}(n-2)\sqrt{\frac{n}{n-1}}H \pm \frac{1}{3}\sqrt{\frac{n}{n-1}(n^2+2n-2)H^2 - 6n},$$

Therefore, we know that

$$Q_H(\|g\|) = (\|g\| - \widetilde{B}_H^-)(-\|g\| + \widetilde{B}_H^+).$$

Clearly, we know that $\|g\| - \widetilde{B}_H^- > 0$. From the assumption of Main Theorem, we infer that $Q_H(\|g\|) \geq 0$ on M^n. This implies that the right-hand side of (3.27) is non-negative. From Proposition 3.1, we know that L is elliptic. Since H obtains its maximum on M^n, from (3.27), we have $H = const.$ on M^n. From (3.27) again, we get $\|g\|^2 Q_H(\|g\|) = 0$. Therefore, we have $\|g\|^2 = 0$ and M^n is totally umbilical, or $Q_H(\|g\|) = 0$. If $Q_H(\|g\|) = 0$, we infer that the equalities hold in (3.27), (3.26) and (3.23). Therefore, we know that

$$S_1 = \|h\|^2 - nH^2, \quad S_2 = S_1 + S_2.$$

From (2.18), this implies that $S_2 = 0$ and $S_1 = 0$. Therefore, we have $\|g\|^2 = \|h\|^2 - nH^2 = 0$ on M^n and M^n is totally umbilical. This completes the proof of Main Theorem.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

INVERSE THEOREM FOR AN ITERATIVE COMBINATION OF BERNSTEIN-DURRMEYER POLYNOMIALS

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Abstract. The Bernstein-Durrmeyer polynomial

$$[M_n(f;t) = (n+1)\sum_{k=0}^{n} p_{n,k}(t) \int_0^1 p_{n,k}(u)f(u)\,du,$$

where $p_{n,k}(t) = \binom{n}{k} t^k(1-t)^{n-k}$, $t \in [0,1]$ defined on $L_B[0,1]$, the space of bounded and integrable functions on $[0,1]$ were introduced by Durrmeyer [5] and extensively studied by Derriennic [3] and other researchers (see [1]-[3], [5], [6], [8]). It turns out that the order of approximation by these operators is, at best, $O(n^{-1})$ however smooth the function may be. In order to improve the rate of approximation we consider an iterative combination $T_{n,k}(f;t)$ of the operators $M_n(f;t)$. This technique was given by Micchelli [9] who first used it to improve the order of approximation by Bernstein polynomials $B_n(f;t)$. In the paper [1] some direct theorems in ordinary and simultaneous approximation for the operators $T_{n,k}(f;t)$ in the uniform norm, have been established. The paper [10] is a study of some direct results in the L_p- approximation by the operators $T_{n,k}(f;t)$. The object of the present paper is to study the corresponding inverse theorem in L_p- approximation by the operators $T_{n,k}(f;t)$.

1. Introduction

For $f \in L_p[0,1], 1 \leqslant p < \infty$ the operators M_n can be expressed as

$$M_n(f;t) = \int_0^1 W_n(u,t)f(u)\,du,$$

Received by the editors: 22.05.2008.

2000 *Mathematics Subject Classification.* 41A27, 41A36.

Key words and phrases. Inverse theorem, L_p- approximation, Steklov means.

where $W_n(u,t) = (n+1)\sum_{k=0}^{n} p_{n,k}(t)p_{n,k}(u)$ is the kernel of the operators.

For $m \in N^0$ (the set of non-negative integers), the mth order moment for the operators M_n is defined as

$$\mu_{n,m}(t) = M_n\left((u-t)^m; t\right).$$

The iterative combination $T_{n,k} : L_p[0,1] \to C^\infty[0,1]$ of the operators is defined as

$$T_{n,k}(f;t) = \left(I - (I - M_n)^k\right)(f;t) = \sum_{r=1}^{k} (-1)^{r+1}\binom{k}{r} M_n^r(f;t),\ k \in N,$$

where $M_n^0 \equiv I$ and $M_n^r \equiv M_n(M_n^{r-1})$ for $r \in N$.

Throughout the present paper we assume that $I = [0,1]$ and $I_j = [a_j, b_j]$, $j = 1,2,3,\ 0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < 1$ and by C we mean the positive constant not necessarily the same at each occurrence.

In [10], we obtained following direct theorem:

Theorem 1. *If $p \geqslant 1, f \in L_p[0,1]$. Then for all n sufficiently large there holds*

$$\|T_{n,k}(f;.) - f\|_{L_p(I_2)} \leqslant C_k\left(\omega_{2k}\Big(f, \frac{1}{\sqrt{n}}, p, I_1.\Big) + n^{-k}\|f\|_{L_p[0,1]}\right), \tag{1.1}$$

where C_k is a constant independent of f and n.

Remark 1. *From above theorem it follows that if $\omega_{2k}(f,\tau,p,I_2) = O(\tau^\alpha)$ as $\tau \to 0$ then $\|T_{n,k}(f,.) - f\|_{L_p(I_2)} = O(n^{-\alpha/2})$ as $n \to \infty$, where $0 < \alpha < 2k$.*

The aim of this paper is to establish a corresponding local inverse theorem for the operators $T_{n,k}(f,t)$ in the L_p-norm i.e. the characterization of the class of functions for which $\|T_{n,k}(f,.) - f\|_{L_p(I_2)} = O(n^{-\alpha/2})$ as $n \to \infty$, where $0 < \alpha < 2k$.

Thus we prove the following theorem (*inverse theorem*):

Theorem 2. *Let $f \in L_p[0,1], 1 \leqslant p < \infty,\ 0 < \alpha < 2k$ and $\|T_{n,k}(f,.) - f\|_{L_p(I_1)} = O(n^{-\alpha/2})$ as $n \to \infty$. Then, $\omega_{2k}(f,\tau,p,I_2) = O(\tau^\alpha)$ as $\tau \to 0$.*

2. **Preliminaries**

In this section we give some results which are useful in establishing our main theorem.

Lemma 1. [1] *For the function $\mu_{n,m}(t)$, we have*

$$\mu_{n,0}(t) = 1, \mu_{n,1}(t) = \frac{(1-2t)}{(n+2)}$$

and for $m \geqslant 1$ there holds the recurrence relation

$$(n+m+2)\mu_{n,m+1}(t) = t(1-t)\left\{\mu'_{n,m}(t) + 2m\mu_{n,m-1}(t)\right\} + (m+1)(1-2t)\mu_{n,m}(t).$$

Consequently,

(i) *$\mu_{n,m}(t)$ is a polynomial in t of degree m;*

(ii) *for every $t \in [0,1], \mu_{n,m}(t) = O\left(n^{[(m+1)/2]}\right)$, where $[\beta]$ is the integer part of β.*

Lemma 2. [8]*For the function $p_{n,k}(t)$, there holds the result*

$$t^r(1-t)^r D^r\left(p_{n,k}(t)\right) = \sum_{\substack{2i+j\leq m\\ i,j\geq 0}} n^i(k-nt)^j q_{i,j,r}(t) p_{n,k}(t),$$

where $D \equiv \frac{d}{dt}$ and $q_{i,j,r}(t)$ are certain polynomials in t independent of n and k.

Lemma 3. [1] *For $k, l \in N$, there holds $T_{n,k}\left((u-t)^l; t\right) = O(n^{-k})$.*

Lemma 4. *If $f \in L_p[0,1]$ then there holds the estimate*

$$\left\| \frac{d^m}{dt^m}\left(T_{n,k}(f;\bullet)\right)\right\|_{L_p[c,d]} \leqslant C\, n^{m/2}\|f\|_{L_p[0,1]},$$

where $[c,d]$ is any closed interval contained in $(0,1)$.

Proof. We have

$$\frac{d^m}{dt^m}\left(M_n^k(f;t)\right) = \frac{d^m}{dt^m}\int_0^1 W_n(u,t) M_n^{k-1}(f;u)\,du$$

$$= (n+1)\sum_{\nu=0}^{n} p_{n,\nu}(t) \sum_{\substack{2i+j\leq m\\ i,j\geqslant 0}} n^i \frac{(\nu-nt)^j q_{i,j,m}(t)}{(t(1-t))^m} \times \int_0^1 p_{n,\nu}(u) M_n^{k-1}(f;u)\,du, \qquad (2.1)$$

Using Holder's inequality for summation, we obtain

$$\frac{d^m}{dt^m}\left(M_n^k(f;t)\right)\Big| \leqslant C(n+1)\sum_{\nu=0}^{n}\sum_{\substack{2i+j\leqslant m\\ i,j\geqslant 0}} p_{n,\nu}(t) n^i |\nu-nt|^j \Big(\int_0^1 p_{n,\nu}(u)\,du\Big)^{1/q}$$

$$\times\left(\int_0^1 p_{n,\nu}(u)\left|M_n^{k-1}(f;u)\right|^p du\right)^{1/p}$$

$$\leqslant C(n+1)^{1-1/q}\sum_{\substack{2i+j\leqslant m\\ i,j\geqslant 0}} n^i\Big(\sum_{\nu=0}^{n} p_{n,\nu}(t)|\nu-nt|^{qj}\Big)^{1/q}$$

$$\times\left(\sum_{\nu=0}^{n} p_{n,\nu}(t)\int_0^1 p_{n,\nu}(u)\left|M_n^{k-1}(f;u)\right|^p du\right)^{1/p}$$

$$\leqslant C(n+1)^{1/p}\sum_{\substack{2i+j\leqslant m\\ i,j\geqslant 0}} n^i.n^{j/2}$$

$$\times\left(\sum_{\nu=0}^{n} p_{n,\nu}(t)\int_0^1 p_{n,\nu}(u)\left|M_n^{k-1}(f;u)\right|^p du\right)^{1/p} \tag{2.2}$$

