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Abstract

We analyze a Bertrand-Edgeworth game in homogeneous product industry, under efficient rationing, constant marginal cost until full capacity utilization, and identical technology across firms. We solve for the equilibrium and establish its uniqueness for capacity configurations in the mixed strategy region of the capacity space such that the capacities of the largest and smallest firm are sufficiently close.

Keywords: Bertrand-Edgeworth competition, mixed strategy equilibrium, almost symmetric oligopoly, Mixed strategy equilibrium

JEL Classification: C72, D43, L13

1 Introduction

The analysis of price competition among capacity-constrained sellers of a homogeneous product (Bertrand-Edgeworth competition) has received considerable attention over recent years. Classic studies of duopoly under efficient rationing and constant (and identical across firms) unit cost below capacity may be found in Kreps and Scheinkman (1983) and Osborne and Pitchik (1986), the latter also establishing uniqueness of equilibrium in the mixed strategy region of the capacity space. More recently, De Francesco and Salvadori (2010) provided a complete characterization of equilibria under triopoly besides pointing out some general properties of equilibria under oligopoly. (For the triopoly, see also Hirata, 2009.) Concerning oligopoly, however, determination of mixed strategy equilibria when the price game does not possess pure strategy equilibria is only available for special cases. In this paper we provide a complete analysis of another, significant case, that of an almost symmetric oligopoly. This complements Vives (1986), who determined the symmetric mixed strategy equilibrium in a symmetric oligopoly while leaving open the question of whether asymmetric mixed strategy equilibria also exist.[1] On the contrary, we prove that the equi-

[1] "Given n firms and restricting attention to symmetric equilibria..." (Vives, 1986, p. 114).

librium we find, which collapses to that of Vives when firms are equally sized, is unique.[2]

2 Preliminaries

There are n firms $(1, 2, ..., n)$ producing an homogeneous commodity with given capacities. For each firm i, production cannot exceed its capacity k_i and marginal cost is identical across the firms and constant until full capacity utilization is reached (with no loss of generality marginal cost is assumed to be 0). Let $k_1 \geqslant k_2 \geqslant ... \geqslant k_n$ and set $K = k_1 + ... + k_n$. Let $D(p)$ be the market demand function: $D(p) > 0$, $D'(p) < 0$, and $D''(p) \leqslant 0$ for $p \in (0, \overline{p})$ and $D(p) = 0$ for $p \geqslant \overline{p}$. Let also $P(x) = D^{-1}(x)$ for $x \in [0, D(0))$ and $P(x) = 0$ for $x \geqslant D(0)$. The efficient rationing rule is assumed to hold; further, in the case of a price tie demand is shared among equally-priced firms in proportion to capacity, such that the residual demand accruing to firm i is $d_i(p_i, p_{-i}) = \max\{0, D(p_i) - \sum_{j:p_j<p_i} k_i\} \times \frac{k_i}{\sum_{r:p_r=p_i} k_r}$ and profit is $\pi_i(p_i, p_{-i}) = p_i \min\{d_i(p_i, p_{-i}), k_i\}$.

Denote by p^c the competitive price. Clearly, $p^c = P(K)$ if $D(0) \geqslant K$ and $p^c = 0$ if $D(0) \leqslant K$. It has been proved (see, for instance, De Francesco and Salvadori, 2010) that $(p_1, ..., p_n) = (p^c, ..., p^c)$ is an equilibrium of the price game if and only if

$$K - k_1 \geqslant D(0) \text{ when } D(0) \leqslant K, \tag{1}$$

or

$$k_1 \leqslant -p^c \left[D'(p)\right]_{p=p^c} \text{ when } D(0) > K. \tag{2}$$

Holding either condition, the firms get the competitive profit at any equilibrium; furthermore, $(p^c, ..., p^c)$ is the unique equilibrium when $D(0) > K$. Failing (1) and (2), no pure strategy equilibrium exists whereas a mixed strategy equilibrium necessarily exists.

