BERS EMBEDDING OF THE TEICHMÜLLER SPACE
OF A ONCE-PUNCTURED TORUS

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ABSTRACT. In this note, we present a method of computing monodromies of
projective structures on a once-punctured torus. This leads to an algorithm
numerically visualizing the shape of the Bers embedding of a one-dimensional
Teichmüller space. As a by-product, the value of the accessory parameter of
a four-times punctured sphere will be calculated in a numerical way as well
as the generators of a Fuchsian group uniformizing it. Finally, we observe the
relation between the Schwarzian differential equation and Heun's differential
equation in this special case.

1. INTRODUCTION

The Teichmüller space was, as its name tells us, invented by O. Teichmüller
around 1940. At this early stage, however, quasiconformal mappings had not been
developed enough, and therefore, his work was difficult for most people to under-
stand. In the late 1950's, L. V. Ahlfors and L. Bers established a firm foundation
to the theory of quasiconformal mappings, which led to a better understanding of
Teichmüller space and the further development of it. Among their contributions,
the idea of simultaneous uniformization due to Bers enabled us to embed the Te-
ichmüller space of a Riemann surface as a bounded domain in the normed vector
space of holomorphic quadratic differentials on it. In the latter half of the last cen-
tury, many authors have revealed a number of remarkable properties of Teichmüller
spaces. Most of them, however, are concerned with the internal geometry of the
spaces. On the one hand, in connection with the deformation of Kleinian groups,
the boundary of the Bers embedding (the so-called Bers boundary) of a Teichmüller
space has been investigated. In spite of much effort in this direction, the shape of the
Bers embedding is still mysterious at present because of its highly transcendental
nature. At least, the supporting evidence that the Bers boundary is fairly irregular
has been reported by, for instance, [22] and [38]. We also note that the Teichmüller
space of finite dimension greater than one is not a symmetric domain. Actually,
the analytic automorphism group of it is the homomorphic image of the mapping
class group on the surface under the discontinuous action on the Teichmüller space

(Royden [34]). As Bers stated in his survey [3], even one-dimensional Teichmüller spaces are exciting one's curiosity enough.

The first attempt to compute a boundary point of a one-dimensional Teichmüller space was made by R. M. Porter [32] and L. Keen, *et al* [13], [12], where they succeeded in drawing several real rays of trace functions ending at cusps in the case of the once-punctured square torus. The present authors are wondering why no one has developed their approach further despite the remarkable progress of computing technology in recent years. We note here that in the 1980's C. McMullen, D. Mumford and D. Wright obtained computer pictures of the Maskit embedding of the Teichmüller space of a once-punctured torus. Since the Maskit embedding had been expected to resemble the Bers embedding in shape, they made several conjectures on the Bers embedding (see, for instance, [25]). Moreover, the Bers boundary of a one-dimensional Teichmüller space was asserted to be a Jordan curve by C. McMullen around 1990. A proof for the assertion in the literature has been given by Y. Minsky as a corollary of his complete description of the space of discrete faithful representations of once-punctured torus groups [27].

Furthermore, using Minsky's Pivot Theorem, H. Miyachi [28] recently proved that each cusp of the one-dimensional Teichmüller space is really cusp-shaped; namely, to the endpoint of each rational pleating ray one can attach a subdomain like cardioid in the Bers embedding.

On the other hand, B. Maskit [24] (see also [21]) discovered projective structures such that the monodromies are quasi-Fuchsian groups but the developing maps are not univalent (for details of the terminology used here, see the next section). Those exotic projective structures have recently been studied intensively by many authors (see, e.g., [26] , [36], [11]). However, the configuration of components of the interior of exotic projective structures on a surface is still far from being clear.

In this article, we will give a practical method of computing the monodromy of a given projective structure by solving a linear ordinary differential equation of the second order which is associated with the Schwarzian differential equation. Employing the bending coordinates, which were developed by L. Keen and C. Series (see, e.g., [15]) and by C. McMullen [26], we explain how we can generate a picture of the Bers embedding with sufficient precision.

Our method can also be applied to obtain the whole picture of exotic components of the discreteness locus of projective structures and, if available, is an algorithm of determining the discreteness of a given Möbius group generated by two elements with parabolic commutator. This will be a main subject of our forthcoming paper [18] with M. Wada and Y. Yamashita.

This article is organized as follows. Section 2 is devoted to the basic definitions of Teichmüller spaces and related notions. In Section 3, we summarize known facts about commensurable Fuchsian groups with given Fuchsian group Γ uniformizing a once-punctured torus. The explicit description of coverings between corresponding quotient surfaces will be helpful to transfer the computation on the once-punctured torus to that on the commensurable four-times punctured sphere in Section 4. In this way, we can avoid using elliptic functions in actual computations. This method can also be used to obtain exact values of the Poincaré density of the once-punctured square torus [37]. Section 5 gives an enumeration of free homotopy classes of (non-peripheral, unoriented) simple closed curves on the topological once-punctured torus by using the notion of Farey neighbours. In Section 6 we describe

the natural "polar coordinates," called the bending coordinates, of the Teichmüller space of a once-punctured torus by following C. McMullen [26]. Our main theorem states that some sort of radial ray, called a rational pleating ray, can be described as a real locus of the entire function representing the trace of monodromy of a simple closed curve corresponding to the ray. Using results in the preceding sections, we can draw numerically all rational pleating rays in a recursive way. As a by-product, we can compute the value of the accessory parameter and a standard generator pair of a Fuchsian group uniformizing a given once-punctured torus as well. In the final section, we investigate the relation between the Schwarzian differential equation in the four-times punctured sphere and Heun's differential equation. We will observe that a simple change of indeterminants transforms the former one to the latter. Thus the monodromy can be computed also by solving Heun's differential equation instead of the Schwarzian differential equation.

2. Basic definitions

Let Γ be a Fuchsian group acting on the unit disk $\mathbb{D}$. In other words, Γ is a discrete subgroup (possibly with torsion elements) of the analytic automorphism group $\mathrm{Aut}(\mathbb{D})$ of $\mathbb{D}$. Note that $\mathrm{Aut}(\mathbb{D})$ is Möbius conjugate to $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}(2, \mathbb{R})$, where $\mathbb{H}$ denotes the upper half plane $\{z \in \mathbb{C}; \mathrm{Im}\, z > 0\}$. We denote by $Q(\mathbb{D}, \Gamma)$ the set of holomorphic quadratic differentials $\varphi(\zeta)d\zeta^2$ for Γ on the unit disk, i.e., φ are holomorphic in $\mathbb{D}$ and satisfy the functional equation $\varphi \circ \gamma \cdot (\gamma')^2 = \varphi$ for each $\gamma \in \Gamma$. (We often identify an element $\varphi(\zeta)d\zeta^2$ in $Q(\mathbb{D}, \Gamma)$ with the corresponding coefficient $\varphi(\zeta)$.)

A quadratic differential $\varphi(\zeta)d\zeta^2 \in Q(\mathbb{D}, \Gamma)$ is called *bounded* if the norm

$$\|\varphi\|_{\mathbb{D}} = \sup_{\zeta \in \mathbb{D}} (1 - |\zeta|^2)^2 |\varphi(\zeta)|$$

is finite. We denote by $B_2(\mathbb{D}, \Gamma)$ the complex Banach space consisting of all bounded holomorphic quadratic differentials for Γ on $\mathbb{D}$ with the norm defined above.

It is well known that $B_2(\mathbb{D}, \Gamma)$ is finite dimensional if and only if the Fuchsian group Γ is cofinite; in other words, finitely generated and of the first kind. If Γ is torsion-free and uniformizes a Riemann surface X of finite analytic type (g, n), where g is the genus of X and n is the number of punctures of X with $2g - 2 + n > 0$, then $\dim B_2(\mathbb{D}, \Gamma) = 3g - 3 + n$. In particular, $\dim B_2(\mathbb{D}, \Gamma) = 1$ if and only if the signature (g, n) is equal to $(1, 1)$ or $(0, 4)$. An element of $B_2(\mathbb{D}, \Gamma)$ is sometimes called a cusp form for a cofinite Γ. Each cusp form admits at most a simple pole at a puncture or a branch point of the quotient Riemann surface (or orbifold).

We now explain a well-known method enabling us to construct a locally univalent meromorphic function on $\mathbb{D}$ for which the Schwarzian derivative coincides with the given holomorphic quadratic differential. For a good account including the historical background, we refer the reader to [9, Chap. 10]. Let $\varphi \in Q(\mathbb{D}, \Gamma)$ be given.

Then consider the following linear homogeneous ordinary differential equation of the second order:

$$(2.1) \qquad\qquad 2\eta'' + \varphi\eta = 0 \quad \text{on} \quad \mathbb{D}.$$

Since $\mathbb{D}$ is simply connected, a unique solution η exists on $\mathbb{D}$ for the given initial data $\eta(0) = a$ and $\eta'(0) = b$. We denote by η_1 and η_2 the special solutions of (2.1) determined by $\eta_1(0) = 0, \eta_1'(0) = 1$ and $\eta_2(0) = 1, \eta_2'(0) = 0$. Then the Wronskian $\eta_1'\eta_2 - \eta_1\eta_2'$ is identically 1. We note that the ratio $f := \eta_1/\eta_2$ has Taylor expansion in the form

$$f(\zeta) = \zeta + a_3\zeta^3 + \cdots$$

near the origin. A simple calculation shows that the Schwarzian derivative $S_f = (f''/f')' - (f''/f')^2/2$ is equal to the given differential φ. Hereafter, we denote by f_φ the meromorphic function $f = \eta_1/\eta_2$ just described above and call it the (normalized) *developing map* of φ when we regard φ as a projective structure on the surface $\mathbb{D}/\Gamma$.

Here we briefly recall facts about the analytic automorphism group $\mathrm{Aut}(\mathbb{D})$ of the unit disk $\mathbb{D}$. Let $\mathrm{SU}(1,1)$ be the subgroup of $\mathrm{SL}(2,\mathbb{C})$ consisting of all matrices U in the form $\begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}$ with $|u|^2 - |v|^2 = 1$. Then the canonical projection $\mathrm{SU}(1,1) \to \mathrm{PSU}(1,1) = \mathrm{SU}(1,1)/\{\pm I\}$ gives the natural homomorphism $P : \mathrm{SU}(1,1) \to \mathrm{Aut}(\mathbb{D})$. In other words, $P(U)(z) = (uz + v)/(\bar{v}z + \bar{u})$. It is well known that $\mathrm{SU}(1,1)$ is conjugate with $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SL}(2,\mathbb{C})$.

We define the pullback $U^*_{-1/2}\eta$ of η under U by

$$U^*_{-1/2}\eta(z) = (\bar{v}z + \bar{u})\eta(P(U)(z)).$$

Here we remark that the function $\bar{v}z + \bar{u}$ is a global branch of $(P(U)')^{-1/2}$.

A direct computation shows that $U^*_{-1/2}\eta$ satisfies differential equation (2.1) with $P(U)^*_2\varphi = \varphi \circ P(U) \cdot (P(U)')^2$ instead of φ. In particular, when $P(U) \in \Gamma$, we see that $U^*_{-1/2}\eta_j$ is again a solution of (2.1) for $j = 1, 2$, and hence, we can write

$$(2.2) \qquad\qquad \begin{aligned} U^*_{-1/2}\eta_1 &= a\eta_1 + b\eta_2, \\ U^*_{-1/2}\eta_2 &= c\eta_1 + d\eta_2, \end{aligned}$$

where a, b, c and d are constants, because η_1 and η_2 form a basis of the vector space of solutions of (2.1). (The relations in (2.2) can be regarded as "additive laws" for the solutions.) We note that we can deduce $ad - bc = 1$ from the fact that the Wronskian of $U^*_{-1/2}\eta_1, U^*_{-1/2}\eta_2$ is identically 1.

Let $\widetilde{\Gamma}$ be the inverse image $P^{-1}(\Gamma)$ of Γ under the projection P. We now define the map $\tilde{\chi}_\varphi : \widetilde{\Gamma} \to \mathrm{SL}(2,\mathbb{C})$ by $\tilde{\chi}_\varphi(\tilde{\gamma}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are the above constants for $U = \tilde{\gamma} \in \widetilde{\Gamma}$. It is easy to see that $\tilde{\chi}_\varphi$ is actually a group homomorphism for each φ. By the holomorphic dependence of solutions on the coefficients, the correspondence $\varphi \mapsto \tilde{\chi}_\varphi(\tilde{\gamma})$ is a holomorphic map from $Q(\mathbb{D}, \Gamma)$ to $\mathrm{SL}(2,\mathbb{C})$, where the space $Q(\mathbb{D}, \Gamma)$ is endowed with the Fréchet space structure determined by the uniform convergence on compacta. Note that χ_0 is nothing but the identity. Furthermore, for a $\tilde{\gamma} \in \widetilde{\Gamma}$, we set

$$(2.3) \qquad\qquad \mathrm{Tr}_{\tilde{\gamma}}(\varphi) = \mathrm{tr}\, \tilde{\chi}_\varphi(\tilde{\gamma}) = a + d.$$

The trace function $\mathrm{Tr}_{\tilde{\gamma}}$ is also holomorphic in φ and plays an important role in the present investigation.

Taking the ratio of relations (2.2), we obtain

$$f_\varphi \circ \gamma = \chi_\varphi(\gamma) \circ f_\varphi$$

on $\mathbb{D}$, where $\gamma = P(\tilde{\gamma})$ and $\chi_\varphi(\gamma)$ is the Möbius transformation induced by $\tilde{\chi}_\varphi(\tilde{\gamma})$, i.e., $(az+b)/(cz+d)$. The element $\chi_\varphi(\gamma)$ is well defined because $(az+b)/(cz+d)$ is independent of the choice of $\tilde{\gamma} \in P^{-1}(\gamma) \subset \widetilde{\Gamma}$. Thus, the map $\chi_\varphi : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ is also a group homomorphism and will be called the *monodromy homomorphism* of Γ associated with $\varphi \in Q(\mathbb{D}, \Gamma)$.

By virtue of the following theorem, we see that by considering only the monodromy homomorphism we would lose nothing in the investigation of deformations of Γ.