Therefore, applying Fubini's theorem, we get

$$\left\|\frac{d^m}{dt^m}\big(M_n^k(f;t)\big)\right\|_{L_p[c,d]} \leqslant C(n+1)^{1/p}n^{m/2}\times$$

$$\left(\int_c^d \sum_{\nu=0}^{n} p_{n,\nu}(t)\int_0^1 \left|M_n^{k-1}(f;u)\right|^p p_{n,\nu}(u)\,du\,dt\right)^{1/p}$$

$$\leqslant C(n+1)^{1/p}n^{m/2}\left\{\sum_{\nu=0}^{n}\left(\int_c^d p_{n,\nu}(t)\,dt\right)\times\right.$$

$$\left.\left(\int_0^1 p_{n,\nu}(u)\left|M_n^{k-1}(f;u)\right|^p du\right)\right\}^{1/p}$$

$$\leqslant Cn^{m/2}\left\{\int_0^1 \sum_{\nu=0}^{n} p_{n,\nu}(u)\left|M_n^{k-1}(f;u)\right|^p du\right\}^{1/p}$$

$$\leqslant Cn^{m/2}\left\|M_n^{k-1}(f;u)\right\|_{L_p[0,1]} \leqslant Cn^{m/2}\|f\|_{L_p[0,1]}. \tag{2.3}$$

Since $T_{n,k}$ are linear combinations of the iterates M_n, and the r.h.s. in (2.3) is independent of k, the lemma follows from (2.3). □

Lemma 5. *If $f \in L_p[0,1]$ is such that $f^{(m-1)} \in AC(I)$ and $f^{(m)} \in L_p(I)$, then*

$$\left\|\frac{d^m}{dt^m}\big(T_{n,k}(f;\bullet)\big)\right\|_{L_p[c,d]} \leqslant M\|f^{(m)}\|_{L_p[0,1]},$$

where $[c,d] \subset (0,1)$.

Proof. It is sufficient to find the estimate for $\frac{d^m}{dt^m}\big(M_n^k(f;\bullet)\big)$. Thus, we have

$$\begin{aligned}
\frac{d^m}{dt^m}\big(M_n^r(f;\bullet)\big) &= \frac{d^m}{dt^m}\big[M_n\big((M_n^{k-1}(f;u_k);u);t\big)\big] \\
&= \sum_{i=0}^{m-1}\frac{f^{(i)}(t)}{i!}\frac{d^m}{dt^m}\big[M_n\big((M_n^{k-1}(u_k-t)^i;u);t\big)\big)\big] \\
&+ \frac{1}{(m-1)!}\frac{d^m}{dt^m}\left[M_n\Big(M_n^{k-1}\Big(\int_t^{u_k}(u_k-w)^{m-1}f^{(m)}(w)\,dw;u\Big);t\Big)\right]
\end{aligned}$$

The term inside the summation is polynomial of degree $(m-1)$ and hence vanish. In order to estimate the second term we break the integral as follows. There exists a non-negative integer $r=r(n)$ such that $r/\sqrt{n}\leqslant \max|u_k-t|\leqslant (r+1)/\sqrt{n}$. Hence, we get

$$\begin{aligned}
I &= \int_0^1 W_n(u_k,u_{k-1})|u_k-t|^{m-1}\left|\int_t^{u_k}\big|f^{(m)}(w)\big|\,dw\right|du_k \\
&\leqslant \sum_{l=0}^{r}\Bigg\{\int_{t+\frac{l}{\sqrt{n}}}^{t+\frac{l+1}{\sqrt{n}}} W_n(u_k,u_{k-1})|u_k-t|^{m-1}\int_t^{t+\frac{l+1}{\sqrt{n}}}\big|f^{(m)}(w)\big|\,dw\,du_k \\
&+ \int_{t-\frac{l+1}{\sqrt{n}}}^{t-\frac{l}{\sqrt{n}}} W_n(u_k,u_{k-1})|u_k-t|^{m-1}\int_{t-\frac{l+1}{\sqrt{n}}}^{t}\big|f^{(m)}(w)\big|\,dw\,du_k\Bigg\} \qquad (2.4)
\end{aligned}$$

Now, $|u_k-t|>l/\sqrt{n}$ and

$$|u_k-t|^{m+3}\leqslant\sum_{s=0}^{m+3}\binom{m+3}{s}|u_k-u_{k-1}|^{m+3-s}|u_{k-1}-t|^s$$

Hence a typical term of (2.4) is estimated as

$$\begin{aligned}
&\leqslant \sum_{r=0}^{m+3}\int_{t+\frac{l}{\sqrt{n}}}^{t+\frac{l+1}{\sqrt{n}}} W_n(u_k,u_{k-1})|u_k-u_{k-1}|^{m+3-r}|u_{k-1}-t|^r \\
&\qquad\times\binom{m+3}{r}\frac{n^2}{l^4}\int_t^{t+\frac{l+1}{\sqrt{n}}}\big|f^{(m)}(w)\big|\,dw\,du_k
\end{aligned}$$

$$\leqslant \sum_{r=0}^{m+3} C\frac{n^2}{l^4}\frac{1}{n^{(m+3-r)/2}}\int\limits_{t}^{t+\frac{l+1}{\sqrt{n}}}\left|f^{(m)}(w)\right|dw$$

Proceeding recursively we reach

$$\frac{d^m}{dt^m}\left[\int\limits_0^1 W_n(u_1,t)|u_1-t|^s\frac{n^2}{l^4}\frac{1}{n^{(m+3-r)/2}}\left(\int\limits_t^{t+\frac{l+1}{\sqrt{n}}} f^{(m)}(w)\,dw\right)du_1\right]$$

$$= (n+1)\sum_{\substack{2i+j\leq m\\ i,j\geqslant 0}}\sum_{\nu=0}^{n} n^i q_{i,j,m}(t)(\nu-nt)^j p_{n,\nu}(t)\left(\int\limits_0^1 p_{n,\nu}(u_1)|u_1-t|^s\,du_1\right)$$

$$\times\frac{n^2}{l^4}\frac{1}{n^{(m+3-r)/2}}\left(\int\limits_t^{t+\frac{l+1}{\sqrt{n}}}\left|f^{(m)}(w)\right|dw\right)$$

Using Holder's inequality and moment estimates for M_n, we obtain

$$\left|\frac{d^m}{dt^m}\left[\int\limits_0^1 W_n(u_1,t)|u_1-t|^s\frac{n^2}{l^4}\frac{1}{n^{(m+3-r)/2}}\left(\int\limits_t^{t+\frac{l+1}{\sqrt{n}}} f^{(m)}(w)\,dw\right)du_1\right]\right|$$

$$\leqslant C\sum_{l=0}^{r}\frac{n^2}{l^4}\frac{n^{m/2-s/2}}{n^{\frac{m+3-s}{2}}}\left(\int\limits_t^{t+\frac{l+1}{\sqrt{n}}}\left|f^{(m)}(w)\right|dw\right)$$

This implies

$$\left\|\frac{d^m}{dt^m}M_n^k(f;t)\right\|_{L_p[x_2',y_2']}\leqslant C\sum_{l=0}^{r}\frac{1}{l^4}(l+1)\left\|f^{(m)}\right\|_{L_p[0,1]}\leqslant C\left\|f^{(m)}\right\|_{L_p[0,1]}$$

This completes the proof of the lemma. □

3. **Proof of the main theorem**

Proof. We prove the theorem by induction on k.

When $k=1$ the operator $T_{n,k}$ becomes the well known Bernstein Durrmeyer operator M_n for which we prove the inverse result. Thus, we prove that

$$\|M_n(f;t)-f(t)\|_{L_p(I_1)} = O\left(n^{-\alpha/2}\right)\Rightarrow \omega_2\left(f,\tau,I_2\right)=O\left(\tau^{\alpha}\right);0<\alpha<2.$$

Let $g \in C_0^\infty$ be such that $supp\, g \in (a_2, b_2)$ with $g(t) = 1$ on I_3. Further, let $\bar{f} = fg$. Now,

$$\left\|\Delta_\tau^2 \bar{f}(t)\right\|_{L_p(I_3)} \leqslant \left\|\Delta_\tau^2(\bar{f}(t) - M_n(\bar{f};t))\right\|_{L_p(I_3)} + \left\|\Delta_\tau^2 M_n(\bar{f};t)\right\|_{L_p(I_3)} = I_1 + I_2. \quad (3.1)$$

In I_1

$$(fg)(t) - M_n\left(f(u)\big(g(t) + (u-t)g'(t) + ...\big);t\right)$$
$$= g(t)\left(f(t) - M_n(f;t)\right) - g'(t)M_n\left(f(u)(u-t);t\right) + ... \quad (3.2)$$

By hypothesis,

$$\|M_n(f;t) - f(t)\|_{L_p(I_1)} = O\left(n^{-\alpha/2}\right). \quad (3.3)$$

and by dual moment estimate,

$$\|M_n\left(f(u)(u-t);t)\right)\|_{L_p(I_1)} = \|f\|/n^{1/2}. \quad (3.4)$$

Now

$$I_2 = \left\|\Delta_\tau^2 M_n(\bar{f};t)\right\|_{L_p(I_1)} \le \tau^2 \left\|\frac{d^2}{dt^2}\left(M_n(\bar{f};t)\right)\right\|_{L_p(I_1)}$$
$$\le \tau^2 \left\|\frac{d^2}{dt^2}\left(M_n(\bar{f} - \bar{f}_\eta;t)\right)\right\|_{L_p(I_1)} + \tau^2 \left\|\frac{d^2}{dt^2}\left(M_n(\bar{f}_\eta;t)\right)\right\|_{L_p(I_1)}$$
$$\le \tau^2 \left(n\ \omega_2(\eta, \bar{f}) + \frac{1}{\eta^2}\ \omega_2(\eta, \bar{f})\right) \quad (3.5)$$
$$\therefore \omega_2(\tau, \bar{f}) \le \frac{M}{n^{1/2}} + \tau^2\left(n + \frac{1}{\eta^2}\right)\omega_2(\eta, \bar{f})$$
$$\Rightarrow \omega_1(\tau, \bar{f}) = O\left(\tau\left|\ln\, \tau\right|\right) \quad (3.6)$$

We use (3.6) in (3.2) and (3.3). Now,

$$\begin{aligned} M_n\left(f(u)(u-t);t)\right) &= M_n\left((f(u) - f(t))(u-t);t)\right) \\ &+ f(t)M_n((u-t);t) \\ &\le M_n\left(|u-t|^2\left|\ln|u-t|\right|;t\right) + O\left(\frac{1}{n}\right) \\ &\le M_n\left(|u-t|^{2-\epsilon};t\right) + O\left(\frac{1}{n}\right) = O\left(\frac{1}{n^{1-\epsilon}}\right). \end{aligned}$$

From,(3.1),(3.2) and (3.3)

$$\omega_2(\tau, \bar{f}) \leq O\left(\frac{M}{n^{1-\epsilon}}\right) + \tau^2\left(n + \frac{1}{\eta^2}\right)\omega_2(\eta, \bar{f})$$

$$\Rightarrow \omega_2(\tau, \bar{f}) = O\left(\tau^{2-\epsilon}\right).$$

Hence theorem is proved for $k = 1$.