In the following, we denote by $(\phi_1(p), ..., \phi_n(p)) = (\phi_i(p), \phi_{-i}(p))$ an equilibrium profile of mixed strategies, where $\phi_i(p) = \Pr(p_i < p)$ is the probability of firm i charging less than p. For brevity, we denote firm i's expected profit at mixed strategy equilibrium $(\phi_i(p), \phi_{-i}(p))$ as Π_i^* (rather than $\Pi_i^*(\phi_i(p), \phi_{-i}(p))$ and denote by $\Pi_i(p, \phi_{-i}(p))$ firm i's expected profit when it charges p against equilibrium strategy profile $\phi_{-i}(p)$ on the part of its rivals. Let S_i be the support of $\phi_i(p)$ and $p_M^{(i)}$ and $p_m^{(i)}$ the supremum and the infimum of S_i, respectively. More precisely, $p \in S_i$ when $\phi_i(\cdot)$ is increasing in p, i. e., when $\phi_i(p+h) > \phi_i(p-h)$ for any $h \in (0, p)$. Besides being non-decreasing, $\phi_i(p)$ is continuous except at p° such that $\Pr(p_i = p^\circ) > 0$. We also define $p_M = \max_i p_M^{(i)}$, $p_m = \min_i p_m^{(i)}$, $M = \{i : p_M^{(i)} = p_M\}$, and $L = \{i : p_m^{(i)} = p_m\}$.

[2] Another special case was provided by Davidson and Deneckere (1984) who analyzed the case of a single large firm and several equally-sized small firms: not dissimilarly from Vives, they focused on equilibria that are symmetric as far as smaller firms are concerned.

Clearly $\Pi_i^* \geqslant \Pi_i(p, \phi_{-i}(p))$ for any $p > 0$; furthermore, $\Pi_i^* = \Pi_i(p, \phi_{-i}(p))$ almost everywhere for $p \in S_i$. The following Proposition lists some general properties of mixed strategy equilibrium to be used in the next section.

Proposition 1 *1.* $p_M = \arg\max_p p(D(p) - \sum_{j\neq 1} k_j)$.

2. $\Pi_i^* = \max_p p(D(p) - \sum_{j\neq 1} k_j)$ *for any* i *such that* $k_i = k_1$.

3. $p_m = \max\{\widehat{p}, \widehat{\widehat{p}}\}$ *where* $\widehat{p} = \Pi_1^*/k_1$ *and* $\widehat{\widehat{p}}$ *is the lower solution of equation* $pD(p) = \Pi_1^*$*;* $p_m^{(i)} = p_m$ *for any* i *such that* $k_i = k_1$.

4. $\Pi_i^* = \Pi_i(p, \phi_{-i}(p)$ *for any* p *internal to* S_i *and any* i*, so that* $\Pr(p_i = p) = 0$ *for any* $p \in (p_m, p_M)$.

All these points were made for the duopoly by Kreps and Scheinkman (1983). For an extension to oligopoly, see De Francesco and Salvadori (2010) and the references contained therein.

3 Almost symmetric oligopoly

When a pure strategy equilibrium does not exist, $p_M < P(k_2 + ... + k_n)$. We define an *almost symmetric oligopoly* as a capacity configuration such that k_1 is so close to k_n that $p_M \leqslant P(k_1 + ... + k_{n-1})$.[3]

Proposition 2 *Let* $k_1 \geqslant k_2 \geqslant ... \geqslant k_n$ *and* $p_M \leqslant P(k_1 + ... + k_{n-1})$*. Then:*

(i) $L = \{1, 2, ..., n\}$ *and* $\Pi_j^* = p_m k_j$ *for any* j.