Theorem A. *Let Γ be a cofinite Fuchsian group. For $\varphi, \psi \in B_2(\mathbb{D}, \Gamma)$, the coincidence $\chi_\varphi = \chi_\psi$ on Γ implies $\varphi = \psi$.*

This result traces back to Poincaré for the case when Γ uniformizes a compact Riemann surface. The generalization above is due to I. Kra [19].

The (Bers embedded) Teichmüller space of a Fuchsian group Γ will be denoted by $T(\Gamma)$ and defined as the set of those holomorphic quadratic differentials $\varphi \in Q(\mathbb{D}, \Gamma)$ for which the developing maps f_φ are univalent and admit quasiconformal extensions to $\widehat{\mathbb{C}}$ compatible with the action of Γ, where a quasiconformal extension F of f_φ is said to be compatible with Γ if for each $\gamma \in \Gamma$ there exists a Möbius transformation γ' such that $F \circ \gamma = \gamma' \circ F$ on $\widehat{\mathbb{C}}$.

Remark 2.1. For simplicity, we have adopted a slightly different definition of the Teichmüller space from the usual one. Our space $T(\Gamma)$ here means, in the standard definition, the Teichmüller space of the Fuchsian group acting on the *exterior* of the unit disk, or, more intuitively, the Teichmüller space of the mirror image of the surface $\mathbb{D}/\Gamma$. Therefore, the reader should refer with special care to another paper. For example, our standard generator pair θ_0, θ_∞ (see Section 3 below) should be replaced by $\theta_0{}^{-1}, \theta_\infty$.

When Γ is the trivial group 1, $T(1)$ is called the universal Teichmüller space. By definition, we see that $T(\Gamma_1) \subset T(\Gamma_2)$ if $\Gamma_1 \supset \Gamma_2$. In particular, the universal Teichmüller space contains all Teichmüller spaces of Fuchsian groups.

The Kraus-Nehari theorem says that $\|\varphi\|_\mathbb{D} \le 6$ if f_φ is univalent. Hence $T(\Gamma)$ is a bounded subset of $B_2(\mathbb{D}, \Gamma)$. We also note that the Ahlfors-Weill theorem implies $\{\varphi \in B_2(\mathbb{D}, \Gamma); \|\varphi\|_\mathbb{D} < 2\} \subset T(\Gamma)$. It is also well known that $T(\Gamma)$ is a connected open subset of $B_2(\mathbb{D}, \Gamma)$. For details, see [10]. By the existence of a conformally natural extension operator from quasisymmetric homeomorphisms of S^1 to quasiconformal ones of $\mathbb{D}$ (see Tukia [39] or Douady-Earle [4]), we can see that

$$(2.4) \qquad\qquad T(\Gamma) = T(1) \cap B_2(\mathbb{D}, \Gamma)$$

and that $T(\Gamma)$ is contractible.

More generally, we consider the set $K(\Gamma)$ of those quadratic differentials $\varphi \in B_2(\mathbb{D}, \Gamma)$ for which the monodromy images $\chi_\varphi(\Gamma)$ are discrete in $\mathrm{PSL}(2,\mathbb{C})$. The following result is due to H. Shiga [35].

Theorem B. *The Bers embedded Teichmüller space $T(\Gamma)$ coincides with the connected component of $\mathrm{Int}\, K(\Gamma)$ which contains the origin.*

Moreover, each element of $\operatorname{Int} K(\Gamma)$ is known to correspond to an isomorphism onto a quasi-Fuchsian group (Shiga and Tanigawa [36]). Thus, the trace function $\operatorname{Tr}_{\tilde{\gamma}}$ for a hyperbolic element $\tilde{\gamma} \in \widetilde{\Gamma}$ satisfies

$$\operatorname{Tr}_{\tilde{\gamma}}(\operatorname{Int} K(\Gamma)) \subset \mathbb{C} \setminus [-2, 2].$$

By virtue of Theorem B, we would see the shape of the Bers embedding if we could mark the grids contained in $K(\Gamma)$ for sufficiently fine mesh in the space $B_2(\mathbb{D}, \Gamma)$. This approach is out of our scope here, however, it will be a basis of our forthcoming paper [18] in the case when $\mathbb{D}/\Gamma$ is a once-punctured torus. It also has the advantage that one can draw a picture (up to translation) even if one does not know about the value of the accessory parameter (see Section 5). To do that, we need an algorithm determining the discreteness of a given Möbius group. Such an algorithm, however, is available only for the special case when the group is generated by two elements with parabolic commutator at present.

Our method here will rely on the internal geometry of the Teichmüller space, and hence, it might be applicable to even higher dimensional cases.

3. Commensurability relations

All one-dimensional Teichmüller spaces are mutually conformally equivalent because they are all simply connected. In particular, the Teichmüller space of a once-punctured torus is biholomorphic to that of a four-times punctured sphere. There is, however, an intrinsic relation between them as is described below.

Let Γ be a Fuchsian group uniformizing a once-punctured torus X. It is a basic fact that Γ has free generators θ_0 and θ_∞ with the properties: (i) both are hyperbolic; (ii) the commutator $[\theta_0, \theta_\infty]$ is parabolic; and (iii) the (signed) intersection number $\theta_0 \cdot \theta_\infty$ equals 1. We call such $(\theta_0, \theta_\infty)$ a *standard generator pair* of Γ.

If a standard generator pair is given, then there exists a unique complex number τ with $\operatorname{Im} \tau > 0$ satisfying the following property: The quotient surface $(\mathbb{C} \setminus \{m + n\tau; m, n \in \mathbb{Z}\})/L_\tau = (\mathbb{C} \setminus L_\tau)/L_\tau$, where L_τ is the lattice group generated by 1 and τ over $\mathbb{Z}$, is conformally equivalent to X in such a way that the curves $t + (1 + \tau)/2$ and $t\tau + (1 + \tau)/2$ in $\mathbb{C}$ parametrized by $t \in [0, 1]$ correspond to θ_0 and θ_∞, respectively.

We call τ the Teichmüller parameter of the (marked) Fuchsian group Γ with standard generator $(\theta_0, \theta_\infty)$. This parameter τ is known to give a global analytic coordinate of the Teichmüller space of Γ onto the upper half plane $\mathbb{H}$ (see [10]).

Now consider the four-times punctured torus Z given as the quotient $(\mathbb{C} \setminus \{(m + n\tau)/2; m, n \in \mathbb{Z}\})/L_\tau = (\mathbb{C} \setminus \frac{1}{2}L_\tau)/L_\tau$. Then the mapping $z \mapsto 2z$ induces a four-sheeted (unbranched) covering $Z \to X$. Note that the covering group of $Z \to X$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

On the other hand, the conformal involution $z \mapsto -z$ induces a 2-sheeted (unbranched) covering $Z \to Y$, where Y is the four-times punctured sphere described as $\widehat{\mathbb{C}} \setminus \{e_0, e_1, e_2, e_3\}$, where $e_0 = \wp(0) = \infty$, $e_1 = \wp(1/2)$, $e_2 = \wp(\tau/2)$, $e_3 = \wp((1 + \tau)/2)$, in terms of the Weierstrass $\wp$-function with period lattice L_τ. We note that $e_1 + e_2 + e_3 = 0$.

Since the covering $Z \to X$ is Galois, we obtain a natural homomorphism from Γ to $\operatorname{Aut}(Z)$. We denote by Γ_0 the kernel of this homomorphism. Then Γ_0 is a Fuchsian group uniformizing Z. By the above remark, we note that $\Gamma/\Gamma_0 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

The composition of the universal covering $\mathbb{D} \to \mathbb{D}/\Gamma_0 \cong Z$ and the above projection $Z \to Y$ is a universal covering of the four-times punctured sphere Y, whose covering group will be denoted by Γ'. Then Γ' is an extension of Γ_0 such that $\Gamma'/\Gamma_0 \cong \mathbb{Z}_2$.

Similarly, by the involution $z \mapsto -z$ of $\mathbb{C} \setminus \{m + n\tau; m, n \in \mathbb{Z}\}$, we obtain a 2-sheeted covering $X \to W$, where W is a Riemann orbifold of signature $(0; 2, 2, 2, \infty)$. In fact, W is realized as the complex plane with ramification of order 2 at the three points e_1, e_2 and e_3 via the covering map $[z] \mapsto \wp(z)$. This orbifold appears also as the quotient space of Y by the action of the group $G(e_0, e_1, e_2, e_3)$ described below.

Let a_0, a_1, a_2 and a_3 be distinct points in the Riemann sphere $\widehat{\mathbb{C}}$ and set $\Omega = \widehat{\mathbb{C}} \setminus \{a_0, a_1, a_2, a_3\}$. Unless these points are in a special position, we cannot permute these in an arbitrary way by a Möbius transformation. The domain Ω, however, always has a special kind of symmetry.

For example, there exists a unique Möbius transformation A_1 swapping a_0 and a_1 and swapping a_2 and a_3. In fact, we take the Möbius transformation A sending a_0, a_1, a_2 to a_1, a_0, a_3, respectively. Since A swaps a_0 and a_1, the transformation A must be an involution, i.e., $A^2 = \text{id}$. Therefore we conclude $A(a_3) = A^{-1}(a_3) = a_2$.

Similarly, we can take Möbius transformations A_2 and A_3 such that

$$A_2(a_0, a_1, a_2, a_3) = (a_2, a_3, a_0, a_1),$$
$$A_3(a_0, a_1, a_2, a_3) = (a_3, a_2, a_1, a_0).$$

We now have the group, denoted by $G(a_0, a_1, a_2, a_3)$, formed by the elements id, A_1, A_2 and A_3, which acts on Ω as analytic automorphisms and is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The quotient space $\Omega/G(a_0, a_1, a_2, a_3)$ is a Riemann orbifold of signature $(0; 2, 2, 2, \infty)$. We remark that the fixed points of A_1 can be expressed by

$$\frac{a_0 a_1 - a_2 a_3 \pm \sqrt{(a_0 - a_2)(a_1 - a_2)(a_0 - a_3)(a_1 - a_3)}}{a_0 + a_1 - a_2 - a_3}.$$

We now return to our case. The covering map $R : Y \to Y/G(e_0, e_1, e_2, e_3) = W$ can be expressed explicitly by

$$R(z) = \frac{z^4 - 2(e_1 e_2 + e_2 e_3 + e_3 e_1)z^2 + 8 e_1 e_2 e_3 z + (e_1 e_2 + e_2 e_3 + e_3 e_1)^2}{4(z - e_1)(z - e_2)(z - e_3)}$$
$$= \frac{z^4 + g_2 z^2/2 + 2 g_3 z + (g_2/4)^2}{4z^3 - g_2 z - g_3},$$

where g_2 and g_3 are well-known constants determined by the lattice L_τ. This rational function comes from the additive law in the theory of elliptic functions, i.e., $\wp(2z) = R(\wp(z))$, and was used to construct an example of complex dynamics such that the Julia set equals the whole sphere $\widehat{\mathbb{C}}$ by Lattès in 1918 (see §4.3 of [2]).

Let Γ_1 be the covering group of the universal cover $\mathbb{D} \to X \to W$ of W. We note that $\Gamma_0 \triangleleft \Gamma \triangleleft \Gamma_1$, $\Gamma_0 \triangleleft \Gamma' \triangleleft \Gamma_1$, $\Gamma_0 = \Gamma \cap \Gamma'$, and that $\Gamma_1 = \langle \Gamma, \Gamma' \rangle$.

In particular, $B_2(\mathbb{D}, \Gamma_1) \subset B_2(\mathbb{D}, \Gamma)$ and $B_2(\mathbb{D}, \Gamma_1) \subset B_2(\mathbb{D}, \Gamma')$, and these are all one-dimensional vector spaces; hence, all are equal. Now relation (2.4) leads to the following result.

Lemma 3.1. *The Bers embedded one-dimensional Teichmüller spaces of the commensurable Fuchsian groups Γ, Γ' and Γ_1 all coincide:*

$$T(\Gamma) = T(\Gamma_1) = T(\Gamma').$$

Therefore, to consider the Bers embedding of the Teichmüller space of a once-punctured torus is equivalent to also considering the corresponding four-times punctured sphere. The latter sometimes has the advantage of making things in the calculation simple, as we shall see in the next section.

4. THE MONODROMY ON A FOUR-TIMES PUNCTURED SPHERE

Let Γ be the Fuchsian group uniformizing a once-punctured torus and Γ' be the Fuchsian group commensurable with Γ uniformizing the corresponding four-times punctured sphere. Although the space $B_2(\mathbb{D}, \Gamma) = B_2(\mathbb{D}, \Gamma')$ is complex one-dimensional, it seems quite hard to calculate numerically the value of functions in this space. On the other hand, an element of $B_2(\mathbb{D}, \Gamma)$ can be regarded as a holomorphic quadratic differential on the quotient space $\mathbb{D}/\Gamma$, which is more tractable than its lift to $\mathbb{D}$. Indeed, R. M. Porter [32] and L. Keen [12] employed this idea and succeeded in computing the real rays of trace functions for several words of special type. In the authors' opinion, the appearance of elliptic functions in their methods has made the computation somewhat difficult.

So, the authors would like to propose an idea using quadratic differentials on the four-times punctured sphere $Y = \mathbb{D}/\Gamma'$ instead of those on the once-punctured torus $X = \mathbb{D}/\Gamma$. Our formulation will need only rational functions, so the involved computation will be much simpler than the other methods. In this section, we will provide a method of computing the monodromy homomorphism $\chi_\varphi : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ up to conjugation by $\mathrm{PSL}(2, \mathbb{C})$, where the conjugation does not depend on φ. In fact, our construction gives global holomorphic mappings μ_0, $\mu_\infty : B_2(\mathbb{D}, \Gamma) \to \mathrm{SL}(2, \mathbb{C})$ such that the homomorphism $\tilde{\chi}'_\varphi : \widetilde{\Gamma} \to \mathrm{SL}(2, \mathbb{C})$ determined by $\tilde{\chi}'_\varphi(\tilde{\theta}_0) = \mu_0(\varphi)$ and $\tilde{\chi}'_\varphi(\tilde{\theta}_\infty) = \mu_\infty(\varphi)$ induces the same homomorphism $\Gamma \to \mathrm{PSL}(2, \mathbb{C})$ as χ_φ up to $\mathrm{PSL}(2, \mathbb{C})$-conjugacy (Theorem 4.3). Here $\tilde{\theta}_r$ is a representative of θ_r in $\mathrm{SU}(1, 1)$ for $r = 0, \infty$. We remark that the argument developed below can also be applied to the case of n-times punctured spheres, $n > 4$.