Now, suppose it is true for a certain k i.e.

$$\omega_{2k}(f, \tau, p, I_2) = O(\tau^{\alpha}) \tag{3.7}$$

Let

$$\|T_{n,k+1}(f, .) - f\|_{L_p(I_1)} = O(n^{-(\alpha+2)/2}) \tag{3.8}$$

We will show that

$$\omega_{2k+2}(f, \tau, p, I_2) = O(\tau^{\alpha+2})$$

Let $a_1 < x_1 < x_2 < x_3 < a_2 < b_2 < y_3 < y_2 < y_1 < b_1$ and $g \in C_0^{\infty}$ be such that $supp\, g \in (x_2, y_2)$ with $g(t) = 1$ on $[x_3, y_3]$. Further, let $\bar{f} = fg$. Then we have

$$\begin{aligned} \left\|\Delta_\tau^{2k+2} T_{n,k+1}(\bar{f}; t)\right\|_{L_p[x_2,y_2]} &\leqslant \left\|\Delta_\tau^{2k+2}(\bar{f}(t) - T_{n,k+1}(\bar{f}; t))\right\|_{L_p[x_2,y_2]} \\ &+ \tau^{2k+2}\left\|\frac{d^{2k+2}}{dt^{2k+2}}\left(T_{n,k+1}(\bar{f} - \bar{f}_{\eta,2k+2}; t)\right)\right\|_{L_p[x_2',y_2']} \\ &+ \tau^{2k+2}\left\|\frac{d^{2k+2}}{dt^{2k+2}}\left(T_{n,k+1}(\bar{f}_{\eta,2k+2}; t)\right)\right\|_{L_p[x_2',y_2']} \end{aligned} \tag{3.9}$$

where $x_2' = x_2$ and $y_2' = y_2 + (2k+2)\tau$.

For the first term, we have the estimate

$$\begin{aligned} &\left\|\Delta_\tau^{2k+2}(\bar{f}(t) - T_{n,k+1}(\bar{f}; t))\right\|_{L_p[x_2,y_2]} \leqslant C\left\|\bar{f}(t) - T_{n,k+1}(\bar{f}; t))\right\|_{L_p[x_2',y_2']} \\ &\leqslant C\left\|f(t)g(t) - T_{n,k+1}\left(f(u)\Big[\sum_{i=0}^{\infty}\frac{g^{(i)}(t)}{i!}(u-t)^i\Big]; t\right)\right\|_{L_p[x_2',y_2']} \\ &\leqslant C\|g\|_{L_p[x_2,y_2]}\,\|f(t) - T_{n,k+1}(f; t)\|_{L_p[x_2',y_2']} \\ &+\|g'\|_{L_p[x_2',y_2']}\,\|T_{n,k+1}(f(u)(u-t); t)\|_{L_p[x_2',y_2']} + \dots \end{aligned} \tag{3.10}$$

Using smoothness of f in second term of (3.10), we get

$$\big\|T_{n,k+1}(f(u)(u-t);t)\big\|_{L_p[x_2',y_2']}$$

$$\begin{aligned}
&\leqslant \quad \Bigg\| \sum_{i=0}^{2k-1} \frac{f^{(i)}(t)}{i!} T_{n,k+1}\big((u-t)^i;t\big) + \frac{1}{(2k-2)!} \\
&\times \quad T_{n,k+1}\Big((u-t)^{2k-1}\Big|\int_t^u \big(f^{(2k-1)}(w)-f^{(2k-1)}(t)\big)\,dw\Big|;t\Big)\Bigg\|_{L_p[x_2',y_2']} \\
&\leqslant \quad O\Big(\frac{1}{n^{k+1}}\Big) + C\sum_{m=1}^{k}\Bigg\| M_n^m\Big(|u-t|^{2k-1}\times \\
&\qquad\qquad \times\Big|\int_t^u \Big|f^{(2k-1)}(w)-f^{(2k-1)}(t)\Big|\,dw\Big|;t\Big)\Bigg\|_{L_p[x_2',y_2']} \\
&\leqslant \quad O\Big(\frac{1}{n^{k+1}}\Big) + C\Bigg\| M_n\Big(|u-t|^{2k-1}\times \\
&\qquad\qquad \times\Big|\int_t^u \Big|f^{(2k-1)}(w)-f^{(2k-1)}(t)\Big|\,dw\Big|;t\Big)\Bigg\|_{L_p[x_2',y_2']} \qquad (3.11)
\end{aligned}$$

Now, we have

$$I=\Big|M_n\Big(|u-t|^{2k-1}\Big|\int_t^u \Big|f^{(2k-1)}(w)-f^{(2k-1)}(t)\Big|\,dw\Big|;t\Big)\Big|^p$$

$$\leqslant \Big(\int_0^1 W_n(u,t)\,du\Big)^{1/p}\Bigg(\int_0^1 W_n(u,t)\Big|\int_t^u \Big|f^{(2k-1)}(w)-f^{(2k-1)}(t)\Big|\,dw\Big|^p du\Bigg)$$

$$\leqslant \int_0^1 W_n(u,t)\Big|\int_t^u dw\Big|^{p/q}\Big|\int_t^u \Big|f^{(2k-1)}(w)-f^{(2k-1)}(t)\Big|\,dw\Big|^p$$

$$\leqslant \int_0^1 W_n(u,t)|u-t|^{(2k-1)p+p/q}\Big|\int_t^u \Big|f^{(2k-1)}(w)-f^{(2k-1)}(t)\Big|\,dw\Big|\,du \qquad (3.12)$$

Now, in order to estimate the quantity in the right, we divide the integral once again as in Lemma 5 and use the moment estimates given in Lemma 1. Thus, from

(3.12) we get the following

$$\begin{aligned}
I &\leqslant \sum_{l=0}^{r}\left\{\int_{t+\frac{l}{\sqrt{n}}}^{t+\frac{l+1}{\sqrt{n}}} \frac{n^2}{l^4}|u-t|^{4+(2k-1)p+p/q}W_n(u,t)\right.\\
&\times \int_{t}^{t+\frac{l+1}{\sqrt{n}}} \left|f^{(2k-1)}(w)-f^{(2k-1)}(t)\right|^p dw\,du\\
&+ \int_{t-\frac{l+1}{\sqrt{n}}}^{t-\frac{l}{\sqrt{n}}} \frac{n^2}{l^4}|u-t|^{4+(2k-1)p+p/q}W_n(u,t)\\
&\times \left.\int_{t-\frac{l+1}{\sqrt{n}}}^{t} \left|f^{(2k-1)}(w)-f^{(2k-1)}(t)\right|^p dw\,du\right\}\\
&\leqslant \sum_{l=0}^{r} C\frac{n^2}{l^4}\frac{1}{n^{2+(2k-1)p/2+p/2q}}\left(\int_{t}^{t+\frac{l+1}{\sqrt{n}}} \left|f^{(2k-1)}(w)-f^{(2k-1)}(t)\right|^p dw\right.\\
&+ \left.\int_{t-\frac{l+1}{\sqrt{n}}}^{t} \left|f^{(2k-1)}(w)-f^{(2k-1)}(t)\right|^p dw\right) \qquad (3.13)
\end{aligned}$$

Now,

$$\int_{x_2'}^{y_2'}\int_{t}^{t+\frac{l+1}{\sqrt{n}}} \left|f^{(2k-1)}(w)-f^{(2k-1)}(t)\right|^p dw\,dt = \int_{0}^{\frac{l+1}{\sqrt{n}}}\int_{x_2'}^{y_2'} \left|f^{(2k-1)}(x+t)-f^{(2k-1)}(t)\right|^p dx\,dt$$

$$= \int_{x_2'}^{y_2'}\int_{0}^{1} \left|f^{(2k-1)}(x+t)-f^{(2k-1)}(t)\right|^p \chi(x)\,dx\,dt \leqslant \int_0^1 x^{\theta p}\chi(x)\,dx$$

(where χ is the characteristic function of $[0,(l+1)/\sqrt{n}]$)

$$= \int_{0}^{1}\int_{x_2'}^{y_2'} \left|f^{(2k-1)}(x+t)-f^{(2k-1)}(t)\right|^p \chi(x)\,dt\,dx \leqslant C\frac{(l+1)^{p\theta+1}}{n^{\frac{p\theta+1}{2}}},\ (\text{where}\, 0<\theta<1). \qquad (3.14)$$

Combining (3.12),(3.13) and (3.14), we get

$$
\begin{aligned}
&\left\| M_n\Big(|u-t|^{2k-1} \int_t^u \Big| f^{(2k-1)}(w) - f^{(2k-1)}(t) \Big|\, dw \Big|; t\Big) \right\|_{L_p[x_2', y_2']} \\
&\qquad\qquad \leqslant \quad C\left\{ \sum_{l=0}^{r} \frac{n^2}{l^4} \frac{1}{n^{2+(2k-1)p/2+p/2q}} \frac{(l+1)^{p\theta+1}}{n^{(p\theta+1)/2}} \right\}^{1/p} \\
&\qquad\qquad \leqslant \quad C\big(n^{-(k+\theta/2)}\big). \qquad (3.15)
\end{aligned}
$$

Similarly the rest terms in (3.10) give the required order.