(ii) There exists an equilibrium where supports of equilibrium strategies are $S_i = [p_m, p_M^{(i)}]$*, each* $i = 1, ..., n$*, where* $p_M^{(i)}$ *is solution to the equation in* p

$$\frac{(p - p_m)\prod_{s\leqslant i} k_s}{p[K - D(p)]k_i^{i-1}} = 1, \tag{3}$$

and distributions are

$$\phi_j(p) = \frac{1}{k_j}\left[\frac{(p - p_m)\prod_{s=1}^{n} k_s}{p(K - D(p))}\right]^{1/(n-1)} \quad p \in [p_m, p_M^{(n)}], \tag{4.j}$$

each $j = 1, ..., n$*,*

$$\phi_j(p) = \frac{1}{k_j}\left[\frac{(p - p_m)\prod_{s=1}^{i} k_s}{p(K - D(p))}\right]^{1/(i-1)} \quad p \in [p_M^{(i+1)}, p_M^{(i)}] \tag{5.j.i}$$

each $j = 1, ..., i,$ *each* $i = 2, ..., n - 1$.

(iii) No other equilibrium exists.

[3] In fact, this condition holds if $D(p_M) - (k_2 + ... + k_n) \geq k_1 - k_n$.

Remark

(a) $p_M^{(i+1)} = p_M^{(i)}$ whenever $k_{i+1} = k_i$, each $i = 2, ..., n$.

(b) $\frac{p-p_m}{p[K-D(p)]}$ is increasing in p;[4] hence $p_M^{(i+1)} < p_M^{(i)}$ whenever $k_{i+1} < k_i$, each $i = 2, ..., n$.

(c) $p_M^{(1)} = p_M^{(2)} = p_M$.

(d) $\phi_j(p_M^{(i)}) = \frac{k_i}{k_j}$ for $j \in \{1, ..., i\}$, each $i = 2, 3, ..., n$.

Proof. (of Proposition 2)

(i) If $\#L < n$, then on a neighbourhood of p_m we would have $\Pi_i(p, \phi_{-i}(p)) = pk_i$ for any $i \in L$, contrary to the constancy of $\Pi_i(p, \phi_{-i}(p))$ in S_i.[5] Therefore, $\Pi_i^* = p_m k_i$ for any i.

(ii) Because of part (i), at any equilibrium, $p_m k_j = \left[\prod_{s\neq j} \phi_s(p)\right] p[D(p) - \sum_{s\neq j} k_s] + \left[1 - \prod_{s\neq j} \phi_s(p)\right] pk_j$ on a neighbourhood of p_m. Hence

$$(p - p_m)k_j = p[K - D(p)] \prod_{s\neq j} \phi_s(p) \text{ for any } j = 1, ..., n, \tag{6}$$

implying that

$$\phi_j(p) = \frac{k_i}{k_j} \phi_i(p) \text{ for any } i, j = 1, ..., n \tag{7}$$

over such a neighbourhood. It follows from (6) and (7) that $(p - p_m)k_j = p[K - D(p)][\phi_j(p)]^{n-1} \prod_{s\neq j} \frac{k_j}{k_s}$. Hence, at any equilibrium, equations (4.j) hold on a neighbouhood of p_m. Since all S_j are assumed to be connected, equations (4.j) hold up to $p = p_M^{(n)}$, namely, the price equating to 1 the RHS of equation (4.n) (and any equation (4.j) such that $k_j = k_n$). Let h be the number of firms with capacity k_n. Since all S_j are assumed to be connected there exists a right neighbourhood of $p_M^{(n)}$ that is contained in $S_1 \cap ... \cap S_{n-h}$. Arguing as above, $p_m k_j = \left[\prod_{s\neq j} \phi_s(p)\right] p[D(p) - \sum_{s\neq j} k_s] + \left[1 - \prod_{s\neq j} \phi_s(p)\right] pk_j$ for any $j \in \{1, ..., n-h\}$ over such a neighbourhood: hence $\phi_j(p) = \frac{k_r}{k_j} \phi_r(p)$ for any $j, r \in \{1, ..., n-h\}$ and equations (5.j.n-h) hold. (Because of Remark 3(d), equations (5.j.i) also hold for $i \in \{n-h+1, ..., n-1\}$, the intervals $[p_M^{(i+1)}, p_M^{(i)}]$ being degenerate.) Iteration of this procedure will finally lead to equations (5.j.2).