Recall the relation $\mathbb{D}/\Gamma' \cong \mathbb{C} \setminus \{e_1, e_2, e_3\}$. For simplicity, by the map $z \mapsto (z - e_2)/(e_1 - e_2)$, we transform the domain to the other one $\mathbb{C} \setminus \{0, 1, \lambda\}$, denoted still by the same letter Y as before, where $\lambda = (e_3 - e_2)/(e_1 - e_2) = \lambda(\tau)$ is known to be a holomorphic universal covering of the domain $\mathbb{C} \setminus \{0, 1\}$ and usually called an elliptic modular function (cf. [1, Chapter 7]).

We denote by p the universal covering $\mathbb{D} \to Y \cong \mathbb{D}/\Gamma'$ constructed above and set $z_0 = p(0)$. Then we can associate the isomorphism, denoted by δ, from Γ' onto $\pi_1(Y, z_0)$ with p by assigning the homotopy class of the image $p([0, \gamma(0)])$ to a given $\gamma \in \Gamma'$.

Let $\rho_Y(z)|dz|$ denote the complete hyperbolic metric on Y of the constant negative curvature -4, i.e., $1/(1 - |\zeta|^2) = \rho_Y(p(\zeta))|p'(\zeta)|$. A holomorphic quadratic differential $\psi(z)dz^2$ on Y is said to be bounded if the norm

$$\|\psi\|_Y := \sup_{z \in Y} |\psi(z)|\rho_Y(z)^{-2}$$

is finite. We denote by $B_2(Y)$ the Banach space of bounded holomorphic quadratic differentials on Y. We remark that a holomorphic quadratic differential ψ is bounded on Y if and only if ψ has at most simple poles at the punctures of Y.

By definition, the spaces $B_2(\mathbb{D}, \Gamma')$ and $B_2(Y)$ are isometrically isomorphic via the pullback $p_2^* : B_2(Y) \to B_2(\mathbb{D}, \Gamma')$ defined by $p_2^* \psi = \psi \circ p \cdot (p')^2$. In particular, $B_2(Y)$ is one-dimensional, too.

On the other hand, the rational function

$$(4.1) \qquad \psi_0(z) = \frac{1}{z(z-1)(z-\lambda)}$$

belongs to $B_2(Y)$, thus, ψ_0 forms a basis of the vector space $B_2(Y)$.

Now we consider the developing map $f_\varphi : \mathbb{D} \to \widehat{\mathbb{C}}$, where φ is the pullback of a given $\psi \in B_2(Y)$ under the projection p. Then the (local) branch g of the function $f_\varphi \circ p^{-1}$ near the basepoint $z_0 = p(0)$ with $g(z_0) = 0$ satisfies the relation

$$(4.2) \qquad p_2^* \psi \left(= \varphi = S_{f_\varphi}\right) = p_2^*(S_g) + S_p.$$

The function g has the expansion

$$(4.3) \qquad g(z) = \frac{1}{p'(0)}(z - z_0) - \frac{p''(0)}{2p'(0)^3}(z - z_0)^2 + \cdots$$

around the point z_0. We should remember the fact that the first two coefficients do not depend on the particular choice of $\varphi = p_2^* \psi$. Here the relation $S_p = -p_2^*(S_{p^{-1}})$ holds and $S_{p^{-1}}$ depends only on the domain Y; in fact, this is independent of the choice of the branch and the covering map p because of the invariance property of the Schwarzian derivative: $S_{A \circ f} = S_f$ for any Möbius transformation A. This holomorphic quadratic differential $S_{p^{-1}}(z)dz^2$ will be written by $\nu_Y(z)dz^2$ in this article and called the *uniformizing connection* of Y (see [20]).

By the behaviour of the universal covering near the puncture, the quadratic differential $\nu_Y(z)dz^2$ is known to have a pole of the second order at every puncture of Y with residue $1/2$. By this observation, we have the following.

Lemma 4.1 ([8, Theorem 3.1], [20, (2.3.1)]). *The uniformizing connection $\nu_Y(z)dz^2$ of Y has the form*

$$\nu_Y(z) = \frac{1}{2z^2(z-1)^2} + \frac{1}{2(z-\lambda)^2} + \frac{c(\lambda)}{z(z-1)(z-\lambda)},$$

where $c(\lambda)$ is a constant determined by λ.

The constant $c(\lambda)$ was called the *accessory parameter* by Poincaré and is known to be difficult to compute in general. I. Kra [20] showed that $c(\lambda)$ is real analytic but not complex analytic in λ.

If a Möbius transformation A leaves Y invariant, we have $\nu_Y = A_2^* \nu_Y$, in particular, ν_Y is invariant under the action of the group $G(0, 1, \infty, \lambda)$. By this invariance, when Y has a good symmetry, the value $c(\lambda)$ can be computed explicitly. For example, we easily see that $c(\lambda)$ is real for a real λ and that $c(\lambda)$ is purely imaginary if $\operatorname{Re}\lambda = 1/2$. In particular, we see that $c(1/2) = 0$. The following examples can be found in [8].

Example 4.1. $c(-1) = 1$. Note that $\lambda(-1 + i) = -1$.

Example 4.2. $c((1 + \sqrt{3}i)/2) = -i/\sqrt{3}$. Note that $\lambda((1 + \sqrt{3}i)/2) = (1 + \sqrt{3}i)/2$.

For the moment, we proceed ahead as if we know the value of the accessory parameter $c(\lambda)$. We will return to this problem in the final part of Section 5.

Since $S_p = -p_2^*(\nu_Y)$, we can see from (4.2) that S_g is a globally defined quadratic differential in Y:

$$(4.4) \qquad\qquad S_g = \psi + \nu_Y.$$

Now we write ψ in the form $\psi = t\psi_0$, where t is a complex number and ψ_0 is given by (4.1), and consider the linear ordinary differential equation on Y of the following form:

$$(4.5) \qquad 2y'' + \left\{ \frac{1}{2z^2(z-1)^2} + \frac{1}{2(z-\lambda)^2} + \frac{t+c(\lambda)}{z(z-1)(z-\lambda)} \right\} y = 0.$$

We remark that I. Laine and T. Sorvali [23] investigated the general differential equation $y'' + C\,y = 0$ with coefficient C meromorphic in a simply connected domain and obtained a nice condition under which the (local) monodromy of the equation becomes trivial (see Corollary 4.2 therein).

Introducing the auxiliary unknown function x, we can transform this equation to the system of linear ordinary differential equations of the first order:

$$(4.6) \qquad\qquad \mathbf{y}' = \mathbf{y}A, \qquad A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2}(\psi + \nu_Y) & 0 \end{pmatrix},$$

where $\mathbf{y} = \begin{pmatrix} x & y \end{pmatrix}$.

For a given point $z_1 \in Y$, let $\mathbf{y}_1 = \begin{pmatrix} x_1 & y_1 \end{pmatrix}$ and $\mathbf{y}_2 = \begin{pmatrix} x_2 & y_2 \end{pmatrix}$ be the (local) solutions of (4.6) near the point z_1 satisfying $\mathbf{y}_1(z_1) = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $\mathbf{y}_2(z_1) = \begin{pmatrix} 0 & 1 \end{pmatrix}$. Then the matrix

$$F_{z_1} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} y_1' & y_1 \\ y_2' & y_2 \end{pmatrix}$$

will be called the fundamental matrix of differential equation (4.6) at z_1. Note that $\det F_{z_1} \equiv 1$. In particular, the matrix F_{z_1} is a holomorphic map from a neighbourhood of z_1 in Y to the complex Lie group $\mathrm{SL}(2, \mathbb{C})$ with $F_{z_1}(z_1) = I$, where I denotes the identity matrix. The matrix F_{z_1} itself can be regarded as an $\mathrm{SL}(2, \mathbb{C})$-valued (local) solution of the differential equation (4.6). An arbitrary (local) solution y of (4.5) near the point z_1 with the initial conditions $y'(z_1) = a$ and $y(z_1) = b$ can be expressed by

$$\begin{pmatrix} y' & y \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} F_{z_1},$$

and vice versa.

The holomorphic map F_{z_1} can be analytically continued along any path α from z_1 to z_2 in Y. The resulting germ at z_2 depends only on the homotopy class $[\alpha]$ of α in Y, and hence, will be denoted by $H_{[\alpha]}$. Since the germ $H_{[\alpha]}$ is an $\mathrm{SL}(2, \mathbb{C})$-valued local solution of (4.6) near the point z_2, there exists a unique constant matrix $B \in \mathrm{SL}(2, \mathbb{C})$ such that $H_{[\alpha]} = BF_{z_2}$. The matrix B is called the transition matrix along $[\alpha]$ and will be denoted by $L[\alpha]$ or, more precisely, $L_t[\alpha]$. In what follows, we will denote by $Y[z_1, z_2]$ the set of homotopy classes of paths connecting z_1 and z_2 in Y.

By definition, we can see the following fundamental properties of the transition matrix:

(i) $H_{[\alpha]} = L[\alpha]F_{z_2}$ for $[\alpha] \in Y[z_1, z_2]$,
(ii) $L[\alpha] = H_{[\alpha]}(z_2)$ for $[\alpha] \in Y[z_1, z_2]$, and
(iii) $L[\alpha]L[\beta] = L[\alpha \cdot \beta]$ for $[\alpha] \in Y[z_1, z_2]$ and $[\beta] \in Y[z_2, z_3]$.

Property (ii) means that the transition matrix $L[\alpha]$ can be computed by (numerically) solving differential equation (4.6) along the path α. (In practical computations, a polygonal line should be taken as α.)

We set $l_t[\alpha] = L_t[\alpha]$ for $[\alpha] \in \pi_1(Y, z_0) = Y[z_0, z_0]$. Then, by property (iii), the map $l_t : \pi_1(Y, z_0) \to \mathrm{SL}(2, \mathbb{C})$ is a group homomorphism and will be called the monodromy homomorphism associated with $\psi = t\psi_0$.

Now we consider the ratio y_1/y_2, which will be denoted by h_{z_1}, of y_1 and y_2 appearing in the fundamental matrix F_{z_1} at z_1. In other words, $h_{z_1} = \Pi \circ F_{z_1}$, where $\Pi : \mathrm{SL}(2, \mathbb{C}) \to \mathbb{C}\mathrm{P}^1 = \widehat{\mathbb{C}}$ is defined by $\Pi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b/d$. Then h_{z_1} satisfies the Schwarzian differential equation (4.4) and has expansion in the form $h(z) = (z - z_1) + b_3(z - z_1)^3 + \cdots$ near the point z_1. In particular, when we take z_0 as the reference point, comparing the above expansion with (4.3), we obtain

$$(4.7) \qquad g = k \circ h, \quad \text{where} \quad k(z) = \frac{p'(0)^2 z}{p'(0) - p''(0)z/2} \quad \text{and} \quad h = h_{z_0}.$$

Note that the Möbius transformation k depends only on the universal covering p of Y and that the function h can be analytically continued to the function $\bar{l}_t[\alpha] \circ h$ along a curve α, where $\bar{l}_t[\alpha]$ is the Möbius transformation induced by $l_t[\alpha]$.

Let us look at the relation between $\chi_{t\varphi_0}$ and l_t for $t \in \mathbb{C}$, where $\varphi_0 = p_2^*(\psi_0) \in B_2(\mathbb{D}, \Gamma')$. First note that the monodromy homomorphism $\chi_{t\varphi_0}$ is defined over the Fuchsian group $\Gamma_1 = \langle \Gamma, \Gamma' \rangle$ because $B_2(\mathbb{D}, \Gamma') = B_2(\mathbb{D}, \Gamma_1)$. We also recall that $\delta : \Gamma' \to \pi_1(Y, z_0)$ is the isomorphism mapping γ to $p([0, \gamma(0)])$. Now we are ready to state the following theorem.

Theorem 4.2. *The monodromies $\chi_{t\varphi_0}$ and $\bar{l}_t$ are essentially the same. More precisely,*

$$(4.8) \qquad\qquad\qquad \chi_{t\varphi_0}(\gamma) = k \circ \bar{l}_t(\delta(\gamma)) \circ k^{-1}$$

for each $t \in \mathbb{C}$ and $\gamma \in \Gamma'$, where $\varphi_0 = p_2^*\psi_0$ and k is the Möbius transformation given in (4.7).

Proof. Set $g = k \circ h$, where $h = h_{z_0}$. Then, as was explained, $f = g \circ p$ near the origin. By the analytic continuation along the image curve $\alpha = \delta(\gamma)$ of the oriented hyperbolic geodesic segment joining 0 and $\gamma(0)$ in the unit disk $\mathbb{D}$, the function h changes to $\bar{l}_t(\alpha) \circ h$. Hence, analytically continuing both sides of $f = k \circ h \circ p$ along this segment, we see $f = A \circ k \circ h \circ p = A \circ g \circ p$ near the point $\gamma(0)$, here we write $A = k \circ \bar{l}_t(\delta(\gamma)) \circ k^{-1}$ for short. In particular, $f \circ \gamma = A \circ g \circ p \circ \gamma = A \circ g \circ p = A \circ f$ near the origin. This implies $\chi_\varphi(\gamma) = A$. $\qquad\qquad\square$

Remark 4.1. The monodromy homomorphism $\chi_\varphi : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ has no natural homomorphic lifts $\Gamma \to \mathrm{SL}(2, \mathbb{C})$ unless we specify a homomorphic section $s : \Gamma \to \mathrm{SU}(1, 1) \subset \mathrm{SL}(2, \mathbb{C})$ of Γ. In contrast, $l_t : \pi_1(Y, z_0) \to \mathrm{SL}(2, \mathbb{C})$ is intrinsically homomorphic. Particularly, the composed map $l_0 \circ \delta : \Gamma' \to \mathrm{SU}(1, 1)$ gives a homomorphic lift of the Fuchsian group Γ'. As we can see from the standard local theory of Fuchsian differential equations around regular singular points, $\mathrm{tr}\,(l_t[\alpha]) = -2$ holds for every simple loop $[\alpha]$ going around a puncture of Y. By this constraint, the lift of Γ' is uniquely determined (note that Γ' is a free group generated by three elements corresponding to punctures). Moreover, Y. Okumura [30] observes under this constraint that $\mathrm{tr}\,(l_0[\alpha]) \leq -2$ holds for each dividing simple loop α in Y (even in more general context). This implies that the homomorphism $l_0 \circ \delta$ has no

homomorphic extensions to $\Gamma_1 = \langle \Gamma, \Gamma' \rangle$. In fact, if such an extension, say $\chi : \Gamma_1 \to$ SL$(2, \mathbb{R})$, exists, then it must satisfy $\mathrm{tr}\,(\chi(\gamma^2)) = \mathrm{tr}\,(\chi(\gamma)^2) = \mathrm{tr}\,(\chi(\gamma))^2 - 2 \geq 2$ for each element $\gamma \in \Gamma$. If we choose γ to correspond to a non-peripheral simple closed curve in X, then $\gamma^2 \in \Gamma'$ corresponds to a dividing loop α in Y, and thus $\mathrm{tr}\,(\chi(\gamma^2)) = \mathrm{tr}\,(l_0[\alpha]) < -2$, which is a contradiction.