By (3.8), (3.11) and (3.15) we obtain the estimate

$$
\begin{aligned}
\left\|\Delta_\tau^{2k+2}(\bar{f}(t) - T_{n,k+1}(\bar{f};t))\right\|_{L_p[x_2,y_2]} \quad &\leqslant \quad C\left\{\frac{1}{n^{k+1}} + \frac{1}{n^{k+\theta/2}}\right\} \\
&\leqslant \quad C\frac{1}{n^{k+\theta/2}}. \qquad (3.16)
\end{aligned}
$$

Combining (3.9), (3.16), Lemma 4 and Lemma 5 and in view of properties of the Steklov means we get

$$
\left\|\Delta_\tau^{2k+2}\bar{f}(t)\right\|_{L_p[x_2,y_2]} \leqslant C\frac{1}{n^{k+\theta/2}} + \tau^{2k+2}\left(n^{k+1} + \frac{1}{\eta^{2k+2}}\right)\omega_{2k+2}(\bar{f},\eta,[x_2,y_2])
$$

Taking $\tau \leqslant r$

$$
\omega_{2k+2}(\bar{f}, r, [x_2, y_2]) = O\big(r^{2k+\theta}\big) \qquad (3.17)
$$

This implies that $\bar{f}^{(2k)}$ exists and belongs to $\operatorname{Lip}\theta$. This is reiterated into second term of (3.10) as

$$
f(u) = \sum_{i=0}^{2k} \frac{f^{(i)}(t)}{i!}(u-t)^i + \frac{1}{(2k-1)!}\int_t^u (u-w)^{2k-1}\left(f^{(2k-1)}(w) - f^{(2k-1)}(t)\right) dw
$$

Thus we get

$$
\left\|T_{n,k+1}(f(u)(u-t);t)\right\|_{L_p[x_2',y_2']} \leqslant \frac{C}{n^{k+1/2+\theta/2}}
$$

This implies $\omega_{2k+2}(\bar{f}, r, p, [x_2, y_2]) = O(r^{2k+1+\theta})$ which further implies

$$
\omega_{2k+2}(f, \tau, p, I_2) = O(\tau^{2k+1+\theta}).
$$

Thus the theorem is completed by induction. $\square$

Acknowledgement. The author (Asha Ram Gairola) is thankful to the "Council of Scientific and Industrial Research", New Delhi, India for financial support to carry out the above work.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

ON THE USE OF ABEL-JENSEN TYPE COMBINATORIAL FORMULAS FOR CONSTRUCTION AND INVESTIGATION OF SOME ALGEBRAIC POLYNOMIAL OPERATORS OF APPROXIMATION

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Abstract. The aim of this paper is to present some Abel-Jensen type combinatorial formulas useful for construction and investigation of some algebraic polynomial linear positive operators of approximation of univariate functions from the space $C[0,1]$.

In the first part of the paper we present the Abel generalization (1.1) of the classical binomial formula. Then we mention the two Abel type formulas (1.2) and (1.3), as well as the Vandermonde-Jensen formula (1.5).

In the second section one extends to factorial powers the preceding formulas. Then we establish the combinatorial identities (2.2)-(2.5), which are used to give the basic polynomials, depending on two non-negative parameters α and β.

In the third section we use these polynomials for construction several linear positive operators, depending on four parameters, associated to function $f \in C[0,1]$. Some particular cases of these operators were investigated by several authors mentioned at the end of the paper. Finally, we want to mention that in the paper [10] of Cheney and Sharma was proved that the operator Q_n reproduces only the constant functions.

In the fourth section are investigated the approximation properties of the operator $Q_m^{\alpha,\beta}$, defined at (3.3). In the last section are given evaluations of the remainder term of the approximation formula (5.1).

Received by the editors: 21.04.2009.

2000 *Mathematics Subject Classification.* 41A20, 41A25, 41A36, 65D32.

Key words and phrases. combinatorial formulas of Abel-Jensen type.

1. **Introduction**

We start with the celebraten generalization of the Newton binomial formula, given in 1826, by the outstanding mathematical genius represented by the Norwegian Niels Henrik Abel [1], namely

$$(u+v)^n = \sum_{k=0}^{n} \binom{n}{k} u(u-k\beta)^{k-1}(v+k\beta)^{n-k}, \tag{1.1}$$

where β is a non-negative parameter.

We mention also the Abel type formulas

$$(u+v+n\beta)^n = \sum_{k=0}^{n} \binom{n}{k} u(u+k\beta)^{k-1}(v+(n-k)\beta)^{n-k}, \tag{1.2}$$

$$(u+v+n\beta)^n = \sum_{k=0}^{n} \binom{n}{k} (u+k\beta)^{k} v(v+(n-k)\beta)^{n-k-1}. \tag{1.3}$$

Jensen [29] has obtained a new symmetrical identity of Abel

$$(u+v(u+v+n\beta))^{n-1} = \sum_{k=0}^{n} \binom{n}{k} u(u+k\beta)^{k-1} v(v+(n-k)\beta)^{n-k-1}. \tag{1.4}$$

In the paper [18] the American mathematician H.W. Gould gave the following generalization of the Vandermonde formula

$$\binom{u+v+n\beta}{n} = \sum_{k=0}^{n} \binom{u+k\beta}{k} \binom{v+(n-k)\beta}{n-k} \frac{v}{v+(n-k)\beta},$$

which can be written, by using the factorial powers, under the form

$$(u+v+n\beta)^{[n]} = \sum_{k=0}^{n} \binom{n}{k} (u+k\beta)^{[k]} v(v+(n-k)\beta)^{[n-k-1]}.$$

The factorial power of a non-negative order n and increment h of u is defined by the formula

$$u^{[n,h]} = u(u-h)\dots(u-(n-1)h), \quad u^{[0,h]} = 1.$$

When $h=1$ we write $u^{[n,1]} = u^{[n]}$.

We shall also consider the generalized Vandermonde-Jensen formula

$$\frac{u+v}{u+v+n\beta}\binom{u+v+n\beta}{n}=\sum_{k=0}^{n}\frac{u}{u+k\beta}\binom{u+k\beta}{k}\frac{v}{v+(n-k)\beta}\binom{v+(n-k)\beta}{n-k}. \tag{1.5}$$

Jensen [29] has made the remark that formula (1.5) and the following formula

$$\binom{u+v}{n}=\sum_{k=0}^{n}\frac{u}{u+k\beta}\binom{u+k\beta}{k}\binom{v-k\beta}{n-k} \tag{1.6}$$

have been given earlier by I.G. Hagen [6] in 1891, but without demonstration.

These two formulas are particular cases of the more general formula

$$\frac{a(u+v-n\beta)+bnu}{u(u+v)(v-n\beta)}\binom{u+v}{n}=\sum_{k=0}^{n}\frac{a+bk}{(u+k\beta)(v-k\beta)}\binom{u+k\beta}{k}\binom{v-k\beta}{n-k},$$

given by Hagen [6] without any proof.

Jensen [29] has given also the new and elegant identity

$$\sum_{k=0}^{n}\binom{u+k\beta}{k}\binom{v-k\beta}{n-k}=\sum_{k=0}^{n}\binom{u+v-k}{n-k}\beta^{k}, \tag{1.7}$$

which can be seen in the book: "Combinatorial Identities" [23] of H.W. Gould.

In order to prove the identity (1.7) we introduce first the following notation

$$G(u,v,n)=\sum_{k=0}^{n}\binom{u+k\beta}{k}\binom{v-k\beta}{n-k}.$$

It is easy to see that we can write successively

$$\begin{aligned}G(u,v,n)&=\sum_{k=0}^{n}\binom{u+k\beta}{k}\binom{v-k\beta}{n-k}\left\{\frac{u}{u+k\beta}+\beta\frac{k}{u+k\beta}\right\}\\&=\sum_{k=0}^{n}\frac{u}{u+k\beta}\binom{u+k\beta}{k}\binom{v-k\beta}{n-k}\\&+\beta\sum_{k=1}^{n}\binom{u-1+k\beta}{k-1}\binom{v-k\beta}{n-k}.\end{aligned}$$

By using an identity (1.6) given in the paper of H.W. Gould ([18], pag. 71), as well as the Vandermonde-type convolution (1.10) from the same paper, we are able to write the equality:

$$\sum_{k=0}^{n} \frac{u}{u+k\beta} \binom{u+k\beta}{k} \binom{v-k\beta}{n-k} = \binom{u+v}{n},$$

so that we can obtain the relation

$$G(u,v,n) = \binom{u+v}{n} + \beta \sum_{k=0}^{n-1} \binom{u-1+\beta+k\beta}{k} \binom{v-\beta-k\beta}{n-1-k}.$$

Consequently we can write the Jensen recurrence formula

$$G(u,v,n) - \beta G(u-1+\beta, v-\beta, n-1) = \binom{u+v}{n}. \tag{1.8}$$

By using this we are able to write

$$\beta G(u-1+\beta, v-\beta, n-1) - \beta^2 G(u-2+2\beta, v-2\beta, n-2) = \beta \binom{u+v-1}{n-1}$$

and by successive application of (1.8) and summing the resulting relations we ultimately obtain

$$G(u,v,n) - \beta^r G(u-r+r\beta, v-r\beta, n-r) = \sum_{k=0}^{r-1} \binom{u+v-k}{n-k} \beta^k.$$

Letting $r = n+1$ we find the relation (1.7), which we intended to prove.