Remark (b) ensures that for each j, $\phi_j'(p) > 0$ throughout $(p_m, p_M^{(j)})$. It must also be checked that, for any j such that $p_M^{(j)} < p_M$, $\Pi_j(p, \phi_{-j}(p)) \leqslant p_m k_j$ for $p_M^{(j)} < p < p_M$. Indeed, should it be $\Pi_j(p', \phi_{-j}(p')) > \Pi_j^*$ for some $p' \in$

[4] This is so if and only if $p^2 D'(p) + p_m[K - D(p) - pD'(p)] > 0$. Note that $pD'(p) = \sum_{j\neq 1} k_j - D(p) + \delta$ with $\delta > 0$ for $p \in (p_m, p_M)$. Hence the required condition becomes $[\Pi_1^* - p(D(p) - \sum_{j\neq 1} k_j)] + [(p - p_m)\delta] > 0$, which holds true since the expression in each square bracket is positive.

[5] This property of mixed strategy equilibria in the given circumstances had already been found by Hirata (2009, p. 7).

$(p_M^{(j)}, p_M)$, we would have

$$p'k_j - p'[K - D(p')] \prod_{s \neq j} \phi_s(p') > \Pi_j^* = \frac{k_j}{k_2} \Pi_2^* = \frac{k_j}{k_2} \left[p'k_2 - p'[K - D(p')] \prod_{s \neq 2} \phi_s(p') \right],$$

and hence

$$\left[p'[K - D(p')] \prod_{s \neq 2} \phi_s(p') \right] \left[\frac{k_j}{k_2} - \phi_2(p') \right] > 0,$$

since $\phi_j(p') = 1$. This inequality is a contradiction since $\phi_2(p') > \phi_2(p_M^{(j)}) = \frac{k_j}{k_2}$.

(iii) Assume that another equilibrium $(\phi_1{}^\circ(p), ..., \phi_n{}^\circ(p))$ exists and let S_1°, $S_2^\circ, ..., S_n^\circ$ be the supports of the equilibrium strategies. Since part (i) holds, if the supports are connected, then $S_i^\circ = S_i$ and $\phi_i^\circ(p) = \phi_i(p)$, each $i = 1, ..., n$, contrary to the assumption. Hence S_h° is not connected for some h.[6] Let $(p^\circ, p^{\circ\circ})$ be a gap in S_h° and with no loss of generality take all supports to be connected in the range $[p_m, p^\circ]$ so that $k_i \phi_i^\circ(p^\circ) = k_j \phi_j^\circ(p^\circ)$ for each i, j such that $p_M^{(i)}, p_M^{(j)} > p^\circ$. Further, assume that $p' \in S_j^\circ$ for some $p' \in (p^\circ, p^{\circ\circ})$ and some firm $j \neq h$. Then, $\Pi_j(p', \phi_{-j}{}^\circ(p')) = \Pi_j^* = p_m k_j$ whereas $\Pi_h(p', \phi_{-h}{}^\circ(p')) \leqslant \Pi_h^* = p_m k_h$, implying that $k_h \phi_h^\circ(p') \leqslant k_j \phi_j^\circ(p')$. Moreover, since $k_h \phi_h^\circ(p^\circ) = k_j \phi_j^\circ(p^\circ)$ and $\phi_h^\circ(p)$ is constant over $(p^\circ, p^{\circ\circ})$ whereas $\phi_j^\circ(p)$ is somewhere increasing over that range, we obtain that $k_h \phi_h^\circ(p^{\circ\circ}) < k_j \phi_j^\circ(p^{\circ\circ})$. Hence a contradiction since either $k_h \phi_h^\circ(p^{\circ\circ}) = k_j \phi_j^\circ(p^{\circ\circ})$ or $k_h \phi_h^\circ(p^{\circ\circ})_j \phi_j^\circ(p^{\circ\circ})$ according to whether $p^{\circ\circ} \in S_j^\circ$ or $p^{\circ\circ} \notin S_j^\circ$, respectively. Since there is no firm $j \neq h$ such that $p' \in S_j^\circ$ for any $p' \in (p^\circ, p^{\circ\circ})$, $(p^\circ, p^{\circ\circ})$ is a gap (or part of a gap) for all i such that $p_M^{(i)} > p^\circ$. This leads to the following contradiction, $\Pi_1(p, \phi_{-1}^\circ(p)) > \Pi_1(p^\circ, \phi_{-1}^\circ(p^\circ)) = \Pi_1^*$ for $p \in (p^\circ, p^{\circ\circ})$.[7] Thus no support can have a gap and hence no other equilibrium exists. ■