As is well known, for determination of a homomorphism $\chi : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$, it is enough to know the values of $\mathrm{tr}\,^2(\chi(\gamma))$ for $\gamma = \theta_0, \theta_\infty$ and $\theta_1 = \theta_0 \theta_\infty$, where (θ_0, θ_1) is a standard generator pair of Γ. Using Theorem 4.2, we can compute $\mathrm{tr}\,^2(\chi_\varphi(\theta_r))$ for $r = 0, 1, \infty$. In fact, since $\delta(\theta_r^2)$ is a dividing loop in Y, we see $\mathrm{tr}\,^2(\chi_\varphi(\theta_r)) = \pm \mathrm{tr}\,(\chi_\varphi(\theta_r^2)) + 2 = -\mathrm{tr}\,(l_t(\delta(\theta_r^2))) + 2$ by the above remark. However, sometimes it is more convenient to have the representation χ_φ itself rather than just the traces. In order to directly compute $\chi_\varphi(\gamma)$ for each $\gamma \in \Gamma$ (up to Möbius conjugation) we use the symmetry of the domain Y which was explained in the previous section.

Given a once-punctured torus $X = \mathbb{D}/\Gamma$ represented by the Teichmüller parameter τ, it suffices to compute the monodromy images of the standard generator pair $(\theta_0, \theta_\infty)$ of Γ. The fact that $\varphi = p_2^* \psi \in B_2(\mathbb{D}, \Gamma_1)$ for each $\psi \in B_2(Y)$ implies that $\psi(z)dz^2$ is invariant under the action of $\Gamma_1/\Gamma' \cong G(0, 1, \infty, \lambda) < \mathrm{Aut}(Y)$, where $\lambda = \lambda(\tau)$. Since $\nu_Y(z)dz^2$ is also invariant, $A_{-1/2}^* y$ becomes a local solution of (4.5) for each local solution y of (4.5) and for $A \in G(0, 1, \infty, \lambda)$ by the same reasoning given in Section 2, where $A_{-1/2}^* y$ is defined by $y \circ A \cdot (A')^{-1/2}$ and $(A')^{-1/2}(z) = rz + s$ for a representative $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ of A. We set for $F_{A(z_1)} = \begin{pmatrix} y_1' & y_1 \\ y_2' & y_2 \end{pmatrix}$,

$$(4.9) \qquad A^*(F_{A(z_1)}) := \begin{pmatrix} (A_{-1/2}^* y_1)' & A_{-1/2}^* y_1 \\ (A_{-1/2}^* y_2)' & A_{-1/2}^* y_2 \end{pmatrix} = M_{z_1}(A) F_{z_1},$$

where the matrix $M_{z_1}(A)$ is defined by

$$M_{z_1}(A) = \begin{pmatrix} 1/(rz_1 + s) & 0 \\ r & rz_1 + s \end{pmatrix}.$$

We consider the simple closed curves $\hat{\beta}_0$ and $\hat{\beta}_\infty$ in Z induced by the curves $s + (1 + \tau)/4$ and $s\tau + (1 + \tau)/4$ parametrized by $s \in [0, 1]$ in $\mathbb{C} \setminus \frac{1}{2}L_\tau$ and we take the reference point z_0 in Y so that the intersection point of $\hat{\beta}_0$ and $\hat{\beta}_\infty$ projects to z_0 under the projection $Z \to Y$. Then the image curves β_0 and β_∞ in Y represent the homotopy classes $\delta(\theta_0^2)$ and $\delta(\theta_\infty^2)$, respectively.

Note that β_0 separates 0 and λ from 1 and ∞. Hence we see that $A_0(z_0) = p(\theta_0(0))$, where $A_0(z) = (z - \lambda)/(z - 1) \in G(0, 1, \infty, \lambda)$ is determined by $A_0(0, \lambda, 1, \infty) = (\lambda, 0, \infty, 1)$. Similarly, noting that β_∞ separates 0 and ∞ from 1 and λ, we have $A_\infty(z_0) = p(\theta_\infty(0))$, where $A_\infty(z) = \lambda/z \in G(0, 1, \infty, \lambda)$ is determined by $A_\infty(0, \infty, 1, \lambda) = (\infty, 0, \lambda, 1)$. For $r = 0, \infty$, let α_r be an oriented curve in Y joining z_0 and $A_r(z_0)$ which is homotopic to $p_*([0, \theta_r(0)])$; in other words, homotopic to the curve coming from the one $s/2 + (1 + \tau)/4$ for $r = 0$ or $s\tau/2 + (1 + \tau)/4$ for $r = \infty$ in Z parametrized by $s \in [0, 1]$.

Now we are in a position to state one of our main results.

Theorem 4.3. *For suitable representatives $\tilde{\theta}_0$ of θ_0 and $\tilde{\theta}_\infty$ of θ_∞ in* SU$(1, 1)$,

$$\tilde{\chi}_{t\varphi_0}(\tilde{\theta}_0) = L_t[\alpha_0] M_{z_0}(A_0), \quad and \quad \tilde{\chi}_{t\varphi_0}(\tilde{\theta}_\infty) = L_t[\alpha_\infty] M_{z_0}(A_\infty)$$

hold for all $t \in \mathbb{C}$ up to $\mathrm{SL}(2, \mathbb{C})$-*conjugacy independent of* t, *where* $\varphi_0 = p_2^*(\psi_0)$ *and* ψ_0 *is given by* (4.1).

Remark 4.2. If the point z_0 and the paths α_0 and α_∞ are replaced by other $z_1 \in Y$, $\alpha_0' \in Y[z_1, A_0(z_1)]$ and $\alpha_\infty' \in Y[z_1, A_\infty(z_1)]$, the above result remains true as far as $\alpha_0' \cdot (A_0)_*(\alpha_0')$ and $\alpha_\infty' \cdot (A_\infty)_*(\alpha_\infty')$ are freely homotopic to β_0 and β_∞ in Y, respectively.

Proof. We set $\varphi = t\varphi_0$ for a fixed $t \in \mathbb{C}$ and let $f = f_\varphi : \mathbb{D} \to \widehat{\mathbb{C}}$ be the normalized developing map for φ. Then $f = k \circ h_{z_0} \circ p$ near the origin of $\mathbb{D}$, where k is the Möbius transformation given by (4.7). Let r represent 0 or ∞. Analytically continuing both sides along the segment $[0, \theta_r(0)]$ in $\mathbb{D}$, we obtain

$$f = k \circ L_t[\alpha_r]^- \circ h_{z_0'} \circ p$$

near the point $\theta_r(0)$, where we set $z_0' = A_r(z_0)$ and denote by M^- the Möbius transformation induced by a matrix $M \in \mathrm{GL}(2, \mathbb{C})$. Noting $h_{z_0'} \circ A_r = M_{z_0}(A_r)^- \circ h_{z_0}$ by (4.9) and $p \circ \theta_r = A_r \circ p$, we have

$$f \circ \theta_r = k \circ L_t[\alpha_r]^- \circ h_{z_0'} \circ p \circ \theta_r = k \circ L_t[\alpha_r]^- \circ h_{z_0'} \circ A_r \circ p$$
$$= k \circ L_t[\alpha_r]^- \circ M_{z_0}(A_r)^- \circ h_{z_0} \circ p = k \circ L_t[\alpha_r]^- \circ M_{z_0}(A_r)^- \circ k^{-1} \circ f$$

near the origin. This implies $\chi_\varphi(\theta_r) = k \circ L_t[\alpha_r]^- \circ M_{z_0}(A_r)^- \circ k^{-1}$. Hence, $\tilde{\chi}_\varphi(\tilde{\theta}_r) = \pm \tilde{k} L_t[\alpha_r] M_{z_0}(A_r) \tilde{k}^{-1}$, where $\tilde{k}$ is a matrix representing k. Since the quantities $\tilde{\chi}_{t\varphi_0}(\tilde{\theta}_r)$ and $L_t[\alpha_r] M_{z_0}(A_r)$ are globally holomorphic in t, we easily see that the above sign is constant. (Note also that the conjugacy map k does not depend on t.) Therefore, we obtain the required equalities by choosing a suitable sign of $\tilde{\theta}_r$ for each r. $\square$

Remark 4.3. The following relation might be useful: *For a point* z_1 *in* Y *and* $A \in G(0, 1, \infty, \lambda)$, *we set* $z_2 = A(z_1)$. *Then we have* $L[\alpha] M_{z_1} = M_{z_2} L[A_*(\alpha)]$ *for any path* α *connecting* z_1 *and* z_2 *in* Y.

We note also that we can use the orbifold W instead of Y in order to compute $\chi_\varphi(\theta_r)$ directly. The present method, however, has the merit that we can "save" the polygonal path used in solving the differential equation numerically. If we try to solve the differential equation along a closed polygonal path, then we would need more segments.

5. Enumeration of simple closed curves

In the preceding section, we have presented a method of computing the monodromy for a given projective structure and $\gamma \in \Gamma$. We explain here how to compute the values of the trace function corresponding to an arbitrary simple closed geodesic in a recursive and algebraic way using the particular values $\mathrm{Tr}_{\tilde{\theta}_0}, \mathrm{Tr}_{\tilde{\theta}_\infty}$ and $\mathrm{Tr}_{\tilde{\theta}_1}$, where $\tilde{\theta}_r$ is a fixed representative of θ_r in $\mathrm{SU}(1, 1)$ for each r. The materials here are, more or less, known although rigorous proofs for those are sometimes difficult to locate. The reader is referred to, e.g., [15], [27] and [40] for further discussions. A good introduction to the Teichmüller space of a torus can also be found in [10].

Since we need only the topological structure of a once-punctured torus for a while, we treat first the topological once-punctured torus described as $\Sigma = (\mathbb{R}^2 \setminus \mathbb{Z}^2)/\mathbb{Z}^2$. We denote by $\langle z \rangle$ the projection of a point $z \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ to Σ. Let $[\alpha]$ and $[\beta]$ be the homotopy classes in Σ with basepoint $x = \langle (1/2, 1/2) \rangle$ which are represented by

the curves $\alpha(t) = x + \langle(t,0)\rangle$ and $\beta(t) = x + \langle(0,t)\rangle$, $0 \le t \le 1$, respectively. Let A and B be the homology classes corresponding to $[\alpha]$ and $[\beta]$. It is well known that the fundamental group $\pi_1(\Sigma, x)$ is a free group generated by $[\alpha]$ and $[\beta]$ and that the homology group $H_1(\Sigma)$ is a free abelian group generated by A and B. Taking A and B as a basis, we may regard $H_1(\Sigma)$ as the lattice group $\mathbb{Z} \oplus \mathbb{Z}$.

The fundamental group and the homology group coincide for the unpunctured torus $\overline{\Sigma} = \mathbb{R}^2/\mathbb{Z}^2$ and both are isomorphic to $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$. In particular, the homomorphism $\pi_1(\Sigma, x) \to \pi_1(\overline{\Sigma}, x)$ induced by the natural inclusion $\Sigma \to \overline{\Sigma}$ is canonically isomorphic to the abelianization $\pi_1(\Sigma, x) \to H_1(\Sigma) = H_1(\overline{\Sigma})$.

A simple closed curve γ in Σ is said to be *non-peripheral* if γ can be continuously shrunken to neither a point nor a puncture in Σ. A non-peripheral simple closed curve γ corresponds to a homology class of the form $pA + qB$, where p and q are relatively prime integers; that is, $\{mp + nq; m, n \in \mathbb{Z}\} = \mathbb{Z}$. We can easily understand this as follows; cutting Σ along γ, we have a once-punctured annulus, which implies that there is another non-peripheral simple closed curve γ' such that homology classes of γ and γ' generate $H_1(\Sigma)$, hence the homology class of γ must be of the above form.

The ratio p/q will be called the *slope* of γ. Note that the slope does not depend on the orientation of the curve.

Remark 5.1. The reader should be careful with the term "slope" when referring to another paper devoted to the subject similar to ours because some authors prefer to call $-q/p$ the slope of γ. The reason is explained by the fact that the pinching deformation of a once-punctured torus along the curve γ corresponds to letting the Teichmüller parameter $\tau \in \mathbb{H}$ tend to the point $-q/p \in \partial\mathbb{H}$ (see [17] for details). In this article, however, we adopt $r = p/q$ as the slope so that r represents the inclination of the vector (p, q).

As might be seen from the linear action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{R}^2$ leaving $\mathbb{Z}^2$ invariant, the mapping class group of Σ is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$. This is a classical result. However, this could be seen more easily with a stronger assertion about the relation between simple closed curves and their slope.

Let us note here that the mapping class group $\mathcal{M}(\Sigma)$ of Σ is defined as $\mathrm{Homeo}^+(\Sigma)/\mathrm{Homeo}_0(\Sigma)$, where $\mathrm{Homeo}^+(\Sigma)$ is the group of orientation-preserving self-homeomorphisms of Σ and $\mathrm{Homeo}_0(\Sigma)$ is the subgroup consisting of all homeomorphisms which are homotopic to the identity in Σ. We also remark that the mapping class group of Σ is isomorphic to the quotient of the automorphism group of $\pi_1(\Sigma, x)$ over the inner automorphism group of $\pi_1(\Sigma, x)$.