Now we want to point out that the identity (1.7) is a counterpart for the Abel-type series:

$$\sum_{k=0}^{n} \frac{(u+k\beta)^k}{k!} \cdot \frac{(v-k\beta)^{n-k}}{(n-k)!} = \sum_{k=0}^{n} \frac{(u+v)^k}{k!} \cdot \beta^{n-k}.$$

One way to prove this is to develop a recurrence relation, or to carry through a limiting process, as was noted in the paper of Gould [18].

Ending this section we mention, with Gian-Carlo Rota and Ronald Mullin ([49], pag. 168 and 195), that the Abel polynomials

$$p_n(x) = x(x-an)^{n-1}$$

are the basic polynomials of the Abel operator $E^a D$ (here D is the differentiation operator and E^a is the shift operator or the translation operator).

We have $DE^a = E^a D : x(x-na)^{n-1} \to nx(x-(n-1)a)^{n-2}$.

Finally, I consider that it is important to mention a generating relation from the monograph of Boas and Buck ([6], pag. 34) for the general difference polynomials

$$e^{xt} = \sum_{n=0}^{\infty} \frac{p_n(x)}{n!} [(e^t - 1)e^{\beta t}]^n,$$

where β is a real parameter.

For $\beta = 0$ we get the Newton binomial polynomials

$$p_n(x) = \binom{x}{n} = x(x-1)\dots(x-n+1)/n!,$$

while for $\beta = -\frac{1}{2}$ we obtain the Stirling interpolation polynomials.

2. Extensions to factorial powers of the Abel-Jensen combinatorial formulas

As we have mentioned above, we denote by $u^{[n,h]}$ the factorial power of order n ($n \geq 0$) and increment h of u, that is

$$u^{[n,h]} = u(u-h)\dots(u-(n-1)h), \quad u^{[0,h]} = 1, \quad u^{[n,1]} = u^{[n]}.$$

By extension to factorial powers the Abel combinatorial formula (1.1) we obtain

$$(u+v)^{[n,h]} = \sum_{k=0}^{n} \binom{n}{k} u(u-k\beta)^{[k-1,h]} (v+k\beta)^{[n-k,h]}, \tag{2.1}$$

where β is a non-negative parameter.

If we replace here $h = -\alpha$, where α is a non-negative parameter, we get the identity

$$(u+v)^{[n,-\alpha]} = \sum_{k=0}^{n} \binom{n}{k} u(u-k\beta)^{[k-1,-\alpha]} (v+k\beta)^{[n-k,-\alpha]}.$$

Now we select $u = x$ and $v = 1 - x$ and we obtain the important identity

$$\sum_{k=0}^{n} \binom{n}{k} x(x-k\beta)^{[k-1,-\alpha]} (1-x+k\beta)^{[n-k,-\alpha]} = 1^{[n,-\alpha]}$$

$$= 1(1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha). \tag{2.2}$$

By using the Abel-Jensen combinatorial formula (1.4) we are able to write

$$\sum_{k=0}^{n} \binom{n}{k} x(x+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+(n-k)\beta) = (1+n\beta)^{[n-1,-\alpha]}. \tag{2.3}$$

According to the combinatorial formula (1.2), we get the identity

$$\sum_{k=0}^{n} \binom{n}{k} x(x+k\beta)^{[k-1,-\alpha]}(1-x+(n-k)\beta)^{[n-k,-\alpha]} = (1+n\beta)^{[n,-\alpha]} \tag{2.4}$$

while from (1.4) we obtain

$$\sum_{k=0}^{n} \binom{n}{k} (x+k\beta)^{[k,-\alpha]}(1-x)(1-x+(n-k)\beta)^{[n-k-1,-\alpha]} = (1+n\beta)^{[n,-\alpha]}. \tag{2.5}$$

By using the combinatorial identities (2.2), (2.3), (2.4) and (2.5) we can introduce the basic polynomials

$$s_{m,k}^{\alpha,\beta}(x) = \frac{1}{1^{[m,-\alpha]}} \sum_{k=0}^{m} \binom{m}{k} x(x-k\beta)^{[k-1,-\alpha]}(1-x+k\beta)^{[m-k,-\alpha]}$$

$$q_{m,k}^{\alpha,\beta}(x) = \frac{1}{(1+m\beta)^{[m-1,-\alpha]}} \sum_{k=0}^{m} \binom{m}{k} x(x+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+(n-k)\beta)^{[m-k,-\alpha]}$$

$$p_{m,k}^{\alpha,\beta}(x) = \frac{1}{(1+m\beta)^{[m,-\alpha]}} \sum_{k=0}^{m} \binom{m}{k} x(x+k\beta)^{[k-1,-\alpha]}(1-x+(m-k)\beta)^{[m-k,-\alpha]}$$

$$r_{m,k}^{\alpha,\beta}(x) = \frac{1}{(1+m\beta)^{[m,-\alpha]}} \sum_{k=0}^{m} \binom{m}{k} (x+k\beta)^{[k,-\alpha]}(1-x)(1-x+(m-k)\beta)^{[m-k-1,-\alpha]}.$$

3. Linear positive operators constructed by means of the basic polynomials considered in the preceding section

For any function $f \in C[0,1]$ we construct linear positive operators, depending on four parameters:

$$\begin{aligned}
(S_m^{\alpha,\beta,\gamma,\delta} f)(x) &= \sum_{k=0}^{m} s_{m,k}^{\alpha,\beta}(x) f\left(\frac{k+\gamma}{m+\delta}\right), \\
(Q_m^{\alpha,\beta,\gamma,\delta} f)(x) &= \sum_{k=0}^{m} q_{m,k}^{\alpha,\beta}(x) f\left(\frac{k+\gamma}{m+\delta}\right), \\
(P_m^{\alpha,\beta,\gamma,\delta} f)(x) &= \sum_{k=0}^{m} p_{m,k}^{\alpha,\beta}(x) f\left(\frac{k+\gamma}{m+\delta}\right), \\
(R^{\alpha,\beta,\gamma,\delta} f)(x) &= \sum_{k=0}^{m} r_{m,k}^{\alpha,\beta}(x) f\left(\frac{k+\gamma}{m+\delta}\right),
\end{aligned} \tag{3.1}$$

where $0 \leq \gamma \leq \delta$.

In the case $\beta = \gamma = \delta = 0$ these operators reduce to the Stancu operator S_m^{α}, introduced and investigated in the paper [55]:

$$(S_m^{\alpha} f)(x) = \sum_{k=0}^{m} s_{m,k}^{\alpha}(x) f\left(\frac{k}{m}\right),$$

where

$$s_{m,k}^{\alpha}(x) = \binom{m}{k} \frac{x^{[k,-\alpha]}(1-x)^{[m-k,-\alpha]}}{1^{[m,-\alpha]}}.$$

This operator was further investigated and applied by several authors. B. Della Vecchia [12], A. Di Lorenzo - M.R. Occorsio [13], F. Frenţiu [17], I. Horova and Budikova [27], G. Mastroianni and G. Occorsio [40], [41], I.A. Rus [50], S. Toader [61] and others.

If we select $\alpha = \gamma = \delta = 0$ then we arrive at the operators of Cheney and Sharma [10] P_m and Q_m.

For $\gamma = \delta = 0$ the operator (3.1) becomes

$$(Q_m^{\alpha,\beta} f)(x) = \sum_{k=0}^{m} q_{m,k}^{\alpha,\beta}(x) f\left(\frac{k}{m}\right), \tag{3.2}$$

where

$$q_{m,k}^{\alpha,\beta}(x) = \frac{\binom{m}{k} x(x+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+(m-k)\beta)^{[m-k-1,-\alpha]}}{(1+m\beta)^{[m-1,-\alpha]}}.$$

When $\alpha = 0$ this operator will be the second operator of Cheney-Sharma [10], defined by the formula

$$(Q_m f)(x;\beta) = \sum_{k=0}^{m} q_{m,k}(x;\beta) f\left(\frac{k}{m}\right),$$

where

$$q_{m,k}(x;\beta) = \binom{m}{k} \frac{x(x+k\beta)^{k-1}(1-x)(1-x+(m-k)\beta)^{m-1-k}}{(1+m\beta)^{m-1}}.$$

It will be easy to prove that this operator is similar with the Bernstein operator B_m, and preserves the linear functions.

4. **Convergence properties of the sequence** $(Q_m^{\alpha,\beta})$

For the convergence of the sequence of operators $Q_m^{\alpha,\beta}$, defined at (3.3), we shall use the classical theorem of Bohman-Korovkin [7], [33], which can be stated as follows:

If we have a sequence of linear positive operators $L_m : C[a,b] \to C[a,b]$ and we have $(L_m s_k)$ converges uniformly to s_k on $[a,b]$ for $k = 0, 1$ and 2, where $s_k(x) = x^k$, then the sequence $(L_m f)$ converges uniformly to f on $[a,b]$ for each $f \in C[a,b]$. In our case $[a,b] = [0,1]$ and we have the operators Q_m, defined at (3.3).

According to Abel-Jensen combinatorial formula (2.3) we can see that $Q_m^{\alpha,\beta} e_0 = e_0$.