To sum up, Proposition 2 determines the equilibrium and establishes its uniqueness in the subset of the mixed strategy region of the capacity space where $p_M \leqslant P(k_1 + ... + k_{n-1})$.[8] It should be emphasized that this is a sufficient condition for uniqueness: the equilibrium is still unique in other (though not all) subsets.[9] This can be seen most simply by showing that the equilibrium is characterized as in Proposition 2 when inequality $p_M \leqslant P(k_1 + ... + k_{n-1})$ is slightly relaxed. Suppose that $p_m \leqslant P(k_1 + ... + k_{n-1})$. Then, by reasoning as in the proof of Proposition 2, at any equilibrium $\#L = n$ and equations

[6] The possibility of equilibria with gaps cannot be ignored. De Francesco and Salvadori (2010) find conditions under which the support of the equilibrium strategy of one firm does have a gap.

[7] Indeed, $d\Pi_1/dp = \prod_{j \neq 1} \phi_j(p) \left[D(p) - \sum_{j \neq 1} k_j + pD'(p) \right] + k_1 \left[1 - \prod_{j \neq 1} \phi_j(p) \right]$: this is positive since $D(p) - \sum_{j \neq 1} k_j + pD'(p) > 0$.

[8] A condition that is met, for example, when $D(p) = 20 - p$, $n = 4$, $k_1 = 6, k_2 = 5, k_3 = 3$, and $k_4 = 2$.

[9] The whole subset of the mixed strategy region where equilibrium is unique has been found, for the triopoly, by De Francesco and Salvadori (2010); in the remaining subset the equilibrium is indeterminate as far as the distributions of smaller firms are concerned.

(4.j) hold on a neighbourhood of p_m. Next, denote by $\widetilde{p}_M^{(i)}$ (each $i = 1, ..., n$) the solution of equation (3) over the range (p_m, p_M), let $l = min\{i : \widetilde{p}_M^{(i)} \leqslant P(k_1 + ... + k_{n-1})\}$, and assume that $p_M \leqslant P(\sum_{j \neq l-1} k_j)$.[10] Then, by reasoning as in the proof of Proposition 2, it turns out that $p_M^{(i)} = \widetilde{p}_M^{(i)}$, each $i = 1, ..., n$, and that distributions are given by equations (4.j) and (5.j.i) throughout $[p_m, p_M]$. This is immediate as far as any $i \in \{l, ..., n\}$ is concerned. As for any $i \in \{1, ..., l-1\}$ (in the event of $l > 2$), $\Pi_i(p, \phi_{-i}(p)) = \left[\prod_{j \neq i} \phi_j(p)\right] p[D(p) - K] + \left[1 - \prod_{j \neq i} \phi_j(p)\right] pk_j$ for $p \in [p_M^{(l)}, p_M]$ since $D(p) > \sum_{j \neq l-1} k_j$. Hence equations (5.j.i) hold throughout that range.

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[10] As can be checked, all this holds if, for example, we let $k_1 = 8$ while leaving other data as in footnote 8.