Proposition 5.1. *The mapping class group of Σ is canonically isomorphic to* $\mathrm{SL}(2, \mathbb{Z})$. *For each rational $r \in \widehat{\mathbb{Q}}$ there exists a non-peripheral simple closed curve in Σ with slope r. If two non-peripheral simple closed curves in Σ have the same slope, they are freely homotopic in Σ up to orientation.*

The latter assertion can be found in [15].

Proof. Let S be the homeomorphism of Σ defined by $S(\langle(s,t)\rangle) = \langle(s,t)\rangle$ for $0 \le t \le 1/2$ and by $S(\langle(s,t)\rangle) = \langle(s + 2t - 1, t)\rangle$ for $1/2 \le t \le 1$, which realizes a Dehn twist around the curve α. Also let T be the rotation $\langle(s,t)\rangle \mapsto \langle(t, -s)\rangle$ by $\pi/2$. Then the induced homomorphisms S_* and T_* on $\pi_1(\Sigma, x)$ satisfy $S_*(\alpha, \beta) = (\alpha, \alpha\beta)$

and $T_*(\alpha, \beta) = (\beta, \alpha^{-1})$, and hence, they act on $H_1(\Sigma)$ as matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, respectively, under the above identification with $\mathbb{Z} \oplus \mathbb{Z}$. It is well known that the above two matrices generate $\mathrm{SL}(2, \mathbb{Z})$, and thus, the natural homomorphism $\mathcal{M}(\Sigma) \to \mathrm{Aut}(H_1(\Sigma)) \cong \mathrm{SL}(2, \mathbb{Z})$ is surjective.

A rational $r \in \widehat{\mathbb{Q}}$ can be written in the form $r = p/q$, where p and q are relatively prime integers. Then there exist integers p' and q' such that $pq' - p'q = 1$. We now take an element f_* in $\mathcal{M}(\Sigma)$ which acts on $H_1(\Sigma)$ as the matrix $\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$. Then $f_*(\alpha)$ has slope r as desired.

We next show the uniqueness part. By means of the action of the mapping class group, it is enough to show that a simple closed curve α^* in $\pi_1(\Sigma, x)$ with slope 0 is necessarily homotopic to α in Σ. Since $\Sigma \setminus \alpha^*$ is a punctured annulus, we can connect both boundary components by a simple arc with endpoints at x. That arc can be regarded as a simple closed curve, say β^*, in Σ so that α^* and β^* form standard generators of $\pi_1(\Sigma, x)$. Furthermore, by repeated application of the Dehn twist along α^*, we can choose β^* so that it is homotopic to β in $\overline{\Sigma}$.

If we take an orientation-preserving self-homeomorphism f of Σ fixing the point x such that $f_*(\alpha) = \alpha^*$ and $f_*(\beta) = \beta^*$, then f_* acts on $\pi_1(\overline{\Sigma}, x)$ trivially. Therefore, f is homotopic to the identity in the unpunctured torus $\overline{\Sigma}$. Let f_t be an isotopy in $\overline{\Sigma}$ connecting the identity and f with $f_0 = \mathrm{id}$ and with $f_1 = f$. Since $\overline{\Sigma} = \mathbb{R}^2/\mathbb{Z}^2$ is an abelian group, $\hat{f}_t = f_t - f_t(0)$ is an isotopy in $\overline{\Sigma}$ fixing the origin $0 = \langle (0,0) \rangle$. Hence $\hat{f}_t|_\Sigma$ is an isotopy in Σ connecting the identity and f, which implies that α^* is freely homotopic to α.

Finally, we show the first assertion in the proposition. It suffices to prove the injectivity of the natural homomorphism $\mathcal{M}(\Sigma) \to \mathrm{Aut}(H_1(\Sigma)) = \mathrm{SL}(2, \mathbb{Z})$. Let f_* be in the kernel of the above homomorphism. We may assume that f fixes x. Then $f_*(\alpha)$ and $f_*(\beta)$ have slope 0 and ∞, respectively. The last argument yields also that f is homotopic to the identity in Σ. Hence f_* is trivial on $\pi_1(\Sigma, x)$. □

We can extract from the proposition the following corollary, which will be utilized effectively below.

Corollary 5.2. *Let (γ_1, γ_2) be a standard generator pair of the fundamental group $\pi_1(\Sigma, x)$ of the topological once-punctured torus Σ. Then $(\gamma_1, \gamma_1\gamma_2)$, $(\gamma_1\gamma_2, \gamma_2)$ and $(\gamma_2, \gamma_1^{-1})$ are all standard generator pairs of $\pi_1(\Sigma, x)$.*

We next recall fundamental facts about Farey triangles (cf. [27]). The reader also finds a good account for Farey sequences in [7] as well as an interesting historical remark.

For three points a, b, c in $\widehat{\mathbb{R}}$, we denote by $\Delta(a, b, c)$ the hyperbolic triangle formed by three hyperbolic geodesics in the upper half plane $\mathbb{H}$ connecting two of the three points a, b and c. Let $\Delta = \Delta(0, 1, \infty)$. Then $\mathbb{H}$ is tessellated by Δ and its conjugates by the modular group $\mathrm{PSL}(2, \mathbb{Z})$. Note that the stabilizer of Δ in $\mathrm{PSL}(2, \mathbb{Z})$ consists of three elements and permutes the vertices of Δ cyclically. Each triangle which is conjugate with Δ by the action of $\mathrm{PSL}(2, \mathbb{Z})$ is called a *Farey triangle*. The initial Farey triangle Δ and its reflection $\Delta' = \Delta(0, -1, \infty)$ in the imaginary axis form a fundamental domain of the modular group $G_2 = \{\pm C \in \mathrm{PSL}(2, \mathbb{Z}); C \equiv I \mod 2\}$ of level 2. We will say that both Δ and Δ' are of level 0. A Farey triangle which

shares a side with that of level 0 will be called of level 1 unless it is of level 0. Similarly, a Farey triangle which shares a side with that of level n will be called of level $n+1$ unless it is of level $\leq n$. It is important to note that the corresponding graph to the above tessellation is a tree, namely, there is no closed circuit.

It is well known that the orbit of 0 under the action of $\mathrm{PSL}(2,\mathbb{Z})$ coincides with $\widehat{\mathbb{Q}}$. We denote by $\tilde{\mathcal{F}}(n)$ the set of rationals which appear as vertices of Farey triangles of level $\leq n$. Set $\mathcal{F}(n) = \tilde{\mathcal{F}}(n) \setminus \tilde{\mathcal{F}}(n-1)$ for $n = 0, 1, \ldots$. For instance, $\mathcal{F}(0) = \{-1, 0, 1, \infty\}$, $\mathcal{F}(1) = \{-2, -1/2, 1/2, 2\}$, $\mathcal{F}(2) = \{-3, -3/2, -2/3, -1/3, 1/3, 2/3, 3/2, 3\}$ and so on. We note that $\#\mathcal{F}(n) = 2^{n+1}$ for $n \geq 1$. An element r in $\mathcal{F}(n)$ will be called of level n and designated by $\mathrm{level}(r) = n$. Two rationals, r_1 and r_2, in $\widehat{\mathbb{Q}}$ are said to be *Farey neighbours* if r_1 and r_2 are distinct vertices of a common Farey triangle. For convenience, we adopt the notation $r_1 \sim r_2$ to mean that r_1 and r_2 are Farey neighbours. It should be remembered that this is not an equivalence relation. Note that if $r_1 \sim r_2$, then $|\mathrm{level}(r_1) - \mathrm{level}(r_2)| \geq 1$ unless $\mathrm{level}(r_1) = \mathrm{level}(r_2) = 0$.

For two rationals $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$ in $\widehat{\mathbb{Q}}$, where p_j and q_j are relatively prime integers, we assign the number $D(r_1, r_2) = |p_1 q_2 - p_2 q_1|$. Note that this is independent of the particular choice of signs of p_j and q_j. The following statement will be useful in our argument below.

Lemma 5.3. *Two rationals r_1 and r_2 are Farey neighbours if and only if $D(r_1, r_2) = 1$.*

Proof. First assume that $D(r_1, r_2) = 1$. Then r_1 and r_2 can be written in the form $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$ so that $p_1 q_2 - p_2 q_1 = 1$. The Möbius transformation $g(z) = (p_1 z + p_2)/(q_1 z + q_2)$ sends ∞ to r_1 and 0 to r_2, which implies that the Farey triangle $g(\Delta)$ contains r_1 and r_2 as vertices. The converse can be seen similarly. □

Let r_1 and r_2 be Farey neighbours with $0 \leq r_1 < r_2 \leq \infty$. Then they are represented in the form $r_j = p_j/q_j$, where p_j and q_j are non-negative integers for $j = 1, 2$ and satisfy $p_2 q_1 - p_1 q_2 = 1$. We define $r_1 \oplus r_2 = r_2 \oplus r_1 = (p_1 + p_2)/(q_1 + q_2)$. Similarly, we define $r_1 \ominus r_2 = r_2 \ominus r_1 = (p_1 - p_2)/(q_1 - q_2)$. For these operations, one can show the following result.

Lemma 5.4. *Let r_1 and r_2 be Farey neighbours with $0 \leq r_1 < r_2 \leq \infty$. Then $\Delta(r_1, r_2, r_1 \oplus r_2)$ and $\Delta(r_1, r_2, r_1 \ominus r_2)$ are Farey triangles sharing the side connecting r_1 with r_2 and the inequality $r_2 < r_1 \oplus r_2 < r_1$ holds.*

Conversely, any rational r with $0 < r < \infty$ can uniquely be decomposed into the form $r = r_1 \oplus r_2$ for rationals $0 \leq r_1 < r_2 \leq \infty$ with $r_1 \sim r_2$. Furthermore, if $r \neq 1$, then $\mathrm{level}(r) = \max\{\mathrm{level}(r_1), \mathrm{level}(r_2)\} + 1$ and $\mathrm{level}(r_1) \neq \mathrm{level}(r_2)$.

Proof. It is immediate to see that $r_1 \oplus r_2 \sim r_j$ and $r_1 \ominus r_2 \sim r_j$ for $j = 1, 2$ and that $r_1 < r_1 \oplus r_2 < r_2$. Hence, the first half of the assertion has been shown. To show the second half, assume that $0 < r < \infty$ is a rational and let $n = \mathrm{level}(r)$. Then there is a sequence of Farey triangles $\Delta_0 = \Delta, \Delta_1, \ldots, \Delta_n$ such that Δ_{k-1} and Δ_k have a common side for $k = 1, \ldots, n$ and that Δ_n has r as its vertex. Set $\Delta_{-1} = \Delta'$ for convenience. Note that Δ_k is of level k and is contained in the right half of the upper half plane $\mathbb{H}$ for $k = 0, 1, \ldots, n$. Let r_1 and r_2 be the other two vertices of Δ_n. Since the side of Δ_n connecting r_1 with r_2 separates r from the left half of $\mathbb{H}$, r lies between r_1 and r_2, and hence, the relation $r = r_1 \oplus r_2$ follows. Assume now that $r \neq 1$. Then $n > 0$ and $\Delta_{n-1} = \Delta(r_1, r_2, r_1 \ominus r_2)$. In particular, $\max\{\mathrm{level}(r_1), \mathrm{level}(r_2)\} \leq n-1$. On the other hand, Δ_{n-1} shares a side

joining $r_1 \ominus r_2$ and, say, r_2 with Δ_{n-2}. We see then that $\text{level}(r_1) = n - 1$ and that $\text{level}(r_2) \leq n - 2$. $\qquad \square$

For each rational r with $0 < r < \infty$, $r \neq 1$, we define r^* by $r^* = r_1 \ominus r_2$, where r_1 and r_2 are as above. Then $r^* < \min\{r_1, r_2\}$ or $r^* > \max\{r_1, r_2\}$. From the above proof, one can also see that $\max\{\text{level}(r^*), \text{level}(r_2)\} = n - 2$ if $\text{level}(r_1) = n - 1$. Moreover, under the assumption that $r_1 < r_2$, the inequality $r_2 < r^*$ holds if and only if $\text{level}(r_1) = n - 1$.

Finally, we extend the above operations for all pairs (r_1, r_2) of Farey neighbours. When $0 \leq -r_1, -r_2 \leq \infty$, we set $r_1 \oplus r_2 = -(-r_1) \oplus (-r_2)$ and $r_1 \ominus r_2 = -(-r_1) \ominus (-r_2)$ unless $\{r_1, r_2\} = \{0, \infty\}$. Also let $r^* = -(-r)^*$ for a rational r with $0 < -r < \infty$ and $r \neq -1$. For $r = \pm 1$, we set $r^* = \mp 1$.

We now return to our case. Let X be a marked once-punctured torus with Teichmüller parameter τ. Let $H : \mathbb{R}^2 \to \mathbb{R}^2 = \mathbb{C}$ be the $\mathbb{R}$-linear map sending $(1, 0)$ to 1 and $(0, 1)$ to τ. Then H induces a homeomorphism h from the topological once-punctured torus Σ onto $X = (\mathbb{C} \setminus L_\tau)/L_\tau$. The terminology above is translated, via $h : \Sigma \to X$, to that for X with standard generator pair $(\theta_0, \theta_\infty)$ of the Fuchsian group Γ uniformizing X. For instance, a non-peripheral simple closed curve γ in X has slope r if and only if $h_*^{-1}(\gamma)$ has slope r. By Proposition 5.1, the conjugacy class of an element θ_r in Γ representing a non-peripheral simple closed curve of slope r is uniquely determined up to inversion. In particular, the squared trace function $\text{Tr}_{\theta_r}^2 : B_2(\mathbb{D}, \Gamma) \to \mathbb{C}$ is determined by its slope. We now explain how to compute the trace functions $\text{Tr}_{\tilde{\theta}_r}$ for *all* $r \in \widehat{\mathbb{Q}}$ and suitable representatives $\tilde{\theta}_r$ of θ_r in $SU(1, 1)$ in a recursive way by using only the values for $r = 0, 1$ and ∞.