In the case of the next test function e_1 we have

$$(Q_m^{\alpha,\beta} e_1)(x) = \frac{1}{(1+\alpha+m\beta)^{[m-1,-\alpha]}} (Z_m^{\alpha,\beta} e_1)(x) \tag{4.1}$$

where

$$(Z_m^{\alpha,\beta} e_1)(x) = \sum_{k=1}^{m} \frac{k}{m} \binom{m}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-k-1,-\alpha]}$$

$$= \sum_{k=1}^{m} \binom{m-1}{k-1} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-1-k,-\alpha]}.$$

By changing the index of summation $k-1=j$, we get

$$(Z_m^{\alpha,\beta}e_1)(x) = x\sum_{j=0}^{m-1}\binom{m-1}{j}(x+\alpha+\beta+j\beta)^{[j,-\alpha]}(1-x)(1-x+(m-1-j)\beta)^{[m-2j,-\alpha]}. \tag{4.2}$$

Now we shall use an extension to factorial powers of the Abel combinatorial formula

$$(u+v+n\beta)^{[n,h]} = \sum_{k=0}^{n}\binom{n}{k}(u+k\beta)^{[k,h]}v(v+(n-k)\beta h)^{[n-k-1,h]}. \tag{4.3}$$

We have to replace here $n=m-1$, $h=-\alpha$, $u=x+\alpha+\beta$, $v=1-x$ and we arrive at the following identity

$$(1+\alpha+m\beta)^{[m-1,-\alpha]}$$

$$= \sum_{k=0}^{m-1}\binom{m-1}{b}(x+\alpha+\beta+b\beta)^{[k,-\alpha]}(1-x)(1-x+(m-k-1)\beta)^{[m-2-k,-\alpha]}.$$

According to (3.4), (3.5) and (3.6) we can write:

$$Q_n^{\alpha,\beta}e_1 = e_1.$$

Consequently, our operator reproduces the linear functions.

Going on to the next test function e_2 we find that

$$(Q_m^{\alpha,\beta}e_2)(x) = \frac{1}{m}\sum_{k=1}^{m}\left[\frac{k}{m}+\frac{k(k-1)}{m}\right]q_{m,k}^{\alpha,\beta}(x)$$

$$= \frac{1}{m}(Q_m^{\alpha,\beta}e_1)(x)+\frac{1}{m}\sum_{k=2}^{m}\binom{m-1}{m-2}x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-k-1,-\alpha]}$$

$$= \frac{x}{m}+\frac{m-1}{m}\sum_{j=0}^{m-2}\binom{m-2}{j}(x+\alpha+2\beta+j\beta)^{[j+1,-\alpha]}(1-x+\alpha+(m-2-j)\beta)^{[m-3-j,-\alpha]}.$$

Now if we use again the extension to factorial powers of the Abel combinatorial formula, we can see that $Q_m^{\alpha,\beta}e_2$ tends uniformly to e_2 on $[0,1]$, when m tends to infinity.

By applying the Bohman-Korovkin [7], [3]] convergence criterion we can state the following result: if $f \in C[0,1]$ and the parameters α and β are non-negative and depend on m such that $\alpha = \alpha(m) \to 0$ and $m\beta(m) \to 0$, when m tends to infinity, then the sequence $(Q_m^{\alpha,\beta} f)$ converges uniformly to f on $[0,1]$.

5. **Evaluations of the remainder term**

Because the operator $Q_m^{\alpha,\beta}$ reproduces the linear functions, we can state that the approximation formula

$$f(x) = (Q_m^{\alpha,\beta} f)(x) + (R_m^{\alpha,\beta} f)(x) \tag{5.1}$$

has the degree of exactness $N = 1$.

Assuming that the function f has a continuous second derivative on the interval $[0,1]$, we can represent the remainder of this formula under the following integral form

$$(R_m^{\alpha,\beta} f)(x) = \int_0^1 G_m^{\alpha,\beta}(t;x) f''(t)dt, \tag{5.2}$$

where

$$G_m^{\alpha,\beta}(t;x) = (R_m^{\alpha,\beta}\varphi_x)(t), \quad \varphi_x(t) = (x-t)_+ = \frac{x-t+|x-t|}{2},$$

understanding that $R_m^{\alpha,\beta}$ operates on φ_x as a function of x.

The above integral representation of the remainder can be obtained if we make use of the well-known theorem of Peano.

For the Peano kernel, associated to our operator, we have

$$G_m^{\alpha,\beta}(t;x) = (x-t)_+ - \sum_{k=0}^{m} q_{m,k}^{\alpha,\beta}(x) \left(\frac{k}{m} - t\right)_+ . \tag{5.3}$$

In order to find an explicit expression of this kernel, we assume that $x \in \left[\frac{s-1}{m}, \frac{s}{m}\right]$ and we can write

$$G_m^{\alpha,\beta}(t;x) = x - t - \sum q_{m,k}^{\alpha,\beta}(x) \left(\frac{k}{m} - t\right) \tag{5.4}$$

for $t \in \left[\frac{j-1}{m}, \frac{j}{m}\right]$, where $1 \leq j \leq s-1$.

If we consider that $t \in \left[\frac{s-1}{m}, x\right]$, then we obtain

$$G_m^{\alpha,\beta}(t;x) = x - t - \sum_{k \geq s} q_{m,k}^{\alpha,\beta}(x) \left(\frac{k}{m} - t\right), \tag{5.5}$$

while for $t \in \left[x, \frac{s}{m}\right]$ we get

$$G_m^{\alpha,\beta}(t;x) = -\sum_{k \geq s} q_{m,k}^{\alpha,\beta}(x) \left(\frac{k}{m} - t\right).$$

In the case $t \in \left[\frac{j-1}{m}, \frac{j}{m}\right]$, where $j > s$, we have

$$G_m^{\alpha,\beta}(t;x) = -\sum_{k \geq j} q_{m,k}^{\alpha,\beta}(x) \left(\frac{k}{m} - t\right).$$

Because the degree of exactness of the formula (5.1) is one, by replacing $f(x) = x - t$, the corresponding remainder vanishes and we obtain

$$x - t - \sum_{k=0}^{m} q_{m,k}^{\alpha,\beta}(x) \left(\frac{k}{m} - t\right)$$
$$= \sum_{k=0}^{j-1} q_{m,k}^{\alpha,\beta}(x) \left(\frac{k}{m} - t\right) + \sum_{k=j}^{m} q_{m,k}^{\alpha,\beta}(x) \left(\frac{k}{m} - t\right).$$

Therefore we can write

$$x - t = \sum_{k=j}^{m} q_{m,k}^{\alpha,\beta}(x) \left(\frac{k}{m} - t\right) = -\sum_{k=0}^{j-1} q_{m,k}^{\alpha,\beta}(x) \left(x - \frac{k}{m}\right).$$

Consequently, the representation (5.4) can be replaced by

$$G_m^{\alpha,\beta}(t;x) = -\sum_{k=0}^{j-1} q_{m,k}^{\alpha,\beta}(x) \left(t - \frac{k}{m}\right),$$

if $t \in \left[\frac{j-1}{m}, \frac{j}{m}\right]$ and $1 \leq j \leq s-1$, while (5.5) can be replaced by

$$G_m^{\alpha,\beta}(t;x) = \sum_{k=0}^{s-1} q_{m,k}^{\alpha,\beta}(x) \left(t - \frac{k}{m}\right),$$

when $t \in \left[\frac{s-1}{m}, x\right]$.

Because on the interval $[0,1]$ we have $G_m^{\alpha,\beta}(t;x) \le 0$, we can apply the mean value theorem to the integral and we obtain

$$(R_m^{\alpha,\beta} f)(x) = f''(\xi) \int_0^1 G_m^{\alpha,\beta}(t;x)dt, \quad \xi \in (0,1), \tag{5.6}$$

under the hypothesis that $f \in C^2[0,1]$.

If in the approximation formula

$$f(x) = (Q_m^{\alpha,\beta} f)(x) + f''(\xi) \int_0^1 G_m^{\alpha,\beta}(t;x)dt$$

we replace $f(x) = e_2(x) = x^2$, we get

$$\int_0^1 G_m^{\alpha,\beta}(t;x)dt = \frac{1}{2}[x^2 - (Q_m^{\alpha,\beta} e_2)(x)] = \frac{1}{2}(R_m^{\alpha,\beta} e_2)(x).$$

Consequently, we can see that the remainder of the approximation formula (5.1) can be expressed under the following form

$$(R_m^{\alpha,\beta} f)(x) = \frac{1}{2}(R_m^{\alpha,\beta} e_2)(x) f''(\xi), \tag{5.7}$$

where $0 < \xi < 1$.

Therefore we can state the following result:

If we have the function $f \in C^2[0,1]$, then the remainder of the approximation formula (5.1) can be represented under the integral form (5.6).

We mention that in the particular case $\alpha = \beta = 0$, when $Q_m = B_m$, the corresponding approximation formula was established by D.D. Stancu in 1963 in the paper [54].

Now we want to make the remark that because $Q_m^\alpha f$ is interpolatory at both sides of the basic interval $[0,1]$, it is clear that $(R_m^\alpha e_2)(x)$ had to contain the factor $x(x-1)$.

Since $R_m f \neq 0$, if $\beta = 0$, for any convex function f of the first order, we can apply a criterion of T. Popoviciu [46] and we can find that the remainder $R_m^\alpha f$ is of a simple form. Therefore we can state the following result:

If the second-order divided differences of the function f are bounded on the interval $[0,1]$, then there exist three distinct points $t_{m,1}, t_{m,2}, t_{m,3}$ in the interval $[0,1]$, *which might depend on f, such that the remainder of the approximation formula (5.1) can be represented under the following form*

$$(R_m^\alpha f)(x) = (R_m^\alpha e_2)(x)[t_{m,1}, t_{m,2}, t_{m,3}; f],$$

where the nodes are certain distinct points of the interval $[0, 1]$.

It is clear that if $f \in C^2[0, 1]$ and we apply the mean-value theorem of divided differences, then we can obtain formula (5.7).

In the case $\alpha = 0$ we can see that we have

$$(R_m f)(x) = \frac{x(x-1)}{2m} f''(\xi),$$

which represents the remainder in the case of the Bernstein approximation operator B_m.

This result was obtained by D.D. Stancu in 1963 in the paper [54].

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BOOK REVIEWS

Charles Chidume, ***Geometric Properties of Banach Spaces and Nonlinear Iterations***, Lectures Notes in Mathematics, Vol. 1965, Springer-Verlag London Limited, 2009, xvii+326 pp; ISBN: 978-1-84882-189-7, e-ISBN: 978-1-84882-190-3, DOI: 10.1007/978-1-84882-190-3.

The classical Banach Contraction Principle asserts that every contraction T on a complete metric space (X, ρ) has a unique fixed point $\bar{x}$ and the Picard iteration $x_{n+1} = Tx_n,\ n \geq 0,$ converges to $\bar{x}$, for every $x_0 \in X$. This result is no longer true for nonexpansive mappings (i.e., such that $\rho(Tx, Ty) \leq \rho(x, y),\ x, y \in X$), even when X is a weakly compact subset of a Banach space E. The study of fixed points for nonexpansive mappings defined on convex subsets of Banach spaces has put in evidence strong connections to the geometric properties of the underlying Banach space - normal structure, rotundity and smoothness properties characterized in terms of various constants and moduli. Also, even if T has a fixed point, the Picard iteration could not converge to the fixed point of T. By a clever modification of Picard iteration, namely $x_{n+1} = \frac{1}{2}(x_n + Tx_n),\ n \geq 0,$ Krasnoselki (1955) succeeded to obtain convergence to the fixed point in some cases. Later extensions to Kransnoselski's idea were given by Mann (1953): $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n,\ n \geq 0,$ (called also the Krasnoselski-Mann iteration), and Ishikawa (1974): $y_n = (1-\beta_n)x_n + \beta_n Tx_n,\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n,\ n \geq 0$. The hypotheses on (α_n) in Mann iteration are: (i) $\alpha_n \in (0, 1),\ \lim_n \alpha_n = 0,$ and (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, with some similar conditions in Ishikawa's method.

The aim of the present book is to give an introduction to this very active area of investigation. The basic results from the geometry of Banach spaces are presented in the first five chapters of the book: 1. *Some geometric properties of Banach spaces*, 2. *Smooth spaces*, 3. *Duality map in Banach spaces*, 4. *Inequalities in uniformly convex spaces*, and 5. *Inequalities in uniformly smooth spaces.*

The study of iterative procedures starts in Chapter 6. *Iterative methods for fixed points of nonexpansive mappings*, and continues along the rest of the chapters (there are 23) with topics as: descent methods for variational inequalities (in Ch. 7), iterative procedures for zeros of generalized accretive operators (Chapters 8 and 9), iterations for pseudo-contractive mappings (Chapters 10 to 12), iterative methods for generalized nonexpansive mappings (Ch. 14), for families of nonexpansive mappings (Chapters 15 to 17), for asymptotically nonexpansive mappings (Chapters 20 and 21) and for nonexpansive semigroups (Ch. 22). Chapter 13 is concerned with applications to Hammerstein integral equations while the last chapter, 23. *Single-valued accretive operators; Applications; Some open questions,* is concerned with continuity conditions implying the single-valuedness of set-valued accretive operators with applications to differential inclusions. This chapter contains also some open problems and recommendations for further reading.

Based on a rich bibliography (561 items) and including many original contributions of the author, the book is of great help for graduate and postgraduate students, as well as for researchers interested in fixed point theory, geometry of Banach spaces and numerical solution of various kinds of equations - operator differential equations, differential inclusions,variational inequalities.

A good companion in reading it could be the recent book by V. Berinde, *Iterative Approximation of Fixed Points*, LNM 1912, Springer, Berlin 2007, dealing with similar topics, with emphasis on the Hilbert space setting. Together, these two books cover a lot of the research work done in the last 20 years in the field of iterative procedures for fixed points.

S. Cobzaş

Leon A. Takhtajan, *Quantum mechanics for mathematicians*, Graduate Studies in Mathematics, Vol. 95, American Mathematical Society, Providence, Rhode Island 2008,x+278 pp, ISBN:978-0-8218-4630-8

From the ancient times, the development of physics stimulated the research in mathematics, leading to the discovery of new areas in mathematics. In this sense, the most known is the creation by L. Schwartz and S. L. Sobolev of distribution theory, as a response to some curious manipulation (at least for mathematicians) of some strange functions, the best known being the Dirac delta function. Up to the early 20th century classical physics, especially classical mechanics, was an integral part of curricula for students in mathematics. The situation is totally different with quantum physics, which uses very advanced mathematical tools, but has never been a part of the curriculum of graduate students in mathematics. In trying to fill this gap the author taught, for more that 14 years, a course on quantum mechanics for students in mathematics at Stony Brook University. In fact, mathematics and physics are strongly interconnected, some topics as, for instance, string theory being chapters of mathematics and physics as well.

The inspiration for this course came for a similar one taught by L. D. Faddeev at Leningrad (now Sankt Petersburg) State University. The AMS published a course by L. D. Faddeev, *Elementary introduction to quantum field theory. Quantum fields and strings: a course for mathematicians,* AMS 1999, and the translation of an another one: L. D. Faddeev and O. A. Yakubovskiĭ, *Lectures on Quantum Mechanics for Mathematics Students* with an appendix by Leon Takhtajan, AMS, 2009, originally published in Russian with Leningrad University Publishers, Leningard 1980. The first part of the book is based mainly on these course, but using more advanced mathematical tools, complete rigorous proofs and going beyond the topics presented there.

The book is divided into two parts: 1. *Foundations,* and 2. *Functional methods and supersymmetry.* The first part starts with a chapter, 1. *Classical mechanics,* where the exposition follows the traditional line, starting with the Lagrangian formalism and introducing the Hamiltonian through the Legendre transform as, for instance, in the classical treatise by V. I. Arnold.

The study of quantum mechanics starts in Chapter 2. *Basic principles of quantum mechanics,* based on the Dirac-von Neumann axioms: an infinite dimensional separable complex Hilbert space $\mathcal{H}$, called the space of states, the family of observables formed of the space of self-adjoint operators on $\mathcal{H}$, and the states $\mathcal{S}$ which are the positive trace class operators M on $\mathcal{H}$ with Tr $M = 1$. Pure states are the projection operators onto the one-dimensional spaces, all of the other states being called mixed states.

The first part continues with Chapters 3. *Schrödinger equation,* and 4. *Spin and identical particles.*

The second part contains the chapters 5. *Path integral formulation of quantum mechanics* (the Feynman approach to quantum mechanics), 6. *Integration in function spaces* (Wiener integration theory, the Feynman-Kac formula), 7. *Fermion systems* (anticommutativity relations, Grassmann algebra), and 8. *Supersymmetry.*

There are a lot of problems spread throughout the book. Some of these require to fill in some proofs, only sketched in the main text, while others refer to supplementary topics, not treated in the main text. For these last ones, exact references are given.

Each chapter ends with a section of *Notes and references,* containing bibliographical references and recommendations for further reading.

The bibliography contains a list of carefully chosen monographs, both classical and modern, on mathematics and physics, survey papers and some fundamental research papers, all being referred in the Notes section of the chapters.

By a cleaver selection of the material and the clear way of exposing it, the book is recommended for graduate students in mathematics looking for applications in physics, as well as for student in physics desiring to be acquainted, in a rigorous but, at the same time, quick and accessible manner, with the basic mathematical tools used in quantum mathematics. The devoted readers can follow the paths indicated by the author in problems and notes, to acquire a master introduction to the area.

Radu Precup

Stephen Lynch, *Dynamical Systems with Applications using MATHEMATICA*®, xvi+484 pp, Birkhäuser, Boston -Basel - Berlin, 2007, ISBN-13: 978-0-8176-4482-6

The book is a good introduction to dynamical systems theory. In the first part, after a short introduction to MATHEMATICA®, differential equations and dynamical systems are considered, with examples taken from mechanical systems, chemical kinetics, electric circuits, interacting species and economics. The second part is devoted to discrete dynamical systems, with many advanced examples from electromagnetic waves, optical resonators, chaos, fractals, neurodynamics.This book presents an original, cheap and powerful solution to the problem of analysis of large data sets.

The theory and applications are presented with the aid of the MATHEMATICA® package. Throughout the book, MATHEMATICA® is viewed as a tool for solving systems or producing exciting graphics. Each chapter contains a subsection with "Mathematica Commands in Text Format". The author suggests that the reader should save the relevant example programs. These programs can then be edited accordingly when attempting to solve the exercises at the end of each chapter. The solution combines C language, data base query and management, statistics and data visualization. The text is aimed at graduate students and working scientists in various branches of applied mathematics, natural sciences and engineering. The material is intelligible to readers with a general mathematical background. Fine details and theorems with proof are kept at a minimum. This book is informed by the research interests of the author which are nonlinear ordinary differential equations, nonlinear optics and fractals. Some chapters include recently published research articles and provide a useful resource for open problems in nonlinear dynamical systems.The book intends to be an alternative to classical statistical books: it does not separate descriptive and inferential An efficient tutorial guide to MATHEMATICA® is included. The knowledge of a computer language would be beneficial but not essential. The MATHEMATICA® programs are kept as simple as possible and the author's experience has shown that this method of teaching using MATHEMATICA® works well with computer laboratory class of small sizes.statistics, simple models are combined into more complex model in a hierarchical way and it is computer oriented. recommend "Dynamical Systems with Applications using MATHEMATICA®" as a good handbook for a diverse readership, for graduates and professionals in mathematics, physics, science and engineering. This book could be considered as a new and extended version of the following books, also written by the same author:

Dynamical Systems with Applications using MATLAB®, ISBN 978-0-8176-4321-8, and *Dynamical Systems with Applications using* MAPLE, ISBN 978-0-8176-4150-4.