We begin by setting $\sigma_r = \text{Tr}_{\tilde{\theta}_r}$ for $r = 0, 1, \infty$, where $\tilde{\theta}_0$ and $\tilde{\theta}_\infty$ are arbitrarily fixed representatives of θ_0 and θ_∞, respectively, and $\tilde{\theta}_1 = \tilde{\theta}_0 \tilde{\theta}_\infty$. Define σ_{-1} by the formula $\sigma_{-1} = \sigma_0 \sigma_\infty - \sigma_1$. If we have defined the functions $\sigma_r : B_2(\mathbb{D}, \Gamma) \to \mathbb{C}$ for all $r \in \tilde{\mathcal{F}}(n)$, we then define σ_r for $r \in \mathcal{F}(n+1)$ by the formula

$$(5.1) \qquad\qquad \sigma_r = \sigma_{r_1} \sigma_{r_2} - \sigma_{r^*},$$

where r_1 and r_2 are Farey neighbours satisfying $r = r_1 \oplus r_2$. Note here that r_1, r_2 and r^* are all in $\tilde{\mathcal{F}}(n)$. In this way, we can define the functions σ_r for all $r \in \widehat{\mathbb{Q}}$.

Proposition 5.5. *The mapping $\sigma_r : B_2(\mathbb{D}, \Gamma) \to \mathbb{C}$ gives the trace function for a non-peripheral simple closed curve of slope r in X for each $r \in \widehat{\mathbb{Q}}$.*

Proof. In order to prove the proposition, we construct elements $\tilde{\theta}_r$ in $\widetilde{\Gamma} = P^{-1}(\Gamma)$ representing non-peripheral simple closed curves of slope r in a concrete way. Note that a similar construction can be found in [15]. We start with $\tilde{\theta}_0, \tilde{\theta}_\infty$ and $\tilde{\theta}_1 = \tilde{\theta}_0 \tilde{\theta}_\infty$, which represents a simple closed curve of slope 1. By Corollary 5.2, (θ_0, θ_1) and $(\theta_1, \theta_\infty)$ are standard generator pairs of Γ, where $\theta_1 = P(\tilde{\theta}_1)$. Suppose that we have defined $\tilde{\theta}_r \in \widetilde{\Gamma}$ for all $r \in \tilde{\mathcal{F}}_+(n) := \tilde{\mathcal{F}}(n) \cap [0, \infty]$ in such a way that the following properties are satisfied:

(1) θ_r represents a simple closed curve of slope r in X for $r \in \tilde{\mathcal{F}}_+(n)$, where $\theta_r = P(\tilde{\theta}_r)$, and

(2) $(\theta_{r_1}, \theta_{r_2})$ is a standard generator pair for each pair of Farey neighbours $r_1, r_2 \in \tilde{\mathcal{F}}_+(n)$ with $0 \leq r_1 < r_2 \leq \infty$.

Then, for $r \in \mathcal{F}(n+1)$ with $0 < r < \infty$, we define $\tilde{\theta}_r$ by

$$\tilde{\theta}_r = \tilde{\theta}_{r_1} \tilde{\theta}_{r_2},$$

where r_1 and r_2 are unique Farey neighbours with $0 \le r_1 < r_2 \le \infty$ satisfying $r = r_1 \oplus r_2$ (see Lemma 5.4). Note also, by Corollary 5.2, that (θ_{r_1}, θ_r) and (θ_r, θ_{r_2}) are standard generator pairs of Γ, where $\theta_r = P(\tilde{\theta}_r)$. In particular, θ_r represents a simple closed curve with slope r. In this way, we extend $\tilde{\theta}_r$ for all $r \in \tilde{\mathcal{F}}_+(n+1)$, while preserving the above properties (1) and (2). By induction, we extend $\tilde{\theta}_r$ for all $r \in \widehat{\mathbb{Q}} \cap [0, \infty]$.

For $r \in \widehat{\mathbb{Q}} \cap (-\infty, 0)$, we can apply the same argument as above by replacing $(\theta_0, \theta_\infty)$ by $(\theta_\infty, \theta_0^{-1})$. More precisely, we set $\tilde{\theta}_0' = \tilde{\theta}_\infty, \tilde{\theta}_\infty' = \tilde{\theta}_0^{-1}$ and $\tilde{\theta}_1' = \tilde{\theta}_0' \tilde{\theta}_\infty' = \tilde{\theta}_\infty \tilde{\theta}_0^{-1}$. Then $\tilde{\theta}_1'$ represents a simple closed curve of slope -1 in X. Also, in the same way as above, we let

$$\tilde{\theta}_r' = \tilde{\theta}_{r_1}' \tilde{\theta}_{r_2}'$$

for $r \in \mathcal{F}(n+1)$ with $0 < r < \infty$ after defining $\tilde{\theta}_s'$ for all $s \in \tilde{\mathcal{F}}_+(n)$. Finally, we set $\tilde{\theta}_{-r} = \tilde{\theta}_r'$ for $r \in \widehat{\mathbb{Q}} \cap (-\infty, 0)$. Then we can easily see that $\theta_{-r} = P(\tilde{\theta}_{-r})$ represents a simple closed curve of slope $-r$ in X.

We are now at the final stage to show the assertion. The key is the general formula

$$\operatorname{tr} AB + \operatorname{tr} A^{-1}B = \operatorname{tr} A \operatorname{tr} B$$

for $A, B \in \operatorname{SL}(2, \mathbb{C})$. Let $\hat{\sigma}_r = \operatorname{Tr}_{\tilde{\theta}_r}$ for $r \in \widehat{\mathbb{Q}}$. Then $\hat{\sigma}_r = \sigma_r$ for all $r \in \tilde{\mathcal{F}}(0)$. We show the same statement for all $r \in \tilde{\mathcal{F}}(n)$ for each n by induction. Suppose that we have shown this up to n. Let $r \in \mathcal{F}(n+1)$ and r_1 and r_2 be unique Farey neighbours with $r_1 < r_2$ satisfying $r = r_1 \oplus r_2$. Since $r_1, r_2, r^* = r_1 \ominus r_2 \in \tilde{\mathcal{F}}(n)$, we see that $\hat{\sigma}_s = \sigma_s$ for $s = r_1, r_2, r^*$. As we observed above, $r_2 = r_1 \oplus r^*$ if $r_2 < r^*$. In this case, $\tilde{\theta}_{r^*} = \tilde{\theta}_{r_1}^{-1} \tilde{\theta}_{r_2}$, and hence,

$$\hat{\sigma}_r(\varphi) = \operatorname{tr} \tilde{\chi}_\varphi(\tilde{\theta}_{r_1} \tilde{\theta}_{r_2}) = \operatorname{tr} \tilde{\chi}_\varphi(\tilde{\theta}_{r_1}) \operatorname{tr} \tilde{\chi}_\varphi(\tilde{\theta}_{r_2}) - \operatorname{tr} \tilde{\chi}_\varphi(\tilde{\theta}_{r_1}^{-1} \tilde{\theta}_{r_2})$$
$$= \sigma_{r_1}(\varphi) \sigma_{r_2}(\varphi) - \sigma_{r^*}(\varphi) = \sigma_r(\varphi).$$

When $r^* < r_1$, the relation $r_1 = r^* \oplus r_2$ holds, and hence, $\tilde{\theta}_{r^*} = \tilde{\theta}_{r_1} \tilde{\theta}_{r_2}^{-1}$. Since $\operatorname{tr} A^{-1}B = \operatorname{tr} AB^{-1}$, we can see the same relation $\hat{\sigma}_r = \sigma_r$ as above. We now conclude that $\operatorname{Tr}_{\tilde{\theta}_r} = \sigma_r$ for all $r \in \widehat{\mathbb{Q}}$ by induction. $\qquad \square$

6. McMullen's bending coordinates of the Bers slice

Following McMullen [26], we describe the bending coordinates of the Teichmüller space $T(\Gamma)$ of a once-punctured torus $X = \mathbb{D}/\Gamma$. For precise definitions for the terminology below, see [26] or references cited there.

Considering the slope, we have identified the space of simple closed geodesics on X with $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. Extending this continuously, we obtain the homeomorphic identification of the projectivized space $\mathbb{P}\mathcal{ML}(X)$ of measured laminations on X with the extended real line $\widehat{\mathbb{R}} \cong S^1$.

Let $\varphi \in T(X) \setminus \{0\}$ be a point in the Bers embedding. The group $\chi_\varphi(\Gamma)$ is then quasi-Fuchsian and its region of discontinuity consists of two quasidisks $f_\varphi(\mathbb{D})$ and $\widehat{\mathbb{C}} \setminus \overline{f_\varphi(\mathbb{D})}$, which are denoted by Ω_φ^+ and Ω_φ^-, respectively. We remark that $\chi_\varphi(\Gamma)$ is never Fuchsian for $\varphi \in T(\Gamma) \setminus \{0\}$. Since f_φ is conformal on $\mathbb{D}$, $\Omega_\varphi^+/\chi_\varphi(\Gamma)$

is always conformally equivalent to X, whereas $X_\varphi^* := \Omega_\varphi^- / \chi_\varphi(\Gamma)$ varies. We denote by $\mathcal{C}_\varphi$ the hyperbolic convex hull of the limit set $\Lambda_\varphi \subset \widehat{\mathbb{C}}$ of $\chi_\varphi(\Gamma)$ in the hyperbolic three-space $\mathbb{H}^3$, where we identify $\widehat{\mathbb{C}}$ with the sphere at infinity of $\mathbb{H}^3$. Then the boundary $\partial\mathcal{C}_\varphi$ in $\mathbb{H}^3$ consists of two connected components $\partial\mathcal{C}_\varphi^\pm$ facing to $\Omega_\varphi^\pm$. We write M_φ for the hyperbolic three manifold $\mathbb{H}^3/\chi_\varphi(\Gamma)$. Then $\mathcal{C}_\varphi/\chi_\varphi(\Gamma)$ is the convex core of M_φ, whose boundary consists of two connected components $\partial\mathcal{C}_\varphi^\pm/\chi_\varphi(\Gamma)$. The nearest point retraction from $\Omega_\varphi^\pm$ to $\partial\mathcal{C}_\varphi^\pm$ induces the homotopy equivalence between $\Omega_\varphi^\pm/\chi_\varphi(\Gamma)$ and $\partial\mathcal{C}_\varphi^\pm/\chi_\varphi(\Gamma)$, and hence, $\partial\mathcal{C}_\varphi^\pm/\chi_\varphi(\Gamma)$ are both topological punctured tori (see for example [5]). The surfaces $\partial\mathcal{C}_\varphi^\pm/\chi_\varphi(\Gamma)$ are endowed with the hyperbolic metric by M_φ and both are pleated surfaces with the natural bending measures on their pleating loci. We denote by $[\mathrm{pl}^\pm(\varphi)]$ the projective class in $\mathbb{PML}(X)$ of the pleating locus of $\mathcal{C}_\varphi^\pm/\chi_\varphi(\Gamma)$. Note that each connected component of $\partial\mathcal{C}_\varphi^\pm$ minus its pleating locus is contained in a hyperbolic plane in $\mathbb{H}^3$. We write $\ell_{\mathrm{pl}^-(\varphi)}(M_\varphi)$ and $\ell_{\mathrm{pl}^-(\varphi)}(X)$ for the hyperbolic length of the bending lamination $\mathrm{pl}^-(\varphi)$ on $\partial\mathcal{C}_\varphi^-/\chi_\varphi(\Gamma)$ and on X, respectively. Now we can state a result of McMullen [26] on the bending coordinates of $T(\Gamma)$.

Theorem C ([26, Theorem 1.5])**.** *The mapping*

$$\Phi : \varphi \mapsto \left([\mathrm{pl}^-(\varphi)], \frac{\ell_{\mathrm{pl}^-(\varphi)}(M_\varphi)}{\ell_{\mathrm{pl}^-(\varphi)}(X)} \right)$$

gives a homeomorphism from $T(\Gamma) \setminus \{0\}$ onto $\mathbb{PML}(X) \times (0,1) = \widehat{\mathbb{R}} \times (0,1)$, where we identify $\mathbb{PML}(X)$ with $\widehat{\mathbb{R}} \cong \mathrm{S}^1$ as above.

For $r \in \widehat{\mathbb{R}}$, we set $\mathcal{P}_r = \Phi^{-1}(\{r\} \times (0,1))$. These proper arcs in $T(\Gamma) \setminus \{0\}$ will be called *pleating rays*. In the special case where for $r \in \widehat{\mathbb{Q}}$, $\mathcal{P}_r$ is called a *rational pleating ray of slope r*. We remark that $\ell_{\mathrm{pl}^-(\varphi)}(X)$ is constant on $\mathcal{P}_r$. In particular, if $r \in \widehat{\mathbb{Q}}$, we have $\ell_{\mathrm{pl}^-(\varphi)}(X) = \mathrm{arccosh}(|\mathrm{tr}\,\theta_r|/2)$ for $\varphi \in \mathcal{P}_r$.

The next result guarantees the computability of the rational pleating rays up to the accessory parameter. For $r \in \widehat{\mathbb{Q}}$, put

$$\mathcal{H}_r := \{\varphi \in B_2(\mathbb{D}, \Gamma) \; ; \; \sigma_r(\varphi) \text{ is real and } \sigma_r(\varphi)^2 > 4\} = \sigma_r^{-1}((-\infty, -2) \cup (2, +\infty))$$

and call it the *hyperbolic locus* of slope r. Note that $0 \in \mathcal{H}_r$ for all $r \in \widehat{\mathbb{Q}}$.

Theorem 6.1. *The rational pleating ray $\mathcal{P}_r$, $r \in \widehat{\mathbb{Q}}$, is the unique connected component of $\mathcal{H}_r \setminus \{0\}$ with endpoint 0 on which $\sigma_r(\varphi)^2$ is strictly smaller than $\sigma_r(0)^2$.*

Besides Theorem C, we need the local pleating theorem due to Keen and Series [14, Theorem 8.1] for the proof of our theorem. We state it here in a specialized form for our aim.