Damian Trif

Dean Corbae, Maxwell B. Stinchcombe and Juraj Zeman, *An Introduction to Mathematical Analysis for Economic Theory and Econometrics*, Princeton University Press, Princeton and Oxford 2009, xxi+671 pp; ISBN: 978-0-691-11867-3,

The book covers a broad spectrum of topics from analysis, functional analysis and measure theory, needed for economic theory and econometrics. Its aim is to bridge the gap existing between the basic mathematical economics tools (calculus, linear algebra, constrained optimization) and the advanced economics texts, as, for instance, N. L. Stokey and R. E. Lucas, *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989, which assume a working knowledge of functional analysis, measure theory and probability. In fact, one of the motivations to write such a textbook comes from the difficulties encountered by the students of the one of the authors to understand the above mentioned book. The present book contains a choice of topics from various areas of mathematics, as lattices, convex analysis, functional analysis, measure theory, probability, that are widely used in economics and econometrics. The book is self-contained in what concerns the mathematical part - almost any theorem used in proving some result is itself proved as well. Another feature of the book is the wealth of examples from economic theory and economics (whose understanding requires an undergraduate basic ground in economics), providing intuition and motivation for grasping the difficult mathematical ideas developed in the book.

The first part of this book (Chapters 1 to 6) was taught in the first-semester Ph.D. core sequence at the University of Pittsburgh and the University of Texas. The second part (Chapters 7 to 11) was taught as a graduate mathematical economics class. Chapters 1 to 6 cover basic mathematics for economics: 1. *Logic*, 2. *Set theory* (including lattices and Tarski's fixed point theorem with application to stable matchings), 3. *The space of real numbers*, 4. *The finite-dimensional metric space of real vectors*, 5. *Finite-dimensional convex analysis* (dealing with finite dimensional normed spaces, Kuhn-Tucker theorem, Lagrange multipliers, etc), and 6. *Metric spaces* (with applications to the space of probabilistic distribution functions on $\mathbb{R}$ equipped with Levy's metric). The set of real numbers is introduced *via* equivalence classes of Cauchy sequences of rational numbers. This construction as well as the notion of completeness (with different meanings in different contexts) form the red thread of the presentation along the book. The logic properties and set operations are presented in parallel, as paradigms of the same idea.

The second part of the book contains the chapters: 7. *Measure spaces and probability* (including convergence in distribution and Skorohod theorem), 8. *The $L^p(\Omega, \mathcal{F}, P)$ and ℓ^p spaces, $p \in [1, \infty]$* (with applications to regression analysis), 9. *Probabilities on metric spaces* (Polish metric spaces, Polish measure spaces, stochastic processes, a proof of the central limit theorem), 10. *Infinite-dimensional convex analysis* (containing an introduction to topological vector spaces and locally convex spaces, including compactness, Alaoglu-Bourbaki theorem, separation and Krein-Milman theorem, Schauder fixed point theorem), and 11. *Expanded spaces* (dealing with the basic constructions of nonstandard analysis).

Each chapter ends with a set of exercises and recommendation for further reading. These refer mainly to books where the topics of the corresponding chapter are treated at large.

By exposing in a self-contained, rigorous but accessible manner a lot of essential results from analysis, functional analysis, measure theory and probability used in economic theory and econometrics, the book is a very useful tool for students specializing in these disciplines. Even students in mathematics and researchers will find the results collected by the authors very useful as well.

S. Cobzaş

Wassim M. Haddad and Vijay Sekhar Chellaboina, *Nonlinear Dynamical Systems and Control, A Lyapunov-Based Approach* Princeton University Press, Princeton and Oxford 2008, xvi+948 pp; ISBN: ,

The book under review is a graduate-level textbook which presents and develop an extensive treatment of stability analysis and control design of nonlinear dynamical systems using the Lyapunov methods.

The book is structured in 14 chapters. The authors introduce the definition of dynamical systems and present a systematic development of the theory of nonlinear differential equations. There are presented the qualitative theory of existence, uniqueness, continuous dependence of solutions on the initial conditions for nonlinear differential equations, the stability theory for nonlinear dynamical systems generated by these nonlinear differential equations, Lyapunov stability theorems for time-invariant nonlinear dynamical systems, invariant set stability theorems, converse Lyapunov theorems and Lyapunov instability theorems. A chapter in advanced stability theory is also included. There are described the partial stability, stability theory for time-varying systems, Lagrange stability, boundedness and ultimate boundedness, input-to-state stability, finite-time stability, semistability and stability theorems via vector Lyapunov functions. The book continues with a chapter regarding the dissipative theory for nonlinear dynamical systems and a chapter regarding the stability and optimality of feedback dynamical systems. The input -output technique for dynamical systems is a tool used in the study of infinite-dimensional systems. The authors present the concept of input-output stability and then establish connections between input-output stability and Lyapunov stability. The nonlinear optimal control problem and the stability and optimality results for backstepping control problems are given in the next chapter. Extension to disturbance rejection and robust control of nonlinear dynamical systems are presented. The last two chapters contain the discrete-time extension of the aforementioned topics.

This book is an excellent textbook addressed to graduate students of applied mathematics, control theorists and engineers studying the stability theory of dynamical systems and controls. It is a rich source of materials for researchers interested in systems theory.

Marcel-Adrian Şerban

Alexander H.W. Schmitt, *Geometric Invariant Theory and Decorated Principal Bundles*, European Mathematical Society, 2008, 389 pp, ISBN 978-3-03719-065-4.

Classifying the objects of a category is a fundamental mathematical problem which allows, whenever it is solved, the control and the manipulation of the objects of that category easily in various purposes. This type of problems is equally important and difficult, in most of the important categories being obtained just partial results. For instance the finitely generated Abelian groups, the one and two dimensional compact manifolds are completely classified, all these results being now classical. For *topological categories*, this problem might be even more difficult than it is for *algebraic* *categories*, due to some properties of objects and morphisms of algebraic nature, which are rather absent in the case of the objects and morphisms of topological nature. However, one can expect advances in the case of the topological categories, thanks to Homotopy Theory and Algebraic Topology, whenever the classification problem advances in the case of some algebraic categories.

The classification of algebraic varieties up to an isomorphism has two faces, one of them consists in classifying the smooth or mildly singular complex projective varieties up to an isomorphism and the second one consists in classifying all closed subvarieties of a certain complex projective space up to projective equivalences. These classification problems are either treated by means of various numerical invariants, such as *the Hilbert polynomial* of polarized varieties or by means of *moduli spaces* (i.e. algebraic varieties whose points are in natural one-to-one correspondence to the set of isomorphism classes of objects with fixed numerical invariants), which generated Mumford's Geometric Invariant Theory as a tool to construct such moduli spaces.

In this book the author constructs more general moduli spaces for semistable projective *ϱ-bumps* of a given topological type as well as moduli spaces for semistable affine pairs with prescribed topological type for the first component.

The book is structured in two large chapters with the following content:

The first chapter deals with Geometric Invariant Theory as developed by Mumford and starts with actions of reductive linear groups on vector spaces or projective spaces. The reductivity of the groups $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $GL_{n_1}(\mathbb{C})\times\cdots\times GL_{n_l}(\mathbb{C})$ is shown at the end of the first section. Next, one parameterizes the space of orbits of an algebraic group action by affine algebraic varieties and formulate the basic properties of such quotients. In section 1.3 one studies the classification of projective hypersurfaces of degree d in the projective space $\mathbb{P}_{n-1}$ up to projective equivalences, while section 1.4 is devoted to the fundamental concepts of Geometric Invariant Theory, which starts by studying *good* and *geometric quotients* and continues with linearizations of group actions. Finally, the last two sections of this first chapter deal with the Hilbert-Mumford criterion and a certain refinement of it, on the existence of one parameter subgroups $\lambda : \mathbb{C}^* \longrightarrow G$ of some reductive group G, linearly represented, such that $\lim_{z\to\infty} \lambda(z)\cdot\nu = 0$, for some special $\nu \in V$. Another problem is the existence of such a one parameter subgroup with fastest possible convergence as well as the uniqueness of such a subgroup.

The second chapter starts with an overview on principal bundles which is necessary to state the classification problem developed within the rest of the chapter and continues with a review on vector bundles on complex algebraic curves. The classification of topological vector bundles on smooth projective curves and the Riemann-Roch theorem for coherent sheaves are presented alongside a discussion on bounded families of vector bundles. The classification problem is stated in terms of projective and affine *ϱ-bumps/swamps*, but is shown how the ϱ-swamp describes a family of hypersurfaces of degree d and an isomorphism is a relative version of projective equivalence, for a particular choice of the representation, namely the general classification problem specializes to the classification of some algebraic varieties. In section 2.4 one obtains, by using *decorated bundles*, the moduli space of semistable principal G-bundles with connected reductive group structure. In section 2.5 one deals with the structure group $G := GL_{r_1}(\mathbb{C})\times\cdots\times GL_{r_t}(\mathbb{C})$ by choosing a faithfull representation $\chi : G \longrightarrow GL(W)$ which allows to reduce the problem of constructing moduli spaces to the case of decorated bundles. One also study the asymptotic behavior of the semistability concept and specialize the abstract obtained results to some concrete situations which allows to see a generalization of a well-known result by King on moduli spaces of quiver representations. Therefore some steps towards the classification problems of principal G bundles with arbitrary reductive group structure have been done and this classification is finalized within the last section 2.7. The itinerary for this purpose is similar to that exposed in previous sections for the classification problem of more particular reductive structure groups, but developed at a superior level in which the role of decorated bundles, for instance, is played by *decorated pseudo-G-bundles*.

The book is very well written and uses the powerful modern mathematical languages of representations, bundles and schemes to treat a very important classification problem with serious implications to the classification problem of algebraic varieties.

Cornel Pintea