Theorem D. *Let $\varphi_0 \in \mathcal{P}_r \cup \{0\}$ for some $r \in \widehat{\mathbb{Q}}$. Then there exists a neighbourhood U of φ_0 in $T(\Gamma)$ such that every element $\varphi \in (\mathcal{H}_r \setminus \{0\}) \cap U$ satisfies either $[\mathrm{pl}^-(\varphi)] = r$ or $[\mathrm{pl}^+(\varphi)] = r$.*

Proof of Theorem 6.1. Let $r \in \widehat{\mathbb{Q}}$. We may assume that $\sigma_r(0) > 2$. Take a point $\varphi \in \mathcal{P}_r$, then the axis of $\chi_\varphi(\theta_r)$ in $\mathbb{H}^3$ lies on the boundary $\partial\mathcal{C}_\varphi^-$ which is invariant under $\chi_\varphi(\theta_r)$. Therefore $\chi_\varphi(\theta_r)$ does not rotate $\mathbb{H}^3$ around the axis; namely, it is hyperbolic (see [14, Proposition 6.4] for details) which implies that $\mathcal{P}_r$ is contained in $\mathcal{H}_r \setminus \{0\}$.
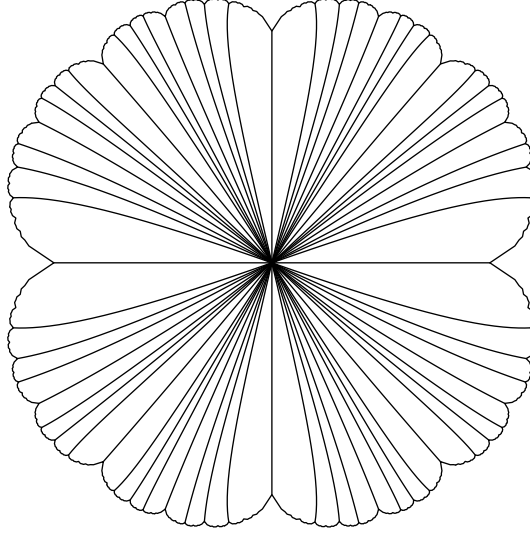
FIGURE 1. Bers embedding of the once-punctured square torus with pleating rays of level ≤ 4

Recall the inequalities due to McMullen [26, Corollary 3.5]

(6.1) $\ell_{\mathrm{pl}^-(\varphi)}(X_\varphi^*) < \ell_{\mathrm{pl}^-(\varphi)}(M_\varphi) < \ell_{\mathrm{pl}^-(\varphi)}(X)$ and

(6.2) $\ell_{\mathrm{pl}^+(\varphi)}(X) < \ell_{\mathrm{pl}^+(\varphi)}(M_\varphi) < \ell_{\mathrm{pl}^+(\varphi)}(X_\varphi^*)$

for every $\varphi \in T(\Gamma) \setminus \{0\}$. Since $\ell_{\mathrm{pl}^-(\varphi)}(M_\varphi) = \operatorname{arccosh}(\sigma_r(\varphi)/2)$ for $\varphi \in \mathcal{P}_r$, we see that $\sigma_r(\varphi) \to 2$ as φ approaches the Bers boundary along $\mathcal{P}_r$ by Theorem C. This means that the endpoint of $\mathcal{P}_r$ corresponds to the cusp pinched along the simple closed geodesic of slope r. In particular, $\mathcal{P}_r$ is closed in $\mathcal{H}_r \setminus \{0\}$.

In order to show that $\mathcal{P}_r$ is a connected component of $\mathcal{H}_r \setminus \{0\}$ ending at the basepoint 0 on which $\sigma_r < \sigma_r(0)$, it suffices to see that $\mathcal{P}_r$ is open in $\mathcal{H}_r \setminus \{0\}$. Suppose that $\varphi_0 \in \mathcal{P}_r$. Note from (6.1) that $2 < \sigma_r(\varphi_0) < \sigma_r(0)$. Therefore, $|\sigma_r| < \sigma_r(0)$ in a sufficiently small neighbourhood U of φ_0 in $T(\Gamma) \setminus \{0\}$. By Theorem D, we can further choose U so that every $\varphi \in \mathcal{H}_r \cap U$ satisfies $[\mathrm{pl}^-(\varphi)] = r$ or $[\mathrm{pl}^+(\varphi)] = r$. If the latter case occurred, (6.2) would yield $\sigma_r(0) < |\sigma_r(\varphi)|$. However, this contradicts the choice of U. Therefore, only the first case occurs, namely, $\mathcal{H}_r \cap U \subset \mathcal{P}_r$. Thus the openness follows. The uniqueness immediately follows from Theorems C and D.   □

We emphasize that the set described in the above theorem can be computed, at least in a numerical way, by tracing the real locus of the function σ_r in the decreasing direction from the origin. Figures 1 and 2 were created in this way. The above theorem shows also that the entire function σ_r has no branch point on the pleating ray $\mathcal{P}_r$ and at the origin. Furthermore, H. Miyachi [28] recently proved that σ_r has no branch point at the cusp corresponding to θ_r. Summarizing the above, we obtain the next result.

Corollary 6.2. *Each rational pleating ray $\mathcal{P}_r$ in $T(\Gamma)$ with its endpoints is a regular analytic simple closed arc and $\sigma_r(\varphi)^2$ is positive real and decreases to 4 when φ*
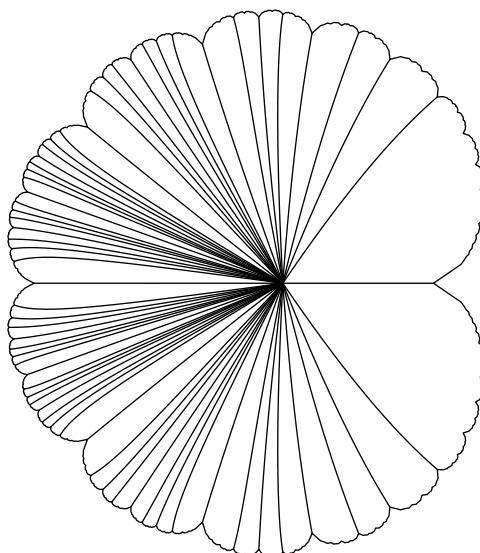
FIGURE 2. Bers embedding of a once-punctured "long" rectangular torus ($\lambda = 0.001$)

moves along the ray toward the cusp on the Bers boundary of $T(\Gamma)$ corresponding to slope r.

We take a closer look at the analytic structure of pleating rays for the later use.

Theorem 6.3. *Any two rational pleating rays intersect transversally at the basepoint unless the tangent vectors have opposite directions there.*

The above assertion seems true even when we allow all pleating rays. We, however, content ourselves here with the above form since we will require only the case of rational pleating rays in this paper. See the final part of this section explaining how to numerically compute the value of the accessory parameter and a standard pair of generators of a Fuchsian group.

We should comment on the similar result for the Maskit embedding of the Teichmüller space of once-punctured tori (see [15], [27] and [40]). The Maskit embedding is a deformation space of terminal regular b-groups of type $(1, 1)$ and it is realized as an unbounded domain in $\mathbb{H}$, where ∞ plays the same role as the basepoint in the Bers embedding even though no groups correspond to ∞. Rational pleating rays can also be considered for the Maskit embedding; among other things, Keen and Series [15, Proposition 3.2] showed that for a point $\mu = s + it$ of the rational pleating ray of slope p/q, the real part s tends to $2p/q$ as $t \to \infty$. This result implies that the asymptotic behaviour of rational pleating rays at the basepoint (∞ for the Maskit embedding) characterizes them.

To prove Theorem 6.3, we need a preliminary result connecting the real analytic structure of the Teichmüller space with the complex analytic one, which might be of independent interest.

Let γ be a hyperbolic element in an arbitrarily given Fuchsian group Γ. We denote by $\ell_\gamma : T(\Gamma) \to \mathbb{R}_+$ the length function on $T(\Gamma)$ corresponding to γ; namely,

for $\varphi \in T(\Gamma)$, $\ell_\gamma(\varphi)$ is the hyperbolic length of the closed geodesic corresponding to γ in the Riemann surface $\Omega_\varphi^-/\chi_\varphi(\Gamma)$, where $\Omega_\varphi^- = \widehat{\mathbb{C}} \setminus \overline{f_\varphi(\mathbb{D})}$. Let $\varepsilon_\gamma(\varphi)$ be the logarithm of the multiplier of $\chi_\varphi(\gamma)$ with positive real part. Therefore, $\sigma_\gamma^2(\varphi) = \operatorname{tr}^2 \chi_\varphi(\gamma) = 2\cosh(\varepsilon_\gamma(\varphi)) + 2$ and $\operatorname{Re}\varepsilon_\gamma(\varphi) > 0$. If we knew the conformal mapping g_φ from $\mathbb{D}$ onto Ω_φ^-, we could relate ℓ_γ to ε_γ by the formula

$$\operatorname{tr}^2\left[g_\varphi^{-1} \circ \chi_\varphi(\gamma) \circ g_\varphi \right] = 2\cosh\ell_\gamma(\varphi) + 2.$$

It is, however, difficult to find g_φ, in general. We can, at least, say something regarding its origin.

Lemma 6.4. *For a fixed $\varphi \in B_2(\mathbb{D}, \Gamma)$, the formula*

$$\left. \frac{d}{dt}\ell_\gamma(t\varphi) \right|_{t=0} = \operatorname{Re} \left. \frac{d}{dt}\varepsilon_\gamma(t\varphi) \right|_{t=0}$$

holds, where the differentiation is taken with respect to the real parameter t.

Proof. We refer the reader to [10] and [29] as standard textbooks for the basic facts used herein. In this proof, we take advantage of the lower half plane $\mathbb{H}^*$ instead of the unit disk $\mathbb{D}$, so that we regard Γ as a Fuchsian group acting on $\mathbb{H}$. This procedure is harmlessly done, as usual, through the Cayley transform $z \mapsto -i(1+z)/(1-z)$.

Let μ be a Beltrami differential for Γ in $\mathbb{H}$ with $\|\mu\|_\infty < 1$. We extend μ to be 0 in $\mathbb{H}^*$. On the other hand, let μ^* denote the Beltrami coefficient for Γ defined by

$$\mu^*(z) = \begin{cases} \mu(z), & z \in \mathbb{H}, \\ \overline{\mu(\bar{z})}, & z \in \mathbb{H}^*. \end{cases}$$

We denote by F^μ and F_μ the quasiconformal self-homeomorphism of $\mathbb{C}$ with Beltrami coefficients μ and μ^*, respectively, normalized so as to fix 0 and 1. Note that F_μ maps $\mathbb{H}$ onto itself while F^μ maps $\mathbb{H}^*$ conformally onto the quasidisk $F^\mu(\mathbb{H}^*)$. (Note also that $F^\mu \circ F_\mu^{-1}$ is a conformal mapping from $\mathbb{H}$ onto $F^\mu(\mathbb{H})$.) Therefore, $F_\mu \Gamma F_\mu^{-1}$ is a Fuchsian group while $F^\mu \Gamma (F^\mu)^{-1}$ is a Kleinian group acting on $F^\mu(\mathbb{H}^*)$. It is known that the mapping $\Phi : \mu \mapsto S_{F^\mu|_{\mathbb{H}^*}}$ is a holomorphic submersion from the open unit ball of the Banach space of Beltrami differentials for Γ onto $T(\Gamma)$.

For a given φ, we take a Beltrami differential μ in such a way that the Fréchet derivative of Φ at the origin sends μ to φ.

Let $\gamma \in \Gamma$ be a hyperbolic element. Conjugating by a suitable Möbius transformation, we may assume that γ is expressed in the form $\gamma(z) = e^{\varepsilon_0} z$, where $\varepsilon_0 > 0$ is a constant. Set $\gamma^t = F^{t\mu} \circ \gamma \circ (F^{t\mu})^{-1}$ and $\gamma_t = F_{t\mu} \circ \gamma \circ F_{t\mu}^{-1}$ for $t \in \mathbb{R}$ small enough. Then, by definition, these Möbius transformations can be written as

$$\gamma^t(z) = e^{\varepsilon(t)} z \quad \text{and} \quad \gamma_t(z) = e^{\varepsilon^*(t)} z.$$

We now recall Gardiner's variation formula for the multiplier of a hyperbolic Möbius transformation under quasiconformal mappings, [6]. As Miyachi remarked in [28, §8.1], the formula is valid not only for Fuchsian groups but also for Kleinian groups.

Gardiner's formula reads as

$$\frac{d}{dt}\varepsilon(t)\Big|_{t=0} = \iint_{A_+} \frac{\mu(z)}{z^2}dxdy, \quad \text{and}$$

$$\frac{d}{dt}\varepsilon^*(t)\Big|_{t=0} = \iint_A \frac{\mu^*(z)}{z^2}dxdy = \iint_{A_+}\left(\frac{\mu(z)}{z^2} + \overline{\frac{\mu(z)}{\bar{z}^2}}\right)dxdy,$$

where $A = \{1 < |z| < e^\varepsilon\}$ and $A_+ = A \cap \mathbb{H}$. Hence, we obtain $(\varepsilon^*)'(0) = 2\text{Re}\,\varepsilon'(0)$.

From the definitions, the basic relations $2\ell_\gamma(\Phi(t\mu)) = \varepsilon^*(t)$ and $\varepsilon_\gamma(\Phi(t\mu)) = \varepsilon(t)$ follow. Noting that $\Phi(t\mu) = t\varphi + O(|t|^2)$ as $t \to 0$, we get the desired relation between ℓ_γ and ε_γ. $\qquad\square$

Proof of Theorem 6.3. The argument and notation herein will be based on a paper of McMullen, [26]. Let $r \in \widehat{\mathbb{Q}}$. Since $\mathcal{P}_r$ is a real locus of an analytic function defined in $\mathbb{C}$, the unit direction vector of $\mathcal{P}_r$ at the origin,

$$\Theta_r = \lim_{\varphi \to 0 \text{ in } \mathcal{P}_r} \|\varphi\|_\mathbb{D}^{-1}\varphi,$$

is well defined. We show that the mapping $r \mapsto \Theta_r$ is injective on $\widehat{\mathbb{Q}}$.

First we observe that Θ_r is parallel to the tangent vector $\frac{d}{dt}\text{gr}_{t\gamma}(X^*)$ at $t = 0$, where $\text{gr}_{t\gamma}(X^*)$ is the point in $T(\Gamma)$ corresponding to the time t grafting of the mirror image $X^* = X_0^*$ of X along the simple closed geodesic representing a hyperbolic element $\gamma \in \Gamma$ of slope r. Let u be the real part of the holomorphic function $\varepsilon_\gamma : T(\Gamma) \to \mathbb{C}$ given above. By the relation $\sigma_\gamma^2(\varphi) = 2\cosh(\varepsilon_\gamma(\varphi)) + 2$, the vector Θ_r has the opposite direction to the gradient ∇u at the origin because u is harmonic. On the other hand, McMullen's theorem [26, Theorem 3.8] states the formula

$$\frac{d}{dt}\text{gr}_{t\gamma}(X^*) = -\nabla\ell_\gamma(X^*),$$

where the gradient is taken with respect to the Weil-Petersson metric. In our case, the Teichmüller space $T(\Gamma)$ is one-dimensional, and therefore, the Weil-Petersson (Kähler) metric is conformally equivalent to the Euclidean metric. Now, the formula $\nabla\ell_\gamma = \nabla u$ deduced from Lemma 6.4 connects these facts to conclude the above claim.

To show the injectivity, we next use the fact that grafting and twisting along γ are related by the relation $\frac{d}{dt}\text{gr}_{t\gamma}(X^*) = i \cdot \frac{d}{dt}\text{tw}_{t\gamma}(X^*)$ at the origin with a suitable orientation for γ (see the proof of Theorem 3.8 in [26]). Therefore, the (positive) earthquake path and the pleating locus (with respect to γ) form a right angle at the basepoint. Hence, it suffices to show that different earthquake paths starting from the basepoint have different directions. This assertion has been proved by Kerckhoff [16, Theorem 3.5] in a more general setting. $\qquad\square$

We conclude this section with a comment on the numerical computation of the accessory parameter and the Fuchsian group uniformizing a given once-punctured torus.

In the above, we have assumed that we know the value of the accessory parameter. However, in practical calculations, we need to know it in advance. We now indicate how to do that by using our methods developed above. Let $\lambda \in \mathbb{C} \setminus \{0, 1\}$

be given. Instead of (4.5), we consider the differential equation

$$(6.3) \qquad 2y'' + \left\{ \frac{1}{2z^2(z-1)^2} + \frac{1}{2(z-\lambda)^2} + \frac{t}{z(z-1)(z-\lambda)} \right\} y = 0.$$

We want to find the value $t = c(\lambda)$ for the given λ. Let α_0 and α_∞ be the same as in Section 4 and let $\hat{L}_t[\alpha_r]$ be the transition matrix of (6.3) along the path α_r for $r = 0, \infty$. We set $A(t) = \hat{L}_t[\alpha_0]M_{z_0}(A_0)$ and $B(t) = \hat{L}_t[\alpha_\infty]M_{z_0}(A_\infty)$, where $M_{z_0}(A_r)$ is given in Section 4. Then $F_0(t) = \operatorname{tr} A(t)$, $F_\infty(t) = \operatorname{tr} B(t)$ and $F_1(t) = \operatorname{tr} A(t)B(t)$ are all entire functions. Let $\Gamma(t)$ be the Möbius group generated by $A(t)$ and $B(t)$. Theorem 4.3 implies that $\Gamma(c(\lambda))$ is Möbius conjugate to the original Fuchsian group Γ uniformizing the once-punctured torus X corresponding to τ where $\lambda = \lambda(\tau)$. Therefore, if $t = c(\lambda)$, the following hold:

$$(6.4) \qquad F_0(t), F_\infty(t), F_1(t) \in (-\infty, -2) \cup (2, +\infty).$$

Conversely, if $t \in \mathbb{C}$ satisfies (6.4), t is a candidate for the value of the accessory parameter $c(\lambda)$. In fact, for t with $(t - c(\lambda))\psi_0 \in T(Y)$, $\Gamma(t)$ is never Fuchsian unless $t = c(\lambda)$. Recalling the fact that $\Gamma(t)$ is conjugate to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ precisely when the values $F_0(t)$, $F_\infty(t)$ and $F_1(t)$ are all real, we see that $\Gamma(t)$ is Fuchsian if and only if (6.4) hold under the assumption that $(t - c(\lambda))\psi_0 \in T(Y)$. Note also that $|t - c(\lambda)| < 2/\|\psi_0\|_Y$ implies $(t - c(\lambda))\psi_0 \in T(Y)$ by the Ahlfors-Weill theorem (see Section 2). Therefore, we can conclude that $t = c(\lambda)$ from condition (6.4) if we are certain that $|t - c(\lambda)| < 2/\|\psi_0\|_Y$.

Hence, if we are given an initial point t_0 which is sufficiently close to $c(\lambda)$, we can construct a sequence t_n tending to the value $c(\lambda)$ as follows. By Theorem 6.3, we can choose two of F_0, F_∞, F_1, say F_0 and F_1, so that the corresponding pleating rays $\mathcal{P}_0$ and $\mathcal{P}_1$ are transversal at the basepoint. Then the point $t = c(\lambda)$ will be determined (at least locally) as the intersection of the real loci of functions F_0 and F_1. Set

$$t_{2j+1} = t_{2j} - i\frac{\operatorname{Im} F_0(t_{2j})}{F_0'(t_{2j})}$$

and

$$t_{2j+2} = t_{2j+1} - i\frac{\operatorname{Im} F_1(t_{2j+1})}{F_1'(t_{2j+1})}$$

for $j = 0, 1, \ldots$. Then, in a similar way to Newton's method, the sequence t_n converges to $c(\lambda)$ if t_0 is sufficiently close to $c(\lambda)$ (see, for example, [31, § 3.5.2]). Note here that the transversality of the pleating rays guarantees the convergence of t_n.

We make a few technical remarks. Since it is difficult to calculate the derivative of F_r practically, we replace it by a suitable difference quotient in the above formulae like the Secant method. It is typical to use $(F_r(t_n) - F_r(t_{n-2}))/(t_n - t_{n-2})$ as the difference quotient provided that $t_n - t_{n-2}$ is sufficiently small.

It is difficult to give the initial point t_0 a priori for a given λ. However, if we know the value of $c(\lambda_0)$ for some λ_0 (e.g., $\lambda_0 = 1/2$), we may choose a finite sequence $\lambda_1, \lambda_2, \ldots, \lambda_n = \lambda$ so that λ_j and λ_{j-1} are close enough for $j = 1, \ldots, n$. Then, we could compute $c(\lambda_j)$ by using the value of $c(\lambda_{j-1})$ as the initial point t_0 for λ_j. In this way, we could reach λ after n-times of this procedure.
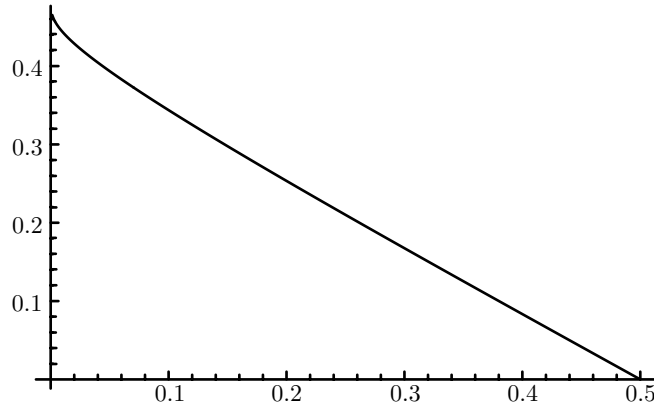
FIGURE 3. Graph of the accessory parameter $c(\lambda)$ in $0 < \lambda < 1/2$

Using the above method, we have drawn the graph of the function $c(\lambda)$ in $0 < \lambda < 1/2$ in Figure 3. Note that Hempel [8] has obtained the asymptotic formula

$$c(\lambda) = \frac{1}{2} - \frac{\pi^2}{2(-\log|\lambda|)^2}(1 + o(1))$$

as $\lambda \to 0$ in $\mathbb{C}$.

7. RELATIONSHIP WITH HEUN'S DIFFERENTIAL EQUATION

In this section, we explain that our differential equation (4.5) can be translated to Heun's differential equation in the standard way. It may be more advantageous to use the latter form because that is more widely studied and the behaviour of coefficients is tamer than the former near the singularities (see [33] and its extensive references).

Heun's equation has the form

$$u'' + \left(\frac{c_0}{z} + \frac{c_1}{z-1} + \frac{c_2}{z-\lambda}\right)u' + \frac{abz - q}{z(z-1)(z-\lambda)}u = 0,$$

where c_0, c_1, c_2, a, b and q are complex parameters satisfying the relation

$$c_0 + c_1 + c_2 = a + b + 1.$$

More generally, a linear differential equation of the form $u'' + Pu' + Qu = 0$ can be transformed locally to the form $y'' + Ry = 0$ by putting $u(z) = \exp(-\int^z P(\zeta)d\zeta/2)y(z)$. In the case when $c_0 = c_1 = c_2 = a = b = 1$ and $q = (1 - t - c(\lambda))/2$, Heun's equation takes the form

$$(7.1) \qquad u'' + \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-\lambda}\right)u' + \frac{2z + t + c(\lambda) - 1}{2z(z-1)(z-\lambda)}u = 0$$

and the above-mentioned transformation

$$u = \frac{y}{\sqrt{z(z-1)(z-\lambda)}}$$

reduces Heun's equation to our equation (4.5). It is an interesting fact that the factor $\omega(z) := 1/\sqrt{z(z-1)(z-\lambda)}$ forms a basis of the one-dimensional vector space of Abelian differentials on the torus defined as the algebraic curve $w^2 =$

$z(z-1)(z-\lambda)$. In particular, $(\omega \circ A) \cdot A' = \pm\omega$ holds for an automorphism A of $Y = \mathbb{C} \setminus \{0, 1, \lambda\}$.

Therefore, for a solution u of (7.1) and $A \in \mathrm{Aut}(Y)$, we can see that $A_{1/2}^* u := (u \circ A)(A')^{1/2}$ is a solution of (7.1) again.

We now set

$$\widetilde{F}_{z_1} = \begin{pmatrix} u_1' & u_1 \\ u_2' & u_2 \end{pmatrix},$$

where u_1 and u_2 are the fundamental solutions of (7.1) at z_1, namely, the solutions satisfying the initial conditions $u_1(z_1) = 0$, $u_1'(z_1) = 1$ and $u_2(z_1) = 1$, $u_2'(z_1) = 0$, respectively. For $[\alpha] \in Y[z_1, z_2]$ we denote by $\widetilde{H}_{[\alpha]}$ the germ at z_2 obtained by analytic continuation of the fundamental matrix $\widetilde{F}_{z_1}$ along the path α in Y. In the same way as in the case of (4.5), we also define the transition matrix $\widetilde{L}[\alpha] = \widetilde{L}_t[\alpha]$ by the relation $\widetilde{H}_{[\alpha]} = \widetilde{L}[\alpha]\widetilde{F}_{z_2}$ for $[\alpha] \in Y[z_1, z_2]$. Then, the similar relations to (i), (ii) and (iii) in Section 3 hold for these quantities. Noting the elementary identity

$$\omega \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \omega(z_1) & 0 \\ \omega'(z_1) & \omega(z_1) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = K(z_1) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

for the fundamental solutions of (4.5) and (7.1) at z_1, where

$$K(z) = \omega(z) \begin{pmatrix} 1 & 0 \\ \mu(z) & 1 \end{pmatrix}, \quad \mu(z) = -\frac{1}{2}\left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-\lambda}\right),$$

we have the fundamental relation $F_{z_1}K = K(z_1)\widetilde{F}_{z_1}$ near $z_1 \in Y$. By analytic continuation, we obtain $H_{[\alpha]}K(z_2) = K(z_1)\widetilde{H}_{[\alpha]}$ for $[\alpha] \in Y[z_1, z_2]$, in particular,

(7.2) $$L[\alpha] = K(z_1)\widetilde{L}[\alpha]K(z_2)^{-1}.$$

Let z_1 be a point in $Y = \mathbb{C} \setminus \{0, 1, \lambda\}$. We now take curves $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 in Y starting from z_1, rounding $0, 1, \lambda$ and ∞, respectively, once anticlockwise and ending at z_1, so that $\pi_1(Y, z_1) = \langle[\varepsilon_1], [\varepsilon_2], [\varepsilon_3], [\varepsilon_4]\rangle$ and $[\varepsilon_1][\varepsilon_2][\varepsilon_3][\varepsilon_4] = 1$. Then, the next result immediately follows.

Theorem 7.1. *Let l_t and $\tilde{l}_t$ be monodromy homomorphisms of differential equations (4.5) and (7.1) from $\pi_1(Y, z_1)$ into $\mathrm{SL}(2, \mathbb{C})$, respectively. We then have $l_t[\varepsilon_j] = -\tilde{l}_t[\varepsilon_j]$ for $j = 0, 1, 2, 3$. In particular, $l_t = \tilde{l}_t$ on the canonical image of the Fuchsian group $\Gamma_0 = \Gamma \cap \Gamma'$.*

Corollary 7.2. *The monodromy group for (4.5) is discrete if and only if the monodromy group for (7.1) is.*

Since $L[\alpha_r]M_{z_0}(A_r)$ is conjugate with $\widetilde{L}[\alpha_r]\widetilde{M}_{z_0}(A_r)$ in $\mathrm{GL}(2, \mathbb{C})$, where $\widetilde{M}_{z_0}(A_r) = K(A_r(z_0))^{-1}M_{z_0}(A_r)K(z_0)$ for $r = 0, \infty$, we obtain the following statement from Theorem 4.3.

Theorem 7.3. *For suitable choices of representatives $\tilde{\theta}_0$ of θ_0 and $\tilde{\theta}_\infty$ of θ_∞ in $\mathrm{SU}(1, 1)$,*

$$\tilde{\chi}_{t\varphi_0}(\tilde{\theta}_0) = \widetilde{L}_t[\alpha_0]\widetilde{M}_{z_0}(A_0),$$

and

$$\tilde{\chi}_{t\varphi_0}(\tilde{\theta}_\infty) = \widetilde{L}_t[\alpha_\infty]\widetilde{M}_{z_0}(A_\infty)$$

for any $t \in \mathbb{C}$ up to $\mathrm{SL}(2, \mathbb{C})$-conjugacy, where $\varphi_0 = p_2^*(\psi_0)$.

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